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The size of some classes of thin sets

by

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Abstract. The size of a class of subsets of the circle is reflected by the family of measures which annihilate all the sets belonging to the given class. For subclasses of U_0, the sets of uniqueness in the wide sense, the corresponding family of annihilating measures always includes $M_0(T)$. We investigate when there are no other annihilating measures, in which case the class of sets is "large". For example, Helson sets are shown not to form a large class, while a closely related class does. The fact that another class of sets, the H-sets, is "small" disproves a conjecture of Rajchman. The class of sets of uniqueness (in the strict sense) is investigated in detail. Tools used include Riesz products and asymptotic distribution.

1. Introduction. Borel subsets of the circle $\boldsymbol{T} = \boldsymbol{R}/\boldsymbol{Z}$ which are called "thin" in harmonic analysis are usually sets of uniqueness in the wide sense, or U_0-sets [10]. Recall that a *U_0-set* is a (Borel) set which has zero measure with respect to every measure belonging to $M_0(\boldsymbol{T}) = \{\mu \in M(\boldsymbol{T})\colon \lim_{|n|\to\infty} \hat{\mu}(n) = 0\}$, where $M(\boldsymbol{T})$ denotes the (finite) complex Borel measures on $\boldsymbol{T}$ and $\hat{\mu}(n) = \int_{\boldsymbol{T}} e^{-2\pi i n t}\, d\mu(t)$. We also denote $M_0(\boldsymbol{T})$ by R. Given two classes of thin sets $\mathscr{C}_1, \mathscr{C}_2 \subset U_0$, we may consider $\mathscr{C}_1$ to be "much larger" than $\mathscr{C}_2$ if there is a measure concentrated on some set from $\mathscr{C}_1$ which annihilates every set in $\mathscr{C}_2$, but not *vice versa*. This is equivalent to the statement $\mathscr{C}_1^{\perp} \subsetneqq \mathscr{C}_2^{\perp}$, where we denote

$$\mathscr{C}^{\perp} = \{\mu \in M(\boldsymbol{T})\colon\ \forall E \in \mathscr{C}\ |\mu|(E) = 0\}.$$

In this case, it is not hard to see that every measure concentrated on a set from $\mathscr{C}_2$ is also concentrated on a countable union of sets from $\mathscr{C}_1$.

Now for any class $\mathscr{C} \subset U_0$, we have $R \subset U_0^{\perp} \subset \mathscr{C}^{\perp}$. In fact, $R = U_0^{\perp}$ [11, 12, 13]; we shall be interested here in seeing whether certain other classes $\mathscr{C}$ share this property ($\mathscr{C}^{\perp} = R$). Such classes are as "large" as U_0 itself. This investigation was begun in [11].

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Riesz products will be our tools for showing the smallness of certain classes. We shall see how Riesz products resemble the measures in R by virtue of the gaps in and the multiplicative structure of their spectrum.

2. $H^{(m)}$-sets. We shall write $e(t)$ for $e^{2\pi it}$. Recall that a Borel set $E \subset \boldsymbol{T}$ is called a *set of uniqueness*, or *U-set*, if the only trigonometric series $\sum_{n=-\infty}^{\infty} c_n e(nt)$ which converges to 0 for all $t \notin E$ is the 0-series: $c_n \equiv 0$. It has long been known that U-sets are U_0-sets and that countable unions of closed U-sets are U-sets ([20, I, Chap. IX, § 6]). Rajchman [18, 19] introduced the first class of uncountable U-sets, which he called *H-sets*. These are Borel sets $E \subset \boldsymbol{T}$ for which there exist a sequence $\{n_k\}$ of positive integers tending to ∞ and a nonempty open arc $I \subset \boldsymbol{T}$ such that for all $x \in E$ and all k, $n_k x \notin I$. It is clear that H-sets are contained in closed H-sets, hence that countable unions of H-sets, denoted H_σ-sets, are also U-sets. The converse, conjectured by Rajchman, was finally shown to be false by Pyatetskii--Shapiro, who introduced the classes $H^{(m)}$ ([17], [1, Chap. XIV, §§ 15, 16], [20, I, Chap. IX, § 6]). We have $H = H^{(1)} \subset H^{(2)} \subset \ldots \subset H^{(m)} \subset H^{(m+1)} \subset \ldots \subset U \subset U_0$, but, for each m, there is an $H^{(m+1)}$-set which cannot be written as a countable union of $H^{(m)}$-sets. It would be interesting to know if $H^{(m+1)}$ is in fact larger than $H^{(m)}$ in the sense given in the introduction.

Rajchman also conjectured (see [3, pp. 85–86]) that $R = H^{\perp}$, but this too is false [11, 12]. In fact, in [11, § III.8], we showed that $R \neq (H^{(m)})^{\perp}$ for any m. Here we shall use an entirely different approach to the problem and shall show that

$$R \neq \big(\bigcup_m H^{(m)}\big)^{\perp}. \tag{1}$$

We recall some definitions. If $V = (v^{(1)}, \ldots, v^{(m)}) \in \boldsymbol{Z}^m$, $\Lambda = (l_1, \ldots, l_m) \in \boldsymbol{Z}^m$ and $x \in \boldsymbol{T}$, we write $V \cdot \Lambda = \sum_{i=1}^{m} v^{(i)} l_i$ and $Vx = (v^{(1)}x, \ldots, v^{(m)}x)$.

Definitions. Let $m \in \boldsymbol{Z}^+$. A sequence $\{V_k\}_{k=1}^{\infty} \subset (\boldsymbol{Z}^+)^m$ of m-tuples of positive integers is called *quasi-independent* if for each fixed $\Lambda \in \boldsymbol{Z}^m$, Λ not the 0-vector, we have $|V_k \cdot \Lambda| \to \infty$ as $k \to \infty$. A Borel set $E \subset \boldsymbol{T}$ is called an *$H^{(m)}$-set* if there is a quasi-independent sequence $\{V_k\} \subset (\boldsymbol{Z}^+)^m$ and a nonempty open set $I \subset \boldsymbol{T}^m$ such that for all $x \in E$ and all k, $V_k x \notin I$. A *box* $I \subset \boldsymbol{T}^m$ is a Cartesian product of arcs $I_j \subset \boldsymbol{T}$: $I = I_1 \times I_2 \times \ldots \times I_m$. A sequence $\{x_k\}_{k=1}^{\infty} \subset \boldsymbol{T}^m$ has the *asymptotic distribution* $\nu \in M(\boldsymbol{T}^m)$, written $\{x_k\} \sim \nu$, if for every box $I \subset \boldsymbol{T}^m$ whose boundary has ν-measure 0, we have

$$\lim_{K \to \infty} \frac{1}{K} \operatorname{card}\{k \leqslant K: x_k \in I\} = \nu I.$$

Recall that Weyl's criterion [20, I, Chap. IV, (4.25)] says that $\{x_k\} \sim \nu$ if and only if for all $l_1, \ldots, l_m \in \boldsymbol{Z}$,

$$\lim_{K \to \infty} \frac{1}{K} \sum_{k \leqslant K} e(l_1 x_k^{(1)} + \ldots + l_m x_k^{(m)}) = \hat{\nu}(-l_1, \ldots, -l_m),$$

where $x_k = (x_k^{(1)}, \ldots, x_k^{(m)})$.

Theorem 1. *Let*

$$\mu = \prod_{k=1}^{\infty} \big(1 + \operatorname{Re}\{\alpha_k e(n_k x)\}\big)$$

be a Riesz product with $|\alpha_k| \leqslant 1$ and $n_{k+1}/n_k \to \infty$. Given any quasi-independent sequence $\{V_j\}_{j=1}^{\infty} \subset (\boldsymbol{Z}^+)^m$, there exist a subsequence $\{V_j'\}$ and a set $D \subset \boldsymbol{Z}^m$ of cardinality at most 3^m such that for μ-almost all x, $\{V_j' x\}_{j=1}^{\infty}$ has an asymptotic distribution ν_x with spectrum in D: $\hat{\nu}_x(\Lambda) = 0$ if $\Lambda \notin D$.

The result (1) now follows: let μ be as in the theorem with $\alpha_k \not\to 0$; then $\mu \notin R$. If E is any $H^{(m)}$-set, there is a quasi-independent sequence $\{V_j\}_{j=1}^{\infty} \subset (\boldsymbol{Z}^+)^m$ such that $\{V_j x\}$ is not dense in $\boldsymbol{T}^m$ for any $x \in E$. But if $\{V_j'\}$ is the subsequence given by the theorem, then for μ-almost every x, $\{V_j' x\}$ has the distribution of a trigonometric polynomial and hence is dense. Therefore $\mu E = 0$ and so $\mu \in \big(\bigcup_m H^{(m)}\big)^{\perp}$.

In order to prove Theorem 1, we need two lemmas (which are easy in the case $m = 1$; indeed, a stronger form of Lemma 3 will be proved for $m = 1$ in the course of proving Theorem 6).

Lemma 2. *Let $d > 0$ and let $\{\Lambda^{(j)}\}_{j=1}^{m+1} \subset [-d, d]^m \cap \boldsymbol{Z}^m$. There is a linear dependence relation*

$$\sum_{j=1}^{m+1} c_j \Lambda^{(j)} = 0$$

with $c_j \in \boldsymbol{Z}$ not all 0 and $|c_j| \leqslant d^m m^{m/2}$.

Proof. Let $\Lambda^{(j)} = (l_1^{(j)}, \ldots, l_m^{(j)})$. Since we have $m+1$ vectors $\Lambda^{(j)}$ in an m-dimensional vector space $\boldsymbol{R}^m$, one of the vectors, say $\Lambda^{(m+1)}$, is linearly dependent on the others:

$$\sum_{j=1}^{m} b_j \Lambda^{(j)} = \Lambda^{(m+1)}. \tag{2}$$

By Cramer's rule, b_j can be written as the quotient of determinants with entries $l_i^{(j)}$. Let c_j be the determinant in the numerator of b_j and let $-c_{m+1}$ be the common determinant of the denominators. Hadamard's inequality,

$$|\det(a_{ij})| \leqslant \prod_i \big(\sum_j |a_{ij}|^2\big)^{1/2},$$

now gives the result when (2) is multiplied through by $-c_{m+1}$, since $|l_i^{(j)}| \leqslant d$. ∎

LEMMA 3. *Let* $\{V_j\}_{j=1}^{\infty} \subset (\mathbf{Z}^+)^m$ *be quasi-independent, let* $\{n_k\}_{k=1}^{\infty}$ *be hyperlacunary (i.e.,* $n_{k+1}/n_k \to \infty$*), let* $L \in \mathbf{Z}^+$*, and let* Δ *be a finite subset of* $\mathbf{Z}$ *containing* 0. *Denote the cardinality of* Δ *by* $|\Delta|$ *and let* D *be any finite subset of* $\mathbf{Z}^m$. *Set*

$$\Omega = \Big\{\sum_{k=1}^{\infty} \varepsilon_k n_k\colon \varepsilon_k \in \Delta \text{ and } \varepsilon_k = 0 \text{ for all but finitely many } k\Big\}. \tag{3}$$

Then for all sufficiently large j, *the number of* $\Lambda \in D$ *such that*

$$|V_j \cdot \Lambda - \Omega| \leqslant L \tag{4}$$

is at most $|\Delta|^m$. *As a function of* $|\Delta|$ *and* m, *this upper bound is best possible.*

We have written $|V_j \cdot \Lambda - \Omega|$ for the distance from $V_j \cdot \Lambda$ to Ω. In proving Theorem 1, we shall use the case $\Delta = \{-1, 0, 1\}$. I am thankful to Hugh L. Montgomery for the argument providing the best bound in Lemma 3.

Note that (4) is equivalent to the system

$$\begin{aligned} &\Big|V_j \cdot \Lambda - \sum_{k=1}^{\infty} \varepsilon_k n_k\Big| \leqslant L, \quad \Lambda \in D, \quad \varepsilon_k \in \Delta, \\ &\varepsilon_k = 0 \text{ for all but finitely many } k. \end{aligned} \tag{5}$$

Proof. We begin by showing that no bound can be better than $|\Delta|^m$. Choose $V_j = (n_{1+j}, n_{2+j}, \ldots, n_{m+j})$ and $D = \Delta^m$. Then for every j, every $\Lambda \in D$ is a solution to (4).

We now prove the rest of the lemma by showing that in some sense the example just given is typical; we show that there exist $k_1, \ldots, k_m$ such that $\varepsilon_{k_1}, \ldots, \varepsilon_{k_m}$ determine the solution $\{\varepsilon_k\}_{k=1}^{\infty}$ to (5), and that for large j, $\{\varepsilon_k\}_{k=1}^{\infty}$ in turn uniquely determines Λ.

Let $M = \max\{|\varepsilon|\colon \varepsilon \in \Delta\}$ and fix j. Let d be the maximum absolute value of the coordinates of Λ over all $\Lambda \in D$. Consider any $m+1$ solutions

$$(\Lambda^{(r)}, \{\varepsilon_k^{(r)}\}_{k=1}^{\infty}), \quad 1 \leqslant r \leqslant m+1,$$

to (5). Let $c_1, \ldots, c_{m+1}$ be as in Lemma 2. Define

$$h^{(r)} = V_j \cdot \Lambda^{(r)} - \sum_{k=1}^{\infty} \varepsilon_k^{(r)} n_k,$$

so that $|h^{(r)}| \leqslant L$. Then

$$\sum_{r=1}^{m+1} c_r h^{(r)} = V_j \cdot \sum_{r=1}^{m+1} c_r \Lambda^{(r)} - \sum_{k=1}^{\infty} \Big(n_k \sum_{r=1}^{m+1} c_r \varepsilon_k^{(r)}\Big)$$
$$= -\sum_{k=1}^{\infty} n_k \delta_k,$$

where $\delta_k = \sum_{r=1}^{m+1} c_r \varepsilon_k^{(r)}$. From our bounds on c_r, $h^{(r)}$, and $\varepsilon_k^{(r)}$, we see that

$$\begin{aligned} \Big|\sum_{k=1}^{\infty} n_k \delta_k\Big| &= \Big|\sum_{r=1}^{m+1} c_r h^{(r)}\Big| \leqslant (m+1) L d^m m^{m/2}, \\ |\delta_k| &= \Big|\sum_{r=1}^{m+1} c_r \varepsilon_k^{(r)}\Big| \leqslant (m+1) M d^m m^{m/2}. \end{aligned} \tag{6}$$

But since $n_{k+1}/n_k \to \infty$, (6) implies that there exists some $k_0 = k_0(L, M, d, m)$ (k_0 does not depend on j) such that $\delta_k = 0$ for all $k \geqslant k_0$. That is, the vectors

$$(\varepsilon_{k_0}^{(r)}, \varepsilon_{k_0+1}^{(r)}, \ldots), \quad 1 \leqslant r \leqslant m+1,$$

are linearly dependent.

We have thus demonstrated that for fixed j,

$$\{\{\varepsilon_k\}_{k=k_0}^{\infty}\colon \{\varepsilon_k\}_{k=1}^{\infty} \text{ is a solution of (5)}\}$$

belongs to an m-dimensional space. There are therefore m coordinates $\varepsilon_{k_1}, \ldots, \varepsilon_{k_m}$ ($k_i \geqslant k_0$) which determine all ε_k, $k \geqslant k_0$. Since there are only $|\Delta|$ choices for each ε_{k_i}, there are at most $|\Delta|^m$ solutions $\{\varepsilon_k\}_{k=k_0}^{\infty}$ to (5). But we claim that for large j, each such solution corresponds to exactly one solution Λ. For let

$$N = \max\Big\{\Big|\sum_{k<k_0} \varepsilon_k n_k\Big|\colon \varepsilon_k \in \Delta - \Delta\Big\},$$

where $\Delta - \Delta = \{\varepsilon - \varepsilon'\colon \varepsilon, \varepsilon' \in \Delta\}$. By quasi-independence of $\{V_j\}$, there exists j_0 such that for each $j \geqslant j_0$, we have

$$\inf\{|V_j \cdot \Lambda|\colon 0 \neq \Lambda \in D - D\} > N + 2L,$$

where $D - D = \{\Lambda_1 - \Lambda_2\colon \Lambda_1, \Lambda_2 \in D\}$. Now suppose that $(\Lambda^{(1)}, \{\varepsilon_k\}_{k=k_0}^{\infty})$, $(\Lambda^{(2)}, \{\varepsilon_k\}_{k=k_0}^{\infty})$ are two solutions of (5) for some $j \geqslant j_0$. Then for some $\varepsilon_k \in \Delta - \Delta$ $(1 \leqslant k < k_0)$,

$$\Big|V_j \cdot (\Lambda^{(1)} - \Lambda^{(2)}) - \sum_{k<k_0} \varepsilon_k n_k\Big| \leqslant 2L.$$

Since $\Lambda^{(1)} - \Lambda^{(2)} \in D - D$, the definition of j_0 implies that $\Lambda^{(1)} - \Lambda^{(2)} = 0$. This establishes the claim and finishes the proof. ■

Proof of Theorem 1. By [11, § III.2] or [13], we may choose $\{V_j'\}_{j=1}^{\infty} \subset \{V_j\}_{j=1}^{\infty}$ so that there exist $\nu_x \in M(\mathbf{T}^m)$, $x \in \mathbf{T}$, such that for any further subsequence $\{V_j''\}$ of $\{V_j'\}$ and for μ-almost all x, $\{V_j'' x\}$ has the asymptotic distribution ν_x. We shall show that $\{V_j'\}$ is the desired subsequence.

Let $f_\Lambda(x) = \hat{\nu}_x(-\Lambda)$, $\Lambda \in \mathbf{Z}^m$. We claim that $\{e(V_j' \cdot \Lambda x)\}_{j=1}^{\infty}$ converges to

$f_A(x)$ weakly in $L^2(\mu)$. For if f is any weak limit point, then f is the weak limit of $\{J^{-1}\sum_{j=1}^{J} e(V_j''\cdot \Lambda x)\}_{J=1}^{\infty}$ for some subsequence $\{V_j''\}$. But by Weyl's criterion and the choice of $\{V_j'\}$, these functions tend pointwise μ-a.e. to f_A. Hence $f = f_A$.

If f_A is not 0 μ-a.e., then there exists an integer L such that $\int e(Lx) f_A(x)\,d\mu(x) \neq 0$. Since

$$\int e(Lx)\, e(V_j'\cdot \Lambda x)\,d\mu(x) \to \int e(Lx) f_A(x)\,d\mu(x),$$

it follows that for all large j, $\hat{\mu}(-L-V_j'\cdot \Lambda) \neq 0$, whence $L+V_j'\cdot \Lambda \in \Omega$, where Ω, as given by (3) (with $\Delta = \{-1, 0, 1\}$), is the spectrum of μ. Thus $\Lambda \in D_{|L|}$, where

$$D_r = \{\Pi \in \mathbf{Z}^m\colon \text{ for all large } j,\ |V_j'\cdot \Pi - \Omega| \leqslant r\}.$$

But the cardinality of D_r is at most 3^m by Lemma 3 and $D_r \subset D_{r+1}$. Hence $D \equiv \bigcup_{r=1}^{\infty} D_r$ also has at most 3^m elements and $f_A = 0$ μ-a.e. if $\Lambda \notin D$. ∎

I am indebted to Bernard Host for one of the ideas used in this proof. It would be very interesting to know if and how Theorem 1 extends to all Riesz products.

3. Helson and related sets. Let $A(\mathbf{T})$ denote the class of functions with absolutely convergent Fourier series. Its dual is $PM(\mathbf{T})$, the pseudomeasures. A closed set $E \subset \mathbf{T}$ is known as a *Helson set* if every function defined and continuous on E can be extended to a function in $A(\mathbf{T})$. By duality, E is Helson if and only if the quantity

$$\alpha(E) = \sup\{\|\mu\|_{M(\mathbf{T})}/\|\mu\|_{PM(\mathbf{T})}\colon 0 \neq \mu \in M(E)\}$$

is finite, where $M(E)$ is the class of measures concentrated on E. Helson showed that such sets are annihilated by R [5, pp. 110–111]. In Theorem 4, we shall exhibit additional measures which annihilate the Helson sets, showing that the class of Helson sets is not "large".

For a closed set E, let $A(E)$ be the restriction of $A(\mathbf{T})$ to E. If $f \in A(E)$ has range in $(-1, 1)$, F is a continuous function on $(-1, 1)$, and E is Helson, then of course $F\circ f \in A(E)$. If, however, the only F on $(-1, 1)$ such that $F\circ f \in A(E)$ for all $f \in A(E)$ with range in $(-1, 1)$ are restrictions of analytic functions, we say that $A(E)$ is analytic or that E is an *analytic set*. The still open Dichotomy Conjecture [5, pp. 402–405] is that all closed sets are either Helson or analytic. We shall show that, in any case, the class of nonanalytic sets is not "large".

It is well known [10, p. 16] that E is Helson if and only if

$$s(E) = \inf\{R(\mu)/\|\mu\|_{M(\mathbf{T})}\colon 0 \neq \mu \in M(E)\}$$

is positive, where, for $\mu \in M(\mathbf{T})$, we write

$$R(\mu) = \limsup_{|n|\to\infty} |\hat{\mu}(n)|.$$

Of course, it is immediate from this criterion and the fact that $\nu \ll \mu \in R \Rightarrow \nu \in R$ [5, Proposition 1.5.1] that Helson sets are U_0-sets. Consider the class of Borel sets E such that

$$s^+(E) = \inf\{R(\mu)/\|\mu\|_{M(\mathbf{T})}\colon 0 \neq \mu \in M^+(E)\}$$

is positive, where $M^+(E)$ consists of the positive measures concentrated on E. Since $s(E) \leqslant s^+(E)$, this class includes the Helson sets. It also includes, for example, the weak Dirichlet sets, these being precisely the sets for which $s^+(E) = 1$ [11, § III.7]. As above, it is a class of U_0-sets. This class, however, is a "large" class (Theorem 7). It should be noted that for any $c > 0$, the class $\{E\colon s^+(E) \geqslant c\}$ is not large ([11, § III.7] or Corollary 12 below).

For $\mu \in M(\mathbf{T})$, write

$$\Sigma_\varepsilon = \{n \in \mathbf{Z}\colon |\hat{\mu}(n)| > \varepsilon\} \quad \text{and} \quad \Sigma_\varepsilon - \Sigma_\varepsilon = \{n-m\colon n, m \in \Sigma_\varepsilon\}.$$

We shall say that $\Sigma_\varepsilon - \Sigma_\varepsilon$ *has arbitrarily long gaps* if its complement in $\mathbf{Z}$ contains arbitrarily long intervals.

THEOREM 4. *Let μ be a measure such that for all $\varepsilon > 0$, $\Sigma_\varepsilon - \Sigma_\varepsilon$ has arbitrarily long gaps. Then $|\mu|(E) = 0$ for all Helson sets E.*

Note that all measures in R trivially satisfy the hypothesis, so that Helson's theorem is a corollary. Furthermore, to exhibit a measure $\mu \notin R$ satisfying the hypothesis, we consider a Riesz product

$$\mu = \prod_{k=1}^{\infty} (1 + \operatorname{Re}\{\alpha_k e(n_k x)\})$$

with $|\alpha_k| \leqslant 1$, $\alpha_k \nrightarrow 0$, and $n_{k+1}/n_k \geqslant q > 5$. Then $\mu \notin R$ and $\Sigma_\varepsilon - \Sigma_\varepsilon$ has the gaps $(2n_k + 2n_{k-1} + \ldots + 2n_1,\ n_{k+1} - 2n_k - \ldots - 2n_1)$ of length

$$n_{k+1} - 4\sum_{j=1}^{k} n_j \geqslant n_{k+1}\Big(1 - 4\sum_{j=1}^{k} q^{-j}\Big) > n_{k+1}\Big(1 - \frac{4}{q-1}\Big),$$

which tends to infinity with k since $q > 5$.

To prove Theorem 4, we shall follow the method used by Kahane and Salem in their proof of Helson's theorem. It depends on the following lemma [5, p. 112]. I am grateful to Carruth McGehee for having brought this method to my attention.

LEMMA 5. *Given $r_1, \ldots, r_M \in \mathbf{Z}$ and $\mu \in M(\mathbf{T})$, there exist $a_1, \ldots, a_M = \pm 1$ such that if $\nu = \sum_{m=1}^{M} a_m e(r_m x)\,\mu$, then $\|\nu\|_{M(\mathbf{T})} \geqslant \sqrt{M/3}\,\|\mu\|_{M(\mathbf{T})}$.*

Proof of Theorem 4. Let E be any set of positive $|\mu|$-measure. We shall show that $\alpha(E)=\infty$. Without loss of generality, we may assume that $\|\mu\|_{M(T)}=1$.

Let M be a positive integer. There is a trigonometric polynomial P such that

$$\|P\mu-(\mu|_E)\|_{M(T)}<M^{-1}.$$

Write $L=\deg P$, $A=\sum_k |\hat{P}(k)|$ and $\varepsilon=\min((MA)^{-1}, M^{-1})$. The hypothesis on Σ_ε allows us to choose inductively $r_1,\ldots,r_M$ such that for $j\neq m$, the distance between r_m-r_j and $\Sigma_\varepsilon-\Sigma_\varepsilon$ is greater than $2L$. With such a choice, the distance between $r_m+\Sigma_\varepsilon$ and $r_j+\Sigma_\varepsilon$ is greater than $2L$, i.e.,

$$(r_m+\Sigma_\varepsilon+[-L,L])\cap(r_j+\Sigma_\varepsilon+[-L,L])=\emptyset.$$

If S_ε is the pseudomeasure equal to $\hat{\mu}$ on Σ_ε and 0 elsewhere, it follows that the spectra of $e(r_m x)PS_\varepsilon$ and $e(r_j x)PS_\varepsilon$ do not intersect. Therefore if $a_m=\pm 1$, we have

$$\Big\|\sum_{m=1}^{M} a_m e(r_m x)PS_\varepsilon\Big\|_{PM}=\sup_m\|a_m e(r_m x)PS_\varepsilon\|_{PM}=\|PS_\varepsilon\|_{PM}$$
$$\leqslant\|P(S_\varepsilon-\mu)\|_{PM}+\|P\mu-(\mu|_E)\|_M+|\mu|(E)$$
$$\leqslant A\varepsilon+M^{-1}+1\leqslant 3,$$

and hence if $v=\sum_{m=1}^{M} a_m e(r_m x)\mu|_E$,

$$\|v\|_{PM}=\Big\|\sum_{m=1}^{M} a_m e(r_m x)[(\mu|_E-P\mu)+(P\mu-PS_\varepsilon)+PS_\varepsilon]\Big\|_{PM}$$
$$\leqslant M\varepsilon+MA\varepsilon+3\leqslant 5.$$

But if a_m are chosen as in the lemma, it follows that $\|v\|_M/\|v\|_{PM}\geqslant\sqrt{M}/(5\sqrt{3})$. Since M is arbitrary and $v\in M(E)$, it follows that $\alpha(E)=\infty$. ∎

Following Kahane and Katznelson [9], we say that a closed set $E\subset T$ satisfies *condition* (R) if there exists K such that for all $N\geqslant 1$, there exist $0\neq v\in M(E)$ and $m\in Z^+$ such that

$$\sup_{p\in Z}\sum_{|n|\leqslant N}|\hat{v}(p-nm)|\leqslant K\|v\|_{PM}.$$

The utility of this condition lies in the fact that all such sets are of analyticity [9]. It immediately follows that nonanalytic sets are U_0-sets. As another corollary, we shall deduce that they do not form a "large" class:

THEOREM 6. *Let*

$$\mu=\prod_{k=1}^{\infty}(1+\operatorname{Re}\{\alpha_k e(n_k x)\})$$

be a hyperlacunary Riesz product: $|\alpha_k|\leqslant 1$, $n_{k+1}/n_k\to\infty$. *Then* $\mu E=0$ *for all nonanalytic sets* E.

Proof. It suffices to show that if $\mu E>0$, then E satisfies condition (R). Indeed, let $v=\mu|_E$ and $N\geqslant 1$. Let Q be a trigonometric polynomial such that $\|v-Q\mu\|_{M(T)}\leqslant\|v\|_M/(2N+1)$. Set Ω to be as in (3) (with $\Delta=\{-1,0,1\}$) and $\Omega'=\Omega+[-L,L]$, where L is the degree of Q. The spectrum of $Q\mu$ is contained in Ω'. Choose $m=n_{k_0}$, where $k\geqslant k_0\Rightarrow n_k/n_{k-1}>10(N+L)$. We claim that for all $p\in Z$,

$$\operatorname{card}(\{p-nm: |n|\leqslant N\}\cap\Omega')\leqslant 3.$$

The proof will then be complete, since it follows that

$$\sum_{|n|\leqslant N}|\hat{v}(p-nm)|\leqslant\|v\|_{PM}+\sum_{|n|\leqslant N}|(Q\mu)\hat{}\,(p-nm)|$$
$$\leqslant\|v\|_{PM}\Big[1+3\Big(1+\frac{1}{2N+1}\Big)\Big]\leqslant 5\|v\|_{PM}.$$

To prove the claim, suppose that $p+nm, p+n'm\in\Omega'$, $|n|\leqslant N$, $|n'|\leqslant N$, $n\neq n'$. Then by subtraction, we can write

$$(n-n')m=\varepsilon_1 n_{k_1}+\varepsilon_2 n_{k_2}+\ldots+l,$$

where $k_1>k_2>\ldots$, $\varepsilon_i=\pm 1,\pm 2$, and $|l|\leqslant 2L$. Dividing by n_{k_1}, we obtain

$$|(n-n')n_{k_0}/n_{k_1}-\varepsilon_1|\leqslant 2(n_{k_1-1}+n_{k_1-2}+\ldots+n_1+L)/n_{k_1}.$$

If $k_1<k_0$, then the right-hand side is less than $1+2L/n_{k_1}$ (using $n_j/n_{j-1}\geqslant 3$ for all j) while the left-hand side is greater than $10(N+L)-1$, an impossibility. Hence $k_1\geqslant k_0$ and the right-hand side is less than 2/5. If $k_1>k_0$, the left-hand side is at least 4/5, whence $k_1=k_0$. Since the left-hand side is then an integer, it must be zero: $n-n'=\varepsilon_1$. In other words, for any n, n' as indicated, $|n-n'|\leqslant 2$. Choosing the largest and smallest n, n' establishes the claim. ∎

We now show that $\{E: s^+(E)>0\}^\perp=R$.

THEOREM 7. *If* $\mu\notin R$, *then there exists a set E of positive $|\mu|$-measure such that* $s^+(E)\geqslant R(|\mu|)/\|\mu\|_M$.

Proof. We may assume that μ is a probability measure. Choose $n_k\to\infty$ so that $\hat{\mu}(n_k)\to\omega R(\mu)$, where $|\omega|=1$, and such that

$$f(t)=\lim_{K\to\infty}\frac{1}{K}\sum_{k=1}^{K}e(-n_k t)$$

exists μ-a.e. ([11, § III.2] or [13]) Since $R(\mu)=\int\bar{\omega}f\,d\mu=\int\operatorname{Re}\{\bar{\omega}f\}\,d\mu$, it follows that

$$E=\{t: f(t)\text{ exists and }\operatorname{Re}\{\bar{\omega}f(t)\}\geqslant R(\mu)\}$$

is not of μ-measure 0.

Now if $v\in M^+(E)$, then

$$R(\mu)\|v\|_M \leqslant \int_E \operatorname{Re}\{\bar{\omega} f\}\,dv = \operatorname{Re}\{\bar{\omega}\lim_{K\to\infty}\int_E K^{-1}\sum_{k=1}^{K} e(-n_k t)\,dv(t)\}$$

$$= \operatorname{Re}\{\bar{\omega}\lim_{K\to\infty}\int_T K^{-1}\sum_{k=1}^{K} e(-n_k t)\,dv(t)\}$$

$$= \operatorname{Re}\{\bar{\omega}\lim_{K\to\infty} K^{-1}\sum_{k=1}^{K}\hat{v}(n_k)\} \leqslant \limsup_{K\to\infty} K^{-1}\sum_{k=1}^{K}|\hat{v}(n_k)|.$$

Hence $R(v) \geqslant R(\mu)\|v\|_M$. ∎

4. Sets of uniqueness. Sets of uniqueness have been under study for over 100 years. During this period, many beautiful results have been obtained, but relatively few examples of U-sets are actually known. Some ingenious theorems show that U is indeed a fairly limited subclass of U_0 (see [5, Chap. 4]). Yet it appears that many more U-sets await to be discovered. As an approach to determining the size of U, the question "is $R = U^{\perp}$?" is particularly fascinating. We have not been able to resolve it, but true or false, the answer would be extremely interesting. Thus, if $R = U^{\perp}$, then for all $\mu \notin R$, there would be a U-set with positive $|\mu|$-measure. This would produce a wealth of U-sets, including necessarily U-sets as yet unknown. On the other hand, if $R \neq U^{\perp}$, then for some $\mu \notin R$, $\mu E = 0$ for all $E \in U$. In other words, the property of not supporting any pseudofunction implies the property of not supporting any part of a certain fixed measure which is not a pseudofunction ! (Recall that the pseudofunctions, $PF(T)$, are those pseudomeasures S such that $\hat{S}(n) \to 0$ as $|n| \to \infty$. A closed set is a U-set if and only if it does not support any pseudofunction [5, Theorem 4.2.1].) We would also obtain new M-sets (i.e., sets not of uniqueness) from such a $\mu \in U^{\perp}\setminus R$, namely, every E of positive $|\mu|$-measure.

Note that by regularity, $U^{\perp} = (\text{closed } U\text{-sets})^{\perp}$. In this section, we first recall some more characterizations of closed U-sets. We shall then show that the class $\mathscr{D}$ of all known U-sets and of all sets satisfying certain conditions known to be sufficient for uniqueness is small: $R \neq \mathscr{D}^{\perp}$. Finally, we shall consider a class larger than but closely related to U.

Note that $A(T)$ is the Banach-space dual of $PF(T)$. Given a closed set $E \subset T$, we define the ideal

$$J(E) = \{f \in A(T): f = 0 \text{ on a neighborhood of } E\}.$$

Then [5, p. 92] E is a U-set if and only if $J(E)$ is weak* dense in $A(T)$. We call E a *U'-set* if $J(E)$ is weak* sequentially dense in $A(T)$. Now U'-sets can be characterized in another way. Define

$$\psi(E) = \inf\left\{\frac{\limsup|\hat{S}(n)|}{\|S\|_{PM}}: 0 \neq S \in PM(E)\right\},$$

where $PM(E)$ consists of the pseudomeasures supported in E. Then $E \in U'$ if and only if $\psi(E) > 0$ [5, Theorems 4.3.2, 4.3.1]. For example, $H^{(m)}$-sets are U'-sets: let $\{V_j\}$ be quasi-independent, $I = I_1 \times I_2 \times \ldots \times I_m$ an open box in T^m, and E a set such that $V_j x \notin I$ for any $x \in E$. Then we claim that

$$\psi(E) \geqslant \prod_{j=1}^{m}\frac{\lambda I_j}{2-\lambda I_j}, \tag{7}$$

where λ is the Lebesgue measure on T. Note that this lower bound goes from 0 to 1 as the volume of I increases from 0 to 1. To prove (7), let f_Q denote the "triangular" function which is zero outside an interval $Q \subset T$ and linear between the endpoints of Q and the midpoint of Q, where it reaches a value of $2/\lambda Q$. Thus $\hat{f}_Q(0) = 1$, $\|f_Q\|_A = 2/\lambda Q$, and so $\|1-f_Q\|_A = (2-\lambda Q)/\lambda Q$. Consider the function

$$g(t_1,\ldots,t_m) = \prod_{j=1}^{m}(1-f_{I'_j}(t_j)),$$

where I'_j is an interval whose closure lies in I_j. Then $\hat{g}(0,\ldots,0) = 0$ and $g = 1$ in a neighborhood of the complement of I. Hence $g(V_j t) = 1$ in a neighborhood of E and for any $S \in PM(E)$,

$$\hat{S}(0) = \langle S, g(V_j t)\rangle = \sum_{\Lambda \neq 0}\hat{S}(V_j\cdot\Lambda)\hat{g}(-\Lambda).$$

Since $\{V_j\}$ is quasi-independent, $|V_j\cdot\Lambda| \to \infty$ as $j \to \infty$ for every $\Lambda \neq 0$ and it follows that

$$|\hat{S}(0)| \leqslant \|g\|_A \limsup_{|n|\to\infty}|\hat{S}(n)|.$$

Since $e(-pt)S \in PM(E)$, the same inequality holds with 0 replaced by any $p \in \mathbf{Z}$, whence

$$\frac{\limsup|\hat{S}(n)|}{\|S\|_{PM}} \geqslant \|g\|_A^{-1} = \prod\|1-f_{I'_j}\|_A^{-1} = \prod\frac{\lambda I'_j}{2-\lambda I'_j}.$$

The inequality (7) results from the fact that $\lambda I'_j$ can be taken arbitrarily close to λI_j.

Similar but more complete results hold when $J(E)$ is replaced by

$$I(E) = \{f \in A(T): f = 0 \text{ on } E\}.$$

A closed set E is called a *U_1-set* [*U'_1-set*] if $I(E)$ is weak* [weak* sequentially] dense in $A(T)$. It turns out [5, Theorem 4.3.4] that every U_1-set is a countable union of U'_1-sets, whence $U_1^{\perp} = (U'_1)^{\perp}$. Unfortunately, such results are not known for U-sets. Evidently, however, $U \subset U_1$ and $U' \subset U'_1$; the inclusions are strict [5, pp. 110–118]. The annihilator of $I(E)$ in $PM(T)$ $= A(T)^*$ (and thus the dual of $A(T)/I(E) = A(E)$) is denoted by $N(E)$. (Note that $M(E) \subseteq N(E) \subseteq PM(E)$; $M(E) = N(E)$ precisely when E is Helson and

$N(E) = PM(E)$ precisely when E obeys synthesis.) Set

$$\eta(E) = \inf\left\{\frac{\limsup|\hat{S}(n)|}{\|S\|_{PM}} : 0 \neq S \in N(E)\right\}.$$

Then $E \in U'_1$ if and only if $\eta(E) > 0$ ([5, p. 98]). For example, Helson sets are U'_1-sets since $s(E) \leqslant \eta(E)$. Indeed, all nonanalytic sets are U'_1-sets. For, as remarked by Graham and McGehee [5, p. 402], if E satisfies condition (R) with $M(E)$ replaced by $N(E)$, then E is analytic (the proof is the same). But if $\eta(E) = 0$, then condition (R) is obviously satisfied. It would be interesting to know whether $R = U_1^{\perp}$ or, in any case, whether $U^{\perp} = U_1^{\perp}$.

The most significant–perhaps the only–known examples of U-sets are countable unions of $H^{(m)}$-sets. These include the Cantor-type symmetric sets which are known to be U-sets [15, pp. 89, 93, 96]. It is easy to see that the example of Bari [2, pp. 102–104] is also an H-set. The only other examples of U-sets known to this author were constructed by McGehee [14], whose work was extended by Meyer [16]. It is unknown whether these are not in fact countable unions of $H^{(m)}$-sets.

To define Meyer's sets, we regard $\boldsymbol{T}$ as $[0, 1) \subset \boldsymbol{R}$. If $T, \varepsilon > 0$ and $\Lambda \subset \boldsymbol{R}$, we say that (T, ε) is *adapted* to Λ if for all functions $a: \Lambda \to \boldsymbol{C}$ sending $\lambda \mapsto a_\lambda$ with $a_\lambda = 0$ for all but finitely many λ, we have

$$\sup_{0 \leqslant x \leqslant T} \Big|\sum_{\lambda \in \Lambda} a_\lambda e(\lambda x)\Big| \geqslant \varepsilon \sup_{x \in \boldsymbol{R}} \Big|\sum_{\lambda \in \Lambda} a_\lambda e(\lambda x)\Big|.$$

Note that $[0, T]$ can be replaced by any interval of length T. Suppose that $E \subset \boldsymbol{R}$ is compact, $\Lambda_k \subset \boldsymbol{R}$ are finite, $l_k \to 0$, (T_k, ε_k) is adapted to Λ_k, and $E \subset \Lambda_k + [-l_k, l_k]$. Then Meyer [16] proved that E is a set of uniqueness in $\boldsymbol{R}$ when

(8) $$\overline{\lim}\,[\varepsilon_k^2 - (2+\varepsilon_k)\pi T_k l_k] > 0.$$

We are interested only in the case $E \subset [0, 1]$. The essential tool in proving Meyer's result is an approximation of the Fourier transform of pseudomeasures with compact support. When we are concerned with measures, this approximation can be improved. This improvement is what will permit us to demonstrate that all such sets are annihilated by Riesz products.

The approximation of pseudomeasures and measures is given by

LEMMA 8. *Suppose that (T, ε) is adapted to Λ and $0 < \pi l T < \varepsilon$. Then $|\lambda - \lambda'| > 2l$ when $\lambda, \lambda' \in \Lambda$ are distinct. If $S \in PM(\Lambda + [-l, l])$, we define $\sigma \in M(\Lambda)$ by*

$$\sigma = \sum_{\lambda \in \Lambda} \langle S, \chi_{[\lambda - l, \lambda + l]} \rangle \delta_\lambda,$$

where δ_λ is the unit mass at λ and χ_I is the indicator function of I. Then for all $x \in \boldsymbol{R}$, we have

(9) $$|\hat{S}(x) - \hat{\sigma}(x)| \leqslant 2\pi l |x| (\varepsilon - \pi l T)^{-1} \|S\|_{PM}.$$

If in addition S is a measure, then

(10) $$|\hat{S}(x) - \hat{\sigma}(x)| \leqslant 2\pi l |x|\, \|S\|_M.$$

Proof. The first statement follows immediately upon consideration of $|e(\lambda x) - e(\lambda' x)|$. Hence σ is well defined. Write

$$\hat{S}(x) = \sum_{\lambda \in \Lambda} a_\lambda(x) e(\lambda x),$$

where the spectrum of a_λ lies in $[-l, l]$. If

$$f(x, y) = \sum_{\lambda \in \Lambda} a_\lambda(x) e(\lambda y),$$

then $f(x, x) = \hat{S}(x)$ and $f(0, x) = \hat{\sigma}(x)$. Fix any y_0; the function $f(x, y_0)$ of x has spectrum in $[-l, l]$ and can be written for any x_0 as

$$f(x, y_0) = f(x_0, y_0) + \left[\frac{\partial}{\partial x} f(x', y_0)\right](x - x_0)$$

for some value x' between x_0 and x. By Bernstein's inequality ([14, p. 149] or [5, p. 418]), $\partial f/\partial x \leqslant 2\pi l \|f\|_\infty$, so that for all x, x_0, y_0,

(11) $$|f(x, y_0) - f(x_0, y_0)| \leqslant 2\pi l |x - x_0|\, \|f\|_\infty.$$

Now when S is a measure, we have

$$\|f\|_\infty \leqslant \sup_x \sum_{\lambda \in \Lambda} |a_\lambda(x)| \leqslant \sum_{\lambda \in \Lambda} |S|(\lambda + [-l, l]) = \|S\|_M.$$

Substitution of this estimate in (11) with $x_0 = 0$ and $y_0 = x$ yields (10).

In general, when S is only a pseudomeasure, we know that for all x_0, there exists $y_0 \in [x_0 - T/2, x_0 + T/2]$ such that $|f(x_0, y_0)| \geqslant \varepsilon \sup_y |f(x_0, y)|$. For this choice of y_0 and $x = y_0$, (11) gives

$$\|S\|_{PM} \geqslant |f(y_0, y_0)| \geqslant |f(x_0, y_0)| - 2\pi l |y_0 - x_0|\, \|f\|_\infty \geqslant \varepsilon \sup_y |f(x_0, y)| - \pi l T \|f\|_\infty.$$

Taking the supremum over x_0 yields

$$\|S\|_{PM} \geqslant (\varepsilon - \pi l T) \|f\|_\infty,$$

which gives (9) when substituted into (11). ∎

Remark. Very slight improvements can be made by using $\partial^2 f/\partial x^2$ (cf. [8, pp. 118–119]); these do not affect the essence of our results.

Meyer's result now follows easily. Given E as above, we may assume without loss of generality that $\varepsilon_k \to \varepsilon$, $\pi T_k l_k \to \beta < \varepsilon^2/(2+\varepsilon)$, and that $\pi T_k l_k < \varepsilon_k$ for all k. For any $S \in PM(E)$, define

$$\sigma_k = \sum_{\lambda \in \Lambda_k} \langle S, \chi_{[\lambda - l_k, \lambda + l_k]} \rangle \delta_\lambda.$$

Choose real numbers $m_k \to \infty$ such that $m_k l_k \to 0$. By hypothesis, there exists $x_k \in [m_k, m_k + T_k]$ such that $|\hat{\sigma}_k(x_k)| \geqslant \varepsilon_k \|\sigma_k\|_{PM}$. By (9), we have

$$\begin{aligned} |\hat{S}(x_k)| &\geqslant |\hat{\sigma}_k(x_k)| - 2\pi l_k |x_k| (\varepsilon_k - \pi l_k T_k)^{-1} \|S\|_{PM} \\ &\geqslant \varepsilon_k \|\sigma_k\|_{PM} - 2\pi l_k (m_k + T_k)(\varepsilon_k - \pi l_k T_k)^{-1} \|S\|_{PM}. \end{aligned}$$

But it also follows from (9) and the fact that $l_k \to 0$ that $\hat{S}(x) = \lim_k \hat{\sigma}_k(x)$, whence $\underline{\lim} \|\sigma_k\|_{PM} \geqslant \|S\|_{PM}$. Therefore

$$\text{(12)} \qquad \overline{\lim_{|x| \to \infty}} |\hat{S}(x)| \geqslant [\varepsilon - 2\beta(\varepsilon - \beta)^{-1}] \|S\|_{PM} = \frac{\varepsilon^2 - (2+\varepsilon)\beta}{\varepsilon - \beta} \|S\|_{PM} > 0.$$

Thus, E does not support any pseudofunctions on $\boldsymbol{R}$ and E is a set of uniqueness. To show that if $E \subset [0, 1]$, then E is a U-set on $\boldsymbol{T}$, i.e.,

$$\overline{\lim_{\substack{|n| \to \infty \\ n \in \boldsymbol{Z}}}} |\hat{S}(n)| > 0 \quad \text{for all } S \in PM(E),$$

we use the following proposition [15, pp. 86–87].

Proposition 9. *For every $\varepsilon > 0$, there exists $C > 0$ such that for all pseudomeasures S on $\boldsymbol{R}$ whose support has diameter at most $1 - \varepsilon$,*

$$\overline{\lim_{n \in \boldsymbol{Z}}} |\hat{S}(n)| \geqslant C \overline{\lim_{x \in \boldsymbol{R}}} |\hat{S}(x)|.$$

Now if $E \subset [0, 1]$, we consider E as being on the circle $\boldsymbol{T}$. Since $E \neq \boldsymbol{T}$, there exists an arc $(h - \varepsilon, h)$ in the complement of E. We may translate E without changing the value of $\psi(E)$, so we may assume that $E \subset [\varepsilon/2, 1 - \varepsilon/2]$. Any element of $PM(\boldsymbol{T})$ with support in E then extends naturally to an element of $PM(\boldsymbol{R})$ with support in E: for $S \in PM(\boldsymbol{T})$, $\operatorname{supp} S \subset E$, define $\tilde{S} \in PM(\boldsymbol{R})$ by $\langle f, \tilde{S} \rangle = \langle f V_\varepsilon, S \rangle$ for $f \in A(\boldsymbol{R})$, where V_ε is the piecewise linear function which is 1 on $[\varepsilon/3, 1 - \varepsilon/3]$ and 0 outside $(0, 1)$. It is easily verified that

$$\|f V_\varepsilon\|_{A(\boldsymbol{T})} \leqslant \|f\|_{A(\boldsymbol{R})} \max_{0 \leqslant x \leqslant 1} \sum_{n \in \boldsymbol{Z}} |\hat{V}_\varepsilon(n + x)|,$$

so that this definition makes sense. It is obvious that $\operatorname{supp} \tilde{S} = \operatorname{supp} S$ and that $\hat{\tilde{S}}(n) = \hat{S}(n)$ for $n \in \boldsymbol{Z}$. It now follows from (12) and Proposition 9 that $\psi(E) > 0$; E is thus a U'-set.

Now suppose $S = \nu \in M^+(E)$, where $E \subset [0, 1]$ is as above. Define σ_k,

m_k, and x_k in the same way and note that $\|\sigma_k\|_{PM} = \|\nu\|_M$. From (10), we have

$$\text{(13)} \qquad |\nu(x_k)| \geqslant [\varepsilon_k - 2\pi l_k (m_k + T_k)] \|\nu\|_M,$$

whence

$$\text{(14)} \qquad \overline{\lim_{x \in \boldsymbol{R}}} |\hat{\nu}(x)| \geqslant (\varepsilon - 2\beta) \|\nu\|_M > \varepsilon \frac{2 - \varepsilon}{2 + \varepsilon} \|\nu\|_M.$$

This already places certain restrictions on the measures which can put mass on E. Note that (13) is valid even if $m_k l_k \nrightarrow 0$. Using this observation with (8) will allow us to establish the following theorem.

Theorem 10. *Let $\mu = \prod (1 + \operatorname{Re} \{\alpha_k e(n_k t)\})$ be a Riesz product with either*

(i) $n_{k+1}/n_k \geqslant 4$ and $R(\mu) = \overline{\lim} \frac{1}{2} |\alpha_k| \leqslant \frac{1}{6}$, or

(ii) $n_{k+1}/n_k \to \infty$ and $R(\mu) \leqslant 1/3$.

Then $\mu E = 0$ for all sets $E \subset [0, 1]$ of the following type: there exist $\Lambda_k \subset \boldsymbol{R}$ finite, $l_k \to 0$, (T_k, ε_k) adapted to Λ_k, $E \subset \Lambda_k + [-l_k, l_k]$, and (8) is satisfied.

We shall need some facts about Riesz products and about the relationship between $\hat{\mu}$ on $\boldsymbol{R}$ and on $\boldsymbol{Z}$ for measures $\mu \in M([0, 1))$.

Lemma 11. *For $\mu \in M([0, 1))$, let $C_\mu = \sup \{R(\nu)/\|\nu\|_M : 0 \neq \nu \ll \mu\}$. Then for $\nu \ll \mu$,*

$$\overline{\lim_{x \in \boldsymbol{R}}} |\hat{\nu}(x)| \leqslant C_\mu \|\nu\|_M.$$

Proof. Let $|x_k| \to \infty$, $x_k = n_k + \delta_k$, $n_k \in \boldsymbol{Z}$, $\delta_k \to \delta \in [0, 1)$. Then $e(-\delta_k t) \to e(-\delta t)$ uniformly on $[0, 1)$, so that for $\nu \ll \mu$,

$$\overline{\lim} |\hat{\nu}(x_k)| = \overline{\lim} \left| \int_0^1 e(-n_k t) e(-\delta t) \, d\nu(t) \right| \leqslant R\big(e(-\delta t) \nu(t)\big) \leqslant C_\mu \|\nu\|_M. \ \blacksquare$$

Corollary 12. *If μ is a Riesz product, then for all $\nu \ll \mu$,*

$$\overline{\lim_{x \in \boldsymbol{R}}} |\hat{\nu}(x)| \leqslant R(\mu) \|\nu\|_M.$$

Proof. For any measure μ, consider the space $L^\infty(\mu)$. Let Γ be the subset $\{e(nt) : n \in \boldsymbol{Z}\}$ and Γ_∞ the set of limit points of Γ in the weak* topology from $L^1(\mu)$. Then with C_μ as in Lemma 11, we easily calculate

$$\begin{aligned} C_\mu &= \sup \{ |\textstyle\int \chi \, d\nu| / \|\nu\|_M : 0 \neq \nu \ll \mu, \ \chi \in \Gamma_\infty \} \\ &= \sup \{ |\textstyle\int \chi f \, d\mu| / \|f\|_{L^1(\mu)} : 0 \neq f \in L^1(\mu), \ \chi \in \Gamma_\infty \} \\ &= \sup \{ \|\chi\|_{L^\infty(\mu)} : \chi \in \Gamma_\infty \}. \end{aligned}$$

Now when μ is a Riesz product, every $\chi \in \Gamma_\infty$ has the form $c\gamma$, where c is a constant and $\gamma \in \Gamma$ ([6, Chap. 5], [7, Chap. 2, Proposition 1], or [5, Theorem 7.3.1]). Since $\chi\bar{\gamma}\in\Gamma_\infty$ for $\chi\in\Gamma_\infty$, $\gamma\in\Gamma$, it follows that $C_\mu = R(\mu)$ and the lemma gives the desired conclusion. (It should be noted that for Riesz products, we actually have an equality: $R(\nu) = R(\mu)\|\nu\|_{PM(T)}$. This follows immediately from the above-mentioned form of Γ_∞.) ∎

LEMMA 13. *Let* $\mu\in M([0, 1])$. *Suppose that* $B_k = \{n\in \mathbf{Z}: a_k \leqslant n \leqslant b_k\}$, $b_k - a_k\to\infty$, *are intervals such that*

$$\lim_{k\to\infty}\sup_{n\in B_k}|\hat{\mu}(n)| = 0.$$

If $B_k' = \{x\in \mathbf{R}: a_k' \leqslant x \leqslant b_k'\}$, $a_k'-a_k\to\infty$, $b_k - b_k'\to\infty$, *are subintervals, then for all* $\nu \ll \mu$,

$$\lim_{k\to\infty}\sup_{x\in B_k'}|\hat{\nu}(x)| = 0.$$

Proof. It suffices to show that if $x_k\in B_k'$ and $|\hat{\nu}(x_k)|\to c$, then $c = 0$. We may assume that $x_k = n_k+\delta_k$, $n_k\in B_k$, $\delta_k\to\delta\in[0, 1)$. If $c\neq 0$, then there exists a trigonometric polynomial P such that $\|P\mu - e(-\delta t)\nu\|_M < c/2$. Put $L = \deg P$. Then

$$c = \lim|\hat{\nu}(x_k)| = \lim\Big|\int_0^1 e(-n_k t)\, e(-\delta t)\, d\nu(t)\Big|$$

$$< \overline{\lim}\Big|\int_0^1 e(-n_k t)\, P(t)\, d\mu(t)\Big| + c/2$$

$$= \overline{\lim}\Big|\sum_{|l|\leqslant L}\hat{P}(l)\,\hat{\mu}(n_k - l)\Big| + c/2 = c/2$$

since $n_k - l\in B_k$ for large k. This is a contradiction. ∎

COROLLARY 14. *Let* $\mu = \prod(1+\mathrm{Re}\{\alpha_k e(n_k x)\})$ *be a Riesz product with* $n_{k+1}/n_k\geqslant 3$, $\{k_j\}$ *a sequence with* $k_j > j$, *and*

$$B_j' = \{x\in \mathbf{R}: n_{k_j}+n_{k_j-1}+\ldots+n_{k_j-j}\leqslant x\leqslant n_{k_j+1}-n_{k_j}-\ldots-n_{k_j-j+1}\}.$$

Then $\lim_{j\to\infty}\sup_{x\in B_j'}|\hat{\nu}(x)| = 0$ *for all* $\nu\ll\mu$.

Proof. It suffices to take

$$B_j = \{n\in \mathbf{Z}: n_{k_j}+n_{k_j-1}+\ldots+n_{k_j-j}-j\leqslant n\leqslant n_{k_j+1}-n_{k_j}-\ldots-n_{k_j-j+1}+j\}$$

in the lemma. ∎

Proof of Theorem 10. Suppose to the contrary that $\mu E\neq 0$. Let $\nu = \mu|_E$. We assume that $\pi T_k l_k\to\beta$, $\varepsilon_k\to\varepsilon$. In case (i), set $\theta = 1/3$ and in case (ii), θ arbitrary but less than 1. For every j, define $k(j)$ by $\theta n_{k(j)}\leqslant T_j < \theta n_{k(j)+1}$. We may assume that $k(j) > j$. Let

$$m_j = n_{k(j)}+n_{k(j)-1}+\ldots+n_{k(j)-j}.$$

It is easily verified that

$$m_j + T_j < n_{k(j)+1} - n_{k(j)} - \ldots - n_{k(j)-j+1}$$

for all j in case (i) and for all sufficiently large j in case (ii). Furthermore, in case (i), we have

$$m_j < n_{k(j)}+n_{k(j)-1}+\ldots+n_1 < n_{k(j)}(1+4^{-1}+4^{-2}+\ldots) = \tfrac{4}{3}n_{k(j)}\leqslant 4T_j,$$

while in case (ii), we have similarly $\overline{\lim}_{j\to\infty} m_j/T_j\leqslant\theta^{-1}$.

Now by Corollary 14,

$$\lim_{j\to\infty}\sup_{x\in[m_j, m_j+T_j]}|\hat{\nu}(x)| = 0.$$

On the other hand, from (13), this limit is at least $\overline{\lim}\,[\varepsilon_j - 2\pi l_j(m_j+T_j)]\,\|\nu\|_M$. In case (i), it follows that $0\geqslant\varepsilon - 10\beta > \varepsilon - (10\varepsilon^2)/(2+\varepsilon)$, or $\varepsilon > 2/9$. Hence by (14),

$$\overline{\lim}_{x\in\mathbf{R}}|\hat{\nu}(x)| > \tfrac{8}{45}\|\nu\|_M > \tfrac{1}{6}\|\nu\|_M,$$

which contradicts Corollary 12. Likewise, in case (ii), $0\geqslant\varepsilon - 2(\theta^{-1}+1)\beta$. Since this is true for all $\theta < 1$, it follows that $0\geqslant\varepsilon - 4\beta$, whence $\varepsilon > 2/3$ and $\overline{\lim}_{x\in\mathbf{R}}|\hat{\nu}(x)| > \frac{1}{3}\|\nu\|_M$, again a contradiction. ∎

With a bit more subtlety in choosing m_j, we can prove the same result for Riesz products μ with $n_{k+1}/n_k\geqslant q > 3$ and $R(\mu)\leqslant\delta = \delta(q)$; however, this method gives $\delta(q)\to 0$ as $q\to 3$.

Note that ultrathin symmetric sets [15, pp. 239, 250] are examples of Meyer's sets described above [15, p. 240]. We now move from examples of U-sets to two known sufficient conditions. If a closed set E is *without true pseudomeasure* (*WTP*), i.e., $PM(E) = M(E)$, then E is, of course, a Helson set. Hence $PF(E) = M_0(E) = \{0\}$ and E is a U-set. Moreover, E is annihilated by Riesz products whose frequencies satisfy $n_{k+1}/n_k\geqslant q > 5$ (Section 2 above).

Our second condition sufficient for uniqueness is more substantial. A closed set E is said to be of *resolution* if every closed subset is of synthesis. Since every WTP set is of synthesis and every closed subset of a WTP set is WTP, it follows that every WTP set is of resolution. Kahane and Katznelson [9] found the following necessary condition to be a set of resolution. If $l > 1$, $r > 0$, and $B\subset\mathbf{R}$, we say that $B\in S_{l,r}$ if B is an infinite union of segments of length l separated by at least rl. A pseudomeasure $T\in PM(\mathbf{T})$ is said to satisfy *condition* (P) if for all positive ε and r, there exist $l > 1$ and $B\in S_{l,r}$ such that $|\hat{T}(n)|\leqslant\varepsilon$ for $n\notin B$. Kahane and Katznelson's theorem is that sets of resolution do not support any nonzero pseudomeasure satisfying condition (P). Since pseudofunctions satisfy (P), it follows that sets of resolution are U-sets. Another consequence is

THEOREM 15. *Let* $\mu=\prod(1+\mathrm{Re}\{\alpha_k e(n_k x)\})$ *be a hyperlacunary Riesz product*: $n_{k+1}/n_k\to\infty$. *Then* $\mu E=0$ *for all sets of resolution*, E.

Proof. If $\mu E\neq 0$, we shall show that $\nu=\mu|_E$ satisfies condition (P). Given $\varepsilon, r>0$, let P be a trigonometric polynomial of degree L such that $\|P\mu-\nu\|_M\leqslant\varepsilon$. Let k_0 be such that for $k\geqslant k_0$, we have

$$n_{k+1}-n_k-\ldots-n_1-L>(2r+1)(n_k+n_{k-1}+\ldots+n_1+L).$$

Set $l=2(n_{k_0}+n_{k_0-1}+\ldots+n_1+L)$ and

$$\Omega_0=\{\pm n_{k_1}\pm n_{k_2}\pm\ldots\pm n_{k_m}\colon k_1>k_2>\ldots>k_m>k_0,\ m\geqslant 1\}\cup\{0\}.$$

We center an interval of length l about each point of Ω_0: $B=\Omega_0+[-l/2,l/2]$. It is clear that $B\in S_{l,r}$ and that $(P\mu)^\wedge(n)=0$ for $n\notin B$. Hence $|\hat{\nu}(n)|\leqslant\varepsilon$ for $n\notin B$. ∎

If $\mathscr{D}$ is the class of countable unions of $H^{(m)}$-sets, Meyer's sets, and sets of resolution, then $\mathscr{D}^\perp\neq R$ since $\mathscr{D}$ is annihilated by all hyperlacunary Riesz products μ with $R(\mu)\leqslant 1/3$. If hyperlacunary Riesz products annihilated all U-sets, they would resemble pseudofunctions quite strongly indeed. Now for any closed set E, $N(E)\supseteq M(E)$ and $\|\mu\|_{PM}=\|\mu\|_M$ for $\mu\in M^+(E)$. Hence $\eta(E)\leqslant s^+(E)$ and $\mathscr{D}\subset U\subset U_1\subset(U_1')_\sigma\subset\{s^+>0\}_\sigma$. We have shown that $\mathscr{D}$ is small while $\{s^+>0\}$ is large. In order to get more information on the size of U, we introduce the following class intermediate between U_1' and $\{s^+>0\}$. Set

$$s_\infty(E)=\inf\{R(\mu)/\|\mu\|_{PM}\colon 0\neq\mu\in M(E)\}.$$

Then $\eta(E)\leqslant s_\infty(E)\leqslant s^+(E)$, so that $U_1'\subset\{s_\infty>0\}\subset\{s^+>0\}$. Although we have not been able to resolve the question whether $\{s_\infty>0\}^\perp=R$, we shall show that $\{s_\infty>0\}$ is a much larger class than $\mathscr{D}$, the class of "known" U-sets. Note that $s(E)\leqslant s_\infty(E)$. The class $\{s_\infty>0\}$ is also much larger than the Helson sets. These statements are consequences of the following theorem.

THEOREM 16. *Let* $\mu=\prod_{k=1}^{\infty}(1+\mathrm{Re}\{\alpha_k e(n_k x)\})$ *be a Riesz product with* $|\alpha_k|\leqslant 1$ *and* $n_{k+1}/n_k\geqslant(3+\sqrt{5})/2=2.618^+$. *Then* μ *is concentrated on a set* E *with* $s_\infty(E)=s^+(E)=R(\mu)$.

Proof. The fact that the product above converges weak* to a probability measure (even without assuming $n_{k+1}/n_k\geqslant 3$) is not hard to show. Now for $k\neq j$, $\hat{\mu}(n_k-n_j)=\hat{\mu}(n_k)\hat{\mu}(-n_j)$. In other words, the random variables $\{e(n_k x)\}_k$ are uncorrelated with respect to the probability measure μ. If $\{n_{k_l}\}$ is a subsequence such that $\hat{\mu}(n_{k_l})\to\omega R(\mu)$, $|\omega|=1$, then by the strong law of large numbers for uncorrelated random variables [4, Theorem IV.5.2, p. 158], μ is concentrated on the set

$$E=\{t\colon \lim_{L\to\infty}L^{-1}\sum_{l=1}^{L}e(-n_{k_l}t)=\omega R(\mu)\}. \tag{15}$$

Since $\mu\in M^+(E)$, we have $s^+(E)\leqslant R(\mu)$. It remains to show that $s_\infty(E)\geqslant R(\mu)$. Now if $\nu\in M(E)$, then for all $m\in\mathbf{Z}$, we have

$$\begin{aligned}R(\mu)|\hat{\nu}(m)|&=\Big|\int_E\lim_{L\to\infty}L^{-1}\sum_{l=1}^{L}e(-n_{k_l}t)\,e(-mt)\,d\nu(t)\Big|\\&=\Big|\lim_{L\to\infty}\int_T L^{-1}\sum_{l=1}^{L}e\big((-n_{k_l}-m)t\big)\,d\nu(t)\Big|\\&=\lim_{L\to\infty}\Big|L^{-1}\sum_{l=1}^{L}\hat{\nu}(n_{k_l}+m)\Big|\leqslant R(\nu).\end{aligned}$$

Hence $R(\mu)\|\nu\|_{PM}\leqslant R(\nu)$, which finishes the proof. ∎

The set E in (15) is, of course, not closed. However, by Egorov's theorem, there exist $\varepsilon_L>0$, $\varepsilon_L\to 0$, such that if

$$F=\{t\colon \forall L\ \Big|L^{-1}\sum_{l=1}^{L}e(-n_{k_l}t)-\omega R(\mu)\Big|\leqslant\varepsilon_L\},$$

then μF is as close to 1 as desired; F is a closed set. Is F a U-set? More generally, if $n_k\uparrow\infty$, $0<|\alpha|\leqslant 1$, $\varepsilon_K\to 0$, is

$$\{t\colon \forall K\ \Big|K^{-1}\sum_{k=1}^{K}e(n_k t)-\alpha\Big|\leqslant\varepsilon_K\}$$

a U-set? This question is extremely interesting not only for the problem at hand ($R\overset{?}{=}U^\perp$), but also because it seems to lie just beyond present techniques for decision.

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A remark on entropy of Abelian groups and the invariant uniform approximation property

by

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Abstract. Let G be a compact Abelian group and let Λ be a finite set of characters on G. We prove that there exists $K \in L_1(G)$ with $\|K\|_1 \leqslant 2$ such that $\hat{K}(\gamma) = 1$ for $\gamma \in \Lambda$ and $\hat{K}(\gamma) \neq 0$ for at most $C^{|\Lambda|}$ characters γ, where C is an absolute constant. In fact, for this type of uniform approximation on G, we obtain more precise estimates in terms of appropriate entropy numbers.

1. Introduction. Let G be a compact Abelian group and Λ a finite subset of the character group $\Gamma = G^*$ (the compactness hypothesis is in fact nonessential and the main result may also be formulated for locally compact groups). Given $\varepsilon > 0$, we consider functions K satisfying the conditions

$$\|K\|_1 < 1+\varepsilon, \tag{1}$$

$$\hat{K}(\gamma) = 1 \quad \text{for each } \gamma \in \Lambda, \tag{2}$$

and where $|\operatorname{supp}\hat{K}|$ (= the size of the support of the Fourier transform of K) is as small as possible. This problem of invariant uniform approximation was considered in [B–P] where an estimate on $|\operatorname{supp}\hat{K}|$ is proved using combinatorial methods.

Associate with Λ the following invariant pseudo-metric on G:

$$d_\Lambda(x, y) = \sup_{\gamma \in \Lambda} |\gamma(x) - \gamma(y)|,$$

and denote by $N_\Lambda(\varrho)$ the corresponding entropy numbers for $\varrho > 0$. The purpose of this note is to show the following fact.

THEOREM 1. *If* $0 < \varepsilon \leqslant 1$, *then there exists* K *satisfying* (1), (2) *and*

$$\log|\operatorname{supp}\hat{K}| \leqslant 8(\log_2(120/\varepsilon))\log N_\Lambda(1/20). \tag{3}$$

In particular, we can find K such that (2), $\|K\|_1 \leqslant 2$ and $|\operatorname{supp}\hat{K}| < C^{|\Lambda|}$ where C is a fixed constant. As has been observed by W. B. Johnson (cf. [F–J–S]), this exponential estimate is the best one can hope for. This is clear from the following example (answering also a question at the end of [B–P]).

Let $G = \{1, -1\}^N$ be the Cantor group and $\Lambda = \{\varepsilon_1, \ldots, \varepsilon_n\}$ the first n Rademacher functions. Assume K fulfills (2). Then by Khintchine's inequali-