The size of some classes of thin sets

by

RUSSELL LYONS* (Stanford, Cal.)

Abstract. The size of a class of subsets of the circle is reflected by the family of measures which annihilate all the sets belonging to the given class. For subclasses of \( U \), the sets of uniqueness in the wide sense, the corresponding family of annihilating measures always includes \( M_\mu(T) \). We investigate when there are no other annihilating measures, in which case the class of sets is "large". For example, Helson sets are shown not to form a large class, while a closely related class does. The fact that another class of sets, the \( H \)-sets, is "small" disproves a conjecture of Rajchman. The class of sets of uniqueness (in the strict sense) is investigated in detail. Tools used include Riesz products and asymptotic distribution.

1. Introduction. Borel subsets of the circle \( T = \mathbb{R}/\mathbb{Z} \) which are called "thin" in harmonic analysis are usually sets of uniqueness in the wide sense, or \( U \)-sets [10]. Recall that a \( U \)-set \( T \) is a (Borel) set which has zero measure with respect to every measure belonging to \( M_\mu(T) = \{ \mu \in M(T) : \lim_{n \to \infty} \mu(n) = 0 \} \), where \( M(T) \) denotes the (finite) complex Borel measures on \( T \) and \( M_\mu(T) = \{ e^{-2\pi i m \mu(t)} : m \in \mathbb{Z} \} \). We also denote \( M_\mu(T) \) by \( R \). Given two classes of thin sets \( \mathcal{C}_1, \mathcal{C}_2 \subset U_0 \), we may consider \( \mathcal{C}_1 \) to be "much larger" than \( \mathcal{C}_2 \) if there is a measure concentrated on some set from \( \mathcal{C}_1 \) which annihilates every set in \( \mathcal{C}_2 \), but not vice versa. This is equivalent to the statement \( \mathcal{C}_2 \preceq \mathcal{C}_1 \), where we denote

\[ \mathcal{C}_2 = \{ \mu \in M(T) : \forall E \in \mathcal{C}_1 \{ \mu(E) = 0 \} \} \]

In this case, it is not hard to see that every measure concentrated on a set from \( \mathcal{C}_2 \) is also concentrated on a countable union of sets from \( \mathcal{C}_1 \).

Now for any class \( \mathcal{C} \subset U_0 \), we have \( R \subset U_0 \subset \mathcal{C} \). In fact, \( R = U_0 \cup \{ \} \) [11, 12, 13]; we shall be interested here in seeing whether certain other classes \( \mathcal{C} \) share this property (\( \mathcal{C}^2 = R \)). Such classes are as "large" as \( U_0 \) itself. This investigation was begun in [11].

* This material is based upon work supported by the National Science Foundation (USA), by the North Atlantic Treaty Organization under a Grant awarded in 1983, and by an American Mathematical Society Research Fellowship.
Riesz products will be our tools for showing the smallness of certain classes. We shall see how Riesz products resemble the measures in $R$ by virtue of the gaps in and the multiplicative structure of their spectrum.

2. $H^{m}$-sets. We shall write $e(i)$ for $e^{2\pi i}$. Recall that a Borel set $E \subset T$ is called a set of uniqueness, or $U$-set, if the only trigonometric series $\sum_{n=-\infty}^{\infty} c_{n}e^{2\pi in}$ which converges to 0 for all $f \neq E$ is the 0-series: $c_{n} = 0$. It has long been known that $U$-sets are $\mathcal{U}_{0}$-sets and that countable unions of closed $U$-sets are $U$-sets [20, I, Chap. IX, §6]. Rajchman [18, 19] introduced the first class of uncountable $U$-sets, which he called $H$-sets. These are Borel sets $E \subset T$ for which there exist a sequence $\{n_{k}\}$ of positive integers tending to $\infty$ and a nonempty open arc $I \subset T$ such that for all $x \in E$ and all $k$, $n_{k}x \notin I$. It is clear that $H$-sets are contained in closed $H$-sets, hence that countable unions of $H$-sets, denoted $H_{\infty}$-sets, are also $U$-sets. The converse, conjectured by Rajchman, was finally shown to be false by Pyatetskii-Shapiro, who introduced the classes $H^{m}$ [17, I, Chap. XIV, §§15, 16], [20, I, Chap. IX, §6]. We have $H = H^{1} \subset H^{2} \subset \cdots \subset H^{m} \subset H^{m+1} \subset \cdots \subset H^{\infty}$, but, for each $m$, there is an $H^{m+1}$-set which cannot be written as a countable union of $H^{m}$-sets. It would be interesting to know if $H^{m+1}$ is in fact larger than $H^{m}$ in the sense given in the introduction.

Rajchman also conjectured [3, pp. 85–86] that $R = H^{1}$, but this too is false [11, 12]. In fact, in [11, §III,B], we showed that $R \neq H^{m+1}$ for any $m$. Here we shall use an entirely different approach to the problem and shall show that

$$R \neq H^{m+1}.$$  

We recall some definitions. If $V = (v^{(1)}, \ldots, v^{(m)}) \in Z^{m}$, $A = (l_{1}, \ldots, l_{m}) \in Z^{m}$ and $x \in T$, we write $V \cdot A = \sum_{i=1}^{m} v^{(i)}l_{i}$ and $Vx = (v^{(1)}x, \ldots, v^{(m)}x)$.

**DEFINITIONS.** Let $m \in Z^{+}$. A sequence $\{V_{k}\}_{k=1}^{\infty} \subset (Z^{+})^{m}$ of $m$-tuples of positive integers is called quasi-independent if for each fixed $A \in Z^{m}$, $A$ not the 0-vector, we have $|V_{k} \cdot A| \to \infty$ as $k \to \infty$. A Borel set $E \subset T$ is called an $H^{m}$-set if there is a quasi-independent sequence $\{V_{k}\}_{k=1}^{\infty} \subset (Z^{+})^{m}$ and a nonempty open set $I \subset T^{m}$ such that for all $x \in E$ and all $k$, $V_{k}x \notin I$. A box $I \subset T^{m}$ is a Cartesian product of arcs $I_{j} \subset T$: $I = I_{1} \times I_{2} \times \cdots \times I_{m}$. A sequence $\{x_{k}\}_{k=1}^{\infty} \subset T^{m}$ has the asymptotic distribution $\nu \in M(T^{m})$, written $[x_{k}] \sim \nu$, if for every box $I \subset T^{m}$ whose boundary has $\nu$-measure 0, we have

$$\lim_{k \to \infty} \frac{1}{k} \cdot \text{card} \{k \in K : x_{k} \in I\} = \nu(I).$$

Recall that Weyl's criterion [20, I, Chap. IV, (4.25)] says that $[x_{k}] \sim \nu$ if and only if for all $l_{1}, \ldots, l_{m} \in Z$,

$$\lim_{k \to \infty} \prod_{k \leq K} \sum_{x \in K} e^{2\pi i(l_{1}x^{(1)}+\cdots+l_{m}x^{(m)})} = \prod_{k \leq K} \sum_{x \in K} e^{2\pi i(l_{1}x_{1}^{(1)}+\cdots+l_{m}x_{m}^{(m)})} = \prod_{k \leq K} \sum_{x \in K} e^{2\pi i(-l_{1}, \ldots, -l_{m})},$$

where $x_{k} = (x_{1}^{(1)}, \ldots, x_{m}^{(m)})$.

**THEOREM 1.** Let

$$\mu = \prod_{A \in U} \{1 + \text{Re} [\langle x, e(x_{A}) \rangle]\}$$

be a Riesz product with $|a_{k}| \leq 1$ and $n_{k+1}/n_{k} \to \infty$. Given any quasi-independent sequence $\{V_{k}\}_{k=1}^{\infty} \subset (Z^{+})^{m}$, there exist a subsequence $\{V_{k}\}$ and a set $D \subset Z^{m}$ of cardinality at most $3^{m}$ such that for $\mu$-almost all $x$, $[V_{k}x]_{k=1}^{\infty}$ has an asymptotic distribution $\nu_{k}$ with spectrum in $D$: $\nu_{k}(A) = 0$ if $A \notin D$.

The result (1) now follows: let $\mu$ be as in the theorem with $a_{k} \to 0$; then $\mu \notin R$. If $E$ is any $H^{m}$-set, there is a quasi-independent sequence $\{V_{k}\}_{k=1}^{\infty} \subset (Z^{+})^{m}$ such that $[V_{k}x]$ is not dense in $T^{m}$ for any $x \in E$. But if $[V_{k}x]$ is the subsequence given by the theorem, then for $\mu$-almost every $x$, $[V_{k}x]$ has the distribution of a trigonometric polynomial and hence is dense. Therefore $\mu B = 0$ and so $\mu \notin H^{m+1}$.

In order to prove Theorem 1, we need two lemmas (which are easy in the case $m = 1$; indeed, a stronger form of Lemma 3 will be proved for $m = 1$ in the course of proving Theorem 6).

**LEMMA 2.** Let $d > 0$ and let $\{A^{(i)}\}_{i=1}^{m+1} \subset [-d, d]^{m+1} \cap Z^{m}$. There is a linear dependence relation

$$\sum_{i=1}^{m+1} c_{i}A^{(i)} = 0$$

with $c_{i} \in Z$ not all 0 and $|c_{i}| \leq d^{m+2}$.

**Proof.** Let $A^{(0)} = (0, \ldots, 0, 0)$. Since we have $m+1$ vectors $A^{(0)}$ in an $m$-dimensional vector space $R^{n}$, one of the vectors, say $A^{(m+1)}$, is linearly dependent on the others:

$$\sum_{j=1}^{m+1} b_{j}A^{(j)} = A^{(m+1)}.$$  

By Cramer's rule, $b_{j}$ can be written as the quotient of determinants with entries $\delta_{l,j}$. Let $c_{j}$ be the determinant in the numerator of $b_{j}$ and let $-c_{m+1}$ be the common determinant of the denominators. Hadamard's inequality,

$$|\det (a_{l,j})| \leq \prod_{l=1}^{n} |a_{l,j}|^{1/2},$$

now gives the result when (2) is multiplied through by $-c_{m+1}$, since $|\delta_{l,j}| \leq d \cdot n$.}
Lemma 3. Let \( \{ Y_j \}_{j=1}^m \subset (\mathbb{Z}^n)^m \) be quasi-independent, let \( \{ n_k \}_{k=1}^n \) be hypercyclic (i.e., \( n_k \to \infty \)), let \( L \in \mathbb{Z}^n \), and let \( A \) be a finite subset of \( \mathbb{Z}^n \) containing 0. Denote the cardinality of \( A \) by \( |A| \) and let \( D \) be any finite subset of \( \mathbb{Z}^m \). Set

\[
\Omega = \{ \sum_{k=1}^n \epsilon_k n_k : \epsilon_k \in A \text{ and } \epsilon_k = 0 \text{ for all but finitely many } k \}.
\]

Then for all sufficiently large \( j \), the number of \( A \in D \) such that

\[
|V_j : A - D| \leq L
\]

is at most \( |A|^m \). As a function of \( |A| \) and \( m \), this upper bound is best possible.

We have written \( |V_j : A - D| \) for the distance from \( V_j : A \) to \( \Omega \). In proving Theorem 1, we shall use the case \( A = \{-1, 0, 1\} \). I am thankful to W. H. Montgomery for the argument providing the best bound in Lemma 3.

Note that (4) is equivalent to the system

\[
|V_j : A - \sum_{k=1}^n \epsilon_k n_k | \leq L, \quad A \in D, \quad \epsilon_k \in A,
\]

(5)

\[
\epsilon_k = 0 \text{ for all but finitely many } k.
\]

Proof. We begin by showing that no bound can be better than \( |D|^m \). Choose \( V_j = \{ n_{j_1}, n_{j_2}, \ldots, n_{j_m} \} \) and \( D = A^m \). Then for every \( j \), every \( A \in D \) is a solution to (4).

We now prove the rest of the lemma by showing that in some sense the example just given is typical; we show that there exist \( k_1, \ldots, k_n \) such that \( \epsilon_{k_1}, \ldots, \epsilon_{k_n} \) determine the solution \( \{ n_k \}_{k=1}^n \) to (5), and that for large \( j \), \( \{ n_k \}_{k=1}^n \) in turn uniquely determines \( A \).

Let \( M = \max \{ |x| : x \in A \} \) and fix \( j \). Let \( d \) be the maximum absolute value of the coordinates of \( A \) over all \( A \in D \). Consider any \( m+1 \) solutions

\[
(A^{(1)}, \ldots, A^{(m+1)}), \quad 1 \leq r \leq m+1,
\]

to (5). Let \( c_1, \ldots, c_{m+1} \) be as in Lemma 2. Define

\[
h^0 = V_j : A^{(0)} - \sum_{k=1}^{m+1} c_k n_k,
\]

so that \( |h^0| \leq L \). Then

\[
\sum_{j=1}^m c_j h^0 = V_j : A^{(0)} - \sum_{k=1}^{m+1} c_k n_k = -\sum_{k=1}^{m+1} c_k \epsilon_k,
\]

where \( \delta_k = \sum_{j=1}^m c_j n_k \). From our bounds on \( c_\epsilon \), \( h^0 \), and \( d^0 \), we see that

\[
|\sum_{k=1}^n \epsilon_k \delta_k| = |\sum_{k=1}^m c_k h^0| \leq (m+1) L d^0 m^m 2^m,
\]

(6)

\[
|\delta_k| = |\sum_{j=1}^m c_j d^0_j| \leq (m+1) M d^0 m^m 2^m.
\]

But since \( n_{k+1}/n_k \to \infty \), (6) implies that there exists some \( k_0 = k_0(L, M, d, m) \) (\( k_0 \) does not depend on \( j \)) such that \( \delta_k = 0 \) for all \( k \geq k_0 \). That is, the vectors

\[
(c_{k_0}^0, c_{k_0+1}^0, \ldots), \quad 1 \leq r \leq m+1,
\]

are linearly dependent.

We have thus demonstrated that for fixed \( j \),

\[
\{ n_k \}_{k=k_0}^n \quad \text{is a solution of (5)}
\]

belongs to an \( m \)-dimensional space. There are therefore \( m \) coordinates \( \epsilon_{k_0}, \ldots, \epsilon_{k_n} \) which determine all \( \epsilon_k, k \geq k_0 \). Since there are only \( |D| \) choices for each \( \epsilon_k \), there are at most \( |D|^m \) solutions \( \{ n_k \}_{k=1}^n \) to (5). But we claim that for large \( j \), each such solution corresponds to exactly one solution \( A \). For let

\[
N = \max \{ |\sum_{k=1}^{k_0} \epsilon_k n_k : \epsilon_k \in A - D \},
\]

where \( A - D = (e - e', e, e' \in A) \). By quasi-independence of \( \{ Y_j \} \), there exists \( j_0 \) such that for each \( j \geq j_0 \), we have

\[
\inf |V_j : A| : 0 \neq A \in D - D > N + 2L,
\]

where \( D - D = \{ A_1 - A_2 : A_1, A_2 \in D \} \). Now suppose that \( A^{(1)}, \{ n_k \}_{k=k_0}^n \), \( A^{(2)}, \{ n_k \}_{k=k_0}^n \) are two solutions of (5) for some \( j \geq j_0 \). Then for some \( \epsilon_k \in D - D (1 \leq k < k_0) \),

\[
|V_j : A^{(1)} - A^{(2)}| - \sum_{k=1}^{k_0} \epsilon_k n_k | \leq 2L.
\]

Since \( A^{(1)} - A^{(2)} \in D - D \), the definition of \( j_0 \) implies that \( A^{(1)} - A^{(2)} = 0 \). This establishes the claim and finishes the proof.

Proof of Theorem 1. By (11), \( V_j \) and \( \{ Y_j \}_{j=1}^m \) may choose \( \{ Y_j \}_{j=1}^m \subset \{ Y_j \}_{j=1}^m \) so that there exist \( r_x \in M(T^m), x \in T \), such that for any further subsequence \( \{ Y_j \} \) of \( \{ Y_j \} \) and for \( \mu \)-almost all \( x \), \( \{ Y_j x \} \) has the asymptotic distribution \( r_x \). We shall show that \( \{ Y_j \} \) is the desired subsequence.

Let \( f_\delta(x) = r_x(-A), A \in \mathbb{Z}^n \). We claim that \( \{ e(Y_j : A)x \}_{j=1}^m \) converges to
is positive, where, for \( \mu \in M(T) \), we write

\[
R(\mu) = \limsup_{|n| \to \infty} |\hat{\mu}(n)|.
\]

Of course, it is immediate from this criterion and the fact that \( \nu \ll \mu \in R \Rightarrow \nu \in R \) [3, Proposition 1.5.1] that Helson sets are \( U_{\infty} \)-sets. Consider the class of Borel sets \( E \) such that

\[
s^+(E) = \inf \{ R(\mu)||\mu||_{M\ast}\; \forall \mu \in M^+(E) \}
\]

is positive, where \( M^+(E) \) consists of the positive measures concentrated on \( E \). Since \( s(E) \leq s^+(E) \), this class includes the Helson sets. It also includes, for example, the weak Dirichlet sets, these being precisely the sets for which \( s^+(E) = 1 \) [11, § III.7]. As above, it is a class of \( U_{\infty} \)-sets. This class, however, is a "large" class (Theorem 7). It should be noted that for any \( \varepsilon > 0 \), the class \( \{ E : s^+(E) \geq \varepsilon \} \) is not large ([11, § III.7] or Corollary 12 below).

For \( \mu \in M(T) \), write

\[
\Sigma = \{ \sigma \in \mathbb{Z} : |\hat{\mu}(\sigma)| > \varepsilon \} \quad \text{and} \quad \Sigma = \{ n : n \in \mathbb{Z} \}.
\]

We shall say that \( \Sigma, \Sigma \) has arbitrarily long gaps if its complement in \( \mathbb{Z} \) contains arbitrarily long intervals.

**Theorem 4.** Let \( \mu \) be a measure such that for all \( \varepsilon > 0 \), \( \Sigma, \Sigma \) has arbitrarily long gaps. Then \( |\mu|(E) = 0 \) for all Helson sets \( E \).

Note that all measures in \( R \) trivially satisfy this property, so that Helson's theorem is a corollary. Furthermore, to exhibit a measure \( \mu \notin R \) satisfying the hypothesis, we consider a Riesz product

\[
\mu = \prod_{i=1}^{\infty} \left( 1 + \Re \{ \lambda \sigma(n_i \chi) \} \right)
\]

with \( |\lambda| < 1 \), \( \sigma \to 0 \), and \( n_i \to n_i \geq q > 5 \). Then \( \mu \notin R \) and \( \Sigma, \Sigma \) has the gaps \( \sum_{i=1}^{k} n_i + 1 \) of length

\[
n_i - 4 \sum_{j=1}^{k} n_j + n_{i+1} \left( 1 - \frac{4}{\sqrt{q - 1}} \right) > n_{i+1} \left( 1 - \frac{4}{\sqrt{q - 1}} \right),
\]

which tends to infinity with \( k \) since \( q > 5 \).

To prove Theorem 4, we shall follow the method used by Kahane and Salem in their proof of Helson's theorem. It depends on the following lemma [5, p. 112]. I am grateful to Garrath McGehee for having brought this method to my attention.

**Lemma 5.** Given \( r_1, \ldots, r_M \in \mathbb{Z} \) and \( \mu \in M(T) \), there exist \( a_1, \ldots, a_M = \pm 1 \) such that if

\[
v = \sum_{n=1}^{M} a_n e^{i \sigma(n \chi)},
\]

then \( |v| \geq \sqrt{M^3 ||\mu||_{M\ast}} \).

**Proof of Theorem 4.** Let \( E \) be any set of positive \( |\mu| \)-measure. We...
shall show that $\alpha(E) = \infty$. Without loss of generality, we may assume that $\|\mu\|_{\text{M}} = 1$.

Let $M$ be a positive integer. There is a trigonometric polynomial $P$ such that
\[ \|P \mu - (\delta_E)\|_{\text{M}} < M^{-1}. \]
Write $L = \deg P$, $A = \sum_{k=0}^{L} \hat{P}(k)$ and $e = \min\{M(AM)^{-1}, M^{-1}\}$. The hypothesis on $\Sigma$ allows us to choose inductively $r_1, \ldots, r_M$ such that for $j \neq m$, the distance between $r_m - r_j$ and $\Sigma_j - \Sigma_m > 2L$. With such a choice, the distance between $r_m + \Sigma_m$ and $r_j + \Sigma_j > 2L$, i.e.,
\[ (r_m + \Sigma_m + [-L, L]) \cap (r_j + \Sigma_j + [-L, L]) = \emptyset. \]
If $S_k$ is the pseudomeasure equal to $\hat{\mu}$ on $\Sigma_k$ and 0 elsewhere, it follows that the spectra of $\phi(r_m) P S_k$ and $\psi(r_j) P S_k$ do not intersect. Therefore if $a_m = \pm 1$, we have
\[ \| \sum_{m=1}^{M} a_m \phi(r_m) P S_k \|_{\text{PM}} = \sup_m \| a_m \phi(r_m) P S_k \|_{\text{PM}} = \| P S_k \|_{\text{PM}} \]
\[ \leq \| P (S_k - \mu) \|_{\text{PM}} + \| P \mu - (\delta_E) \|_{\text{M}} + |\mu| (E) \]
\[ \leq A e + M^{-1} + 1 \leq 3, \]
and hence if $\nu = \sum_{m=1}^{M} a_m \phi(r_m) \mu_S$, then
\[ \| \nu \|_{\text{PM}} = \| \sum_{m=1}^{M} a_m \phi(r_m) \|_{\text{PM}} \leq \| P \mu - (\delta_E) \|_{\text{M}} + |\mu| (E) \]
\[ \leq A e + M^{-1} + 1 \leq 3. \]
But if $a_m$ are chosen as in the lemma, it follows that $\|\nu\|_M \leq 3$.

The utility of this condition lies in the fact that all such sets are of analyticity [9]. It immediately follows that nonanalytic sets are $U_2$-sets. As another corollary, we shall deduce that they do not form a "large" class:

**Theorem 6.** Let
\[ \mu = \prod_{n=1}^{\infty} \left(1 + \text{Re} \{a_n e(n_k)\}\right) \]
be a hyperlacunary Riesz product: $|a_k| \leq 1$, $n_{k+1}/n_k \to \infty$. Then $\mu E = 0$ for all nonanalytic sets $E$.

**Proof.** It suffices to show that if $\mu E > 0$, then $E$ satisfies condition (R). Indeed, let $v = \mu_E$ and $N > 1$. Let $Q$ be a trigonometric polynomial such that $|v - Q|_{\text{M}} \leq \|v\|_M / (2N + 1)$. Set $\Omega$ to be as in (3) (with $\delta = (-1, 0, 1)$) and $\Omega' = \Omega + [-L, L]$, where $L$ is the degree of $Q$. The spectrum of $Q \mu$ is contained in $\Omega'$. Choose $m = n_k$, where $k \geq k_0 \gg n_k/n_k - 1 > 10(N + L)$. We claim that for all $p \in \mathbb{Z}$,
\[ \text{card}(\{p - nm : |n| \leq N\} \cap \Omega') \leq 3. \]
The proof will then be complete, since it follows that
\[ \sum_{|n| \leq N} |f(p - nm)| \leq \|v\|_M + \sum_{|n| \leq N} |Q\mu'|(p - nm) \]
\[ \leq \|v\|_M \left[1 + 3 \left(1 + \frac{1}{2N + 1} \right) \right] \leq 5 \|v\|_M. \]

To prove the claim, suppose that $p + nm, p + n'm \in \Omega'$, $|n| \leq N$, $|n'| \leq N$, $n \neq n'$. Then by subtraction, we can write
\[ (n - n') m = e_1 n_k + e_2 n_k + \ldots + e_l, \]
where $k_1 > k_2 > \ldots, e_i = \pm 1, \pm 2$, and $|\nu| \leq 2L$. Dividing by $n_k$, we obtain
\[ |n - n'| n_{2l} / n_{k-1} - e_1 \leq 2(n_{k-1} + n_{k-2} + \ldots + n_1 + n_1) / n_k. \]
If $k_1 < k_0$, then the right-hand side is less than $1 + 1/2L n_k$ (using $n_{k-1} \gg 3$ for all $L$) while the left-hand side is greater than $10(N + L - 1)$, an impossibility. Hence $k_1 > k_0$ and the right-hand side is less than $2/5$. If $k_1 > k_0$, the left-hand side is at least $4/5$, whence $k_1 = k_0$. Since the left-hand side is then an integer, it must be zero: $n - n' = e_1$. In other words, for any $n, n'$ as indicated, $|n - n'| \leq 2$. Choosing the largest and smallest $n, n'$ establishes the claim.

We now show that $\{E : s^2(E) > 0\} = R$.

**Theorem 7.** If $\mu \notin R$, then there exists a set $E$ of positive $\mu$-measure such that $s^2(E) > R(\mu) / \|\mu\|_M$.

**Proof.** We may assume that $\mu$ is a probability measure. Choose $n_k \to \infty$ so that $\mu(n_k) \to \infty R(\mu)$, where $|\nu| = 1$, and such that
\[ f(t) = \lim_{K \to \infty} \frac{1}{K} \sum_{k=-K}^{K} \nu(-n_k t) \]
exists $\mu$-a.e. (11, § III.2) or (13)) Since $R(\mu) = \int \nu f df \mu = \int \text{Re} \{\nu f\} df \mu$, it follows that
\[ E = \{t : f(t) \text{ exists and } \text{Re} \{\nu f(t)\} \geq R(\mu)\} \]
is not of $\mu$-measure 0.

Now if $\nu M^*(E)$, then
\[ R(\psi) := \{ \varphi \mid \varphi \text{satisfies some condition} \} \]

\[ = \text{Re} \left[ \varphi \lim_{K \to \infty} \sum_{k=1}^{K} e(-n_k) \psi((t)) \right] \]

\[ = \text{lim sup}_{K \to \infty} \sum_{k=1}^{K} \psi(n_k) \leq \limsup_{K \to \infty} \sum_{k=1}^{K} |\psi(n_k)|, \]

Hence \( R(\psi) \geq R(\|\psi\|) \).

**4. Sets of uniqueness.** Sets of uniqueness have been under study for over 100 years. During this period, many beautiful results have been obtained, but relatively few examples of \( U \)-sets are actually known. Some ingenious theorems show that \( U \) is indeed a fairly limited subclass of \( U_0 \) (see [5, Chap. 4]).

Yet it appears that many more \( U \)-sets await to be discovered. As an approach to determining the size of \( U \), the question “is \( \lambda U \)?” is particularly fascinating. We have not been able to resolve it, but true or false, the answer would be extremely interesting. Thus, if \( \lambda U = U \), then for all \( \mu \in R \), there would be a \( U \)-set with positive \( \mu \)-measure. This would produce a wealth of \( U \)-sets, including necessarily \( U \)-sets as yet unknown. On the other hand, if \( \lambda U \neq U \), then for some \( \mu \in R \), \( \mu = 0 \) for all \( E \in U \). In other words, the property of not supporting any pseudomeasure implies the property of not supporting any part of a certain fixed measure which is not a pseudofunction \( f \). (Recall that the pseudofunctions, \( \mathcal{P}(T) \), are those pseudomeasures \( S \) such that \( S(n) \to 0 \) as \( |n| \to \infty \). A closed set is a \( U \)-set if and only if it does not support any pseudofunction [5, Theorem 4.2.1].) We would also obtain new \( U \)-sets (i.e., sets not of uniqueness) from such \( \mu \in U \)-sets.

Note that by regularity, \( U \) is the class of all known \( U \)-sets and of all sets satisfying certain conditions known to be sufficient for uniqueness is small; \( \lambda U \). Finally, we shall consider a class larger than but closely related to \( U \).

By regularity, \( U \) is the class of all known \( U \)-sets and of all sets satisfying certain conditions known to be sufficient for uniqueness is small; \( \lambda U \). Finally, we shall consider a class larger than but closely related to \( U \).

We call \( E \) a \( U \)-set if \( J(E) \) is weak* sequentially dense in \( A(T) \). Now \( U \)-sets can be characterized in another way. Define

\[ \psi(E) = \inf \left\{ \limsup_{\|S\| \to \infty} |S(n)| : 0 \neq S \in \text{PM}(E) \right\}. \]

where \( \text{PM}(E) \) consists of the pseudomeasures supported in \( E \). Then \( E \in U \) if and only if \( \psi(E) > 0 \) [5, Theorems 4.3.2, 4.3.1]. For example, \( \text{E}(\mu) \) are \( \text{U} \)-sets; let \( \{V_j\} \) be quasi-independent, \( t = t_1 \times t_2 \times \ldots \times t_m \) an open box in \( T^m \), and \( E \) a set such that \( V_j \not\in E \) for any \( x \in E \). Then we claim that

\[ \psi(E) \geq \prod_{j=1}^{m} \frac{1}{1 - \lambda_j}, \]

where \( \lambda \) is the Lebesgue measure on \( T \). Note that this lower bound goes from 0 to 1 as the volume of \( I \) increases from 0 to 1. To prove (7), let \( f_0 \) denote the “triangular” function which is zero outside an interval \( Q \subseteq T \) and linear between the endpoints of \( Q \) and the midpoint of \( Q \), where it takes a value of \( 2/|Q| \). Thus \( \int_{Q} f_0 = 1, \|f_0\|_2 = 2/|Q|, \) and so \( \|1 - f_0\|_2 = (2 - |Q|)/|Q| \). Consider the function

\[ g(t_1, \ldots, t_m) = \prod_{j=1}^{m} \left( 1 - f_0(t_j) \right), \]

where \( I_j \) is an interval whose closure lies in \( I_j \). Then \( g(0, \ldots, 0) = 0 \) and \( g = 1 \) in a neighborhood of the complement of \( I \). Hence \( g(V_j) = 1 \) in a neighborhood of \( E \) and for any \( \mu \in \text{PM}(E) \),

\[ \|S \| = \sum_{j \neq k} S_j \cdot \mu_j \cdot \delta_j \cdot (\lambda_j - \lambda_k). \]

Since \( \{V_j\} \) is quasi-independent, \( \|V_j\| \to \infty \) as \( j \to \infty \) for every \( \lambda \neq 0 \) and it follows that

\[ |S(0) - S| \leq \|S\|_\text{PM} \text{ lim sup}_{\psi \to \infty} |S(n)|. \]

Since \( e(-pt)S \in \text{PM}(E) \), the same inequality holds with \( p \neq 0 \), whence

\[ \limsup_{\psi \to \infty} |S(n)| \leq \|S\|_\text{PM} \text{ lim sup}_{\psi \to \infty} |S(n)|. \]

The inequality (7) results from the fact that \( \lambda_j \) can be taken arbitrarily close to \( \lambda_j \).

Similar but more complete results hold when \( J(E) \) is replaced by

\[ I(E) = \{ f \in A(T) : f = 0 \text{ on } E \}. \]

A closed set \( E \) is called a \( U \)-set if \( E \) is weak* sequentially dense in \( A(T) \). It turns out [5, Theorem 4.3.4] that every \( U \)-set is a countable union of \( U \)-sets, whence \( U \) is a countable union of \( U \)-sets. Unfortunately, such results are not known for \( U \)-sets. Evidently, however, \( U \subseteq U \) and \( U \subseteq U \); the inclusions are strict [5, pp. 110-118]. The annihilator of \( J(E) \) in \( \text{PM}(T) \) is \( A(T) \) and the dual of \( A(T)/E \) is \( A(E) \) is denoted by \( N(E) \). (Note that \( M(E) \subseteq N(E) \subseteq \text{PM}(E) \); \( M(E) = N(E) \) precisely when \( E \) is Helson and
$N(E) = PM(E)$ precisely when $E$ obeys synthesis. Set

$$
\eta(E) = \inf \left\{ \limsup \frac{|S(n)|}{|S||PM|} : 0 < S \in N(E) \right\}.
$$

Then $E \in U_1$ if and only if $\eta(E) > 0$ ([5, p. 98]). For example, Helson sets are $U_1$-sets since $s(E) \leq \eta(E)$. Indeed, all nonanalytic sets are $U_1$-sets. For, as remarked by Graham and McGehee [5, p. 402], if $E$ satisfies condition (R) with $M(E)$ replaced by $N(E)$, then $E$ is analytic (the proof is the same). But if $\eta(E) = 0$, then condition (R) is obviously satisfied. It would be interesting to know whether $R = U_1$ or, in any case, whether $U_1 = U_1^\perp$.

The most significant—perhaps the only known examples of $U$-sets are countable unions of $H^{\omega_0}$-sets. These include the Cantor-type symmetric sets which are known to be $U$-sets [15, pp. 89, 93, 96]. It is easy to see that the example of Bari [2, pp. 102–104] is also an $H$-set. The only other examples of $U$-sets known to this author were constructed by McGehee [14], whose work was extended by Meyer [16]. It is unknown whether these are not in fact countable unions of $H^{\omega_0}$-sets.

To define Meyer's sets, we regard $T$ as $[0, 1) \subset \mathbb{R}$. If $T \cap 0 + A \subset R$, we say that $(T, \varepsilon)$ is adapted to $A$ if for all functions $a: A \to C$ sending $\lambda \mapsto a_\lambda$ with $a_\lambda = 0$ for all but finitely many $\lambda$, we have

$$
\sup_{\lambda \in A \cap T} \sum_{k \in \mathbb{N}} a_\lambda e(\lambda k) \geq \varepsilon \sup_{\lambda \in A \cap T} \sum_{k \in \mathbb{N}} a_\lambda e(\lambda k).
$$

Note that $[0, T)$ can be replaced by any interval of length $T$. Suppose that $E \subset \mathbb{R}$ is compact, $A_\varepsilon \subset \mathbb{R}$ are finite, $k_\varepsilon = 0$, $(T, \varepsilon)$ is adapted to $A_\varepsilon$, and $E \subset A_\varepsilon + [-k_\varepsilon, k_\varepsilon]$. Then Meyer [16] proved that $E$ is a set of uniqueness in $\mathbb{R}$ when

$$
\lim_{n \to \infty} \left( \int_{-k_\varepsilon}^{k_\varepsilon} |x|^2 e^{-n|x|} dx \right) = 0.
$$

We are interested only in the case $E \subset [0, 1)$. The essential tool in proving Meyer's result is an approximation of the Fourier transform of pseudomeasures with compact support. When we are concerned with measures, this approximation can be improved. This improvement is what will permit us to demonstrate that all such sets are annihilated by Riesz products.

The approximation of pseudomeasures and measures is given by

**Lemma 6.** Suppose that $(T, \varepsilon)$ is adapted to $A$ and $0 < \pi T < \varepsilon$. Then

$$
|\lambda - \lambda| > 2t
$$

when $\lambda, \lambda \in A$ are distinct. If $S \in PM(A + [-1, 1])$, we define $\sigma \in M(A)$ by

$$
\sigma = \sum_{k \in \mathbb{N}} \left( S(x_{\lambda} - k + \varepsilon) \right) \delta_\lambda,
$$

where $\delta_\lambda$ is the unit mass at $\lambda$ and $x_{\lambda}$ is the indicator function of $I$. Then for all $x \in \mathbb{R}$, we have

$$
|S(x) - \sigma(x)| \leq 2\pi T |x| e^{-\pi T} ||S||_{PM}.
$$

If in addition $S$ is a measure, then

$$
|S(x) - \sigma(x)| \leq 2\pi T |x| ||S||_M.
$$

**Proof.** The first statement follows immediately upon consideration of $|e(\lambda x) - e(\lambda' x)|$. Hence $\sigma$ is well defined. Write

$$
\sigma(x) = \sum_{k \in \mathbb{N}} a_k(x) e(\lambda x),
$$

where the spectrum of $a_k$ lies in $[-1, 1]$. If

$$
f(x, y) = \sum_{k \in \mathbb{N}} a_k(x) e(\lambda x),
$$

then $f(x, y) = \sigma(x)$ and $f(0, x) = \sigma(x)$. Fix any $x_0$; the function $f(x, y_0)$ of $x$ has spectrum in $[-1, 1]$ and can be written for any $x_0$ as

$$
f(x, y_0) = f(x_0, y_0) + \frac{\partial}{\partial x} f(x', y_0) (x - x_0)
$$

for some value $x'$ between $x_0$ and $x$. By Bernstein's inequality ([14, p. 149] or [5, p. 418]), $\partial f/\partial x \leq 2\pi T ||f||_M$, so that for all $x$, $y_0$, $x_0$,

$$
|f(x, y_0) - f(x_0, y_0)| \leq 2\pi T |x - x_0| ||f||_M.
$$

Now when $S$ is a measure, we have

$$
||f||_M \geq \sup_x \sum_{k \in \mathbb{N}} |a_k(x)| \leq \sum_{k \in \mathbb{N}} |S| (e + [-1, 1]) = ||S||_M.
$$

Substitution of this estimate in (11) with $x_0 = 0$ and $y_0 = x$ yields (10).

In general, when $S$ is only a pseudomeasure, we know that for all $x_0$, there exists $y_0 \in [x_0 - T/2, x_0 + T/2]$ such that $|f(x_0, y_0)| \geq \varepsilon \sup_y |f(x_0, y)|$.

For this choice of $y_0$ and $x = y_0$, (11) gives

$$
||S||_{PM} \geq ||f(y_0, y)| = ||f(x_0, y)| = 2\pi T |y_0 - x_0| ||f||_M
$$

$$
\geq \varepsilon \sup_y |f(x_0, y)| = \pi T ||f||_M.
$$

Taking the supremum over $x_0$ yields

$$
||S||_{PM} \geq (\pi T) ||f||_M,
$$

which gives (9) when substituted into (11). 

**Remark.** Very slight improvements can be made by using $\partial^2 f/\partial x^2$ (cf. [8, pp. 118–119]); these do not affect the essence of our results.
Meyer's result now follows easily. Given $E$ as above, we may assume without loss of generality that $\varepsilon_k \to 0$, $\pi \varepsilon_k \to \beta < \varepsilon/(2 + \varepsilon)$, and that $\pi \varepsilon_k \to \varepsilon_k$ for all $k$. For any $S \in PM(E)$, define

$$\sigma_k = \sum_{k \in \mathbb{Z}} \langle S, n_k - 1, 1 \rangle \delta_k.$$  

Choose real numbers $m_k \to \infty$ such that $m_k \varepsilon_k \to 0$. By hypothesis, there exists $x_k \in [m_k, m_k + T_k]$ such that $|\delta_x(x_k)| = \sigma_k ||S||_{PM}$. By (9), we have

$$|S(x_k)| \geq |\delta_x(x_k) - 2m_k| = |m_k - m_k + T_k|^{-1} |S||S|_{PM}$$

$$\geq m_k ||S||_{PM} - 2m_k - (m_k + T_k)(m_k - m_k + T_k)|^{-1} |S||S|_{PM}.$$  

But it also follows from (9) and the fact that $k \to 0$ that $S(x_k) = \lim \delta_x(x_k)$, whence $\lim ||S||_{PM} = ||S||_{PM}$. Therefore

$$\lim_{|x| \to \infty} |S(x)| = \lim_{|x| \to \infty} |x| - \frac{2m_k}{m_k + T_k} = \frac{2m_k}{m_k + T_k} > 0.$$  

Thus, $E$ does not support any pseudofunctions on $R$ and $E$ is a set of uniqueness. To show that if $E = [0, 1]$, then $E$ is a $U$-set on $T$, i.e.,

$$\lim_{|x| \to \infty} |S(x)| = 0 \text{ for all } S \in PM(E),$$

we use the following proposition [15, pp. 86-87].

**Proposition 9.** For every $\varepsilon > 0$, there exists $C > 0$ such that for all pseudomeasures $S$ on $R$ whose support has diameter at most $1 - \varepsilon$,

$$\lim_{|n| \to \infty} |S(n)| = C \lim_{|x| \to \infty} |S(x)|.$$  

Now if $E \subset [0, 1]$, we consider $E$ as being on the circle $S$. Since $E \subset \mathbb{R}$, there exists an arc $(\pi + \varepsilon, \pi)$ in the complement of $E$. We may translate $E$ without changing the value of $\psi(E)$, so we may assume that $E \subset [\pi/2, \pi - \pi/2]$. Any element of $PM(T)$ with support in $E$ then extends naturally to an element of $PM(\mathbb{R})$ with support in $E$: for $S \in PM(T)$, suppose $S \subset E$, define $\hat{S} \in PM(\mathbb{R})$ by $\langle f, \hat{S} \rangle = \langle fV, S \rangle$ for $f \in A(\mathbb{R})$, where $V$ is the piecewise linear function which is 1 on $[\pi/2, \pi - \pi/2]$ and 0 outside $[0, 1]$. It is easily verified that

$$||fV||_{A(\mathbb{R})} \leq ||f||_{A(\mathbb{R})} \max_{0 \leq \varepsilon \leq \sqrt{2}} \sum_{|n| \leq \sqrt{2}} |f|_n(n + \varepsilon)|.$$  

so that this definition makes sense. It is obvious that $\hat{S} = \hat{S}$ for $n \in \mathbb{Z}$. It now follows from (12) and Proposition 9 that $\psi(E) > 0$; $E$ is thus a $U$-set.

Now suppose $S = \varepsilon M^*(E)$, where $E \subset [0, 1]$ is as above. Define $\sigma_k$, $m_k$, and $x_k$ in the same way and note that $||S||_{PM} = ||S||_{M}$. From (10), we have

$$|S(x_k)| \geq |\delta_x(x_k) - 2m_k| = |m_k - m_k + T_k|^{-1} |S||S|_{PM}$$

whence

$$\lim_{x \to \mathbb{R}} \frac{1}{|x|} |S(x)| \geq \frac{2m_k}{m_k + T_k} > 0.$$  

This already places certain restrictions on the measures which can put mass on $E$. Note that (13) is valid even if $m_k \varepsilon_k \to 0$. Using this observation with (8) will allow us to establish the following theorem.

**Theorem 10.** Let $\mu = \prod_{\mathbb{R}} \mathbb{R}$ be a Riesz product with either

(i) $n_{\mu} / n_k \geq 4$ and $R(\mu) = \lim_{|n| \to \infty} |S| = 0$, or

(ii) $n_{\mu} / n_k \to \infty$ and $R(\mu) = 1/3$.

Then $\mu = 0$ for all sets $E \subset [0, 1]$ of the following type: there exist $A_k \subset \mathbb{R}$ for $k \in \mathbb{Z}$, $k \to 0$, $(\mathbb{Q}, \mathbb{Q})$ adapted to $A_k$, $E \subset A_k + [-b_k, b_k]$, and (8) is satisfied.

We shall need some facts about Riesz products and about the relationship between $\mu$ and $R$ on $T$ and on $Z$ for measures $\mu \in M([0, 1])$.

**Lemma 11.** For $\mu \in M([0, 1])$, let $C_\mu = \sup \{R(v) : v \in \mu, v \leq \mu\}$. Then for $v \leq \mu$,

$$\lim_{x \to \mathbb{R}} \frac{1}{|x|} |S(x)| \leq C_\mu |S||S|_{M}.$$  

**Proof.** Let $|x_k| \to \infty$, $x_k = n_k - \delta_k$, $n_k \to \infty$, $x_k \to 0 \subset [0, 1]$. Then $e^{-\delta_k} e^{-\delta_k} \to e^{-\delta_k}$ uniformly on $[0, 1]$, so that for $v \leq \mu$,

$$\lim_{x \to \mathbb{R}} \frac{1}{|x|} |S(x)| = \lim_{x \to \mathbb{R}} \frac{1}{|x|} e^{-\delta_k} e^{-\delta_k} \delta_k \leq R(e^{-\delta_k} e^{-\delta_k}) \leq C_\mu |S||S|_{M}.$$  

**Corollary 12.** If $\mu$ is a Riesz product, then for all $v \leq \mu$,

$$\lim_{x \to \mathbb{R}} \frac{1}{|x|} |S(x)| \leq R(v) |S||S|_{M}.$$  

**Proof.** For any measure $\mu$, consider the space $L^2(\mu)$. Let $\Gamma$ be the subset of $\{e : e \in \mathbb{R} \subset [0, 1] \}^*$ the set of limit points of $\Gamma$ in the weak topology from $L^2(\mu)$. Then with $C_\mu$ as in Lemma 11, we easily calculate

$$C_\mu = \sup \{ ||f\mu||_{L^2(\mu)} : 0 \neq f \in \mu, \chi \in \Gamma \}

= \sup \{ ||f\mu||_{L^2(\mu)} : 0 \neq f \in \mu, \chi \in \Gamma \}

= \sup \{ ||\mu||_1 : \chi \in \Gamma \}.$$  

Now when $\mu$ is a Riesz product, every $\chi \in \Gamma$ has the form $c_\chi$, where $c$ is a constant and $\gamma \in \Gamma$ [6, Chap. 5], [7, Chap. 2, Proposition 13], or [5].
Theorem 7.3.1]. Since $\chi_\nu \in \Gamma_\varphi$ for every $\varphi \in \Gamma^\varphi$, it follows that $C_\varphi = R(\varphi)$ and the lemma gives the desired conclusion. (It should be noted that for Riesz products, we actually have an equality: $R(\nu) = R(\varphi) \|\nu\|_M$. This follows immediately from the above-mentioned form of $\Gamma_\varphi$.)

**Lemma 13.** Let $\mu \in M([0, 1])$. Suppose that $B_k = \{n \in \mathbb{Z}: a_k < n \leq b_k\}$, $a_k \to -\infty$, $b_k \to \infty$, are intervals such that

$$\lim_{k \to \infty} \sup_{n \in B_k} \|\mu(n)\| = 0.$$

If $B_k = \{x \in R: a_k < x \leq b_k\}$, $a_k \to -\infty$, $b_k \to \infty$, are subintervals, then for all $\nu \ll \mu$,

$$\lim_{k \to \infty} \sup_{x \in B_k} \|\nu(x)\| = 0.$$

**Proof.** It suffices to show that if $x_k \in B_k$ and $\|\nu(x_k)\| \to c$, then $c = 0$. We may assume that $x_k = n_k + \delta_k$, $n_k \in B_k$, $\delta_k \to 0$ as $k \to 0$. If $c \neq 0$, then there exists a trigonometric polynomial $P$ such that $\|P - \nu\|_M < c/2$. Put $L = \deg P$. Then

$$c = \lim_{k \to \infty} \|\nu(x_k)\| = \lim_{k \to \infty} \frac{1}{L} \int \left| e(-n_k + \delta_k) e(-\delta_k) d\nu(n) \right|$$

$$< \lim_{k \to \infty} \frac{1}{L} \int \left| e(-n_k) P(n) d\nu(n) \right| + c/2$$

$$= \lim_{k \to \infty} \sum_{|n| \leq L} P(n) \langle \mu(n) \rangle + c/2 = c/2$$

since $n_k \in B_k$ for large $k$. This is a contradiction. ■

**Corollary 14.** Let $\mu = \prod (1 + \text{Re}(\alpha z))$ a sequence with $k_j \to j$, and

$$B_j = \{x \in R: a_j + \delta_j < x \leq b_j + \delta_j \}.$$

Then $\lim_{j \to \infty} \sup_{x \in B_j} \|\nu(x)\| = 0$ for all $\nu \ll \mu$.

**Proof.** It suffices to take

$$B_j = \{n \in \mathbb{Z}: n_j + \delta_j < n \leq n_j + \delta_j + j \}$$

in the lemma. ■

**Proof of Theorem 10.** Suppose to the contrary that $\mu \ll \nu$. Let $\nu = \mu_{\theta}$. We assume that $n_k \to \infty$, $\nu_k \to \nu$. In case (i), set $\theta = 1/3$ and in case (ii), $\theta$ arbitrary but less than $1$. For every $j$, define $k(j)$ by $\theta n_{k(j)} < \theta n_{k(j) + 1}$. We may assume that $k(j) > j$. Let

$$m_j = n_{k(j)} + n_{k(j) + 1} + \ldots + n_{k(j) + j}.$$

It is easily verified that

$$m_j + T_j < n_{k(j) + 1} - n_{k(j) + 1} + \ldots - n_{k(j) + 1}$$

for all $j$ in case (i) and for all sufficiently large $j$ in case (ii). Furthermore, in case (i), we have

$$m_j < n_{k(j) + 1} + n_{k(j) + 1} + \ldots + n_{k(j) + 1} (1 + 4^{-1} + 4^{-2} + \ldots) = \frac{1}{2} n_{k(j)} \leq 4 T_j,$$

while in case (ii), we have similarly $\lim_{j \to \infty} m_j / T_j \leq \theta^{-1}$. Now by Corollary 14,

$$\lim_{j \to \infty} \sup_{x \in B_j} \|\nu(x)\| = 0.$$

On the other hand, from (13), this limit is at least $\lim_{j \to \infty} \frac{1}{2} m_j / T_j \|\nu\|_M$. In case (i), if $j > 1$, then $\frac{1}{2} m_j / T_j \|\nu\|_M$ is finite, hence the method gives $\|\nu\|_M < 0$ as $\theta < 1$. This is true for all $\theta < 1$, it follows that $\|\nu\|_M < 0$, whence $\|\nu\|_M < 0$ and $\|\nu\|_M < 0$ again a contradiction. ■

With a bit more subtlety in choosing $m_j$, we prove the same result for Riesz products $\mu$ with $n_{k(j) + 1} n_j \geq q > 3$ and $R(\mu) \leq \delta = \delta(q)$; however, this method gives $\|\nu\|_M < 0$ as $\theta < 1$.

Note that ultrathin symmetric sets [15, pp. 239, 250] are examples of Meyer's sets described above [15, p. 240]. We now move from examples of U-sets to two known sufficient conditions. If a closed set $E$ is without true pseudomeasure (WTP), i.e., $PM(E) = M(E)$, then $E$ is, of course, a Helson set. Hence $PM(E) = M(E) = (0)$ and $E$ is a U-set. Moreover, $E$ is annihilated by Riesz products whose frequencies satisfy $n_{k(j) + 1} n_j \geq q > 5$ (Section 2 above).

Our second condition sufficient for uniqueness is more substantial. A closed set $E$ is said to be of resolution if every closed subset of $E$ is synthesis. Since every WTP set is of synthesis and every closed subset of a WTP set is WTP, it follows that every WTP set is of resolution. Kahane and Katznelson [9] found the following necessary condition to be a set of resolution. If $l > 0$, $r = 0$, and $E \subset \mathbb{R}$, we say that $E \subset S_{l,r}$ if $B$ is an infinite union of segments of length $l$ separated by at least $r l$. A pseudomeasure $T \subset PM(T)$ is said to satisfy condition (P) if for all positive $\varepsilon$ and $r$, there exist $l > 1$ and $E \subset S_{l,r}$ such that $|T| < \varepsilon$ for $n \notin B$. Kahane and Katznelson's theorem is that sets of resolution do not support any nonzero pseudomeasure satisfying condition (P). Since pseudofunctions satisfy (P), it follows that sets of resolution are U-sets. Another consequence is
Theorem 15. Let \( \mu = \prod (1 + \text{Re} \{a_{k} a_{n}(x, x)\}) \) be a hyperlacunary Riesz product: \( n_{k+1}/n_{k} \to \infty \). Then \( \mu E = 0 \) for all sets of resolution, \( E \).

Proof. If \( \mu E \neq 0 \), we shall show that \( n_{k} \) satisfies condition (P). Given \( \varepsilon > 0 \), let \( P \) be a trigonometric polynomial of degree \( L \) such that \( \|P\|_{\infty} \leq \varepsilon \).

Let \( k_{0} \) be such that for \( k > k_{0} \), we have

\[
n_{k+1} - n_{k} - \ldots - n_{L} > (2\pi + 1) \left(n_{k+1} + \ldots + n_{k+L}\right).
\]

Set \( I = (n_{k+1} + \ldots + n_{k+L}) \) and

\[
\Omega_{k} = \{ \pm n_{k+1} \pm n_{k+2} \pm \ldots \pm n_{k+L}; k_{1} = 0, k_{2} = 0, \ldots, k_{k_{r}} = 0, m > 1 \} \cup \{0\}.
\]

We center an interval of length \( 1 \) about each point of \( \Omega_{k} \); \( B = \Omega_{k} + [-1/2, 1/2] \). It is clear that \( B \in \mathcal{S}_{\nu} \), and that \( (P\mu)^{n} = 0 \) for \( n \notin B \). Hence \( \|\nu|n| \| \leq \varepsilon \) for \( n \notin B \).

If \( \mathcal{S} \) is the class of countable unions of \( \mathcal{H}_{\alpha}^{m} \)-sets, Meyer's sets, and sets of resolution, then \( \mathcal{S} \) is annihilated by all hyperlacunary Riesz products \( \mu \) with \( R(\mu) = 1/3 \). If hyperlacunary Riesz products annihilated all \( U \)-sets, they would resemble pseudofunctions quite strongly indeed. Now for any closed set \( E, N(E) \subset M(E) \) and \( \|\phi\|_{PM} = \|\phi\|_{M} \) for \( \mu \in M^{*}(E) \). Hence \( \eta(E) \subset s^{*}(E) \) and \( \mathcal{S} \subset U \subset U_{1} \subset (U_{1})_{c} \subset s^{*}(E) \). We have shown that \( \eta \) is small while \( s^{*} > 0 \) is large. In order to get more information on the size of \( U_{1} \), we introduce the following class intermediate between \( U_{1} \) and \( s^{*} > 0 \).

Set

\[
s_{\alpha}(E) = \inf \{R(\mu)||\phi||_{PM}; 0 \neq \mu \in M(E)\}.
\]

Then \( \eta(E) \subset s_{\alpha}(E) \subset s^{*}(E) \), so that \( U_{1} \subset \{s_{\alpha} > 0\} \subset \{s^{*} > 0\} \). Although we have not been able to resolve the question whether \( \{s_{\alpha} > 0\} = R \), we shall show that \( \{s_{\alpha} > 0\} \) is a much larger class than \( \mathcal{S} \), the class of "known" \( U \)-sets. Note that \( \mathcal{S} \subset \{s_{\alpha} > 0\} \). The class \( \{s_{\alpha} > 0\} \) is also much larger than the Henle sets. These statements are consequences of the following theorem.

Theorem 16. Let \( \mu = \prod (1 + \text{Re} \{a_{k} a_{n}(x, x)\}) \) be a Riesz product with \( \|a_{k}\|_{1} = 1 \) and \( n_{k+1}/n_{k} > 3(3 + \sqrt{5})/2 = 2.618^{*} \). Then \( \mu \) is concentrated on a set \( E \) with \( s_{\alpha}(E) = s^{*}(E) = R(\mu) \).

Proof. The fact that the product above converges weak* to a probability measure (even without assuming \( n_{k+1}/n_{k} \to 3 \)) is not hard to show. Now for \( k \neq n_{k} \), \( \mu_{n_{k}}(E) = \mu_{n_{k}}(\mathcal{S} \subset E) \). In other words, the random variables \( \{a_{n}(x)\} \) are uncorrelated with respect to the probability measure \( \mu \). If \( \{n_{k}\} \) is a subsequence such that \( \mu_{n_{k}}(E) \to \mu_{n}(E), |n_{k}| = 1 \), then by the strong law of large numbers for uncorrelated random variables [4, Theorem IV.5.2, p. 158], \( \mu \) is concentrated on the set

\[
E = \{t: \lim_{L \to \infty} \sum_{L=1}^{L} e(-n_{k} t) = \omega R(\mu)\}.
\]

References

A remark on entropy of Abelian groups and the invariant uniform approximation property

by

J. BOURGAIN (Bures-sur-Yvette)

Abstract. Let G be a compact Abelian group and let A be a finite set of characters on G. We prove that there exists K ∈ L_p(G) with |K|_p < 2 such that K(γ) = 1 for γ ∈ A and K(γ) ≠ 0 for at most C^p characters γ, where C is an absolute constant. In fact, for this type of uniform approximation on G, we obtain more precise estimates in terms of appropriate entropy numbers.

1. Introduction. Let G be a compact Abelian group and A a finite subset of the character group I = G^* (the compactness hypothesis is in fact nonessential and the main result may also be formulated for locally compact groups). Given ε > 0, we consider functions K satisfying the conditions

\[ |K|_1 < 1 + ε, \]
\[ K(γ) = 1 \quad \text{for each } γ ∈ A, \]

and where |supp K| (= the size of the support of the Fourier transform of K) is as small as possible. This problem of invariant uniform approximation was considered in [B-P] where an estimate on |supp K| is proved using combinatorial methods.

Associate with A the following invariant pseudo-metric on G:

\[ d_ε(x, y) = \sup_{γ ∈ A} |γ(x) - γ(y)|, \]

and denote by N_ε(g) the corresponding entropy numbers for g > 0. The purpose of this note is to show the following fact.

Theorem 1. If 0 < ε < 1, then there exists K satisfying (1), (2) and

\[ \log |supp K| ≤ δ |log_2(120/ε)|log N_ε(1/20). \]

In particular, we can find K such that (2), |K|_1 < 2 and |supp K| < C/ε where C is a fixed constant. As has been observed by W. B. Johnson [cf. (F-J-S)], this exponential estimate is the best one can hope for. This is clear from the following example (answering also a question at the end of [B-P]):

Let G = \{-1\}^n be the Cantor group and A = \{α_1, ..., α_n\} the first n Rademacher functions. Assume K fulfills (2). Then by Khintchine’s inequi-