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Integration and the Feynman–Kac formula

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Abstract. Main problems concerning the Feynman–Kac formula for the Schrödinger group, rather than the diffusion semigroup, pertain to integration with respect to set functions having everywhere infinite variation. An integration theory which does not presuppose finite variation but still leads to a complete space of integrable functions is described. It is modelled on the theory of the Daniell–Stone integral. The resulting notions and techniques are then used to express the superpositions of some semigroups of operators, including those which describe the motion of a quantum-mechanical particle in a potential force field.

In the title, the themes of this note are announced in the order of their formal treatment although the order in which they naturally arise is reverse. For, only those problems of integration are considered here which have direct bearing on the Feynman–Kac formula. Let us describe briefly how they come about.

Let E be a Banach space, $L(E)$ the space of bounded linear operators on E and $S: [0, \infty) \to L(E)$ a continuous semigroup of operators.

Let Λ be a locally compact Hausdorff space, $\mathscr{B}(\Lambda)$ the σ-algebra of Baire sets in Λ and $P: \mathscr{B}(\Lambda) \to L(E)$ a spectral measure. For a Baire function W on Λ, we denote by

$$P(W) = \int_{\Lambda} W\,dP$$

the operator defined by

$$P(W)\varphi = \int_{\Lambda} W(x)\,P(dx)\,\varphi$$

for every $\varphi \in E$ for which the right-hand side exists as the integral with respect to an E-valued measure.

Given such a function W, let

$$T(t) = \exp\bigl(tP(W)\bigr),$$

for every $t \geqslant 0$, assuming that $T(t) \in L(E)$ and that the resulting map $T: [0, \infty) \to L(E)$ is a continuous semigroup of operators.

The semigroups S and T are interpreted as describing evolution processes in which an element φ of the space E is transformed, in time from 0 to

$t \geqslant 0$, into the elements $S(t)\varphi$ and $T(t)\varphi$, respectively. Our problem is to determine the element of the space E into which a given element φ evolves in a time $t \geqslant 0$ if both these processes go on simultaneously. In other words, we wish to construct a semigroup U which describes the superposition of the processes described by the semigroup S and the semigroup T, respectively.

This problem is traditionally formulated in terms of differential equations. Let A be the infinitesimal generator of the semigroup S. The infinitesimal generator of the semigroup T is the operator $P(W)$. Hence, the semigroup U is the solution of the initial-value problem

(0.1) $$\dot U(t) = AU(t) + P(W)U(t), \quad t > 0; \quad U(0+) = I,$$

provided a solution exists. Or, for a given $\varphi \in E$, let $u(t) = U(t)\varphi$, $t \geqslant 0$. Then the E-valued function u is the solution of the problem

(0.2) $$\dot u(t) = Au(t) + P(W)u(t), \quad t > 0; \quad u(0+) = \varphi.$$

However, in many situations, the superposition, U, of the semigroups S and T exists, and can be taken as a good description of the corresponding evolution process but the problem (0.1) does not have a solution or the problem (0.2) does not have a solution for some $\varphi \in E$.

EXAMPLE 0.1. Let $\Lambda = \boldsymbol{R}^3$. Let $E = M(\boldsymbol{R}^3)$, the Banach space of all scalar σ-additive measures on $\mathscr{B}(\boldsymbol{R}^3)$ with the total variation norm. Let

$$p(t,x) = \frac{1}{(2\pi t)^{3/2}} \exp\left(-\frac{|x|^2}{2t}\right)$$

for every $t > 0$ and $x \in \boldsymbol{R}^3$. Let $S(0) = I$, the identity operator, and

$$\big(S(t)\varphi\big)(B) = \int_B dx \int_{\boldsymbol{R}^3} p(t, x-y)\,\varphi(dy),$$

for every $t > 0$, $\varphi \in E$ and $B \in \mathscr{B} = \mathscr{B}(\boldsymbol{R}^3)$. Then $S\colon [0, \infty) \to L(E)$ is a continuous semigroup of operators.

For any $B \in \mathscr{B}$, let $P(B)$ be the operator of forming the indefinite integral of the characteristic function of the set B. That is, $\big(P(B)\varphi\big)(X) = \varphi(B \cap X)$, for every $\varphi \in E$ and $X \in \mathscr{B}$.

The semigroup S is a mathematical model of spatially homogeneous diffusion. Imagine the space $\boldsymbol{R}^3$ filled with a solvent into which some soluble substance is added. The initial distribution of the substance is represented by a measure $\varphi \in E$. Then $S(t)\varphi$ is the distribution of this substance after it was left to diffuse freely in the solvent for a time $t \geqslant 0$.

Imagine further that the substance undergoes also a process of reaction so that its distribution and also the total amount is changing in time but the environment is not affected by this process. The rate of the reaction is proportional to the concentration of the substance with the coefficient of proportion, W, varying from place to place. The semigroup T is then interpreted by saying that $T(t)\varphi$ is the distribution of the substance initially distributed according to the law φ provided the process of reaction went on for a time $t \geqslant 0$ while the diffusion process was suspended.

We want to know the distribution, $U(t)\varphi$, of the substance, initially distributed according to the law φ, after both processes, the diffusion and the reaction, went on simultaneously for a time $t \geqslant 0$.

The density, $u(t, x)$, of the distribution is the solution of the initial-value problem

$$\dot u(t,x) = \tfrac{1}{2}\Delta u(t,x) + W(x)u(t,x), \quad t > 0, x \in \boldsymbol{R}^3;$$
$$\lim_{t \to 0+} \int_B u(t,x)\,dx = \varphi(B), \quad B \in \mathscr{B},$$

provided of course that a solution exists.

EXAMPLE 0.2. Let $\Lambda = \boldsymbol{R}^3$. Let $E = L^2(\boldsymbol{R}^3)$. Let

$$p(t,x) = \frac{1}{(2\pi i t)^{3/2}} \exp\left(-\frac{|x|^2}{2it}\right),$$

for every $t \neq 0$ and $x \in \boldsymbol{R}^3$. Let the semigroup $S\colon [0, \infty) \to L(E)$ be given by $S(0) = I$ and

$$\big(S(t)\varphi\big)(x) = \int_{\boldsymbol{R}^3} p(t, x-y)\,\varphi(y)\,dy, \quad x \in \boldsymbol{R}^3,$$

for every $\varphi \in E \cap L^1(\boldsymbol{R}^3)$, $t \neq 0$.

For every $B \in \mathscr{B}(\boldsymbol{R}^3)$, let $P(B)$ be the operator of pointwise multiplication by the characteristic function of B.

Then the semigroup S, which is in fact a unitary group, describes the motion of a free nonrelativistic quantum-mechanical particle with 3 degrees of freedom. We wish to know the motion of such a particle in a force field with a given potential V. That is, given an initial state, $\varphi \in E$, of such a particle, we want to know its states $u(t) = U(t)\varphi$ at times $t \geqslant 0$. It is known that u is the solution of the Schrödinger equation

$$\dot u(t,x) = \frac{i}{2}\Delta u(t,x) + iV(x)u(t,x), \quad t > 0, \quad u(0+, x) = \varphi(x), \quad x \in \boldsymbol{R}^3.$$

To construct the desired semigroup U, we introduce the following objects.

For a $t \geqslant 0$, let Y_t be the set of continuous paths $v\colon [0, t] \to \Lambda$ in Λ based on the interval $[0, t]$. Let $\mathscr{P}_t$ be the family of sets

$$Y = \{v \in Y_t\colon v(t_j) \in B_j,\ j = 1, 2, \ldots, n\}$$

for arbitrary $n = 1, 2, \ldots$, $0 \leqslant t_1 < t_2 < \ldots < t_{n-1} < t_n \leqslant t$ and $B_j \in \mathscr{B}(\Lambda)$, $j = 1, 2, \ldots, n$. Let

$$M_t(Y) = S(t - t_n)P(B_n)S(t_n - t_{n-1})P(B_{n-1}) \ldots P(B_2)S(t_2 - t_1)P(B_1)S(t_1)$$

for any such set Y. The resulting set function $M_t\colon \mathscr{P}_t \to L(E)$ is obviously additive. Then an obvious and well-known heuristic argument suggests that U is given by the Feynman–Kac formula

$$U(t) = \int_{Y_t} \Big[\exp\Big(\int_0^t W(v(s))\,ds\Big)\Big] M_t(dv),$$

for any $t \geq 0$. A formal proof, under the appropriate assumptions, can be based on the Trotter–Kato formula. None the less, the following alternative argument retains its attractiveness for several reasons.

For a $t > 0$, let

$$f(s,v) = W(v(s))\exp\Big(\int_0^s W(v(r))\,dr\Big), \quad s \in [0,t], \quad v \in Y_t.$$

Note that

$$\int_{Y_t} f(s,v)\,M_t(dv) = S(t-s)\,P(W)\,U(s)$$

for $s \in [0, t]$. Therefore, by the Fubini theorem,

$$\begin{aligned} U(t) - S(t) &= \int_{Y_t} \Big[\exp\Big(\int_0^t W(v(r))\,dr\Big) - 1\Big] M_t(dv) \\ &= \int_{Y_t} \Big(\int_0^t f(s,v)\,ds\Big) M_t(dv) = \int_0^t \Big(\int_{Y_t} f(s,v)\,M_t(dv)\Big)\,ds \\ &= \int_0^t S(t-s)\,P(W)\,U(s)\,ds. \end{aligned}$$

The integral equation obtained corresponds to the initial-value problem (0.1).

The question remains of course whether this calculation can be justified. A framework of concepts, whose centerpiece is a suitable notion of integral with respect to M_t, is needed which would give the indicated operations a good meaning and which would allow the conditions for their correct use to be formulated.

In the case of Example 0.1, there is such a framework readily available. Indeed, for every $Y \in \mathscr{P}_t$, the operator norm $\|M_t(Y)\|$ is equal to the Wiener measure, $w(Y)$, of the set Y. Consequently, the set function $M_t\colon \mathscr{P}_t \to L(E)$ generates a unique continuous linear map of the space $L^1(w)$ into $L(E)$. Hence, a sufficient condition for the validity of the proposed argument is, for example, the integrability of the function f on $[0, t] \times Y_t$ with respect to the product of the one-dimensional Lebesgue measure and the Wiener measure w. Then the Fubini theorem indeed can be used. This is of course classical. (See [2] in the present context.)

In Example 0.2, the situation is not so simple. There does not exist a finite σ-additive measure m on $\mathscr{P}_t$ such that $\|M_t(Y)\| \leq m(Y)$, for every $Y \in \mathscr{P}_t$. Consequently, M_t does not generate a continuous linear map from an L^1-space into $L(E)$.

The efforts to overcome these difficulties lead to many interesting new notions and results. Indeed, the literature on the subject is too vast to be surveyed here. However, these results seem to depend too closely on the specific features of Example 0.2. On the other hand, it is not unreasonable to expect that there might exist a theory which is sufficiently general to be applicable in both situations considered here and in other situations of interest and which is also rich enough to give meaningful results when applied to specific cases. At the same time, as noted, to produce such a theory, it is necessary to reinterpret the basic notions of the integration theory.

Accordingly, in the first part of this note, we introduce an abstract space Ω, a vector space $\mathscr{K}$ of real- or complex-valued functions on Ω and a seminorm γ on $\mathscr{K}$. The seminorm γ is said to be integrating if the completion of $\mathscr{K}$ with respect to it can be represented as a space of functions on Ω containing $\mathscr{K}$ as a dense subspace. Then the corresponding Banach space is represented as a space of equivalence classes of functions on Ω. A typical example of such a seminorm is of course the L^1-norm with respect to a σ-additive measure. But also other examples have already been studied in the literature. The L^p-norms, for any $p \in [1, \infty]$, are perhaps the most prominent ones. The norm in the space of functions integrable with respect to a Banach-valued measure, defined as the total semivariation of indefinite integral, is another example ([1], IV.10).

The study of such seminorms can thus be viewed as a natural genralization of the classical integration theory. We take the treatment of the Daniell integral by M. H. Stone [6] as a model for the generalized integration theory. In fact, we only need to free the Stone procedure from its dependence on the lattice properties of integrated functions to obtain a viable theory well suited for our purpose. At any rate, the proposed theory differs from the classical integration theory most prominently in that the space of integrable functions is not necessarily a lattice.

In the second part we return to the construction of the superposition of two semigroups or, more generally, two evolutions using the Feynman–Kac formula. The suggested argument of interchanging the order of two integrations is fully justified there. The working of the general theory is then illustrated by the examples of perturbed diffusion and of motion of a particle in a force field.

1. Integration.

A. Let Ω be a nonempty set; it will be usually referred to as the space Ω. The subsets of Ω and their characteristic functions will be freely identified in the notation. So, if $\mathscr{P}$ is a semiring of subsets of Ω, then the vector space $\mathrm{sim}(\mathscr{P})$ consisting of $\mathscr{P}$-simple functions can be defined as the linear hull of $\mathscr{P}$. To be sure, $f \in \mathrm{sim}(\mathscr{P})$ if and only if there exist a natural number n, real or complex numbers c_j and sets $X_j \in \mathscr{P}$, $j = 1, 2, \ldots, n$, such that

$$f = \sum_{j=1}^{n} c_j X_j.$$

Similarly, if E is a vector space and $\mu\colon \mathscr{P} \to E$ an additive set function, then there exists a unique linear map from $\mathrm{sim}(\mathscr{P})$ into E whose values coincide with μ on $\mathscr{P}$. This linear map is without ambiguity denoted again by μ and its value, $\mu(f)$, on an element f of $\mathrm{sim}(\mathscr{P})$ is of course called the *integral* of the function f with respect to μ.

Let $\mathscr{K}$ be a vector space of scalar (real or complex) valued functions on a space Ω. A seminorm γ on $\mathscr{K}$ is said to be *integrating* if

(1.1) $$\lim_{n\to\infty} \gamma\Big(\sum_{j=1}^{n} f_j\Big) = 0$$

for any functions $f_j \in \mathscr{K}$, $j = 1, 2, \ldots$, such that

(1.2) $$\sum_{j=1}^{\infty} \gamma(f_j) < \infty$$

and

(1.3) $$\sum_{j=1}^{\infty} f_j(\omega) = 0$$

for every $\omega \in \Omega$ for which

(1.4) $$\sum_{j=1}^{\infty} |f_j(\omega)| < \infty.$$

Example 1.1. A measure, i, in the space Ω is specified by specifying a real-valued nonnegative σ-additive set function on a ring or even a semiring of subsets of Ω. Let $1 \leqslant p < \infty$. The vector space of all (individual) functions f on Ω such that the function $f|f|^{p-1}$ is i-integrable is denoted by $\mathscr{L}^p(i)$. Let $\mathscr{K}$ be a vector subspace of $\mathscr{L}^p(i)$ and let

$$\gamma(f) = \Big(\int_{\Omega} |f|^p \, di\Big)^{1/p}$$

for every $f \in \mathscr{K}$. Then, by the Beppo Levi theorem, γ is an integrating seminorm on $\mathscr{K}$.

Example 1.2. Let $\mathscr{K}$ be a vector space of bounded functions on Ω and let

$$\gamma(f) = \|f\|_{\infty} = \sup\{|f(\omega)|\colon \omega \in \Omega\}$$

for every $f \in \mathscr{K}$. Then γ is an integrating seminorm on $\mathscr{K}$. In fact, γ is a norm on $\mathscr{K}$.

Example 1.3. Let $\mathscr{S}$ be a σ-algebra of sets in a space Ω. Let E be a Banach space and $\mu\colon \mathscr{S} \to E$ a σ-additive vector measure. Let $\mathscr{K}$ be a vector space of μ-integrable functions. (See e.g. [1], IV.10.) For any $f \in \mathscr{K}$, let $\gamma(f)$ be equal to the total semivariation of the indefinite integral of the function f with respect to μ. Then γ is an integrating seminorm on $\mathscr{K}$.

Let γ be an integrating seminorm on a vector space $\mathscr{K}$ of functions on Ω.

A scalar-valued function f on Ω is said to be *integrable* with respect to γ if there exist functions $f_j \in \mathscr{K}$, $j = 1, 2, \ldots$, satisfying the condition (1.2) such that

(1.5) $$f(\omega) = \sum_{j=1}^{\infty} f_j(\omega)$$

for every $\omega \in \Omega$ for which the inequality (1.4) holds.

The set of all functions integrable with respect to γ is denoted by $\mathscr{L}(\gamma)$. Given a function $f \in \mathscr{L}(\gamma)$, let

$$\gamma_*(f) = \inf \sum_{j=1}^{\infty} \gamma(f_j),$$

where the infimum is taken over all choices of functions $f_j \in \mathscr{K}$, $j = 1, 2, \ldots$, satisfying the condition (1.2) such that the equality (1.5) holds for every $\omega \in \Omega$ for which the inequality (1.4) does.

Proposition 1.4. *$\gamma_*(f) = \gamma(f)$ for every $f \in \mathscr{L}(\gamma)$.*

Proof. Evidently, $\gamma_*(f) \leqslant \gamma(f)$, for every $f \in \mathscr{L}(\gamma)$. For the proof of the reverse inequality, let $f \in \mathscr{L}(\gamma)$ and $\varepsilon > 0$. Then there are functions $f_j \in \mathscr{K}$, $j = 1, 2, \ldots$, such that (1.5) holds for every $\omega \in \Omega$ for which (1.4) does and

$$\sum_{j=1}^{\infty} \gamma(f_j) < \gamma_*(f) + \varepsilon.$$

Then

$$\lim_{n\to\infty} \gamma\Big(f - \sum_{j=1}^{n} f_j\Big) = 0,$$

because γ is an integrating seminorm. Hence, for a sufficiently large n,

$$\gamma(f) \leqslant \gamma\Big(\sum_{j=1}^{n} f_j\Big) + \varepsilon \leqslant \sum_{j=1}^{n} \gamma(f_j) + \varepsilon.$$

Consequently, $\gamma(f) \leqslant \gamma_*(f) + 2\varepsilon$.

In view of this proposition, we write $\gamma(f) = \gamma_*(f)$ for every $f \in \mathscr{L}(\gamma)$, without causing any confusion.

B. Let γ be an integrating seminorm on a vector space $\mathscr{K}$ of scalar-valued functions on a space Ω.

PROPOSITION 1.5. *If $f_j \in \mathscr{L}(\gamma)$, $j = 1, 2, \ldots$, are functions satisfying the condition* (1.2) *and f is a function on Ω such that* (1.5) *holds for every $\omega \in \Omega$ for which* (1.4) *does, then $f \in \mathscr{L}(\gamma)$ and*

$$\lim_{m\to\infty} \gamma\Big(f - \sum_{j=1}^{m} f_j\Big) = 0. \tag{1.6}$$

Proof. For every $j = 1, 2, \ldots$, let $f_{jn} \in \mathscr{K}$, $n = 1, 2, \ldots$, be functions such that

$$\sum_{n=1}^{\infty} \gamma(f_{jn}) < \gamma(f_j) + 2^{-j}$$

and

$$f_j(\omega) = \sum_{n=1}^{\infty} f_{jn}(\omega)$$

for every $\omega \in \Omega$ for which $\sum_{n=1}^{\infty} |f_{jn}(\omega)| < \infty$. Then $\sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \gamma(f_{jn}) < \infty$ and

$$f(\omega) = \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} f_{jn}(\omega)$$

for every $\omega \in \Omega$ such that

$$\sum_{j=1}^{\infty} \sum_{n=1}^{\infty} |f_{jn}(\omega)| < \infty.$$

Therefore, $f \in \mathscr{L}(\gamma)$. Moreover, for every $m = 1, 2, \ldots$,

$$f(\omega) - \sum_{j=1}^{m} f_j(\omega) = \sum_{j=m+1}^{\infty} \sum_{n=1}^{\infty} f_{jn}(\omega)$$

for every $\omega \in \Omega$ such that $\sum_{j=m+1}^{\infty} \sum_{n=1}^{\infty} |f_{jn}(\omega)| < \infty$. Consequently,

$$\gamma\Big(f - \sum_{j=1}^{m} f_j\Big) \leqslant \sum_{j=m+1}^{\infty} \sum_{n=1}^{\infty} \gamma(f_{jn}) \to 0$$

as $m \to \infty$.

This is a form of the Beppo Levi theorem in the present context. It will be slightly reformulated as Theorem 1.9.

COROLLARY 1.6. *$\mathscr{L}(\gamma)$ is a vector space and γ is a seminorm on it. The corresponding normed space is complete and $\mathscr{K}$ represents its dense subspace.*

This corollary shows the importance of γ being an integrating seminorm.

For, $\mathscr{K}$ has an abstract completion with respect to any seminorm. But, if the seminorm γ is integrating, then the completion has a concrete representation whose elements are γ_*-equivalence classes of functions on Ω.

EXAMPLE 1.7. Let $\Omega = (0, 1]$ and let $\mathscr{P} = \{(u, v]: 0 \leqslant u \leqslant v \leqslant 1\}$. Let $\mathscr{K} = \operatorname{sim}(\mathscr{P})$ and

$$\gamma(f) = \lim_{\omega\to 0+} |f(\omega)|,$$

for every $f \in \mathscr{K}$. Then γ is a seminorm on $\mathscr{K}$. If we now define $\mathscr{L}(\gamma)$ disregarding the fact that the seminorm γ is not integrating, then we discover that $\gamma_*(f) = 0$ for every $f \in \mathscr{L}(\gamma)$. So, the space $\mathscr{L}(\gamma)$ is useless.

A function f on Ω is said to be *γ-null* or *γ-negligible* if $f \in \mathscr{L}(\gamma)$ and $\gamma(f) = 0$. A subset of Ω is said to be γ-null if its characteristic function is γ-null.

We shall use the customary jargon referring to a γ-null set by saying that γ-almost all points of Ω belong to its complement.

COROLLARY 1.8. *A function f is γ-null if and only if there exist functions $f_j \in \mathscr{L}(\gamma)$, $j = 1, 2, \ldots$, satisfying the condition* (1.2) *such that*

$$\sum_{j=1}^{\infty} |f_j(\omega)| = \infty$$

for every $\omega \in \Omega$ for which $f(\omega) \neq 0$.

THEOREM 1.9. *Let the functions $f_j \in \mathscr{L}(\gamma)$, $j = 1, 2, \ldots$, satisfy the condition* (1.2) *and let f be a function on Ω such that the equality* (1.5) *holds for γ-almost every $\omega \in \Omega$. Then $f \in \mathscr{L}(\gamma)$ and the equality* (1.6) *holds.*

C. Let $\mathscr{K}$ be a vector space of scalar-valued functions on a space Ω. Let E be a Banach space. Let $\mu: \mathscr{K} \to E$ be a linear map.

We say that a seminorm γ on $\mathscr{K}$ *integrates for the map* μ if it is integrating and there exists a number $k \geqslant 0$ such that

$$\|\mu(f)\| \leqslant k\gamma(f)$$

for every $f \in \mathscr{K}$.

This condition of course means that the map $\mu: \mathscr{K} \to E$ is continuous in the topology induced on $\mathscr{K}$ by the seminorm γ. It is the relative topology on $\mathscr{K}$ inherited from $\mathscr{L}(\gamma)$. Because the space E is complete and $\mathscr{K}$ is dense in $\mathscr{L}(\gamma)$, there exists a unique continuous linear map $\mu_\gamma: \mathscr{L}(\gamma) \to E$ such that $\mu_\gamma(f) = \mu(f)$ for every $f \in \mathscr{L}(\gamma)$. Because of the uniqueness, it is not necessary to distinguish between μ and μ_γ and so we write

$$\mu(f) = \int_\Omega f\, d_\gamma\mu = \int_\Omega f(\omega)\,\mu(d_\gamma\omega) = \mu_\gamma(f)$$

for every $f \in \mathscr{L}(\gamma)$. The element $\mu_\gamma(f)$ of the space E is of course called the *integral* of the function f with respect to μ (and γ).

EXAMPLE 1.10. Let i be a measure in the space Ω. Let $\mathscr{K} = \mathscr{L}^1(i)$ and let γ be the seminorm on $\mathscr{K}$ defined in Example 1.1 for $p = 1$. Then γ integrates for i interpreted as a linear scalar-valued function on $\mathscr{K}$. By the Beppo Levi theorem, $\mathscr{L}(\gamma) = \mathscr{L}^1(i)$ and $i_\gamma(f) = i(f)$ is the integral of any function $f \in \mathscr{L}(\gamma)$ with respect to i in the standard sense of the word.

The following proposition suffices for our purposes as a source of integrating seminorms and as a test of integrability with respect to such seminorms.

PROPOSITION 1.11. *Let i be a measure in the space Ω, let $1 \leqslant p < \infty$, and let $\mathscr{K}$ be a vector subspace of $\mathscr{L}^p(i)$. Let F be a Banach space and let v: $\mathscr{K} \to F$ be a linear map. Let G be a topological vector space and Ψ: $F \to G$ an injective continuous linear map such that the composite map $\Psi \circ v$: $\mathscr{K} \to G$ is closed. Let*

$$\gamma(f) = \big(i(|f|^p)\big)^{1/p} + \|v(f)\|$$

for every $f \in \mathscr{K}$.

Then γ is a seminorm on $\mathscr{K}$, integrating for the map v and such that $\mathscr{L}(\gamma) = \mathscr{K}$.

Proof. Let $f_j \in \mathscr{K}$, $j = 1, 2, \ldots$, be functions satisfying the condition (1.2) and let f be a function on Ω such that the equality (1.5) holds for every $\omega \in \Omega$ for which the inequality (1.4) does. Then

$$\sum_{j=1}^{\infty} \big(i(|f_j|^p)\big)^{1/p} < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} \|v(f_j)\| < \infty .$$

Let

$$g_n = \sum_{j=1}^{n} f_j$$

for every $n = 1, 2, \ldots$ Then $f \in \mathscr{L}^p(i)$, $g_n \to f$ in $\mathscr{L}^p(i)$ and there exists an element φ of F such that $v(g_n) \to \varphi$ as $n \to \infty$. Hence, $\Psi\big(v(g_n)\big) \to \Psi(\varphi)$, by the continuity of Ψ. So, $f \in \mathscr{K}$ and $\Psi\big(v(f)\big) = \Psi(\varphi)$, because the map $\Psi \circ v$ is closed. Consequently, $v(f) = \varphi$, because the map Ψ is injective. The statements now follow readily.

Only the case of $p = 1$ and of G being the space F under a topology coarser than its norm topology will be used in the sequel.

EXAMPLE 1.12. Let a be a (finite) real-valued continuous function on the two-point compactification, $[-\infty, \infty]$, of the real line such that $a(-\infty) = 0$ and $a(x) \neq 0$ for some $x \in (-\infty, \infty)$. Our aim is to produce a reasonably rich class of functions f on $(-\infty, \infty)$ for which the integral

$$\int_{-\infty}^{\infty} f\,da = \int_{-\infty}^{\infty} f(x)\,a(dx)$$

can be defined in a "natural" way.

Let $C_0\big((-\infty, \infty]\big)$ be the Banach space of all functions continuous on $[-\infty, \infty]$ and vanishing at $-\infty$ under the usual sup-norm. Let $F = c\big(C_0\big((-\infty, \infty]\big)\big)$ be the space of all sequences $\varphi = \{\varphi_n\}_{n=1}^{\infty}$ of elements of $C_0\big((-\infty, \infty]\big)$ uniformly convergent on $[-\infty, \infty]$, equipped with the norm

$$\|\varphi\| = \sup\{|\varphi_n(x)|\colon x \in [-\infty, \infty], n = 1, 2, \ldots\}, \qquad \varphi \in F.$$

Let i be the measure on the real line such that $i(dx) = (1+x^2)\exp(-\tfrac{1}{2}x^2)\,dx$. That is, $\mathscr{L}^1(i)$ consists of all measurable functions f on $(-\infty, \infty)$ such that

$$i(|f|) = \int_{-\infty}^{\infty} |f(x)|(1+x^2)\exp(-\tfrac{1}{2}x^2)\,dx < \infty .$$

Let

$$k_n(x) = \sqrt{\frac{n}{2k}}\exp(-\tfrac{1}{2}nx^2)$$

for every $x \in (-\infty, \infty)$ and $n = 1, 2, \ldots$

Given a function $f \in \mathscr{L}^1(i)$, let

$$\begin{aligned} v_n(f)(x) &= \int_{-\infty}^{x} f(z)\Big(\int_{-\infty}^{\infty} k_n'(z-y)\,a(y)\,dy\Big)dz \\ &= -n^{3/2}(2\pi)^{-1/2}\int_{-\infty}^{x} f(z)\Big[\int_{-\infty}^{\infty}(z-y)\exp\big(-\tfrac{1}{2}n(z-y)^2\big)a(y)\,dy\Big]dz \\ &= n^{3/2}(2\pi)^{-1/2}\int_{-\infty}^{\infty} a(y)\Big[\int_{-\infty}^{x}(y-z)\exp\big(-\tfrac{1}{2}n(y-z)^2\big)f(z)\,dz\Big]dy, \end{aligned}$$

for every $x \in [-\infty, \infty]$ and every $n = 1, 2, \ldots$

Then clearly $v_n(f) \in C_0\big((-\infty, \infty]\big)$, for every $n = 1, 2, \ldots$

Let $\mathscr{K}$ be the set of all functions $f \in \mathscr{L}^1(i)$ such that the sequence $v(f) = \{v_n(f)\}_{n=1}^{\infty}$ belongs to F. Then $\mathscr{K}$ is a vector space and v: $\mathscr{K} \to F$ a closed linear map. Because this map is closed, the space $\mathscr{K}$ is complete with respect to the seminorm

$$\gamma(f) = i(|f|) + \|v(f)\|, \qquad f \in \mathscr{K}.$$

Now, let

$$\int_{-\infty}^{\infty} f\,da = \lim_{n\to\infty}\big(v_n(f)(\infty)\big)$$

for every $f \in \mathscr{K}$. The integral so defined is continuous on $\mathscr{K}$ with respect to γ.

D. Let $\mathscr{L}(\gamma)$ be the space of all functions on Ω which are integrable with respect to an integrating seminorm γ. Let β be a seminorm on a vector space F such that the induced normed space is complete.

A function $\Phi\colon \Omega \to F$ is said to be *Bochner integrable* with respect to γ if there exist elements φ_j of F and functions $f_j \in \mathscr{L}(\gamma)$, $j = 1, 2, \ldots$, such that

$$\sum_{j=1}^{\infty} \gamma(f_j)\,\beta(\varphi_j) < \infty \tag{1.7}$$

and

$$\lim_{n\to\infty} \beta\Big(\Phi(\omega) - \sum_{j=1}^{n} f_j(\omega)\,\varphi_j\Big) = 0 \tag{1.8}$$

for every $\omega \in \Omega$ such that

$$\sum_{j=1}^{\infty} |f_j(\omega)|\,\beta(\varphi_j) < \infty. \tag{1.9}$$

Let

$$\gamma_\beta(\Phi) = \inf \sum_{j=1}^{\infty} \gamma(f_j)\,\beta(\varphi_j),$$

for any such function Φ, where the infimum is taken over all choices of $\varphi_j \in F$ and $f_j \in \mathscr{L}(\gamma)$, $j = 1, 2, \ldots$, satisfying the condition (1.7) such that (1.8) holds for every $\omega \in \Omega$ for which (1.9) does.

PROPOSITION 1.13. *Let $\Phi_n\colon \Omega \to F$, $n = 1, 2, \ldots$, be functions Bochner integrable with respect to γ such that*

$$\sum_{n=1}^{\infty} \gamma_\beta(\Phi_n) < \infty$$

and let $\Phi\colon \Omega \to F$ be a function such that

$$\lim_{n\to\infty} \beta\Big(\Phi(\omega) - \sum_{j=1}^{n} \Phi_j(\omega)\Big) = 0$$

for γ-almost every $\omega \in \Omega$. Then the function Φ is Bochner integrable with respect to γ and

$$\lim_{n\to\infty} \gamma_\beta\Big(\Phi - \sum_{j=1}^{n} \Phi_j\Big) = 0.$$

Proof. It is analogous to that of Proposition 1.5, therefore the details are omitted.

Let us now suppose that the seminorm γ integrates for a scalar-valued linear function i on $\mathscr{L}(\gamma)$. Then the *integral* with respect to i of a function $\Phi\colon \Omega \to F$ Bochner integrable with respect to γ is defined to be the element

$$\int_\Omega \Phi(\omega)\, i(d_\gamma \omega) = \sum_{j=1}^{\infty} i(f_j)\,\varphi_j$$

of the space F, where the $f_j \in \mathscr{L}(\gamma)$ are functions and the $\varphi_j \in F$ vectors, $j = 1, 2, \ldots$, satisfying the condition (1.7) such that (1.8) holds for every $\omega \in \Omega$ for which (1.9) does. The integral is defined uniquely modulo the seminorm β.

E. Let Ξ and Y be two spaces and let $\Omega = \Xi \times Y$.

Given a function f on Ω and a point $\xi \in \Xi$, by $f(\xi, \cdot)$ is of course denoted the function $v \mapsto f(\xi, v)$, $v \in Y$. The meaning of $f(\cdot, v)$ for any given $v \in Y$ is analogous. If g is a function on Ξ and h a function on Y, then $f = g \otimes h$ is the function on Ω such that $f(\omega) = g(\xi)\,h(v)$ for every $\omega = (\xi, v)$, $\xi \in \Xi$, $v \in Y$.

Let $\mathscr{L}(\alpha)$ be the space of all functions on Ξ which are integrable with respect to a given integrating seminorm α and $\mathscr{L}(\beta)$ be the space of all functions on Y integrable with respect to an integrating seminorm β.

Let $\mathscr{K}$ be the vector space of all functions f on Ω for which there exist functions $g_j \in \mathscr{L}(\alpha)$ and $h_j \in \mathscr{L}(\beta)$, $j = 1, 2, \ldots$, such that

$$\sum_{j=1}^{\infty} \alpha(g_j)\,\beta(h_j) < \infty \tag{1.10}$$

and

$$f(\omega) = \sum_{j=1}^{\infty} g_j(\xi)\,h_j(v) \tag{1.11}$$

for every $\omega = (\xi, v)$, $\xi \in \Xi$, $v \in Y$, such that

$$\sum_{j=1}^{\infty} |g_j(\xi)\,h_j(v)| < \infty. \tag{1.12}$$

Let

$$\gamma(f) = \inf \sum_{j=1}^{\infty} \alpha(g_j)\,\beta(h_j)$$

for every $f \in \mathscr{K}$, where the infimum is taken over all choices of functions $g_j \in \mathscr{L}(\alpha)$ and $h_j \in \mathscr{L}(\beta)$, $j = 1, 2, \ldots$, satisfying the condition (1.10) such that the equality (1.11) holds for every $\omega = (\xi, v)$, $\xi \in \Xi$, $v \in Y$, for which the inequality (1.12) does.

It is immediate that γ is a seminorm on $\mathscr{K}$. We will call it the *product* of the seminorms α and β and will write $\gamma = \alpha \otimes \beta$.

PROPOSITION 1.14. *Let $f \in \mathscr{K}$. Let $g_j \in \mathscr{L}(\alpha)$ and $h_j \in \mathscr{L}(\beta)$, $j = 1, 2, \ldots$, be functions satisfying the condition* (1.10) *such that the equality* (1.11) *holds for every $\omega = (\xi, v)$, $\xi \in \Xi$, $v \in Y$, for which the inequality* (1.12) *does. Then*

$$\lim_{n\to\infty} \gamma\Big(f - \sum_{j=1}^{n} g_j \otimes h_j\Big) = 0.$$

Furthermore, for α-almost every $\xi \in \Xi$, the function $f(\xi, \cdot)$ belongs to $\mathscr{L}(\beta)$ and, if $\Phi\colon \Xi \to \mathscr{L}(\beta)$ is a function such that $\Phi(\xi) = f(\xi, \cdot)$ for α-almost every $\xi \in \Xi$, then Φ is Bochner integrable with respect to α and $\alpha_\beta(\Phi) = \gamma(f)$. Moreover, if $\Phi_j(\xi) = g_j(\xi) h_j$, for every $\xi \in \Xi$ and $j = 1, 2, \ldots$, then

$$\lim_{n\to\infty} \alpha_\beta\Big(\Phi - \sum_{j=1}^{n} \Phi_j\Big) = 0.$$

Similarly, for β-almost every $v \in Y$, the function $f(\cdot, v)$ belongs to $\mathscr{L}(\alpha)$ and, if $\Psi\colon Y \to \mathscr{L}(\alpha)$ is a function such that $\Psi(v) = f(\cdot, v)$ for β-almost every $v \in Y$, then Ψ is Bochner integrable with respect to β and $\beta_\alpha(\Psi) = \gamma(f)$. Moreover, if $\Psi_j(v) = h_j(v) g_j$, for every $v \in Y$, $j = 1, 2, \ldots$, then

$$\lim_{n\to\infty} \beta_\alpha\Big(\Psi - \sum_{j=1}^{n} \Psi_j\Big) = 0.$$

Proof. It follows from the definitions in an almost routine manner.

PROPOSITION 1.15. *The product, $\gamma = \alpha \otimes \beta$, of the seminorms α and β is an integrating seminorm on $\mathscr{K}$ and $\mathscr{L}(\gamma) = \mathscr{K}$.*

Proof. The fact that γ is integrating follows from Proposition 1.14. Then the equality $\mathscr{L}(\gamma) = \mathscr{K}$ can be deduced by an argument similar to that used in the proof of Proposition 1.5.

Assume now that the seminorm α integrates for a scalar-valued linear function i on $\mathscr{L}(\alpha)$. Assume further that E is a Banach space and that $\varkappa\colon \mathscr{L}(\beta) \to E$ is a linear map and the seminorm β integrates for it.

PROPOSITION 1.16. *There exists a unique linear map $\mu\colon \mathscr{L}(\gamma) \to E$ such that*

(i) *$\mu(g \otimes h) = i(g)\varkappa(h)$ for every $g \in \mathscr{L}(\alpha)$ and $h \in \mathscr{L}(\beta)$; and*

(ii) *The seminorm γ integrates for μ.*

THEOREM 1.17. *Let $f \in \mathscr{L}(\alpha \otimes \beta)$.*

Then $f(\cdot, v) \in \mathscr{L}(\alpha)$, for β-almost every $v \in Y$, and, if h is a function on Y such that

$$h(v) = \int_{\xi\in\Xi} f(\xi, v)\, i(d_\alpha \xi)$$

for β-almost every $v \in Y$, then $h \in \mathscr{L}(\beta)$ and

$$\int_\Omega f\, d_{\alpha\otimes\beta}\mu = \int_Y h(v)\, \varkappa(d_\beta v).$$

Also, $f(\xi, \cdot) \in \mathscr{L}(\beta)$, for α-almost every $\xi \in \Xi$, and, if $g\colon \Xi \to E$ is a function such that

$$g(\xi) = \int_{v\in Y} f(\xi, v)\, \varkappa(d_\beta v)$$

for α-almost every $\xi \in \Xi$, then the function g is Bochner integrable with respect to α and

$$\int_\Omega f\, d_{\alpha\otimes\beta}\mu = \int_\Xi g(\xi)\, i(d_\alpha \xi).$$

2. The Feynman–Kac formula.

A. Let E be a Banach space.

Let $S\colon \{(t, s)\colon 0 \leqslant s \leqslant t < \infty\} \to L(E)$ be a map such that

(i) $S(t, t) = I$, the identity operator, for every $t \geqslant 0$;

(ii) $S(t, r) = S(t, s) S(s, r)$ for any r, s and t such that $0 \leqslant r \leqslant s \leqslant t < \infty$;

and

(iii) S is continuous in the strong operator topology of $L(E)$.

Such a map is called an *evolution* or a *propagator* in the space E. If $S(t, s) = S(t-s, 0)$, for any $0 \leqslant s \leqslant t < \infty$, then we speak of a *semigroup* or a *dynamical propagator* and write without ambiguity $S(t) = S(t, 0)$, for every $t \geqslant 0$.

Let Λ be a locally compact Hausdorff space. Let $P\colon \mathscr{B}(\Lambda) \to L(E)$ be a spectral measure. That is, P is σ-additive in the strong operator topology, $P(\Lambda) = I$ and $P(B \cap C) = P(B)P(C)$ for any $B \in \mathscr{B}(\Lambda)$ and $C \in \mathscr{B}(\Lambda)$.

If W is a $\mathscr{B}(\Lambda)$-measurable scalar-valued function on Λ, let

$$P(W) = \int_\Lambda W(x)\, P(dx)$$

be the, possibly unbounded, operator whose domain is the vector space of all elements φ of the space E such that the function W is integrable with respect to the E-valued measure $B \mapsto P(B)\varphi$, $B \in \mathscr{B}(\Lambda)$, and whose value at any such φ is equal to the integral

$$P(W)\varphi = \int_\Lambda W(x)\, P(dx)\, \varphi$$

with respect to this measure.

For every $t \geqslant 0$, let Y_t be a set of maps $v\colon [0, t] \to \Lambda$ to be called paths. We assume that, for every $s \in (0, t]$, $\{v(s)\colon v \in Y_t\} = \Lambda$ and that the set of the restrictions of elements of Y_t to the interval $[0, s]$ is equal to Y_s.

Of main interest are the cases when $Y_t = \Lambda^{[0,t]}$, or Y_t consists of all continuous paths $v\colon [0, t] \to \Lambda$, or ones which are càdlàg, etc.

Let $\mathscr{P}_t$ be the family of all sets

$$Y = \{v \in Y_t\colon v(t_j) \in B_j,\ j = 1, 2, \ldots, k\} \tag{2.1}$$

corresponding to arbitrary $k = 1, 2, \ldots$, numbers $0 \leqslant t_1 < t_2 < \ldots < t_{k-1} < t_k \leqslant t$ and sets $B_j \in \mathscr{B}(\Lambda)$, $j = 1, 2, \ldots, k$. Then $\mathscr{P}_t$ is a semialgebra of sets in Y_t.

Let $\mathscr{K}_t$ be a vector space of scalar-valued functions on Y_t such that

$\mathrm{sim}(\mathscr{P}_t) \subset \mathscr{K}_t$. Let $M_t\colon \mathscr{K}_t \to L(E)$ be a linear map such that

(2.2) $$M_t(Y) = S(t,t_k)P(B_k)S(t_k,t_{k-1})P(B_{k-1})\dots P(B_2)S(t_2,t_1)P(B_1)S(t_1,0)$$

for every set $Y \in \mathscr{P}_t$ given in the form (2.1). Let β_t be a seminorm on $\mathscr{K}_t$ integrating for M_t, and $\mathscr{L}(\beta_t)$ the corresponding space of functions integrable with respect to β_t.

Let $\mathscr{L}(\alpha_t)$ be the space of all functions on the interval $[0, t]$ which are integrable with respect to the one-dimensional Lebesgue measure and let

$$\alpha_t(g) = \int_{[0,t]} |g(s)|\,ds$$

for every $g \in \mathscr{L}(\alpha_t)$.

Let W be a function on $[0, \infty) \times \Lambda$ such that, for every $t \geqslant 0$, the integral in

(2.3) $$e_t(v) = \exp\Big(\int_0^t W(r, v(r))\,dr\Big)$$

exists for β_t-almost every $v \in Y_t$ and the function e_t so defined belongs to $\mathscr{L}(\beta_t)$. Let

(2.4) $$U(t) = \int_{Y_t} e_t(v)\,M_t(d_{\beta_t}v)$$

for every $t \geqslant 0$.

The operator $U(t)$ can be interpreted in the following way. Let

$$T(t,s) = \exp\Big(\int_s^t P(W(r,\cdot))\,dr\Big),$$

for $0 \leqslant s \leqslant t < \infty$, assuming that these operators belong to $L(E)$ and the resulting map $T\colon \{(s, t)\colon 0 \leqslant s \leqslant t < \infty\} \to L(E)$ is an evolution in the space E. Then, for any given $\varphi \in E$, the element

(2.5) $$u(t) = U(t)\varphi$$

of the space E is the result of the simultaneous action of the evolutions S and T during the time interval $[0, t]$ on the element φ.

Under some additional assumptions on W, the function $t \to U(t)$, $t \geqslant 0$, satisfies a Duhamel-type integral equation. Indeed, given a $t \geqslant 0$, let

(2.6) $$f_t(s,v) = W(s, v(s))\exp\Big(\int_0^s W(r, v(r))\,dr\Big)$$

for every $s \in [0, t]$ and $v \in Y_t$.

THEOREM 2.1. *Let $t > 0$. If $f_t \in \mathscr{L}(\alpha_t \otimes \beta_t)$, then*

(2.7) $$U(t) = S(t,0) + \int_0^t S(t,s)P(W(s,\cdot))U(s)\,ds.$$

Proof. First note that

$$\int_0^t f_t(s,v)\,ds = \exp\Big(\int_0^t W(r,v(r))\,dr\Big) - 1,$$

for every $v \in Y_t$ such that $f_t(\cdot, v) \in \mathscr{L}(\alpha_t)$, and

$$\int_{Y_t} f_t(s,v)\,M_t(d_{\beta_t}v) = S(t,s)P(W(s,\cdot))U(s)$$

for every $s \in [0, t]$ such that $f_t(s,\cdot) \in \mathscr{L}(\beta_t)$. Therefore, by Theorem 1.17,

$$\begin{aligned} U(t)S(t,0) &= \int_{Y_t}\Big[\exp\Big(\int_0^t W(r,v(r))\,dr\Big) - 1\Big]M_t(d_{\beta_t}v) \\ &= \int_{Y_t}\Big(\int_0^t f_t(s,v)\,ds\Big)M_t(d_{\beta_t}v) = \int_0^t\Big(\int_{Y_t} f_t(s,v)\,M_t(d_{\beta_t}v)\Big)ds \\ &= \int_0^t S(t,s)P(W(s,\cdot))U(s)\,ds. \end{aligned}$$

If the equation (2.7) is satisfied and $\varphi \in E$, then the function $t \mapsto u(t)$, $t \geqslant 0$, defined by (2.5) satisfies the integral equation

(2.8) $$u(t) = S(t,0)\varphi + \int_0^t S(t,s)P(W(s,\cdot))u(s)\,ds.$$

B. The integral equation (2.7) corresponds to an initial-value problem and traditionally that initial-value problem is used instead of (2.7). Namely, given a $t > 0$, let

$$A(t)\varphi = \lim_{r\to 0} r^{-1}\big(S(t+r,t)\varphi - \varphi\big)$$

for every $\varphi \in E$ such that this limit exists in E. Then the formal differentiation of (2.7) gives

(2.9) $$\dot U(t) = A(t)U(t) + P(W(t,\cdot))U(t), \quad t > 0; \quad U(0+) = I.$$

Similarly, for a given $\varphi \in E$, the integral equation (2.8) corresponds to the initial-value problem

(2.10) $$\dot u(t) = A(t)u(t) + P(W(t,\cdot))u(t), \quad t > 0; \quad u(0+) = \varphi.$$

But, of course, it often happens that the $L(E)$-valued function $t \mapsto U(t)$, $t \geqslant 0$, is well defined by (2.4) but the integral equation (2.7) is not satisfied because, for too many $s \in [0, t]$, the operator $P(W(s,\cdot))$ is unbounded. And then, even if the integral equation (2.7) is satisfied the initial-value problem (2.9) might not be.

For a given $\varphi \in E$, the integral equation (2.8) could be satisfied even if the integral equation (2.7) does not have a solution which is an $L(E)$-valued function. But still, it is often possible to define the E-valued function $t \mapsto u(t)$, $t \geqslant 0$, by (2.4) and (2.5), while it is not possible to determine it from (2.8). And even if the equation (2.8) determines an E-valued function, the initial-value problem (2.10) might not have a solution.

So, the E-valued function u defined by (2.5) can be taken for a generalized solution of the initial-value problem (2.10).

The usefulness of the differential equation (2.10) for the construction of a superposition of the evolutions S and T can of course be widened by considering solutions in some extended sense. For example, we can take solutions in the sense of distributions. Or, we can pick up a locally convex space $\tilde{E}$ which contains E continuously such that the operators $S(t,s)$, $0 \leqslant s \leqslant t < \infty$, and $P(B)$, $B \in \mathscr{B}(\Lambda)$, have continuous extensions onto the whole of $\tilde{E}$, and consider $\tilde{E}$-valued solutions of (2.10). The integral equations (2.8) are perhaps more susceptible of this approach.

However, one of the main reasons for studying the Feynman–Kac formula (2.4) is that it enables us to construct a superposition of the evolutions S and T also in situations when the initial-value problems (2.10) or the integral equations (2.8) do not have classical solutions.

So, the problem of the construction of a superposition of evolutions S and T is reduced to the problem of a choice of the spaces of functions $\mathscr{K}_t$ and of the seminorms β_t. This choice is of course not necessarily unique. What is more, if $\mathscr{K}_t$ is sufficiently large, then the choice of the linear map $M_t\colon \mathscr{K}_t \to L(E)$ consistent with the requirement (2.2) is not necessarily unique. This corresponds to the fact that, for some "potentials" W, there may be no canonical way for constructing a superposition of the evolutions S and T.

C. Let $d \geqslant 1$ be an integer. We are going to specialize the situation of Section 2A by taking $\Lambda = \mathbf{R}^d$ and $E = M(\mathbf{R}^d)$, the Banach space of all scalar σ-additive measures on $\mathscr{B} = \mathscr{B}(\mathbf{R}^d)$ with the total variation norm.

Given a set $B \in \mathscr{B}$, let $P(B) \in L(E)$ be the operator of restriction to the set B. That is, $(P(B)\varphi)(X) = \varphi(B \cap X)$ for every set $X \in \mathscr{B}$ and every $\varphi \in E$. Then $P\colon \mathscr{B} \to L(E)$ is a spectral measure.

Let D be a strictly positive real number and let

$$p_D(t,x) = \frac{1}{(4\pi D t)^{d/2}} \exp\left(-\frac{|x|^2}{4Dt}\right),$$

for every $t > 0$ and $x \in \Lambda$. Let $S(0) = I$ and

$$(S(t)\varphi)(B) = \int_B dx \int_\Lambda p_D(t, x-y)\,\varphi(dy)$$

for every $t > 0$, $B \in \mathscr{B}$ and $\varphi \in E$. Then $S\colon [0, \infty) \to L(E)$ is the Poisson semigroup of operators which can be interpreted as a mathematical description of homogeneous diffusion in $\mathbf{R}^d$ with the diffusion coefficient D. Its infinitesimal generator is the operator

$$A = D\Delta,$$

where Δ is the Laplacian in $\mathbf{R}^d$.

Given a $t \geqslant 0$, let Y_t be the set of all continuous paths $v\colon [0, t] \to \Lambda$. Because S is a semigroup, the formula (2.2) takes the form

$$\text{(2.11)} \qquad M_t(Y) = S(t-t_k)P(B_k)S(t_k - t_{k-1})\ldots P(B_2)S(t_2-t_1)P(B_1)S(t_1),$$

for every set $Y \in \mathscr{P}_t$ given by (2.1) with some integer $k \geqslant 1$, numbers t_j such that $0 \leqslant t_1 < t_2 < \ldots < t_{k-1} < t_k \leqslant t$ and sets $B_j \in \mathscr{B}$, $j = 1, 2, \ldots, k$.

If $\varphi \in E$ is a nonnegative measure of norm 1, that is, a probability measure, and

$$i(Y) = \|M_t(Y)\varphi\| = (M_t(Y)\varphi)(\Lambda),$$

for every $Y \in \mathscr{P}_t$, then the set function i generates a probability measure in the space Y_t, namely the Wiener measure of variance $2D$ per unit time with initial distribution φ. Let

$$w_t(Y) = (M_t(Y)\delta_0)(\Lambda) = \|M_t(Y)\|,$$

for every $Y \in \mathscr{P}_t$, where δ_0 is the unit mass measure centralized at the origin of Λ. Then the seminorm β_t defined by

$$\beta_t(h) = \int_{Y_t} h(v)\,w_t(dv)$$

for every $h \in \mathscr{L}^1(w_t)$ integrates for the linear map $M_t\colon \operatorname{sim}(\mathscr{P}_t) \to L(E)$.

Let W be a real-valued function on $[0, \infty) \times \Lambda$ such that, for every $t \geqslant 0$, the Feynman–Kac functional e_t, defined by (2.3) for w_t-almost every $v \in Y_t$, is w_t-integrable. This happens, for example, if $W(t,\cdot) = W(0,\cdot)$ for every $t \geqslant 0$ and $W(0,\cdot)$ is a function bounded above and continuous on the complement of a set of capacity 0 in Λ.

Define the operator $U(t)$ by (2.4), for every $t \geqslant 0$.

The resulting operator-valued function $t \mapsto U(t)$, $t \geqslant 0$, can be interpreted as a mathematical description of the superposition of diffusion with a creation/destruction process whose rate is proportional to the concentration of the diffusing substance with the coefficient of the proportion equal to W. So, if the distribution of the substance at time $t = 0$ is represented by the measure $\varphi \in E$, then, at any time $t \geqslant 0$, its distribution is represented by the measure $u(t) = U(t)\varphi$.

Now, if the functions $W(t,\cdot)$, $t > 0$, also happen to be integrable with respect to the Gaussian measures on $\mathbf{R}^d$, then the assumptions of Theorem 2.1 are satisfied and, hence, U satisfies the integral equation

$$U(t) = S(t) + \int_0^t S(t-s)P(W(s,\cdot))U(s)\,ds,$$

for every $t \geqslant 0$. Consequently, u satisfies the integral equation

$$u(t) = S(t)\varphi + \int_0^t S(t-s)P(W(s,\cdot))u(s)\,ds, \tag{2.12}$$

for every $t \geqslant 0$. But, for every $t > 0$, the measure $u(t)$ has a density; let us denote it by $x \mapsto u(t, x)$, $x \in \boldsymbol{R}^d$. By (2.12), this density satisfies the equation

$$u(t,x) = \int_{\boldsymbol{R}^d} p_D(t, x-y)\varphi(dy) + \int_0^t \int_{\boldsymbol{R}^d} p_D(t-s, x-y)W(s,y)u(s,y)\,dy\,ds,$$

for $x \in \boldsymbol{R}^d$ and $t > 0$. This equation represents the initial-value problem

$$u(t,x) = D\Delta u(t,x) + W(t,x)u(t,x), \quad t > 0, x \in \boldsymbol{R}^d;$$

$$\lim_{t\to 0+} \int_B u(t,x)\,dx = \varphi(B), \quad B \in \mathscr{B}.$$

If $d \geqslant 2$, it is very easy to produce a function W on $[0, \infty) \times \Lambda$ such that $U(t)$ is well defined by (2.9) for every $t \geqslant 0$, but, for many $\varphi \in E$, the integral equation (2.12) does not have a solution. Then of course this initial-value problem does not have a solution either. For example, $W(t, x) = |x|^{-d}$, $t \in [0, \infty)$, $x \in \Lambda$, $x \neq 0$, is such a function. In such cases, we can take the function $t \mapsto u(t)$, $t \geqslant 0$, defined by (2.5) for a generalized solution. However, there is no need to consider the integral equation (2.12) and the initial-value problem at all because this function u is a perfectly good solution of the original problem of the evolution of an element φ of E under the simultaneous influence of the propagators S and T.

D. Let $d \geqslant 1$ be an integer. Let $\Lambda = \boldsymbol{R}^d$. Let $E = L^2(\boldsymbol{R}^d)$.

For a set $B \in \mathscr{B} = \mathscr{B}(\boldsymbol{R}^d)$, let $P(B)$ be the operator of pointwise multiplication by the characteristic function of the set B. That is, $P(B)\varphi = B\varphi$, for every $\varphi \in E$. Then $P: \mathscr{B} \to L(E)$ is a spectral measure.

Let m be a strictly positive number. Let $S(0) = I$ and, for every $t \neq 0$, let $S(t) \in L(E)$ be the operator such that

$$(S(t)\varphi)(x) = \left(\frac{m}{2\pi t i}\right)^{d/2} \int_\Lambda \varphi(y) \exp\left(\frac{m\mathrm{i}}{2t}|x-y|^2\right), \quad x \in \Lambda,$$

for every $\varphi \in L^1 \cap L^2(\Lambda)$. The root is determined from the branch which assigns positive real values to positive real numbers. It is well known that such an operator exists and the resulting map $t \mapsto S(t)$, $t \in \boldsymbol{R}$, is a unitary group of operators, the Schrödinger group, whose infinitesimal generator is the operator

$$A = \frac{\mathrm{i}}{2m}\Delta,$$

where Δ is the Laplacian on $\Lambda = \boldsymbol{R}^d$.

For every $\tau > 0$, the resolvent $(I - \tau A)^{-1}$ is a convolution-type integral operator belonging to $L(E)$. More precisely, there exists a function $K(\cdot\,;\tau) \in L^1(\Lambda)$ such that, if $\varphi \in E$, then

$$((I-\tau A)^{-1}\varphi)(x) = \int_\Lambda K(x-y;\tau)\varphi(y)\,dy$$

for almost every $x \in \Lambda$ in the sense of the d-dimensional Lebesgue measure. The kernel $K(\cdot\,;\tau)$ is the inverse Fourier transform of the function $\zeta \mapsto 2m(\tau \mathrm{i}\zeta^2 + 2m)^{-1}$, $\zeta \in \Lambda$, that is

$$K(z;\tau) = \frac{2m}{(2\pi)^{d/2}} \int_\Lambda \frac{\exp(\mathrm{i}\zeta\cdot z)}{\tau \mathrm{i}\zeta^2 + 2m}\,d\zeta, \tag{2.13}$$

for almost every $z \in \Lambda$, with the integral understood in the sense of tempered distributions or simply calculated using the Fourier transform of an approximate unit in $L^1(\Lambda)$.

If $d = 1$, 2 or 3, then $K(\cdot\,;\tau) \in L^2(\Lambda)$. For $d = 1$ or 3, it is easy to calculate (2.13) explicitly. So, for $d = 3$, we have

$$K(z;\tau) = \frac{m}{2\pi \mathrm{i}\tau|z|}\exp\left(-(1+\mathrm{i})\sqrt{m/\tau}\,|z|\right), \quad z \in \boldsymbol{R}^3, z \neq 0.$$

In general, $K(\cdot\,;\tau)$ can be expressed in terms of the Hankel functions ([5], Appendix IV.C.).

For every $t \geqslant 0$, let Y_t be the set of all continuous paths $v: [0, t] \to \Lambda$.

Because S is a semigroup, the formula (2.2) takes the form (2.11) for every $Y \in \mathscr{P}_t$ given by (2.1). We are going to construct a seminorm β_t integrating for linear extensions of M_t. The construction is closely related to the modification of the Feynman integral recently proposed by Michel L. Lapidus [3]. In fact, it mimicks his approach to the Lie–Trotter formula for unitary groups of operators [4].

Let i_t be the Wiener measure on Y_t of variance m^{-1} per unit of time with the initial distribution standard normal.

Let $F = l^\infty(L(E))$ be the space of all bounded sequences of operators in $L(E)$ equipped with the norm

$$\|\{T_n\}_{n=1}^\infty\| = \sup\{\|T_n\|: n = 1, 2, \ldots\}, \quad \{T_n\}_{n=1}^\infty \in F.$$

Let F_0 be the subspace of F consisting of those sequences of operators which are convergent in the weak operator topology.

Let LIM be a Banach limit on the space l^∞ of all bounded sequences of complex numbers. That is to say, LIM is a translation-invariant continuous linear functional on l^∞ assigning nonnegative real values to nonnegative real elements of l^∞, whose value at any convergent sequence is equal to its limit in the usual sense of the word.

Then LIM generates a continuous linear map of F into $L(E)$, denoted by the same symbol, defined by

$$\langle (\mathrm{LIM}\,\{T_n\}_{n=1}^{\infty})\varphi, \psi\rangle = \lim\{\langle T_n\varphi, \psi\rangle\}_{n=1}^{\infty}, \qquad \varphi\in E, \quad \psi\in E,$$

for every $\{T_n\}_{n=1}^{\infty}\in F$.

Let $t>0$ and let $n\geqslant 1$ be an integer.

Let $\pi_{t,n}\colon Y_t\to \Lambda^{2^n}$ be the map such that

$$\pi_{t,n}(v) = \big(v(2^{-n}t), v(2\cdot 2^{-n}t), v(3\cdot 2^{-n}t), \ldots, v(t)\big) = \big(v(j2^{-n}t)\big)_{j=1}^{2^n},$$

for every $v\in Y_t$.

Now, for a function $f\in L^1(i_t)$, let $i_t(f|\pi_{t,n})$ be a version of the conditional i_t-expectation of f given $\pi_{t,n}$. That is, $i_t(f|\pi_{t,n})$ is a function on Λ^{2^n} such that

$$\int_Z i_t(f|\pi_{t,n})(\zeta)(i_t\circ\pi_{t,n}^{-1})(d\zeta) = \int_{\pi_{t,n}^{-1}(Z)} f(v)\,i_t(dv),$$

for every $Z\in\mathscr{B}(\Lambda^{2^n})$. So, $i_t(f|\pi_{t,n})$ is defined $i_t\circ\pi_{t,n}^{-1}$-almost uniquely on Λ^{2^n} and, hence, uniquely almost everywhere with respect to the $2^n d$-dimensional Lebesgue measure.

Let

$$K_n(x_0, x_1, \ldots, x_{2^n}; t) = \prod_{j=1}^{2^n} K(x_j - x_{j-1}; 2^{-n}t)$$

for any points $x_j\in\Lambda$, $j=0,1,\ldots,2^n$.

Let $\mathscr{K}_{t,n}$ be the vector space of all functions $f\in\mathscr{L}^1(i_t)$ for which there exists an operator $N_{t,n}(f)\in L(E)$ such that

$$\langle N_{t,n}(f)\varphi, \psi\rangle$$
$$= \int_{\Lambda^{2^n+1}} \varphi(x_0) K_n(x_0, x_1, \ldots, x_{2^n}; t)\, i_t(f|\pi_{t,n})(x_1, \ldots, x_n)\overline{\psi(x_n)}\,dx_0\,dx_1\ldots dx_{2^n},$$

for every $\varphi\in E$ and $\psi\in E$.

Let $t>0$. Let $\mathscr{K}_t$ be the space of all functions $f\in\mathscr{L}^1(i_t)$ such that $f\in\mathscr{K}_{t,n}$, for every $n=1,2,\ldots$, and the sequence of operators $N_t(f) = \{N_{t,n}(f)\}_{n=1}^{\infty}$ belongs to F. Let $\mathscr{K}_t^0$ be the subspace of $\mathscr{K}_t$ consisting of functions $f\in\mathscr{K}_t$ for which $N_t(f)\in F_0$.

Let

$$\beta_t(f) = i_t(|f|) + \|N_t(f)\|$$

for every $f\in\mathscr{K}_t$. Because $N_t\colon\mathscr{K}_t\to F$ is a closed linear map, Proposition 1.11 implies that β_t is an integrating seminorm on $\mathscr{K}_t$ such that $\mathscr{L}(\beta_t)=\mathscr{K}_t$. Obviously, the seminorm β_t integrates for N_t.

Obviously, $\mathscr{K}_t^0$ is a closed subspace of $\mathscr{K}_t$.

Now, because LIM: $F\to L(E)$ is a continuous linear map, the seminorm

β_t integrates for the linear map $\mathrm{LIM}\circ N_t\colon\mathscr{K}_t\to L(E)$. Furthermore, because

$$\lim_{n\to\infty}(I-2^{-n}tA)^{2^n} = \exp(tA),$$

for every $t\geqslant 0$, in the strong operator topology, it is straightforward that the characteristic function of every set $Y\in\mathscr{P}_t$ given by (2.1) with dyadic rational t_j/t, $j=1,2,\ldots,k$, belongs to $\mathscr{K}_t^0$ and that

$$(\mathrm{LIM}\circ N_t)(Y) = M_t(Y)$$

for every such set Y. This equality then follows for every $Y\in\mathscr{P}_t$. Consequently, $\mathrm{LIM}\circ N_t$ is a linear extension of M_t onto $\mathscr{K}_t=\mathscr{L}(\beta_t)$.

Let us conclude by noting that the space $\mathscr{L}(\beta_t)$ is rather large. In particular, it includes every i_t-measurable function h on Y_t such that $|h(v)|\leqslant 1$, for every $v\in Y_t$. So, if V is a real-valued function on Λ continuous on the complement of a set of capacity 0, then the Feynman–Kac functional

$$e_t(v) = \exp\Big(i\int_0^t V\big(v(r)\big)\,dr\Big), \qquad v\in Y_t,$$

belongs to $\mathscr{L}(\beta_t)$.

The interesting question when a function belongs to $\mathscr{K}_t^0$ could be dealt with using the methods of [4]; it will be pursued elsewhere.

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