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Boundary values of vector-valued harmonic functions considered as operators

by

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Abstract. It is proved that the spaces of the boundary values of B-valued harmonic functions in $h^p_{w,B}(D)$ and $h^p_B(D)$ may be interpreted as spaces of operators. It is also shown that the Radon–Nikodým property is the exact condition on B to make $L^p_B(T)$ be the boundary values space of $h^p_B(D)$.

§ 0. Introduction. In the following we will denote by D the open unit disc in C, by B a (real or complex) Banach space and by $(T, \mathfrak{B}, m)$ the usual measure space on the torus T.

Let us recall that $h^p(D)$ denotes the space of harmonic functions on D such that

$$(0.1)\qquad \|F\|_{h^p} = \sup_{0<r<1} \left(\frac{1}{2\pi}\int_0^{2\pi} |F(re^{it})|^p\,dt\right)^{1/p} < +\infty \quad \text{if } 1 \leqslant p < \infty,$$

$$(0.2)\qquad \|F\|_{h^\infty} = \sup\{|F(z)|: z \in D\} < \infty \quad \text{if } p = \infty.$$

It is well known (see [5]) that the space of boundary values of functions in $h^p(D)$ is $L^p(T)$ if $1 < p \leqslant \infty$ and $M(T)$, the regular Borel measures, if $p = 1$. This means that every function in $h^p(D)$ is the Poisson integral of a function in $L^p(T)$ if $1 < p \leqslant \infty$ or of a measure in $M(T)$ if $p = 1$.

In this paper we are interested in studying this subject when the functions take values in a Banach space B.

In order to define the analogous concepts for B-valued functions there are two different ways to follow: we can consider the space of B-valued functions on D such that ξF belongs to $h^p(D)$ for every ξ in B^*, denoted by $h^p_{w,B}(D)$, or the space of B-valued harmonic functions on D which satisfy the above definition with the absolute value replaced by the norm in B, denoted by $h^p_B(D)$.

We shall prove here that $h^p_{w,B}(D)$ is isometric via Poisson integral to the space of bounded operators between $C(T)$ and B, $L(C(T), B)$, for $p = 1$, and between $L^{p'}(T)$ and B, $L(L^{p'}(T), B)$, for $1 < p \leqslant \infty$ where $1/p + 1/p' = 1$.

It will also be shown that $h^p_B(D)$ can be identified with a class of operators defined by Dinculeanu (see [3]). We shall connect this Dinculeanu space with the space of p-summing operators by finding an equivalent formulation of it in terms of a class of operators we shall call positive p-summing operators.

Finally, the interpretation of the functions in $h_B^p(D)$ as Poisson integrals of certain operators will enable us to obtain Bukhvalov and Danilevich's result (see [1]) which asserts that the Radon–Nikodým property of B is a necessary and sufficient condition for $h_B^p(D)$ to be isometric to $L_B^p(T)$ for any $1 < p \leqslant \infty$.

§ 1. Preliminary definitions and results.

Definition 1 ([6]). Let A and B be Banach spaces and $1 \leqslant p < \infty$. An operator S in $L(A, B)$ is called *p-summing* if there exists a constant C such that for every n in N and $x_1, x_2, \ldots, x_n$ in A,

$$(1.1)\qquad \Big(\sum_{i=1}^{n} \|S(x_i)\|_B^p\Big)^{1/p} \leqslant C \sup\Big\{\Big(\sum_{i=1}^{n} |\langle \xi, x_i\rangle|^p\Big)^{1/p} \colon \|\xi\|_{B^*} \leqslant 1\Big\}.$$

We shall denote by $\pi^p(A, B)$ the space of p-summing operators from A to B and the norm in it will be the infimum of the constants satisfying (1.1).

When we deal with $A = L^q(T)$ $(1 \leqslant q < \infty)$ there is a space of operators in $L(L^q(T), B)$ defined by Dinculeanu [3] which will play an important role in the following.

Definition 2. Given p, p' such that $1 < p \leqslant \infty$ and $1/p + 1/p' = 1$ and S in $L(L^{p'}(T), B)$ we set

$$(1.2)\qquad |||S|||_p = \sup\Big\{\sum_{i=1}^{n} |\alpha_i|\, \|T(\chi_{E_i})\|_B\Big\}$$

where the supremum is taken over all simple functions $\psi = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$ with $\|\psi\|_{L^{p'}} \leqslant 1$.

We shall denote by $\mathscr{L}(L^{p'}(T), B)$ the space of operators S in $L(L^{p'}(T), B)$ with $|||S|||_p < \infty$. The space $\mathscr{L}(L^{p'}(T), B)$ with the norm $|||\ |||_p$ becomes a Banach space.

The next result will show the relationship between $\mathscr{L}(L^{p'}(T), B)$ and $L^p(T)$ $(1/p + 1/p' = 1)$.

Proposition 3. *Let S be an operator in $L(L^{p'}(T), B)$ with $1 \leqslant p' < \infty$ and $1/p + 1/p' = 1$.*

S belongs to $\mathscr{L}(L^{p'}(T), B)$ if and only if there exists a positive function g in $L^p(T)$ such that for every function ψ in $L^{p'}(T)$,

$$(1.3)\qquad \|S(\psi)\|_B \leqslant \int |\psi(t)| g(t)\, dt$$

Moreover, in this case g can be chosen such that

$$|||S|||_p = \|g\|_{L^p}.$$

Proof. Suppose S belongs to $\mathscr{L}(L^{p'}(T), B)$ and consider the B-valued measure G defined by S as follows: $G(E) = S(\chi_E)$ for every measurable set E in $\mathfrak{B}$. From (1.2) it is not difficult to see that G is m-continuous and has bounded variation, so $|G|$, the variation of G, is a finite positive measure which is m-continuous. Now the Radon–Nikodým theorem implies that there is a positive function g in $L^1(T)$ such that

$$(1.4)\qquad |G|(E) = \int_E g(t)\, dt \quad \text{for every } E \text{ in } \mathfrak{B}.$$

To do the direct implication we have to prove that g belongs to $L^p(T)$, $\|g\|_{L^p} = |||S|||_p$ and (1.3) is satisfied.

Let us compute $\|g\|_{L^p}$ as follows:

$$\|g\|_{L^p} = \sup\Big\{\Big|\int\Big(\sum_{i=1}^{n} \alpha_i \chi_{E_i}\Big) g\Big| \colon \Big\|\sum_{i=1}^{n} \alpha_i \chi_{E_i}\Big\|_{L^{p'}} \leqslant 1\Big\}$$
$$\leqslant \sup\Big\{\sum_{i=1}^{n} |\alpha_i|\, |G|(E_i) \colon \Big\|\sum_{i=1}^{n} \alpha_i \chi_{E_i}\Big\|_{L^{p'}} \leqslant 1\Big\}.$$

Now given $\psi = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$ observe that

$$\sum_{i=1}^{n} |\alpha_i|\, |G|(E_i)$$
$$= \sum_{i=1}^{n} |\alpha_i| \sup\Big\{\sum_{j=1}^{k_i} \|G(A_{i,j})\|_B \colon \{A_{i,j}\}_j \text{ a partition of } E_i\Big\}$$
$$\leqslant \sup\Big\{\sum_{i=1}^{n} \sum_{j=1}^{k_i} |\beta_{i,j}|\, \|G(A_{i,j})\|_B \colon \Big\|\sum_{i,j} \beta_{i,j} \chi_{A_{i,j}}\Big\|_{L^{p'}} = \|\psi\|_{L^{p'}}\Big\}.$$

These two inequalities obviously imply that $\|g\|_{L^p} \leqslant |||S|||_p$.

On the other hand, by Hölder's inequality we have

$$(1.5)\qquad \sum |\alpha_i|\, \|S(\chi_{E_i})\|_B \leqslant \sum |\alpha_i| \int_{E_i} g(t)\, dt \leqslant \|g\|_{L^p} \Big\|\sum \alpha_i \chi_{E_i}\Big\|_{L^{p'}}$$

and this clearly implies that $|||S|||_p \leqslant \|g\|_{L^p}$.

To finish this implication we have only to prove (1.3). From (1.4) it clearly holds for $\psi = \chi_E$ with $E \in \mathfrak{B}$. Hence it follows for simple functions $\psi = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$ since

$$\Big\|S\Big(\sum_{i=1}^{n} \alpha_i \chi_{E_i}\Big)\Big\|_B \leqslant \sum_{i=1}^{n} |\alpha_i|\, |G|(E_i).$$

Now the density of the simple functions in $L^{p'}(T)$ $(1 \leqslant p' < \infty)$ and the continuity of S lead us to (1.3).

The converse is immediate by making the same computation as in (1.5). ■

Let us connect functions with operators in the following sense:

The spaces $L^p_B(T)$ of B-valued measurable functions f with $\|f\|_{L^p_B} = (\int \|f(t)\|^p_B\,dt)^{1/p}$ for $1 \leqslant p < \infty$ (with the obvious modification for $p = \infty$) may be interpreted as spaces of operators as we shall see below.

In order to unify the results we shall use the following notation:

$X_q = L^q(T)$ for $1 \leqslant q < \infty$,

$X_\infty = C(T)$, the space of continuous function on T.

Now given $1 \leqslant p, p' \leqslant \infty$ with $1/p + 1/p' = 1$, we shall denote by J_p the embedding

$$J_p\colon L^p_B(T) \to L(X_{p'}, B)$$

given by

(1.6) $$J_p(f)(\psi) = \int f(t)\psi(t)\,dt \qquad (\psi \in X_{p'},\ f \in L^p_B(T))$$

It is obvious that $\|J_p(f)\|_B \leqslant \|f\|_{L^p_B}$.

Furthermore, operators in $J_p(L^p_B(T))$ are compact for $1 \leqslant p < \infty$. This fact can be proved by observing that operators corresponding to simple functions are obviously compact and the simple functions are dense in $L^p_B(T)$.

According to Proposition 3, by taking $g(t) = \|f(t)\|_B$, $J_p(L^p_B(T))$ is actually included in $\mathscr{L}(L^{p'}(T), B)$ and we can state the following

COROLLARY 4. *For* $1 < p \leqslant \infty$, $L^p_B(T)$ *is isometrically embedded in* $\mathscr{L}(L^{p'}(T), B)$.

§ 2. $h^p_{w,B}(D)$ spaces.

DEFINITION 5. Let $1 \leqslant p \leqslant \infty$, and let B be a Banach space. We shall denote by $h^p_{w,B}$ the space

(2.1) $$h^p_{w,B}(D) = \{F\colon D \to B\colon \xi \cdot F \in h^p(D) \text{ for all } \xi \in B^*\}$$

where $\xi \cdot F(z) = \langle \xi, F(z) \rangle$.

First of all we shall define a norm in it as follows:

PROPOSITION 6. *For each* $1 \leqslant p \leqslant \infty$,

(2.2) $$\|F\|_{w,p} = \sup\{\|\xi \cdot F\|_{h^p}\colon \|\xi\|_{B^*} \leqslant 1\}$$

is a norm in $h^p_{w,B}(D)$.

Proof. We shall show that for every F in $h^p_{w,B}(D)$ $\|F\|_{w,p}$ is finite. The other properties of norm are straightforward. Let us consider $\varphi_F\colon B^* \to h^p(D)$ defined by $\varphi_F(\xi) = \xi \cdot F$ for all $\xi \in B^*$. If we show that φ_F has closed graph then φ_F will be continuous and therefore $\|\varphi_F\| = \|F\|_{w,p}$ will be finite. Let $\{\xi_n\}$ be a sequence converging to zero in B^* and let us suppose $\xi_n \cdot F$ converges to g in $h^p(D)$. Thus, writing $g_r(\theta) = g(re^{i\theta})$ and $h^{(n)}_r(\theta) = \langle \xi_n, F(re^{i\theta}) \rangle$ for $0 < r < 1$, we deduce that $h^{(n)}_r$ converges to g_r in $L^p(T)$ and therefore there exists a subsequence $n_k(r) = n_k$ such that

(2.3) $$g_r(\theta) = \lim_{k \to \infty} h^{(n_k)}_r(\theta) \qquad \theta\text{-a.e.}$$

Since $\langle \xi_{n_k}, F(re^{i\theta}) \rangle = h^{(n_k)}_r(\theta)$ converges to zero, by (2.3) we get $g_r(\theta) = 0$ θ-a.e. Finally, since g_r is continuous for any r, $g(z) = 0$ for all $z \in D$. ■

Let us remark that every F in $h^p_{w,B}(D)$ is a B-valued harmonic function and therefore $F_r(\theta) = F(re^{i\theta})$ is a continuous function on T with values in B. Hence F_r can be considered as an operator according to (1.6) and in this sense we can present the space $h^p_{w,B}(D)$ in an analogous way to the classical one.

PROPOSITION 7. *For each* p, $1 \leqslant p \leqslant \infty$, $h^p_{w,B}(D)$ *is the space of B-valued harmonic functions F on D such that*

$$\sup_{0<r<1} \|J_p(F_r)\|_{L(X_{p'},B)} = \|F\|_{w,p}$$

is finite.

Proof.

$$\begin{aligned}
\sup_{0<r<1} \|J_p(F_r)\|_{L(X_{p'},B)}
&= \sup_{0<r<1} \sup_{\|\psi\|_{X_{p'}} \leqslant 1} \Big\|(2\pi)^{-1} \int_0^{2\pi} F(re^{i\theta})\psi(e^{i\theta})\,d\theta\Big\|_B \\
&= \sup_{0<r<1} \sup_{\|\psi\|_{X_{p'}} \leqslant 1} \sup_{\|\xi\|_{B^*} \leqslant 1} \Big|(2\pi)^{-1} \int_0^{2\pi} \langle \xi, F(re^{i\theta}) \rangle \psi(e^{i\theta})\,d\theta\Big| \\
&= \sup_{\|\xi\|_{B^*} \leqslant 1} \sup_{0<r<1} \|(\xi \cdot F)_r\|_{L^p} = \sup_{\|\xi\|_{B^*} \leqslant 1} \|\xi \cdot F\|_{h^p} = \|F\|_{w,p}.
\end{aligned}$$

Our next goal is to look for an isometry which allows us to identify $h^p_{w,B}(D)$ with a space defined on the boundary T of D. To do this we shall extend the concept of Poisson integral to operators.

For each $z = re^{i\theta}$ in D let us denote by P_z the function on T given by $P_z(t) = P_r(\theta - t)$ where

$$P_r(\theta) = \frac{1 - r^2}{1 + r^2 - 2r\cos\theta}$$

is the Poisson kernel on T. Observe that P_z belongs to X_p for all p, $1 \leqslant p \leqslant \infty$. This allows us to give the following

DEFINITION 8. Given $S \in L(X_p, B)$, $1 \leqslant p \leqslant \infty$, the *Poisson integral* of S, denoted by $\mathscr{P}(S)$, will be the B-valued function F on D defined by

$$F(z) = S(P_z).$$

We can now establish the following

THEOREM 9. *For each* p, $1 \leq p \leq \infty$, *and* $1/p+1/p' = 1$, $h^p_{w,B}(D)$ *is isometric to* $L(X_{p'}, B)$.

Proof. We shall prove that the Poisson integral defines an isometry between $L(X_{p'}, B)$ and $h^p_{w,B}(D)$ for every p, $1 \leq p \leq \infty$.

Thus, we consider $\mathscr{P}$: $L(X_{p'}, B) \to h^p_{w,B}(D)$ defined by $\mathscr{P}(S) = F$ where $F(z) = S(P_z)$ for every S belonging to $L(X_{p'}, B)$.

Given $S \in L(X_{p'}, B)$ and $\xi \in B^*$, it is clear that $\xi \cdot S \in (X_{p'})^*$. By recalling now that the Poisson integral maps isometrically $(X_{p'})^*$ onto $h^p(D)$, and since $\xi \cdot F$ is the Poisson integral of $\xi \cdot S$ we have

$$\|S\|_{L(X_{p'},B)} = \sup_{\|\psi\|_{X_{p'}} \leq 1} \|S(\psi)\|_B = \sup_{\|\psi\|_{X_{p'}} \leq 1} \sup_{\|\xi\|_{B^*} \leq 1} |\langle \xi, S(\psi)\rangle|$$
$$= \sup_{\|\xi\|_{B^*} \leq 1} \|\xi \cdot S\|_{(X_{p'})^*} = \sup_{\|\xi\|_{B^*} \leq 1} \|\xi \cdot F\|_{h^p} = \|F\|_{w,p}.$$

Hence we have only to show that $\mathscr{P}$ is onto. First we shall give a proof for $1 < p \leq \infty$. Let us take a function F in $h^p_{w,B}(D)$. Since $\xi \cdot F$ belongs to $h^p(D)$ for every $\xi \in B^*$, by a classical result there is a function f_ξ in $L^p(T)$ such that

$$\xi \cdot F(re^{i\theta}) = P_r * f_\xi(\theta). \tag{2.4}$$

We then consider the operator S in $L(L^{p'}(T), B^{**})$ given by

$$\langle S(\psi), \xi\rangle = \int f_\xi(t)\psi(t)\,dt \quad \text{for every } \xi \in B^* \text{ and } \psi \in L^{p'}(T). \tag{2.5}$$

From (2.4) we can see that $f_{\xi+\xi'} = f_\xi + f_{\xi'}$ in $L^p(T)$ and consequently $S(\psi)$ is a linear form on B^* for every ψ in $L^{p'}(T)$. The continuity of $S(\psi)$ can be obtained from Hölder's inequality as the following computation shows:

$$|\langle S(\psi), \xi\rangle| \leq \|f_\xi\|_{L^p}\|\psi\|_{L^{p'}} = \|\xi \cdot F\|_{h^p}\|\psi\|_{L^{p'}} \tag{2.6}$$
$$\leq \|\xi\|_{B^*}\|F\|_{w,p}\|\psi\|_{L^{p'}}.$$

Hence $\|S(\psi)\|_{B^{**}} \leq \|F\|_{w,p}\|\psi\|_{L^{p'}}$ and so $\|S\| \leq \|F\|_{w,p}$.

Since $S \in L(L^{p'}(T), B^{**})$ we can consider $\mathscr{P}(S)$ as a function with values in B^{**} and we claim that $\mathscr{P}(S)(z) = F(z)$ for all $z \in D$.

Indeed, take $z = re^{i\theta}$ in D and ξ in B^*. From (2.5) and (2.4) by taking $\psi = P_z$ we have

$$\langle \mathscr{P}(S)(z), \xi\rangle = \int f_\xi(t)P_r(\theta - t)\,dt = \xi \cdot F(z) = \langle \xi, F(z)\rangle.$$

To finish the proof it is sufficient to see that the range of S is actually contained in B. Take ψ in $L^{p'}(T)$ and recall that $P_r * \psi$ converges to ψ in $L^{p'}(T)$ as $r \to 1$, so, since S is continuous, we have $S(\psi) = \lim_{r\to 1} S(P_r * \psi)$. According to Hille's theorem about Bochner integration (see [2], p. 47) and since $P_r(-t) = P_r(t)$ we have

$$S(P_r * \psi(\cdot)) = S\left(\int P_r(\cdot - t)\psi(t)\,dt\right) = \int S(P_{re^{it}})\psi(t)\,dt = \int F(re^{it})\psi(t)\,dt.$$

Consequently, $S(P_r * \psi) \in B$, and so $S(\psi) \in B$ as well.

The proof for $p = 1$ is essentially the same. First, for every $\xi \in B^*$ there exists a measure μ_ξ in $M(T) = (C(T))^*$ such that

$$\xi \cdot F(re^{i\theta}) = P_r * \mu_\xi(\theta) = \int P_r(\theta - t)\,d\mu_\xi(t). \tag{2.4'}$$

The operator in this case will be given by

$$\langle S(\psi), \xi\rangle = \int \psi(t)\,d\mu_\xi(t) \quad (\psi \in C(T), \xi \in B^*). \tag{2.5'}$$

The whole proof can now be repeated, a decisive fact being that $P_r * \psi$ converges to ψ in $C(T)$ too.

From this theorem we can deduce the following

COROLLARY 10. *For* $1 \leq p \leq \infty$, $(h^p_{w,B}(D), \|\ \|_{w,p})$ *is a Banach space.*

We have just found a characterization of the "boundary values" of functions in $h^p_{w,B}(D)$ in terms of operators. Now we are going to find out what sort of convergence there exists between an operator S on the boundary and the operators given by F_r where $F = \mathscr{P}(S)$. Given S in $L(X_{p'}, B)$ and $F = \mathscr{P}(S)$, let us denote by S_r the operator $J_p(F_r)$.

THEOREM 11. *With the above notation,*

(1) $\|S_r\|_{L(X_{p'},B)}$ *is increasing to* $\|S\|_{L(X_{p'},B)}$ *as* $r \nearrow 1$.

(2) *For each* p, $1 \leq p \leq \infty$, S_r *converges strongly to* S *as* $r \to 1$.

(3) *For each* p, $1 < p < \infty$, S_r *converges to* S *in the norm topology if and only if* S *is a compact operator.*

Proof. (1) It is immediate that

$$\xi \cdot S_r = (\xi \cdot S) * P_r = (\xi \cdot F)_r.$$

Hence

$$\|S_r\|_{L(X_{p'},B)} = \sup_{\|\xi\|_{B^*} \leq 1} \|\xi \cdot S_r\|_{(X_{p'})^*} = \sup_{\|\xi\|_{B^*} \leq 1} \|(\xi \cdot F)_r\|_{L^p}.$$

For $1 \leq p < \infty$ Hardy's convexity theorem (see [4]) implies that $\|(\xi \cdot F)_{r_1}\|_{L^p} \leq \|(\xi \cdot F)_{r_2}\|_{L^p}$ for $r_1 \leq r_2$ and therefore

$$\|S_{r_1}\|_{L(X_{p'},B)} \leq \|S_{r_2}\|_{L(X_{p'},B)}.$$

The case $p = \infty$ is due to the maximum principle for harmonic functions.

Now we have

$$\|S\|_{L(X_{p'},B)} = \sup_{\|\xi\|_{B^*} \leq 1} \|\xi \cdot S\|_{(X_{p'})^*}$$
$$= \sup_{\|\xi\|_{B^*} \leq 1} \sup_{0<r<1} \|\xi \cdot S_r\|_{(X_{p'})^*}$$
$$= \sup_{0<r<1} \|S_r\|_{L(X_{p'},B)} = \lim_{r\to 1} \|S_r\|_{L(X_{p'},B)}.$$

(2) This is a direct consequence of two facts: first, $P_r * \psi$ converges to ψ in $X_{p'}$ for all $\psi \in X_{p'}$, and secondly, $S_r(\psi) = S(P_r * \psi)$.

(3) Let $0 < r < 1$ and $1 < p < \infty$. Since $S_r = J_p(F_r)$ it follows that S_r is a compact operator, and hence S will also be compact if S_r converges to S in the norm topology. Conversely, assume that S is a compact operator from $L^{p'}(\boldsymbol{T})$ to B where $1/p + 1/p' = 1$ and consider $\psi_r\colon L^p(\boldsymbol{T}) \to L^p(\boldsymbol{T})$ defined by $\psi_r(\varphi) = P_r * \varphi$ for all $\varphi \in L^p(\boldsymbol{T})$. Since $\psi_r(\varphi)$ converges to φ for each φ in $L^p(\boldsymbol{T})$,

$$\psi_r(\varphi) \to \varphi \quad \text{uniformly on compact sets} \quad \text{as } r \to 1. \tag{2.7}$$

Observe that since the adjoint operator of S, S^*, is compact from B^* to $L^p(\boldsymbol{T}) = (L^{p'}(\boldsymbol{T}))^*$ it follows that $\{S^*(\xi)\colon \|\xi\|_{B^*} \leqslant 1\}$ is a relatively compact set in $L^p(\boldsymbol{T})$. Now from (2.7), $\psi_r \cdot S^*(\xi)$ converges to $S^*(\xi)$ uniformly on $\|\xi\|_{B^*} \leqslant 1$. On the other hand, the adjoint of ψ_r, $(\psi_r)^*$, is also defined by $(\psi_r)^*(\varphi) = P_r * \varphi$ for all $\varphi \in L^{p'}(\boldsymbol{T})$, and since $\psi_r \cdot S^* = (S \cdot \psi_r)^* = (S_r)^*$, $(S_r)^*$ converges to S^* in the norm topology and therefore S_r also converges to S in the same way.

Remarks 12. In order to complete the third part of Theorem 11 in the cases $p = 1$ and $p = \infty$, we can make the following observations:

(1) Taking $B = \boldsymbol{R}$, there is a function f in $L^\infty(\boldsymbol{T})$, which coincides with $L(L^1(\boldsymbol{T}), \boldsymbol{R})$, such that $S_r = P_r * f$ does not converge to f in $L^\infty(\boldsymbol{T})$.

(2) Taking $B = \boldsymbol{R}$, there is a measure δ (the Dirac delta at 0) in $L(C(\boldsymbol{T}), \boldsymbol{R})$ such that $P_r = P_r * \delta$ does not converge to δ in $M(\boldsymbol{T})$.

§ 3. $h_B^p(D)$ spaces.

DEFINITION 13. Let $1 \leqslant p < \infty$ and let B be a Banach space. We shall denote by $h_B^p(D)$ the space of B-valued harmonic functions F such that

$$\|F\|_p = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \|F(re^{i\theta})\|_B^p \, d\theta \right)^{1/p} < +\infty. \tag{3.1}$$

For $p = \infty$, $h_B^\infty(D)$ will be the space of B-valued bounded harmonic functions F on D, with the norm

$$\|F\|_\infty = \sup \{\|F(z)\|_B\colon z \in D\}. \tag{3.1'}$$

It is easy to show that $\|\cdot\|_p$ is a norm in $h_B^p(D)$ for all $1 \leqslant p \leqslant \infty$.

Before dealing with the boundary values spaces of these spaces let us establish some immediate properties:

(3.2) If $1 \leqslant p \leqslant q \leqslant \infty$ then $h_B^q(D) \subseteq h_B^p(D)$. Moreover, if $F \in h_B^q(D)$ then $\|F\|_q \leqslant \|F\|_p$.

(3.3) For each $1 \leqslant p < \infty$, $h_B^p(D) \subseteq h_{w,B}^p(D)$. Moreover, if $F \in h_B^p(D)$ then $\|F\|_{w,p} \leqslant \|F\|_p$.

(3.4) $h_B^\infty(D) = h_{w,B}^\infty(D)$ and $\|\cdot\|_{w,\infty} = \|\cdot\|_\infty$.

(3.5) $\|F_r\|_{L_B^p}$ is increasing and $\|F\|_p = \lim_{r\to 1} \|F_r\|_{L_B^p}$.

The proofs of (3.2), (3.3) and (3.4) are straightforward. To prove (3.5) it suffices to realize that $g(z) = \|F(z)\|_B^p$ is a subharmonic function for $1 \leqslant p < \infty$, and then to use Hardy's convexity theorem. The case $p = \infty$ is again a consequence of the maximum principle for B-valued harmonic functions.

Now we shall prove in a different way the following known result:

PROPOSITION 14. *For each* $1 \leqslant p \leqslant \infty$, $(h_B^p(D), \|\cdot\|_p)$ *is a Banach space.*

Proof. The case $p = \infty$ is already proved in Corollary 10 by using (3.4). Let $1 \leqslant p < \infty$ and let us take a Cauchy sequence $\{F_n\}$ in $h_B^p(D)$. From (3.3) and Corollary 10, there is a function F in $h_{w,B}^p(D)$ such that $F = \mathscr{P}(S)$ for some S in $L(X_{p'}, B)$. Moreover, if $F_n = \mathscr{P}(T_n)$ then we have

$$\|T_n - S\|_{L(X_{p'},B)} \to 0 \quad \text{as } n \to \infty. \tag{3.6}$$

Since $\{F_n\}$ is a Cauchy sequence, given $\varepsilon > 0$ there is a positive integer $n_0(\varepsilon)$ such that

$$\sup_{0<r<1} \|(F_n)_r - (F_m)_r\|_{L_B^p} < \varepsilon \quad \text{for all } n, m \geqslant n_0(\varepsilon). \tag{3.7}$$

Then we fix $0 < r < 1$ and observe that

$$\sup_{t\in \boldsymbol{T}} \|F_m(re^{it}) - F(re^{it})\|_B = \sup_{t\in \boldsymbol{T}} \|(T_m - S)(P_{re^{it}})\|_B \leqslant \|T_m - S\|_{L(X_{p'},B)} \|P_r\|_{X_{p'}}. \tag{3.8}$$

Now from (3.8) we can see that $(F_m)_r$ converges to F_r in $L_B^p(\boldsymbol{T})$ as $m \to \infty$. Finally, fix $n \geqslant n_0(\varepsilon)$ and take the limit in (3.7) as $m \to \infty$. It follows that F belongs to $h_B^p(D)$ and F_m converges to F in $h_B^p(D)$. ■

Let us remark that, as usual, we can obtain functions in $h_B^p(D)$ simply by considering the Poisson integrals of functions in $L_B^p(\boldsymbol{T})$. So given $f \in L_B^p(\boldsymbol{T})$, $1 \leqslant p \leqslant \infty$, if we take

$$F(re^{i\theta}) = P_r * f(\theta) = \int f(t) P_r(\theta - t)\, dt$$

it is not difficult to deduce that F belongs to $h_B^p(D)$ and moreover $\|F\|_p = \|f\|_{L_B^p}$.

When we deal with $\boldsymbol{R}$-valued functions and $1 < p \leqslant \infty$ the converse is also true: If F belongs to $h^p(D)$ there exists a function f in $L^p(\boldsymbol{T})$ such that $F = \mathscr{P}(f)$. The next example shows that this does not remain valid in the B-valued setting.

EXAMPLE. Given $1 < p < \infty$ and $B = L^1(\boldsymbol{T})$ we consider $F(z) = P_z$. It is not difficult to see that $F \in h_{L^1(\boldsymbol{T})}^p(D)$ for all p, $1 \leqslant p \leqslant \infty$, but is not the Poisson integral of any function in $L_{L^1(\boldsymbol{T})}^1(\boldsymbol{T})$. In fact, F is the Poisson integral of the operator $I: L^{p'}(T) \to L^1(T)$ defined by $I(\psi) = \psi$ for every $\psi \in L^{p'}(T)$ $(1/p'+1/p = 1)$. The operator I cannot be represented by any function, i.e. there does not exist any $f \in L^p_{L^1(T)}(T)$ such that $J_p(f) = I$, since I is not compact. To see this, it suffices to take

$$f_n = \sum_{j=0}^{2^{n-1}-1} \chi_{[2j/2^n,(2j+1)/2^n)}$$

for $n = 1, 2, \ldots$ and to observe that $\|f_n\|_{L^{p'}} = (1/2)^{1/p'}$ and $\|f_n - f_m\|_{L^1} = 1/2$ for every $n \neq m$, so $\{f_n\}$ does not have any convergent subsequence. ■

Since $h^p_B(D)$ is included in $h^p_{w,B}(D)$, we have to look for the boundary values of functions in $h^p_B(D)$ among the operators in $L(X_{p'}, B)$.

Definition 15. For $1 \leq p \leq \infty$ we shall denote by $h^p_B(T)$ the space of operators S in $L(X_{p'}, B)$ such that $\mathscr{P}(S)$ belongs to $h^p_B(D)$. with the norm

$$\|S\|_{h^p_B} = \|\mathscr{P}(S)\|_p.$$

We are going to give a characterization of these spaces in three steps: $p = \infty$, $1 < p < \infty$ and $p = 1$.

According to Theorem 9 and (3.4) we have

Proposition 16. *$h^\infty_B(T) = L(L^1(T), B)$ with equal norms.*

Theorem 17. *Let $1 < p < \infty$ and $1/p+1/p' = 1$. Then $h^p_B(T) = \mathscr{L}(L^{p'}(T), B)$ with equal norms.*

Proof. Let S be an operator in $h^p_B(T)$ and take a simple function $\psi = \sum_{i=1}^n \alpha_i \chi_{E_i}$ with $\|\psi\|_{L^{p'}} \leq 1$.

Since $\chi_{E_i} * P_r$ converges to χ_{E_i} in $L^{p'}(T)$ for all $i \in \{1, 2, \ldots, n\}$ and S is continuous we can write

$$\|S(\chi_{E_i})\|_B = \lim_{r\to 1} \|S(\chi_{E_i} * P_r)\|_B. \tag{3.9}$$

Denoting by F the function $\mathscr{P}(S)$ and using (3.9), Hille's theorem and Hölder's inequality we can compute as follows:

$$\begin{aligned}\sum_{i=1}^n |\alpha_i| \|S(\chi_{E_i})\|_B &= \lim_{r\to 1} \sum_{i=1}^n |\alpha_i| \|S(\chi_{E_i} * P_r)\|_B \\ &= \lim_{r\to 1} \sum_{i=1}^n |\alpha_i| \Big\|S\Big(\int_{E_i} P_r(\cdot - t)\, dt\Big)\Big\|_B \\ &= \lim_{r\to 1} \sum_{i=1}^n |\alpha_i| \Big\| \int_{E_i} S(P_{re^{it}})\, dt\Big\|_B \\ &\leq \sup_{0<r<1} \int |\psi(t)|\, F(re^{it})\, dt \leq \sup_{0<r<1} \|F_r\|_{L^p_B} \\ &= \|S\|_{h^p_B}.\end{aligned}$$

Therefore $|||S|||_p \leq \|S\|_{h^p_B}$.

Conversely, given S in $\mathscr{L}(L^{p'}(T), B)$ and $z = re^{it}$ in D, by (1.3) there exists $g \in L^p(T)$ such that

$$\|S(P_z)\|_B \leq P_r * g(t). \tag{3.10}$$

By integrating in (3.10) and using Minkowski's inequality we obtain

$$\begin{aligned}\|S\|_{h^p_B} &= \sup_{0<r<1} \Big(\int \|S(P_{re^{it}})\|^p_B\, dt\Big)^{1/p} \\ &\leq \sup_{0<r<1} \|P_r * g\|_{L^p} \leq \|g\|_{L^p} = |||S|||_p.\end{aligned}$$

Theorem 18. *$h^1_B(T) = \pi^1(C(T), B)$ with equal norms.*

Proof. Let S be a 1-summing operator from $C(T)$ to B. According to Pietsch's factorization theorem (see [6], p. 41) there is a measure ν in $M(T)$ such that $\|\nu\|_{M(T)} = 1$ and

$$\|S(\psi)\|_B \leq \|S\|_{\pi^1} \int |\psi(t)|\, d\nu(t) \quad \text{for all } \psi \in C(T). \tag{3.11}$$

Using now (3.11) and writing $z = re^{i\theta}$ it follows that

$$\|S(P_z)\|_B \leq \|S\|_{\pi^1} (P_r * \nu(t))$$

and therefore $F = \mathscr{P}(S)$ satisfies

$$\begin{aligned}\|F_r\|_{L^1_B} &= \int \|S(P_{re^{it}})\|_B\, dt \\ &\leq \|S\|_{\pi^1} \|P_r * \nu\|_{L^1} \leq \|S\|_{\pi^1}.\end{aligned}$$

So we have $\|S\|_{h^1_B} \leq \|S\|_{\pi^1}$.

To see the converse let S be an operator in $h^1_B(T)$ and $\psi_1, \psi_2, \ldots, \psi_n$ functions in $C(T)$. We claim that

$$\sum_{i=1}^n \|S(\psi_i)\|_B \leq \|S\|_{h^1_B} \Big\|\sum_{i=1}^n |\psi_i|\Big\|_{C(T)}. \tag{3.12}$$

Indeed, since $\psi_i * P_r$ converges to ψ_i in $C(T)$ for all i, we can write

$$\begin{aligned}\sum_{i=1}^n \|S(\psi_i)\|_B &= \lim_{r\to 1} \sum_{i=1}^n \|S(P_r * \psi_i)\|_B \\ &= \lim_{r\to 1} \sum_{i=1}^n \Big\|S\Big(\int P_r(\cdot - t)\, \psi_i(t)\, dt\Big)\Big\|_B \\ &\leq \lim_{r\to 1} \sum_{i=1}^n \int \|S(P_{re^{it}})\|_B\, |\psi_i(t)|\, dt \\ &\leq \Big(\sup_{0<r<1} \int \|S(P_{re^{it}})\|_B\, dt\Big) \Big\|\sum_{i=1}^n |\psi_i|\Big\|_{C(T)} \\ &\leq \|S\|_{h^1_B} \Big\|\sum_{i=1}^n |\psi_i|\Big\|_{C(T)}.\end{aligned}$$

It is proved in [2], p. 162, that

$$\Big\|\sum_{i=1}^{n} |\psi_i|\Big\|_{C(T)} = \sup\Big\{\sum_{i=1}^{n} \Big|\int \psi_i(t)\,d\mu(t)\Big| : \|\mu\|_{M(T)} \leqslant 1\Big\}.$$

This last inequality together with (3.12) implies that $\|S\|_{\pi^1} \leqslant \|S\|_{h_B^1}$. ■

There is a way of unifying the results about boundary values of $h_B^p(D)$ for all p, $1 \leqslant p \leqslant \infty$. This can be done by defining a class of operators slightly different from the p-summing operators.

DEFINITION 19. As above, X_p denotes $L^p(T)$ if $1 \leqslant p < \infty$ and $C(T)$ for $p = \infty$. Given $1 \leqslant p < \infty$ and $1/p + 1/p' = 1$, an operator S in $L(X_{p'}, B)$ will be called *positive p-summing* if there exists a constant C such that for every $n \in N$ and for every family of positive functions $\psi_1, \psi_2, \ldots, \psi_n$ in $X_{p'}$,

$$(3.14)\qquad \Big(\sum_{i=1}^{n} \|S(\psi_i)\|_B^p\Big)^{1/p} \leqslant C \sup_{\|\xi\|_{(X_{p'})^*} \leqslant 1} \Big(\sum_{i=1}^{n} |\langle \xi, \psi_i\rangle|^p\Big)^{1/p}.$$

An operator S in $L(L^1(T), B)$ is called *positive ∞-summing* if there is a constant C such that for every positive function ψ in $L^1(T)$,

$$(3.15)\qquad \|S(\psi)\|_B \leqslant C \sup\Big\{\Big|\int \psi(t)\,\varphi(t)\,dt\Big| : \|\varphi\|_{L^\infty} \leqslant 1\Big\}.$$

We shall denote by $\pi^{p,+}(X_{p'}, B)$ the space of positive p-summing operators from $X_{p'}$ to B and the norm in it will be given by the infimum of the constants satisfying (3.14) or (3.15) according as $1 \leqslant p < \infty$ or $p = \infty$.

The following result will connect these operators with Theorems 16, 17 and 18.

PROPOSITION 20. (1) $\pi^{\infty,+}(L^1(T), B) = L(L^1(T), B)$.

(2) $\pi^{p,+}(L^{p'}(T), B) = \mathscr{L}(L^{p'}(T), B)$ $(1 < p < \infty)$.

(3) $\pi^{1,+}(C(T), B) = \pi^1(C(T), B)$.

Proof. (1) Being a positive ∞-summing operator simply means that $\|S(\psi)\|_B \leqslant C\|\psi\|_{L^1}$ for every $\psi \geqslant 0$ and some constant C. But if $\psi \in L^1(T)$ we can write $\psi = \psi^+ - \psi^-$ with $\psi^+, \psi^- \geqslant 0$ and then

$$\|S(\psi)\|_B \leqslant C(\|\psi^+\|_{L^1} + \|\psi^-\|_{L^1}) = C\,\||\psi|\|_{L^1} = C\|\psi\|_{L^1}.$$

Therefore every positive ∞-summing operator from $L^1(T)$ to B belongs to $L(L^1(T), B)$.

(2) Let S be an operator in $\pi^{p,+}(L^{p'}(T), B)$ and let $\psi = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$ be a function with $\|\psi\|_{L^{p'}} \leqslant 1$. Since $\psi_i = m(E_i)^{-1/p'}\chi_{E_i} \geqslant 0$, Hölder's inequality implies that

$$\sum_{i=1}^{n} |\alpha_i|\,\|S(\chi_{E_i})\|_B = \sum_{i=1}^{n} |\alpha_i|\,m(E_i)^{1/p'}\|S(\psi_i)\|_B$$

$$\leqslant \Big(\sum_{i=1}^{n} \|S(\psi_i)\|_B^p\Big)^{1/p}$$

$$\leqslant \|S\|_{\pi^{p,+}} \sup_{\|\varphi\|_{L^p} \leqslant 1} \Big(\sum_{i=1}^{n} \Big|\int \psi_i \varphi\Big|^p\Big)^{1/p}$$

$$\leqslant \|S\|_{\pi^{p,+}} \sup_{\|\varphi\|_{L^p} \leqslant 1} \Big(\sum_{i=1}^{n} \Big|\int_{E_i} \varphi(t)\,m(E_i)^{-1/p'}\,dt\Big|^p\Big)^{1/p}$$

$$\leqslant \|S\|_{\pi^{p,+}} \sup_{\|\varphi\|_{L^p} \leqslant 1} \Big(\sum_{i=1}^{n} \int_{E_i} |\varphi(t)|^p\,dt\Big)^{1/p}$$

$$\leqslant \|S\|_{\pi^{p,+}}.$$

Therefore $|||S|||_p \leqslant \|S\|_{\pi^{p,+}}$.

Conversely, let S be an operator in $\mathscr{L}(L^{p'}(T), B)$ and $\psi_1, \psi_2, \ldots, \psi_n \geqslant 0$ in $L^{p'}(T)$. By Proposition 1 there exists a function g in $L^p(T)$, $g \geqslant 0$, such that

$$\|S(\psi_i)\|_B \leqslant \int \psi_i(t)\,g(t)\,dt \quad \text{and} \quad \|g\|_{L^p} = |||S|||_p.$$

Therefore,

$$\Big(\sum_{i=1}^{n} \|S(\psi_i)\|_B^p\Big)^{1/p} \leqslant \Big(\sum_{i=1}^{n} \Big|\int \psi_i(t)\,g(t)\,dt\Big|^p\Big)^{1/p}$$

$$= |||S|||_p \left(\sum_{i=1}^{n} \left|\int \psi_i(t)\frac{g(t)}{|||S|||_p}\,dt\right|^p\right)^{1/p}$$

$$= |||S|||_p \sup_{\|\varphi\|_{L^p} \leqslant 1} \Big(\sum_{i=1}^{n} \Big|\int \psi_i(t)\,\varphi(t)\,dt\Big|^p\Big)^{1/p}.$$

Thus $\|S\|_{\pi^{p,+}} \leqslant |||S|||_p$.

(3) In [2], p. 162, it is proved that $\pi^1(C(T), B)$ coincides with the space of B-valued regular measures with bounded variation, but if the proof is looked over, it can be observed that only positive functions are really used there, so this space of measures coincides with $\pi^{1,+}(C(T), B)$ too. ■

This last result allows us to state the following unified result:

COROLLARY 21. *If* $1 \leqslant p \leqslant \infty$ *then* $h_B^p(T) = \pi^{p,+}(X_{p'}, B)$.

The last question we are going to answer is the following:

What conditions have to be imposed on B to make $h_B^p(T)$ be the space $J_p(L_B^p(T))$?

The condition on B was found by Bukhvalov and Danilevich [1]. It is the well-known Radon–Nikodým property. Let us prove the same result by using our techniques of operators.

THEOREM 22. *Let* $1 < p \leqslant \infty$. *Then* $h_B^p(T) = J_p(L_B^p(T))$ *if and only if B has the Radon–Nikodým property.*

Proof. Theorem 5 in [2], p. 63, states that the Radon–Nikodým property of B is equivalent to Riesz's representation theorem, i.e. $L(L^1(T), B) = J_\infty(L_B^\infty(T))$ if and only if B has the Radon–Nikodým property. Hence the case $p = \infty$ follows immediately from Proposition 16. Let us assume then that $1 < p < \infty$ and $h_B^p(T) = J_p(L_B^p(T))$. To show that B has the Radon–Nikodým property let us take an operator S in $h_B^\infty(T)$ (which obviously is in $h_B^p(T)$) and show that S belongs to $J_\infty(L_B^\infty(T))$. By assumption there is a function f in $L_B^p(T)$ such that $S(\psi) = \int \psi(t) f(t)\,dt$ for all ψ in $L^{p'}(T)$. We have only to prove that f belongs to $L_B^\infty(T)$. Since

$$\|S(\chi_E)\|_B = \Big\|\int_E f(t)\,dt\Big\|_B \leqslant \|S\|\, m(E)$$

for all measurable sets E, using Lebesgue's differentiation theorem and denoting by $I_\varepsilon(t)$ the interval $(t-\varepsilon, t+\varepsilon)$ for $t \in T$, we obtain

$$\|f(t)\|_B = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \Big\| \int_{I_\varepsilon(t)} f(s)\,ds \Big\|_B \leqslant \|S\|$$

for almost all t in T.

Conversely, suppose B has the Radon–Nikodým property and S belongs to $h_B^p(T)$. Set $G(E) = S(\chi_E)$ for all measurable sets E. Then, as in Proposition 3, G is m-continuous and has bounded variation. Thus there is a function f in $L_B^1(T)$ satisfying $S(\chi_E) = \int_E f(t)\,dt$ for all measurables sets E. Finally, since $S \in \mathscr{L}(L^{p'}(T), B)$, an argument like that in Proposition 3 shows again that f belongs to $L_B^p(T)$. From this it is clear that $S = J_p(f)$ with $f \in L_B^p(T)$, and so the proof is finished. ∎

This last theorem can be used to deduce that spaces $h_B^p(D)$ and $h_{w,B}^p(D)$ are not the same in general as we shall show in the following

Corollary 23. *For each $1 \leqslant p < \infty$ there exists a Banach space B such that $h_B^p(D)$ is strictly contained in $h_{w,B}^p(D)$.*

Proof. *Case* $p = 1$. Let us take $B = C(T)$ and $I\colon C(T) \to C(T)$ given by $I(\psi) = \psi$. Then $F = \mathscr{P}(I)$ is a function in $h_{w,B}^1(D)$. On the other hand, since 1-summing operators map unconditionally convergent series into absolutely convergent ones, the Dvoretzky–Rogers theorem (see [6], p. 67) implies that I is not 1-summing and therefore F does not belong to $h_B^1(D)$.

Case $1 < p < \infty$. Let us take $B = L^{p'}(T)$ with $1/p + 1/p' = 1$ and $I\colon L^{p'}(T) \to L^{p'}(T)$ given by $I(\psi) = \psi$. The function $F = \mathscr{P}(I)$ belongs to $h_{w,B}^p(D)$ but it cannot belong to $h_B^p(D)$ since $L^{p'}(T)$ has the Radon–Nikodým property and in that case from Theorem 22 there would exist some $f \in L_B^p$ such that $I = J_p(f)$; but this is a contradiction since the operator I is not compact.

References

[1] A. V. Bukhvalov and A. A. Danilevich, *Boundary properties of analytic and harmonic functions with values in a Banach space* (in Russian), Mat. Zametki 31 (1982), 203–214; English transl. Math. Notes 31 (1982), 104–110.

[2] J. Diestel and J. J. Uhl, *Vector Measures*, Math. Surveys 15, Amer. Math. Soc., 1977.

[3] N. Dinculeanu, *Vector Measures*, Pergamon Press, New York 1967.

[4] P. L. Duren, *Theory of H^p-Spaces*, Academic Press, New York 1970

[5] P. Koosis, *Introduction to H_p-Spaces*, London Math. Soc. Lecture Note Ser. 40, Cambridge Univ. Press, 1980.

[6] A. Pietsch, *Nuclear Locally Convex Spaces*, Springer, Berlin 1972.

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