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Added in proof (May 1987). V. Müller pointed out to the author an example of a noncommutative Banach algebra for which the left and right approximate point spectra have the projection property. Hence the conjecture on p. 284 is false (see the author's forthcoming paper *On the projection property of approximate point joint spectra*, Comment. Math., vol. 28).

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Note on a theorem by Reshetnyak-Gurov

by

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Abstract. The paper gives a sharp estimate for the L^p-norm of functions whose mean oscillation in every cube is at most a fixed multiple, ε, of the mean value of the function in that cube. The estimate improves an earlier result of Reshetnyak–Gurov as $\varepsilon \to 0$.

In their paper [2] Reshetnyak and Gurov study functions with a mean oscillation which in every cube is not greater than a fixed multiple of the mean value of the function in that particular cube. Their result has been used by Bojarski [1] in a study of the stability of inverse Hölder inequalities.

A cube in $\boldsymbol{R}^n$ will always mean a cube with sides parallel to the axes. We let $|E|$ denote the Lebesgue measure of the set E and prove the following theorem:

THEOREM. *Let q be any positive number, ε a number in the range $0 < \varepsilon < (3 \cdot 2^{1/q})^{-1}$ and f a vector-valued function $f\colon \Omega \to \boldsymbol{R}^m$, $\Omega \subset \boldsymbol{R}^n$. Suppose that for every cube Q in Ω there exists a vector f_Q in $\boldsymbol{R}^m$ such that*

$$\frac{1}{|Q|} \int_Q |f(x) - f_Q|^q \, dx \leqslant \varepsilon^q |f_Q|^q. \tag{1}$$

Then f has to be a function in $L^p_{\text{loc}}(\Omega)$ for $q \leqslant p < c_1 \varepsilon^{-1}$. For these values of p we have for every cube Q in Ω

$$\int_Q |f(x) - f_Q|^p \, dx < c_2 |f_Q|^{p-q} \varepsilon^{p-q} \int_Q |f(x) - f_Q|^q \, dx. \tag{2}$$

c_1 may be taken as $(q \ln 2)(6 \cdot 2^{n+1/q})^{-1}$ and c_2 depends only on p, q and n.

Remark 1. This constitutes an improvement of the result in [2] in that it contains a factor $\varepsilon \log(a(q)/\varepsilon)$ instead of ε on the right-hand side of (2) and also requires q to be at least 1.

Remark 2. It is easy to find an example showing that (2) gives the best possible order as ε tends to zero.

Proof. Let Q be an arbitrary cube in Ω and put

$$E_1 = \{x \in Q;\ |f(x) - f_Q| > \delta |f_Q|\},$$

where $\delta > 0$ will be specified later. From (1) we conclude

$$|E_1| \leqslant (\varepsilon/\delta)^q |Q|.$$

We cover E_1' (= the set of density points of E_1) with disjoint cubes $\{Q_\nu\}_1^\infty$ dyadic with respect to Q such that

(3) $$2^{-n-1}|Q_\nu| \leqslant |E_1 \cap Q_\nu| < \tfrac{1}{2}|Q_\nu|, \quad \nu = 1, 2, \ldots$$

(see Wik [3]). δ will later be chosen so large that $|E_1| < \frac{1}{2}|Q|$. For $x \in Q_\nu \setminus E_1$ we have $|f(x) - f_Q| < \delta |f_Q|$ and since, by (3), $|Q_\nu \setminus E_1| > \frac{1}{2}|Q_\nu|$, we have *a fortiori*

$$\big|\{x \in Q_\nu;\ |f(x)| < (1+\delta)|f_Q|\}\big| > \tfrac{1}{2}|Q_\nu|$$

and either

$$|f_{Q_\nu}| < (1+\delta)|f_Q|$$

or

(4) $$\int_{Q_\nu} |f(x) - f_{Q_\nu}|^q\,dx \geqslant \int_{Q_\nu} \big||f_{Q_\nu}| - |f(x)|\big|^q\,dx \geqslant \tfrac{1}{2}|Q_\nu|\big(|f_{Q_\nu}| - (1+\delta)|f_Q|\big)^q.$$

Also, by assumption,

(5) $$\int_{Q_\nu} |f(x) - f_{Q_\nu}|^q\,dx \leqslant \varepsilon^q |f_{Q_\nu}|^q |Q_\nu|.$$

(4) and (5) combine to

$$|f_{Q_\nu}| - (1+\delta)|f_Q| < 2^{1/q}\varepsilon |f_{Q_\nu}|,$$

i.e.

(6) $$|f_{Q_\nu}| < a|f_Q|, \quad \text{where } a = \frac{1+\delta}{1-2^{1/q}\varepsilon}.$$

Thus in either case (6) is valid.

By (1) and (3) we also have for $\nu = 1, 2, \ldots,$

$$\big|\{x \in Q_\nu;\ |f(x) - f_{Q_\nu}| < 2^{1/q}\varepsilon |f_{Q_\nu}|\}\big| > \tfrac{1}{2}|Q_\nu|,$$

$$\big|\{x \in Q_\nu;\ |f(x) - f_Q| < \delta |f_Q|\}\big| > \tfrac{1}{2}|Q_\nu|.$$

Thus there are points x in Q_ν satisfying both

$$|f(x) - f_{Q_\nu}| < 2^{1/q}\varepsilon |f_{Q_\nu}| \quad \text{and} \quad |f(x) - f_Q| < \delta |f_Q|.$$

Together with (6) this gives us

(7) $$|f_{Q_\nu} - f_Q| < \delta |f_Q| + 2^{1/q}\varepsilon |f_{Q_\nu}| < (\delta + 2^{1/q}\varepsilon a)|f_Q| = b|f_Q|,$$

where $b = \delta + 2^{1/q}\varepsilon a$.

We now follow the same procedure with each of the cubes Q_ν. Put

$$E_2 = \bigcup_\nu \{x \in Q_\nu;\ |f(x) - f_{Q_\nu}| > \delta |f_{Q_\nu}|\}$$

and cover E_2' with disjoint cubes, $\bigcup_1^\infty Q_\nu^{(2)}$, where $Q_\nu^{(2)}$ is a dyadic subcube of Q_μ for some μ, with properties vis-à-vis Q_μ corresponding to (3). In the same way as above we obtain for every ν

(8) $$|f_{Q_\nu^{(2)}}| < a|f_{Q_\mu}| < a^2 |f_Q|.$$

Also, in analogy with (7) we get $|f_{Q_\nu^{(2)}} - f_{Q_\mu}| < b|f_{Q_\mu}| < ab|f_Q|$ and

(9) $$|f_{Q_\nu^{(2)}} - f_Q| \leqslant |f_{Q_\nu^{(2)}} - f_{Q_\mu}| + |f_{Q_\mu} - f_Q| < b(1+a)|f_Q|.$$

By (1) and (3)

$$|E_2| \leqslant (\varepsilon/\delta)^q \big|\bigcup Q_\mu\big| \leqslant (\varepsilon/\delta)^q \cdot 2^{n+1}|E_1|.$$

Proceeding in the same manner we obtain the estimates

(10) $$|f_{Q_\nu^{(k)}}| \leqslant a^k |f_Q|,$$

(11) $$|f_{Q_\nu^{(k)}} - f_Q| \leqslant b(1 + a + \ldots + a^{k-1})|f_Q|,$$

(12) $$|E_k| \leqslant [(\varepsilon/\delta)^q \cdot 2^{n+1}]^{k-1}|E_1| \leqslant (\varepsilon/\delta)^{qk} \cdot 2^{(n+1)(k-1)}|Q|,$$

where

(13) $$E_k = \bigcup \{x \in Q_\nu^{(k-1)};\ |f(x) - f_{Q_\nu^{(k-1)}}| > \delta |f_{Q_\nu^{(k-1)}}|\}$$

and $\{Q_\nu^{(k)}\}$ is a set of disjoint dyadic cubes covering E_k'. Obviously

(14) $$\int_Q |f(x) - f_Q|^p\,dx \leqslant \int_{Q\setminus E_1} |f(x) - f_Q|^p\,dx + \sum_{k=1}^\infty \int_{\bigcup_\nu Q_\nu^{(k)} \setminus E_{k+1}} |f(x) - f_Q|^p\,dx,$$

where $Q_\nu^{(1)}$ is to be interpreted as Q_ν. On the set $\bigcup Q_\nu^{(k)} \setminus E_{k+1}$ we have, by the triangle inequality, (13) and (11)

$$|f(x) - f_Q| \leqslant |f(x) - f_{Q_\nu^{(k)}}| + |f_{Q_\nu^{(k)}} - f_Q| \leqslant \delta |f_{Q_\nu^{(k)}}| + b\frac{a^k - 1}{a-1}|f_Q|.$$

Since δ is less than b, it follows from (10) that

$$|f(x) - f_Q| < b\frac{a^{k+1}-1}{a-1}|f_Q| \quad \text{on} \quad \bigcup Q_\nu^{(k)} \setminus E_{k+1}.$$

Using the fact that $\big|\bigcup Q_\nu^{(k)}\big| \leqslant 2^{n+1}|E_k|$ and (12) we obtain

$$\int_{Q_\nu^{(k)} \setminus E_{k+1}} |f(x) - f_Q|^p\,dx < \left(b\frac{a^{k+1}-1}{a-1}\right)^p \cdot 2^{n+1}\left[\left(\frac{\varepsilon}{\delta}\right)^q \cdot 2^{n+1}\right]^{k-1} |f_Q|^p |E_1|.$$

It follows that the series in (14) converges if $(\varepsilon/\delta)^q \cdot 2^{n+1} \cdot a^p < 1$. We are still free to make our choice of δ. We put

(15) $$\delta = 2 \cdot 2^{(n+1)/q} \cdot \varepsilon.$$

Then we have convergence if $a^p < 2^q$. This is true if, for example,

$$\varepsilon < (3\cdot 2^{1/q})^{-1} \quad \text{and} \quad p < \frac{q\ln 2}{4\cdot 2^{(n+1)/q}}\cdot\frac{1}{\varepsilon}.$$

The sum on the right-hand side of (14) is then less than

$$c(n, p, q)\cdot\varepsilon^p\cdot|f_Q|^p|E_1|.$$

Since

$$\int_{E_1} |f-f_Q|^q\,dx \geqslant \delta^q|f_Q|^q|E_1|$$

we have

$$\int_{E_1} |f-f_Q|^p\,dx \leqslant c(n, p, q)\,\varepsilon^{p-q}|f_Q|^{p-q}\int_{E_1} |f-f_Q|^q\,dx. \tag{16}$$

Furthermore, for $p > q$ we have the trivial estimate

$$\int_{Q\setminus E_1} |f-f_Q|^p\,dx \leqslant \delta^{p-q}|f_Q|^{p-q}\int_{Q\setminus E_1} |f-f_Q|^q\,dx. \tag{17}$$

We add the results of the inequalities (16) and (17) to obtain the inequality (2), which is thus proved.

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Correction to "Walsh equiconvergence for best l_2-approximates"

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by

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On p. 524, line 7 from below, it is written that (2.4) is true for $|z| < \varrho^{ls/(s-1)}$. This is not correct and should be replaced by

$$\text{for} \quad |z| < \min\{\varrho^{1+lrs/(rs-r+1)}, \varrho^{ls/(s-1)}\}.$$

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