

Approximate point joint spectra and multiplicative functionals

by

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Abstract. We show that a unital Banach algebra has a nonzero multiplicative linear functional whose kernel consists of joint left [respectively right] topological divisors of zero if and only if the left [right] approximate point spectrum $\tau_l(a_1, \dots, a_n)$ [$\tau_r(a_1, \dots, a_n)$] is nonempty for every finite set of elements a_1, \dots, a_n in the algebra.

§ 1. Introduction. Let A be a unital complex Banach algebra. The unit of A will be denoted by 1. The *left approximate point spectrum* of an n -tuple (a_1, \dots, a_n) of elements in A , denoted by $\tau_l^A(a_1, \dots, a_n)$ or simply by $\tau_l(a_1, \dots, a_n)$ if there is no confusion, is defined to be the subset of C^n consisting of those $(\lambda_1, \dots, \lambda_n)$ for which there exists a sequence (u_k) of elements in A such that $\|u_k\| = 1$ for all k and $\lim_k \|(a_j - \lambda_j)u_k\| = 0$ for $j = 1, \dots, n$. (Here, $a_j - \lambda_j$ stands for $a_j - \lambda_j 1$.) The *right approximate point spectrum* $\tau_r(a_1, \dots, a_n)$ is defined in a similar manner. The *joint approximate point spectrum* $\tau^A(a_1, \dots, a_n)$, or simply written as $\tau(a_1, \dots, a_n)$, is defined to be their union:

$$\tau(a_1, \dots, a_n) = \tau_l(a_1, \dots, a_n) \cup \tau_r(a_1, \dots, a_n).$$

If $A = B(X)$ where X is a complex Banach space, then

$$\tau_l^A(a_1, \dots, a_n) = \{(\lambda_1, \dots, \lambda_n) \in C^n: \inf_{\|x\|=1} \sum_{j=1}^n \|(a_j - \lambda_j)x\| = 0\}$$

(and is usually denoted by $\sigma_\pi(a_1, \dots, a_n)$, and called the “approximate point spectrum”), while

$$\tau_r^A(a_1, \dots, a_n) = \{(\lambda_1, \dots, \lambda_n) \in C^n: \sum_{j=1}^n (a_j - \lambda_j)X \neq X\}$$

(and is usually denoted by $\sigma_d(a_1, \dots, a_n)$, and called the “defect spectrum”). (See [4], p. 95.)

Let us note that for an arbitrary n -tuple (a_1, \dots, a_n) of elements in A the following inclusions hold true:

$$\tau_l^A(a_1, \dots, a_n) \subset \sigma_l^A(a_1, \dots, a_n), \quad \tau_r^A(a_1, \dots, a_n) \subset \sigma_r^A(a_1, \dots, a_n),$$

where $\sigma_l^A(a_1, \dots, a_n)$ and $\sigma_r^A(a_1, \dots, a_n)$ denote the left joint spectrum and the right joint spectrum of the n -tuple (a_1, \dots, a_n) respectively. (See [3] or [4] for their definitions.)

The approximate point spectra are always compact subsets of \mathbb{C}^n . However, they can be void (see [4]). The main result of [3] says that if the joint spectrum $\sigma(a_1, \dots, a_n) = \sigma_l(a_1, \dots, a_n) \cup \sigma_r(a_1, \dots, a_n)$ is nonempty for an arbitrary n -tuple (a_1, \dots, a_n) of elements in A with $n = 1, 2, \dots$, then A has a (nonzero) multiplicative (linear) functional. In view of the above-mentioned inclusions the same result is true for the joint approximate point spectrum. Conversely, if A has a multiplicative functional φ , then $(\varphi(a_1), \dots, \varphi(a_n)) \in \sigma(a_1, \dots, a_n)$ for an arbitrary n -tuple (a_1, \dots, a_n) . However, this is not true for the joint approximate point spectrum. A suitable example will be given in the last section. It is clear that we have to assume something about this multiplicative functional. To explain this assumption we need the following concept.

DEFINITION. A subset S of a Banach algebra A consists of *joint left topological divisors of zero* if for every finite subset $\{x_1, \dots, x_n\}$ of S there exists a sequence (z_k) of elements in A such that $\|z_k\| = 1$ for every k and $\lim_k \|x_j z_k\| = 0$ for $j = 1, \dots, n$. The definition of a subset consisting of joint right topological divisors of zero is similar (cf. [8], p. 65). The fact that a subset S of A consists of joint left [respectively right] topological divisors of zero will be shortly denoted by $S \in l_1(A)$ [$r_1(A)$].

Now, let us observe that, if A has a multiplicative functional φ with kernel consisting of joint left topological divisors of zero, then $\tau_1(a_1, \dots, a_n)$ is always nonempty; in fact, in that case we have

$$(\varphi(a_1), \dots, \varphi(a_n)) \in \tau_1(a_1, \dots, a_n).$$

Of course the same observation can be made for the right approximate point spectrum.

The main purpose of the present paper is to show the converse of this fact:

THEOREM. If $\tau_l(a_1, \dots, a_n)$ [respectively $\tau_r(a_1, \dots, a_n)$] is nonempty for an arbitrary n -tuple (a_1, \dots, a_n) of elements in a Banach algebra A , with $n = 1, 2, \dots$, then A has a multiplicative functional φ with $\ker \varphi$ consisting of joint left [right] topological divisors of zero.

§ 2. Main results. We start with the following:

LEMMA. If a function $\varphi: A \rightarrow \mathbb{C}$ is such that

$$(\varphi(x), \varphi(y), \varphi(z)) \in \tau_1(x, y, z) \quad [\tau_r(x, y, z)]$$

for arbitrary elements x, y, z in A , then it is linear and multiplicative.

Proof. We shall give the proof for the left approximate point spectrum. The argument for the right spectrum is analogous. To avoid brackets let us denote $\varphi(x) = \lambda_x$ for each x in A . Now taking arbitrary x, y in A and α, β in \mathbb{C} we have $(\lambda_{\alpha x + \beta y}, \lambda_x, \lambda_y) \in \tau_1(\alpha x + \beta y, x, y)$. This implies that there exists a sequence (z_k) such that $\|z_k\| = 1$ for each k and $(\alpha x + \beta y - \lambda_{\alpha x + \beta y}) z_k \rightarrow 0$, $(x - \lambda_x) z_k \rightarrow 0$, $(y - \lambda_y) z_k \rightarrow 0$ as $k \rightarrow \infty$. We have

$$\begin{aligned} |\lambda_{\alpha x + \beta y} - \alpha \lambda_x - \beta \lambda_y| &= |\lambda_{\alpha x + \beta y} - \alpha \lambda_x - \beta \lambda_y| \|z_k\| \\ &= \|(\lambda_{\alpha x + \beta y} - \alpha \lambda_x - \beta \lambda_y) z_k\| \\ &= \|(\lambda_{\alpha x + \beta y} - (\alpha x + \beta y)) z_k + \alpha (x - \lambda_x) z_k + \beta (y - \lambda_y) z_k\| \\ &\leq \|(\lambda_{\alpha x + \beta y} - (\alpha x + \beta y)) z_k\| + |\alpha| \|(x - \lambda_x) z_k\| + |\beta| \|(y - \lambda_y) z_k\|. \end{aligned}$$

The right-hand side of this inequality tends to zero as $k \rightarrow \infty$, therefore $\lambda_{\alpha x + \beta y} = \alpha \lambda_x + \beta \lambda_y$ which means that the functional φ is linear.

To see that it is multiplicative let us take arbitrary x and y in A . Since $(\lambda_{xy}, \lambda_x, \lambda_y) \in \tau_1(xy, x, y)$ there exists a sequence (u_k) with $\|u_k\| = 1$ for all k and such that $(xy - \lambda_{xy}) u_k \rightarrow 0$, $(x - \lambda_x) u_k \rightarrow 0$, $(y - \lambda_y) u_k \rightarrow 0$ as $k \rightarrow \infty$. As before we have

$$\begin{aligned} |\lambda_{xy} - \lambda_x \lambda_y| &= |\lambda_{xy} - \lambda_x \lambda_y| \|u_k\| = \|(\lambda_{xy} - \lambda_x \lambda_y) u_k\| \\ &= \|(\lambda_{xy} - xy) u_k + x (y - \lambda_y) u_k + \lambda_y (x - \lambda_x) u_k\| \\ &\leq \|(\lambda_{xy} - xy) u_k\| + \|x\| \|(y - \lambda_y) u_k\| + |\lambda_y| \|(x - \lambda_x) u_k\|. \end{aligned}$$

Since the right-hand side of the above inequality tends to zero as $k \rightarrow \infty$ we have $\lambda_{xy} = \lambda_x \lambda_y$. ■

Remark. Using the Kowalski-Słodkowski generalization of the Gleason-Kahane-Żelazko theorem (see [5]) we can show that it is enough to take arbitrary two elements instead of three in the assumption of the lemma. Indeed, if we assume $(\varphi(x), \varphi(y)) \in \tau_1(x, y)$ for all x and y in A , then $(\varphi(0), \varphi(0)) \in \tau_1(0, 0) = \{(0, 0)\}$ which implies that $\varphi(0) = 0$ and by the "one-way spectral mapping theorem" for τ_1 (see [4], p. 100) we get $\varphi(x) - \varphi(y) \in \tau_1(x - y) \subset \sigma(x - y)$. Hence by the Kowalski-Słodkowski theorem the functional φ is linear and multiplicative. However, we gave the proof of the lemma with the assumption for three elements because it is very elementary while the proof of the Kowalski-Słodkowski theorem requires deep results on Lipschitz mappings in Fréchet spaces.

It should also be noted that the lemma is no longer true if we replace two by one element in the assumption (to see this it is enough to associate to each 2×2 complex matrix one of its eigenvalues).

Proof of the theorem. We assume that $\tau_l(a_1, \dots, a_n)$ is always nonempty for finitely many elements a_1, \dots, a_n in A . Let an arbitrary n -tuple (a_1, \dots, a_n) be fixed from now on. For each m -tuple (b_1, \dots, b_m) of elements

in A , we write $\varrho(b_1, \dots, b_m)$ for the set of all n -tuples $(\lambda_1, \dots, \lambda_n)$ in C^n such that

$$(\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_m) \in \tau_1(a_1, \dots, a_n, b_1, \dots, b_m)$$

for some $(\mu_1, \dots, \mu_m) \in C^m$. By our assumption $\varrho(b_1, \dots, b_m)$ is a nonempty compact subset of C^n . It is obvious that

$$\varrho(b_1, \dots, b_m, c_1, \dots, c_p) \subset \varrho(b_1, \dots, b_m) \cap \varrho(c_1, \dots, c_p).$$

Therefore $\{\varrho(b_1, \dots, b_m)\}$, where (b_1, \dots, b_m) runs through all m -tuples in A^m and m runs through all positive integers, is a family of compact sets with the finite intersection property, and hence its intersection is nonempty. Let $\tau_{1,\mu}(a_1, \dots, a_n)$ denote this intersection, i.e. the set of all $(\lambda_1, \dots, \lambda_n)$ in C^n such that whenever $\{b_1, \dots, b_m\}$ is a finite subset of A , $(\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_m) \in \tau_1(a_1, \dots, a_n, b_1, \dots, b_m)$ for some μ_1, \dots, μ_m in C . It is a nonempty compact subset of C^n (for an arbitrary n -tuple (a_1, \dots, a_n) in A^n).

Now, we shall show that $\tau_{1,\mu}$ has the following "projection property":

$$P_n^{n+m} \tau_{1,\mu}(a_1, \dots, a_n, a_{n+1}, \dots, a_{n+m}) = \tau_{1,\mu}(a_1, \dots, a_n)$$

where P_n^{n+m} is the canonical projection from C^{n+m} onto C^n which sends $(\lambda_1, \dots, \lambda_n, \lambda_{n+1}, \dots, \lambda_{n+m})$ to $(\lambda_1, \dots, \lambda_n)$. It is enough to prove this for $m = 1$. The rest will follow by an easy induction argument. Let an arbitrary $(n+1)$ -tuple (a_1, \dots, a_{n+1}) be fixed from now on. It is obvious that

$$P_n^{n+1} \tau_{1,\mu}(a_1, \dots, a_{n+1}) \subset \tau_{1,\mu}(a_1, \dots, a_n).$$

To establish the converse inclusion let us take an arbitrary $(\lambda_1, \dots, \lambda_n)$ from $\tau_{1,\mu}(a_1, \dots, a_n)$. For each m -tuple (b_1, \dots, b_m) of elements in A we write $\omega(b_1, \dots, b_m)$ for the set of all complex numbers λ such that

$$(\lambda_1, \dots, \lambda_n, \lambda, \mu_1, \dots, \mu_m) \in \tau_1(a_1, \dots, a_n, a_{n+1}, b_1, \dots, b_m)$$

for some $(\mu_1, \dots, \mu_m) \in C^m$. The set $\omega(b_1, \dots, b_m)$ is always nonempty since $(\lambda_1, \dots, \lambda_n) \in \tau_{1,\mu}(a_1, \dots, a_n)$. Moreover, it is easy to see that $\omega(b_1, \dots, b_m)$ is compact and

$$\omega(b_1, \dots, b_m, c_1, \dots, c_p) \subset \omega(b_1, \dots, b_m) \cap \omega(c_1, \dots, c_p)$$

for all $b_1, \dots, b_m, c_1, \dots, c_p$ in A . So the family $\{\omega(b_1, \dots, b_m)\}$, where (b_1, \dots, b_m) runs through all m -tuples in A^m and m runs through all positive integers, has the finite intersection property. Therefore its intersection is nonempty. Taking λ_{n+1} from this intersection we get $(\lambda_1, \dots, \lambda_n, \lambda_{n+1}) \in \tau_{1,\mu}(a_1, \dots, a_n, a_{n+1})$.

To complete the proof of the theorem we shall use Kuratowski-Zorn's lemma. For this purpose let us denote by \mathcal{P} the family of all ordered pairs.

(X, φ) , where X is a (nonempty) subset of A and $\varphi: X \rightarrow C$ is a function such that

$$(\varphi(x_1), \dots, \varphi(x_n)) \in \tau_{1,\mu}(x_1, \dots, x_n)$$

for every finite subset $\{x_1, \dots, x_n\}$ of X . The family \mathcal{P} is nonvoid (on each finite subset of A one can construct a function with the above-mentioned property) and partially ordered by the following relation:

$$(X_1, \varphi_1) \leq (X_2, \varphi_2) \quad \text{if and only if} \quad X_1 \subset X_2$$

and φ_2 restricted to X_1 coincides with φ_1 .

It is standard that every totally ordered subfamily of \mathcal{P} has an upper bound. Hence, by Kuratowski-Zorn's lemma \mathcal{P} has a maximal element (X_0, φ_0) . We claim that $X_0 = A$.

Suppose to the contrary that there exists an element a in A which does not belong to X_0 . For an arbitrary n -tuple (x_1, \dots, x_n) of elements in X_0 we write $\delta(a, x_1, \dots, x_n)$ for the set of all complex numbers λ such that

$$(\lambda, \varphi_0(x_1), \dots, \varphi_0(x_n)) \in \tau_{1,\mu}(a, x_1, \dots, x_n).$$

The projection property of $\tau_{1,\mu}$ implies that $\delta(a, x_1, \dots, x_n)$ is always nonempty. It is obvious that it is compact and moreover

$$\delta(a, x_1, \dots, x_n, y_1, \dots, y_m) \subset \delta(a, x_1, \dots, x_n) \cap \delta(a, y_1, \dots, y_m)$$

for arbitrary elements $x_1, \dots, x_n, y_1, \dots, y_m$ in X_0 and all positive integers n, m . Thus the family $\{\delta(a, x_1, \dots, x_n)\}$, where (x_1, \dots, x_n) runs through all n -tuples in X_0^n and n runs through all positive integers, has the finite intersection property. Let a number λ be in the intersection of this family. Now, if we define $\varphi(x) = \varphi_0(x)$ for x in X_0 and $\varphi(a) = \lambda$, then the pair $(X_0 \cup \{a\}, \varphi)$ belongs to \mathcal{P} . This contradicts the maximality of (X_0, φ_0) . Therefore $X_0 = A$. Since $\tau_{1,\mu}(a_1, \dots, a_n)$ is always contained in $\tau_1(a_1, \dots, a_n)$ the lemma concludes the proof for the left approximate point spectrum. The proof for the right spectrum is similar. ■

COROLLARY 1. *The joint approximate point spectrum $\tau(a_1, \dots, a_n)$ is nonempty for every finite set a_1, \dots, a_n of elements in A if and only if there exists on A a multiplicative functional φ such that $\ker \varphi$ consists of either joint left or joint right topological divisors of zero.*

Proof. The "if" part is obvious. To prove the "only if" part let us assume that $\tau(a_1, \dots, a_n) \neq \emptyset$ for an arbitrary n -tuple $(a_1, \dots, a_n) \in A^n$ and each n . This means that always either $\tau_1(a_1, \dots, a_n) \neq \emptyset$ or $\tau_r(a_1, \dots, a_n) \neq \emptyset$. If $\tau_1(a_1, \dots, a_n)$ is always nonvoid we are done. If, on the other hand, $\tau_1(a_1, \dots, a_n) = \emptyset$ for some elements a_1, \dots, a_n in A , then we have $\tau_r(a_1, \dots, a_n) \neq \emptyset$. Actually, we then obtain $\tau_r(a_1, \dots, a_n, b_1, \dots, b_m)$

$\neq \emptyset$ for every m -tuple $(b_1, \dots, b_m) \in A^m$ and each m , since if $\tau_r(a_1, \dots, a_n, b_1, \dots, b_m) = \emptyset$ then $\tau_l(a_1, \dots, a_n, b_1, \dots, b_m) \neq \emptyset$, and the inclusion

$$P_n^{n+m} \tau_l(a_1, \dots, a_n, b_1, \dots, b_m) \subset \tau_l(a_1, \dots, a_n)$$

implies that $\tau_l(a_1, \dots, a_n) \neq \emptyset$, which is impossible. From this we can conclude that $\tau_r(c_1, \dots, c_p) \neq \emptyset$ for arbitrary elements c_1, \dots, c_p in A and every p . ■

From the proof of the theorem we have the following:

COROLLARY 2. For each n -tuple (a_1, \dots, a_n) of elements in A and every positive integer n

$$\tau_{l,\mu}(a_1, \dots, a_n) = \{(\varphi(a_1), \dots, \varphi(a_n)): \varphi \text{ is a multiplicative functional on } A \text{ with } \ker \varphi \in l_1(A)\}.$$

COROLLARY 3. The left approximate point spectrum τ_l has the projection property if and only if

$$\tau_l(a_1, \dots, a_n) = \{(\varphi(a_1), \dots, \varphi(a_n)): \varphi \text{ is a multiplicative functional on } A \text{ with } \ker \varphi \in l_1(A)\}$$

for an arbitrary n -tuple $(a_1, \dots, a_n) \in A^n$ and each positive integer n .

Proof. The projection property of τ_l implies that τ_l and $\tau_{l,\mu}$ coincide. So Corollary 2 gives the desired formula for τ_l . The converse implication is obvious. ■

It is evident that analogous results (to those of Corollaries 2 and 3) are true for the right approximate point spectrum.

COROLLARY 4. The joint approximate point spectrum τ has the projection property if and only if

$$\tau(a_1, \dots, a_n) = \{(\varphi(a_1), \dots, \varphi(a_n)): \varphi \text{ is a multiplicative functional on } A \text{ with } \ker \varphi \in l_1(A) \cup l_r(A)\}$$

for an arbitrary n -tuple $(a_1, \dots, a_n) \in A^n$ and each positive integer n .

Proof. Suppose that τ has the projection property and take arbitrary n -tuples $(a_1, \dots, a_n) \in A^n$ and $(\lambda_1, \dots, \lambda_n) \in \tau(a_1, \dots, a_n)$. Without loss of generality we can assume that $(\lambda_1, \dots, \lambda_n) \in \tau_l(a_1, \dots, a_n)$. Then either $(\lambda_1, \dots, \lambda_n) \in \tau_{l,\mu}(a_1, \dots, a_n)$ or $(\lambda_1, \dots, \lambda_n) \notin \tau_{l,\mu}(a_1, \dots, a_n)$. If the first possibility occurs, then, by Corollary 2, there exists a multiplicative functional φ on A such that its kernel consists of joint left topological zero divisors and $\varphi(a_1) = \lambda_1, \dots, \varphi(a_n) = \lambda_n$. So assume that $(\lambda_1, \dots, \lambda_n) \notin \tau_{l,\mu}(a_1, \dots, a_n)$. Then there exists an m -tuple (b_1, \dots, b_m) of elements in A such that

$$(\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_m) \notin \tau_l(a_1, \dots, a_n, b_1, \dots, b_m)$$

for arbitrary complex numbers μ_1, \dots, μ_m . The projection property of τ implies that there exist scalars $\lambda_{n+1}, \dots, \lambda_{n+m}$ such that

$$(\lambda_1, \dots, \lambda_n, \lambda_{n+1}, \dots, \lambda_{n+m}) \in \tau_r(a_1, \dots, a_n, b_1, \dots, b_m).$$

Moreover, for an arbitrary p -tuple (c_1, \dots, c_p) of elements in A we have

$$(\lambda_1, \dots, \lambda_n, \lambda_{n+1}, \dots, \lambda_{n+m}, \mu_1, \dots, \mu_p) \in \tau_r(a_1, \dots, a_n, b_1, \dots, b_m, c_1, \dots, c_p)$$

for some $(\mu_1, \dots, \mu_p) \in C^p$ (since otherwise we would get

$$(\lambda_1, \dots, \lambda_n, \lambda_{n+1}, \dots, \lambda_{n+m}) \in \tau_l(a_1, \dots, a_n, b_1, \dots, b_m)$$

which is impossible). This means that

$$(\lambda_1, \dots, \lambda_n, \lambda_{n+1}, \dots, \lambda_{n+m}) \in \tau_{r,\mu}(a_1, \dots, a_n, b_1, \dots, b_m),$$

and as before there exists a multiplicative functional φ on A with kernel consisting of joint right topological divisors of zero and such that $\varphi(a_1) = \lambda_1, \dots, \varphi(a_n) = \lambda_n$.

It is evident that if for every finite set $\{a_1, \dots, a_n\}$ we have

$$\tau(a_1, \dots, a_n) = \{(\varphi(a_1), \dots, \varphi(a_n)): \varphi \text{ is a multiplicative functional on } A \text{ with } \ker \varphi \in l_1(A) \cup l_r(A)\},$$

then τ has the projection property. ■

Remark. If the algebra A is commutative, then always $\tau_l(a_1, \dots, a_n) = \tau_r(a_1, \dots, a_n) = \tau(a_1, \dots, a_n)$, and by a result of M.-D. Choi, Ch. Davis, Z. Słodkowski and W. Żelazko (see [1] and [7]) the approximate point spectrum has the projection property. In this case we obviously have $\tau(a_1, \dots, a_n) = \{(\varphi(a_1), \dots, \varphi(a_n)): \varphi \text{ is a multiplicative functional on } A \text{ with } \ker \varphi \text{ consisting of joint topological zero divisors}\}.$

By a result of V. Müller (see [6]) a multiplicative functional has kernel consisting of joint topological divisors of zero if and only if it can be extended to a multiplicative functional on every superalgebra of A . The set of such functionals is called the cortex of the algebra A , in symbols $\text{cor } A$.

Thus we see that

$$\tau(a_1, \dots, a_n) = \{(\varphi(a_1), \dots, \varphi(a_n)): \varphi \in \text{cor } A\}$$

for each n -tuple (a_1, \dots, a_n) of elements in a commutative Banach algebra A and every positive integer n .

On the other hand, if the left approximate point spectrum τ_l has the projection property on A then the algebra $A/\text{rad } A$ must be commutative (where $\text{rad } A$ stands for the radical of A). Indeed, by Corollary 3 we have $\tau_l((ab-ba)c) = \{0\}$ for arbitrary elements a, b, c in A . Since $\partial\sigma((ab-ba)c) \subset \tau_l((ab-ba)c)$ (see [4], p. 103 or [8], p. 57) we have $\sigma((ab-ba)c)$

$= \{0\}$, and so $ab - ba \in \text{rad } A$. The same observation can be made for the spectra τ_r and τ .

We do not know any example of a noncommutative Banach algebra on which the left [right] approximate point spectrum has the projection property. Hence we conjecture that if it is so, then the algebra must be commutative. An example given at the end of the paper will show that the commutativity of the algebra modulo its radical does not suffice for the approximate point spectra τ_l , τ_r , and τ to have the projection property (while it is a necessary and sufficient condition for the joint spectrum σ , and also for σ_l and σ_r , to have such property; see [3]).

§ 3. **Examples.** In this final section we give the previously announced examples.

EXAMPLE 1. It will be shown that an algebra can have multiplicative functionals but the joint approximate point spectrum of some n -tuples can be empty. The idea of this example is due to C.-K. Fong (see [2]).

Let A be the algebra $M_5(\mathbb{C})$ of all 5×5 matrices with complex entries. Take the following two elements of A :

$$a_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad a_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then we have $a_1^3 = a_2^3 = 0$. Hence $\sigma^A(a_1) = \sigma^A(a_2) = \{0\}$. This implies that $\sigma^A(a_1, a_2) \subset \{(0, 0)\}$ (cf. [4], p. 92). But if we take

$$a_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \text{and} \quad a_4 = \begin{bmatrix} 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

then we shall get $a_1 a_3 + a_2 a_1 = 1$ (identity matrix) and $a_2 a_1 + a_4 a_2 = 1$. Therefore $\sigma^A(a_1, a_2) = \emptyset$. Now, let us take B to be the subalgebra of A generated by a_1 , a_2 , and 1. If we assign to each element b of B the entry of b in the third row and the third column we shall get a multiplicative functional φ on B . Since $\varphi(a_1) = \varphi(a_2) = 0$, we get $(0, 0) \in \sigma^B(a_1, a_2)$. However, we have $\tau^B(a_1, a_2) \subset \tau^A(a_1, a_2) \subset \sigma^A(a_1, a_2)$ (see [4], p. 92). Therefore $\tau^B(a_1, a_2) = \emptyset$.

EXAMPLE 2. Now we describe an algebra in which the joint approximate point spectrum is always nonempty but it does not have the projection property.

Let A be the algebra of all 3×3 upper triangular matrices with complex entries

$$a = \begin{bmatrix} a(11) & a(12) & a(13) \\ 0 & a(22) & a(23) \\ 0 & 0 & a(33) \end{bmatrix}.$$

It is known (see [4], pp. 94–95) that for all a_1, \dots, a_n in A we have the equality $\sigma(a_1, \dots, a_n) = \{(a_1(jj), \dots, a_n(jj)) : j = 1, 2, 3\}$, while $(a_1(11), \dots, a_n(11)) \in \tau_l(a_1, \dots, a_n)$ and $(a_1(33), \dots, a_n(33)) \in \tau_r(a_1, \dots, a_n)$. Hence $\tau_l(a_1, \dots, a_n)$, $\tau_r(a_1, \dots, a_n)$, and $\tau(a_1, \dots, a_n)$ are always nonvoid. However, if we take for example

$$a = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{bmatrix},$$

then it is a matter of easy computations to show that $\tau_l(a, b) = \{(2, 2)\}$, $\tau_r(a, b) = \{(3, 3)\}$ and $\tau_l(a) = \tau_r(a) = \{2, 1, 3\}$. Thus $(1, 1) \notin \tau(a, b)$ which means that the spectra τ_l , τ_r , and τ do not have the projection property on this algebra.

Let us also note that the algebra A is commutative modulo the radical.

It may be interesting to observe that on the algebra of all 2×2 upper triangular matrices the joint approximate point spectrum coincides with the joint spectrum, and therefore it has the projection property, while the left and right approximate point spectra do not have this property.

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Added in proof (May 1987). V. Müller pointed out to the author an example of a noncommutative Banach algebra for which the left and right approximate point spectra have the projection property. Hence the conjecture on p. 284 is false (see the author's forthcoming paper *On the projection property of approximate point joint spectra*, *Comment. Math.*, vol. 28).

Note on a theorem by Reshetnyak–Gurov

by

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Abstract. The paper gives a sharp estimate for the L^p -norm of functions whose mean oscillation in every cube is at most a fixed multiple, ε , of the mean value of the function in that cube. The estimate improves an earlier result of Reshetnyak–Gurov as $\varepsilon \rightarrow 0$.

In their paper [2] Reshetnyak and Gurov study functions with a mean oscillation which in every cube is not greater than a fixed multiple of the mean value of the function in that particular cube. Their result has been used by Bojarski [1] in a study of the stability of inverse Hölder inequalities.

A cube in \mathbb{R}^n will always mean a cube with sides parallel to the axes. We let $|E|$ denote the Lebesgue measure of the set E and prove the following theorem:

THEOREM. Let q be any positive number, ε a number in the range $0 < \varepsilon < (3 \cdot 2^{1/q})^{-1}$ and f a vector-valued function $f: \Omega \rightarrow \mathbb{R}^m$, $\Omega \subset \mathbb{R}^n$. Suppose that for every cube Q in Ω there exists a vector f_Q in \mathbb{R}^m such that

$$(1) \quad \frac{1}{|Q|} \int_Q |f(x) - f_Q|^q dx \leq \varepsilon^q |f_Q|^q.$$

Then f has to be a function in $L^p_{\text{loc}}(\Omega)$ for $q \leq p < c_1 \varepsilon^{-1}$. For these values of p we have for every cube Q in Ω

$$(2) \quad \int_Q |f(x) - f_Q|^p dx < c_2 |f_Q|^{p-q} \varepsilon^{p-q} \int_Q |f(x) - f_Q|^q dx.$$

c_1 may be taken as $(q \ln 2)(6 \cdot 2^{n+1/q})^{-1}$ and c_2 depends only on p, q and n .

Remark 1. This constitutes an improvement of the result in [2] in that it contains a factor $\varepsilon \log(a(q)/\varepsilon)$ instead of ε on the right-hand side of (2) and also requires q to be at least 1.

Remark 2. It is easy to find an example showing that (2) gives the best possible order as ε tends to zero.

Proof. Let Q be an arbitrary cube in Ω and put

$$E_1 = \{x \in Q; |f(x) - f_Q| > \delta |f_Q|\},$$