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Initial value Abelian theorems for the distributional Stieltjes transform

by

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Abstract. In the first part we investigate the L-quasiasymptotic at zero introduced by B. I. Zav'yalov [15]. In the second part we use the L-quasiasymptotic to prove some Abelian type theorems for distributions.

Introduction. The notion of quasiasymptotic behaviour of tempered distributions was introduced by B. I. Zav'yalov in [15] and investigated in several papers of Yu. N. Drozhzhinov and B. I. Zav'yalov [4], [5], ... In those papers the authors deeply investigated the quasiasymptotic behaviour at infinity of distributions from the space (S'_Γ) of tempered distributions with support in a cone $\Gamma \subset \boldsymbol{R}^n$. In this paper we give, first, several assertions concerning the L-quasiasymptotic behaviour at 0^+ for elements from (S'_+) $((S'_+) = (S'_\Gamma),\ \Gamma = \bar{\boldsymbol{R}}^n_+)$. The motivation for our investigations made in the first part are theorems of Abelian type for the distributional Stieltjes transform, which we shall give in the second part. In the one-dimensional case the theorems presented include the known results of this type and the new ones, because the quasiasymptotic behaviour at 0^+ is more convenient than the ordinary asymptotic behaviour of a distribution, especially to prove Abelian type theorems for the Stieltjes transform.

1. Notation. For $a = (a_1, \ldots, a_n) \in \boldsymbol{R}^n$ and $x = (x_1, \ldots, x_n) \in \boldsymbol{C}^n$ ($\boldsymbol{C}$ is the set of complex numbers), we write

$$ax = (a_1 x_1, \ldots, a_n x_n); \quad x/a = (x_1/a_1, \ldots, x_n/a_n), \quad a_i \neq 0;$$

$$x^a = \prod_{i=1}^{n} x_i^{a_i}, \quad x_i \neq 0 \text{ if } a_i < 0;$$

$$a \geqslant 0 \ (a > 0) \Leftrightarrow a_i \geqslant 0 \ (a_i > 0),\ i = 1, \ldots, n;$$

$$a \to 0^+ \ (\infty) \Leftrightarrow a_i \to 0^+ (\infty), \quad i = 1, \ldots, n;$$

$$|a| = \sum_{i=1}^{n} |a_i|; \quad \|a\|^2 = \sum_{i=1}^{n} a_i^2.$$

N denotes the set of natural numbers and $N_0 = N \cup \{0\}$. For $p \in N_0^n$, D^p is the operator of partial derivative of a distribution: $D_t^p = D_{t_1}^{p_1} \ldots D_{t_n}^{p_n}$, $t = (t_1, \ldots, t_n)$, $p = (p_1, \ldots, p_n)$.

If $\tau \in \boldsymbol{R}$, $f_\alpha(\tau) = \theta(\tau)\tau^{\alpha-1}/\Gamma(\alpha)$, $\alpha > 0$, and $f_\alpha(\tau) = D^k f_{\alpha+k}(\tau)$, $\alpha \leqslant 0$, $\alpha + k > 0$, $k \in N$, where θ is the Heaviside function. Now, if $a, t \in \boldsymbol{R}^n$ we put $f_a(t) = \prod_{i=1}^n f_{a_i}(t_i)$. We know that

$$(D_t^p f)(t) = \Big(\prod_{i=1}^n f_{-p_i}(t_i)\Big) * f, \qquad p = (p_1, \ldots, p_n) \in N_0^n.$$

A numerical function $L(\tau)$ continuous on $(0, \infty)$ is *slowly varying at zero* if $\tau^{-\gamma} L(\tau)$ is bounded on $[b, \infty)$, $b > 0$, for every $\gamma > 0$, positive in $(0, a)$, $a > 0$, and satisfies the following condition: $\lim L(u\tau)/L(\tau) = 1$ as $\tau \to 0^+$, $u > 0$. We denote by $L(t)$, $t \in \boldsymbol{R}^n$, the product $L_1(t_1) \ldots L_n(t_n)$, where all the L_i are slowly varying at zero.

2. L-quasiasymptotic at zero.

DEFINITION 1. A distribution $f \in (S'_+)$ *has an L-quasiasymptotic at zero of power* $a \in \boldsymbol{R}^n$ *with limit* $g \in (S'_+)$, $g \neq 0$, if

$$\lim_{k\to\infty} \frac{k^a}{L(e/k)} f(t/k) = g(t) \quad \text{in } (S'), \quad e = (1, \ldots, 1) \in \boldsymbol{R}^n,$$

i.e. for every $\varphi \in (S)$

$$\lim_{k\to\infty} \frac{k^a}{L(e/k)} \langle f(t/k), \varphi(t)\rangle = \langle g(t), \varphi(t)\rangle.$$

We write shortly: $f \overset{k}{\sim} g$, $t \to 0^+$ (L, a).

Let us remark that in the case $n = 1$ and $L \equiv 1$ this definition can be deduced from the definition of quasiasymptotic at zero given in [15] with $-a$ instead of a.

DEFINITION 2. If for a numerical function f there exist an $a \in \boldsymbol{R}^n$, a slowly varying function L and a subset $D \subset \boldsymbol{R}^n_+$, $\operatorname{meas}(\boldsymbol{R}^n_+ \setminus D) = 0$, such that

$$\lim_{t\to 0^+} \frac{f(t)}{t^a L(t)} = C, \qquad C \neq 0,\ t \in D,$$

then we write $f(t) \sim Ct^a L(t)$, $t \to 0^+$, and we say that *f has an asymptotic at zero.*

Similarly to [4] one can prove that if, in Definition 1, $a > 0$, then g is a continuous function and $g = Cf_{a+e}$.

It can happen that a regular distribution has an L-quasiasymptotic at zero but has no asymptotic. Such an example is given by the distribution

$$(2+\sin(1/t))_+ = \begin{cases} 2+\sin(1/t), & t > 0, \\ 0, & t < 0. \end{cases}$$

We shall only show that for every $\varphi \in (S)$

$$\lim_{k\to\infty} \langle (\sin(k/x))_+, \varphi(x)\rangle = 0;$$

hence $(2+\sin(1/t))_+ \overset{k}{\sim} 2$, $t \to 0^+$ $(1, 0)$. Indeed,

$$\langle (\sin(k/x))_+, \varphi(x)\rangle = \int_0^\infty \sin(k/x)\varphi(x)\,dx = k\int_0^\infty \sin(1/t)\varphi(kt)\,dt$$

$$= k\int_0^\infty \int_{1/t}^\infty \sin u \frac{du}{u} [\varphi(kt) + kt\varphi'(kt)]\,dt = \int_0^\infty \Psi_k(t)\,dt.$$

The function $\Psi_k(t)$ has the following properties: it belongs to $L^1(0, \infty)$ for every $k \in N$; $|\Psi_k(t)| \leqslant C/(1+t^2)$, $t \in [0, \infty)$; $\Psi_k(0) = 0$ for every $k \in N$ and $\lim \Psi_k(t) = 0$ as $k \to \infty$. Using Lebesgue's theorem we have the assertion.

The next example $((1/\sqrt{t})\sin(a/t))_+$, $a > 0$, shows that a regular distribution can have neither asymptotic nor L-quasiasymptotic. For the L-quasiasymptotic it is enough to use the function $\eta(t)e^{-t} \in (S)$ where $\eta(t) \in C^\infty$, $\operatorname{supp}\eta \subset [-1, \infty)$, $\eta(t) = 1$ if $t \geqslant 0$. By the relation from [9], p. 173, we obtain

$$\lim_{k\to\infty} k^\alpha \langle \sqrt{k/t}\sin(ka/t), \eta(t)e^{-t}\rangle = \lim_{k\to\infty} k^{\alpha+1} \int_0^\infty \sqrt{1/u}\sin(a/u)e^{-ku}\,du$$

$$= \lim_{k\to\infty} k^{\alpha+1}\sqrt{\pi/k}\, e^{-\sqrt{2ka}} \sin\sqrt{2ka} = 0, \qquad \alpha \in \boldsymbol{R}.$$

3. Assertions on L-quasiasymptotic.

PROPOSITION 1. *Let* $f \in (S'_+)$ *and* $f \overset{k}{\sim} g$, $t \to 0^+$ (L, a). *Then*

(1) $(f_{b+e} * f) \overset{k}{\sim} (f_{b+e} * g)$, $\quad t \to 0^+ (L, a+b+e)$, $\quad a, b \in \boldsymbol{R}^n$, $e = (1, \ldots, 1)$.

Proof. Let us suppose that $\alpha(x)$ and $\beta(x)$ are smooth functions equal to 1 in an ε-neighbourhood of $\bar{\boldsymbol{R}}^n_+$ and equal to zero outside a 2ε-neighbourhood of $\bar{\boldsymbol{R}}^n_+$, $\varepsilon > 0$. Using the definition of convolution given by V. S. Vladimirov [14] we obtain for $\varphi \in (S)$:

$$\lim_{k\to\infty} \frac{k^{b+a+e}}{L(e/k)} \langle (f_{b+e} * f)(t/k), \varphi(t)\rangle$$

$$= \lim_{k\to\infty} \frac{k^{b+a+2e}}{L(e/k)} \langle (f_{b+e} * f)(t), \varphi(kt)\rangle$$

$$= \lim_{k\to\infty} \frac{k^{b+a+2e}}{L(e/k)} \langle f_{b+e}(x) \times f(y), \alpha(kx)\beta(ky)\varphi(kx+ky)\rangle$$

$$= \lim_{k\to\infty} \frac{k^a}{L(e/k)} \langle f_{b+e}(u)\alpha(u), \langle f(v/k), \beta(v)\varphi(u+v)\rangle\rangle$$

$$= \langle f_{b+e}(u)\alpha(u), \langle g(v), \beta(v)\varphi(u+v)\rangle\rangle = \langle f_{b+e} * g, \varphi\rangle.$$

We used here the continuity of the direct product. ■

If $b = -e - e_i$ where $e_i = (0, 0, \ldots, 1, \ldots, 0)$, Proposition 1 gives the connection between the partial derivative and the L-quasiasymptotic because $f_{-e_i} * f = (\partial/\partial x_i) f$.

Following [12] we say that $f \in L^1_{\mathrm{loc}}$ defines a *regular tempered distribution*, denoted again by f, if, for every $\varphi \in (S)$, $f\varphi \in L^1$ and $\langle f, \varphi\rangle = \int_{\boldsymbol{R}^n} f(t)\varphi(t)\,dt$. It is proved in [12] that $f \in L^1_{\mathrm{loc}}$ defines a regular tempered distribution if and only if, for some $m \in N_0$, $f(t)(1+\|t\|^2)^{-m/2} \in L^1$.

For the proof of the next theorem we need the following theorem from [1] (Theorem 6; see also footnote on p. 82, and [11]).

Theorem A. *If $L(\tau)$ is a function slowly varying at 0^+ and if $h(\tau)$ is a function defined on $(0, \tau)$ such that*

$$\int_{0^+}^{\infty} \tau^{\nu} |h(\tau)|\, d\tau < \infty \quad \text{for } -\alpha < \nu < \beta, \quad \alpha > 0, \beta > 0,$$

then

$$\int_{0^+}^{\infty} h(\tau) L(\lambda\tau)\, d\tau \sim L(\lambda) \int_{0^+}^{\infty} h(\tau)\, d\tau, \quad \lambda \to 0^+.$$

Theorem 1. *Suppose that $f \in L^1_{\mathrm{loc}}$, $\operatorname{supp} f \subset \boldsymbol{R}^n_+$, defines a regular tempered distribution and that $f(t) \sim Ct^b L(t)$, $t \to 0^+$, for some $b > -e$. Then $f(t) \overset{k}{\sim} Ct^b$, $t \to 0^+$ (L, b).*

Proof. From the assumption of the theorem it follows that $f(t) = Ct^b L(t)(1+\varepsilon(t))$, where $\varepsilon(t) L(t)$ is a locally integrable function such that $\varepsilon(t) \to 0$ as $t \to 0^+$. Thus with the notation $0 = (0, \ldots, 0)$ and $\infty = (\infty, \ldots, \infty)$, we have for $\varphi \in (S)$

$$\left|\frac{k^b}{L(e/k)}\Big[\Big(\int_0^{\eta k} + \int_{\eta k}^{\infty}\Big) f(t/k)\varphi(t)\,dt\Big] - C\int_0^{\infty} t^b \varphi(t)\,dt\right|$$

$$= \left|\frac{1}{L(e/k)}\int_0^{\eta k} Ct^b(1+\varepsilon(t/k)) L(t/k)\varphi(t)\,dt + \frac{k^b}{L(e/k)}\int_{\eta k}^{\infty} f(t/k)\varphi(t)\,dt - C\int_0^{\infty} t^b\varphi(t)\,dt\right|$$

$$\leqslant \left|\frac{1}{L(e/k)}\int_0^{\infty} Ct^b L(t/k)\varphi(t)\,dt - \int_0^{\infty} Ct^b \varphi(t)\,dt\right| + \left|\frac{k^{b+e}}{L(e/k)}\int_{\eta}^{\infty} f(t)\varphi(tk)\,dt\right| + \frac{\varepsilon'}{L(e/k)}\int_0^{\infty} |t^b L(t/k)\varphi(t)|\,dt + \frac{1}{L(e/k)}\left|\int_{\eta k}^{\infty} Ct^b L(t/k)\varphi(t)\,dt\right|$$

where $\varepsilon' = C\sup\{|\varepsilon(t)|: 0 \leqslant t_i \leqslant \eta_i,\ i = 1, \ldots, n\}$, $\eta > 0$.

Theorem A implies

$$(2) \qquad \left|\frac{1}{L(e/k)}\int_0^{\infty} Ct^b L(t/k)\varphi(t)\,dt - \int_0^{\infty} Ct^b\varphi(t)\,dt\right| \to 0, \quad k \to \infty,$$

$$(3) \qquad \frac{\varepsilon'}{L(e/k)}\int_0^{\infty} |t^b L(t/k)\varphi(t)|\,dt \to \varepsilon' \int_0^{\infty} t^b|\varphi(t)|\,dt, \quad k \to \infty.$$

Let $m \in N^n_0$ be such that $m > b+e$. From Theorem A it follows that

$$(4) \qquad \left|\frac{1}{L(e/k)}\int_{\eta k}^{\infty} Ct^b L(t/k)\varphi(t)\,dt\right| \leqslant \frac{C_1}{L(e/k)}\int_{\eta k}^{\infty} \frac{|L(t/k)|}{t^{m-b}}\,dt < \frac{C_2}{(m-b-e)(\eta k)^{m-b-e}} \to 0, \quad k \to \infty.$$

Since f is a regular tempered distribution, we can find a $q \in N^n_0$ and a suitable C_3 such that

$$(5) \qquad \left|\frac{1}{t^{q+e} L(e/k)}\int_{\eta}^{\infty} \frac{f(t)}{t^{q+b+2e}}(tk)^{q+b+2e}\varphi(tk)\,dt\right| \leqslant \frac{C_3}{k^{q+e} L(e/k)}\int_{\eta}^{\infty} \frac{|f(t)|}{t^{q+b+2e}}\,dt \to 0, \quad k \to \infty.$$

From (2)–(5) the assertion follows. ■

Theorem 1 implies

Corollary 1. *Suppose that $F \in L^1_{\mathrm{loc}}$ and that $\operatorname{supp} F \subset \{x \in \boldsymbol{R}^n: 0 \leqslant x \leqslant a,\ a > 0\}$. If $F \sim t^b L(t)$, $t \to 0^+$, $b > -e$, and if $m \in N_0$, $m > |b|+n$, then the sequence $k^b L(e/k)^{-1} F(t/k)$ converges to t^b in $(S^{m\prime})$.*

Proof. For every $m \in N_0$, F belongs to $(S^{m\prime})$ and $t^b \in (S^{m\prime})$ for $b > -e$ and $m > |b|+n$. The proof of Theorem 1 is valid also if we suppose that $\varphi \in (S^m)$. ○

Theorem 2. *Let $f \in (S'_+)$. Then $f \overset{k}{\sim} Cf_{b+e}$, $t \to 0^+$ (L, b) if and only if there exists $a \in \boldsymbol{R}^n_+$, $a > -b-2e$, such that $F = f_{a+e} * f \in L^1_{\mathrm{loc}}$, F defines a regular tempered distribution and $F(t) \sim Cf_{b+a+2e}(t) L(t)$, $t \to 0^+$. For $F(t)$ we know that $\operatorname{supp} F(t) \subset \bar{\boldsymbol{R}}^n_+$ and $|F(t)| \leqslant C(1+\|t\|)^m t^a$, $\min_{1 \leqslant i \leqslant n} a_i > m$.*

Proof. The proof of this theorem is analogous to the proof of Theorem 1 in [4].

If $F = f_{a+e} * f \sim Cf_{b+a+2e} L$, $t \to 0^+$, then Theorem 1 implies that $F \overset{k}{\sim} Cf_{b+a+2e}$, $t \to 0^+$ $(L, b+a+e)$. Since $f_{-(a+e)} * F = f$, Proposition 1 implies $f \overset{k}{\sim} Cf_{b+e}$, $t \to 0^+$ (L, b).

Let us now prove the necessity. The sequence $k^b L(e/k)^{-1} f(t/k)$ is from (S'_+) and converges in (S'). Thus there exists $m \in N$ such that this sequence converges in $(S^{m\prime}_+)$ to $g \in (S^{m\prime}_+)$. We known that for a suitable $a \in \boldsymbol{R}^n_+$, $a_i > m$,

$f_{a+e}(e-t)$ belongs to (S^m_+). We have

$$(6)\qquad \lim_{k\to\infty}\frac{k^b}{L(e/k)}\langle f(t/k), f_{a+e}(e-t)\rangle = C\langle f_{b+e}(t), f_{a+e}(e-t)\rangle = Cf_{a+b+2e}(e).$$

The number a can be chosen in such a way that $\langle f(x), f_{a+e}(t-x)\rangle$ defines a continuous function for $t\in \boldsymbol{R}^n_+$. By putting $t/k = u$ we obtain

$$\lim_{k\to\infty}\frac{k^b}{L(e/k)}\langle f(t/k), f_{a+e}(e-t)\rangle = \lim_{k\to\infty}\frac{k^{b+a+e}}{L(e/k)}\langle f(u), f_{a+e}(e/k-u)\rangle = \lim_{k\to\infty}\frac{k^{b+a+e}}{L(e/k)}(f_{a+e}*f)(e/k).$$

Thus (6) implies

$$\lim_{k\to\infty}\frac{k^{b+a+e}}{L(e/k)}(f_{a+e}*f)(e/k) = Cf_{a+b+2e}(e).$$

It remains to prove that F defines a regular tempered distribution. As $a_i > m$, for $t\in\boldsymbol{R}^n_+$ and $x\in\boldsymbol{R}^n_+$ we have

$$F(t) = (f_{a+e}*f)(t) = \langle f(x), f_{a+e}(t-x)\rangle.$$

Since $f\in(S^{m\prime})$, for a suitable C we have (see [14], p. 95)

$$\begin{aligned}|F(t)| &\leqslant \|f\|_{-m}\|f_{a+e}(t-x)\|_m\\ &\leqslant \|f\|_{-m}\sup\{(1+\|x\|)^m|D^\alpha f_{a+e}(t-x)|:\ x\in\boldsymbol{R}^n_+,\ |\alpha|\leqslant m\}\\ &\leqslant C(1+\|t\|)^m t^a.\end{aligned}$$

The proof is complete. ∎

In the following proposition we shall show that the L-quasiasymptotic at zero is a local property of a distribution.

Proposition 2. *Suppose that f_1 and f_2 belong to (S'_+) and that $f_1 = f_2$ in some neighbourhood of zero. If $f_1 \overset{k}{\sim} g$, $t\to 0^+$ (L, b), then $f_2 \overset{k}{\sim} g$, $t\to 0^+$ (L, b) too.*

Proof. As $f_1 - f_2 = 0$ in some neighbourhood of zero, it follows that there exists $A\in\boldsymbol{R}^n_+$ such that $\operatorname{supp}(f_1-f_2)\subset\prod_{i=1}^n[A_i, \infty)$. Theorem 2 implies that there exists $a\in\boldsymbol{R}^n_+$ such that $H = f_{a+e}*(f_1-f_2)$ is an L^1_{loc}-function which defines a regular tempered distribution with support in $\prod_{i=1}^n[A_i, \infty)$. Since

$$D^{a+e}H = f_{-(a+e)}*(f_{a+e}*(f_1-f_2)) = f_1-f_2$$

we have

$$\begin{aligned}\lim_{k\to\infty}\frac{k^b}{L(e/k)}\langle (f_1-f_2)(t/k), \varphi(t)\rangle &= \lim_{k\to\infty}\frac{k^{b+a+e}}{L(e/k)}\langle D_t^{a+e}H(t/k), \varphi(t)\rangle\\ &= (-1)^{|a+e|}\lim_{k\to\infty}\frac{k^{a+b+e}}{L(e/k)}\int_{kA}^{\infty}H(t/k)\,D^{a+e}\varphi(t)\,dt\\ &= (-1)^{|a+e|}\lim_{k\to\infty}\frac{k^{b+a+2e}}{L(e/k)}\int_{A}^{\infty}H(u)\,\varphi^{(a+e)}(uk)\,du.\end{aligned}$$

Since $\varphi\in(S)$, the last integral tends to zero (see the proof of (5) in Theorem 1). ∎

Examples. At the end of this part, as an illustration, we give a few examples for the quasiasymptotic at zero in the one-dimensional case.

1. For $k\in N_0$, $\delta^{(k)} \overset{k}{\sim} f_{-k}$, $t\to 0^+$ $(1, -k-1)$.

2. $(x^\lambda \ln^m x)_+$, $\mathrm{Re}\,\lambda > -1$, $m\in N_0$, is a regular distribution and $(x^\lambda\ln^m x)_+ \overset{k}{\sim} \Gamma(\lambda+1)$, $f_{\lambda+1}$, $x\to 0^+$ $(\ln^m x, \lambda)$.

3. $(x^\lambda\ln^m x)_+$, $-n > \mathrm{Re}\,\lambda > -n-1$, $m\in N_0$, is a distribution defined by the process of regularization (see [6], p. 338). In this case

$$(x^\lambda\ln^m x)_+ \overset{k}{\sim} \Gamma(\lambda+1)D^n f_{\lambda+n+1}(x), \quad x\to 0^+ \quad (\ln^m x, \lambda).$$

4. For the distribution $(x^{-n}\ln^m x)_+$, $n\in N$, $m\in N_0$ (see [6], p. 339) the L-quasiasymptotic is

$$(x^{-n}\ln^m x)_+ \overset{k}{\sim} \frac{1}{(1+m)(n-1)!}(-1)^n f_{-n+1}(x), \qquad x\to 0^+ \quad (\ln^{m+1}x, -n).$$

4. Definition of the Stieltjes transform of a distribution. First we introduce the following notation: For a given $s\in(\boldsymbol{C}\setminus\bar{\boldsymbol{R}}_-)^n$, $A(s)$ will denote the family of smooth functions η with the properties: $0\leqslant\eta(t)\leqslant 1$; $\eta(t) = 1$ if t belongs to the ε-neighbourhood of $\bar{\boldsymbol{R}}^n_+$; $\eta(t) = 0$ outside the 2ε-neighbourhood of $\bar{\boldsymbol{R}}^n_+$, where ε is an arbitrary number such that $0 < 2\varepsilon < |\mathrm{Re}\, s_i|$, $i = 1, \ldots, n$; $|D^p\eta(t)|\leqslant c_p$, $p\in N^n_0$, $t\in\boldsymbol{R}^n$.

We notice that for a given $r\in N^n_0$ and $s\in(\boldsymbol{C}\setminus\bar{\boldsymbol{R}}_-)^n$ the set of functions

$$\frac{\eta(t)\exp(-\omega, t)}{(s+t)^{r+e}}, \qquad \eta\in A(s),\ \omega\in\operatorname{int}\bar{\boldsymbol{R}}^n_+,\ e = (1, \ldots, 1),$$

is a subset of (S). Here $(x, y) = x_1y_1 + \ldots + x_ny_n$ and $\operatorname{int}A$ is the set of interior points of A. ($(s+t)^{r+e} = \prod_{i=1}^n (s_i+t_i)^{r_i+1}$ takes positive values for $s_i > 0$, $i = 1, \ldots, n$.)

Definition 3. The *Stieltjes transform of index $r\in\boldsymbol{R}^n$ (S_r-transform)* of a distribution $f \in (S'_+)$ is the function $\hat{f}^r(s)$ defined on some subset D of the set $(C\setminus\bar{R}_-)^n$ by the limit

$$(7)\qquad \hat{f}^r(s) = \lim_{\omega\to 0^+}\left\langle f(t), \frac{\eta(t)\exp(-\omega, t)}{(s+t)^{r+e}}\right\rangle, \qquad \eta\in A(s),\ \omega\in \operatorname{int}\bar{R}^n_+,$$

if it exists for $s\in D$.

Obviously, the limit in (7) does not depend on the function η from $A(s)$; in this sense η can be omitted.

For every $f\in(S'_+)$ there exists $r_0\in R^n$ such that for every $r\geqslant r_0$ the S_r-transform of f is defined.

A little more restrictive is the definition of the Stieltjes transform given by J. Lavoine and O. P. Misra [8] in the case $n=1$. They start from a subspace $J'(r)$ of the space (S'_+) in one dimension. First we extend this definition to the case $n>1$.

Definition 4. Let $J'(r)$, $r\in(R\setminus -N)^n$, denote the space of distributions T having supports in $\bar{R}^n_+$ and admitting the decomposition $T=S+D^kF$, where F is a function with support in the set $B=\{x\in\bar{R}^n_+ : \|x\|\geqslant\alpha,\ \alpha\in R^+\}$, $F(x)x^{-r-k-e}$ is summable and S is a distribution having support in $\bar{R}^n_+\setminus\operatorname{int}B$.

Now if $n=1$ and $T=S+D^kF\in J'(r)$, the Stieltjes transform of T defined by J. Lavoine and O. P. Misra [8] is

$$(\mathscr{S}_r T)(s) = \langle S_x, (x+s)^{-r-1}\rangle + \frac{\Gamma(r+k+1)}{\Gamma(r+1)}\int_{\|x\|\geqslant\alpha} F(x)(x+s)^{-r-k-1}\,dx.$$

If $f\in J'(r_0)$, then we can prove that for all $r\geqslant r_0$ the Stieltjes transform $\hat{f}^r(s)$, given by Definition 3, exists and

$$\hat{f}^r(s) = \hat{S}^r(s) + (r+e)_k \int_{\bar{R}^n_+} \frac{F(t)}{(s+t)^{r+k+e}}\,dt, \qquad s\in(C\setminus\bar{R}_-)^n.$$

Here we have used the notation $(r+e)_p=(r_1+1)\dots(r_1+p_1)\dots(r_n+1)\dots(r_n+p_n)$, $(r_i+1)_0=1$.

For a distribution $S\in(S'_+)$ with support in a compact $K\subset\bar{R}^n_+$, the Stieltjes transform is

$$\hat{S}^r(s) = \lim_{\omega\to 0}\left\langle S(t), \frac{\eta(t)}{(s+t)^{r+e}}\exp(-\omega, t)\right\rangle = \left\langle S(t), \frac{\xi(t)}{(s+t)^{r+e}}\right\rangle$$

where $\xi\in C^\infty$, $\operatorname{supp}\xi\subset K^{2\varepsilon}$ and $\xi(t)=1$, $t\in K^\varepsilon$; K^ε is the ε-neighbourhood of K.

By some appropriate examples we shall show that our S_r-transform has a meaning for $r=-n$, $n\in N$, too and that for some distributions the set of parameters r for which our S_r-transform exists contains as a proper subset the set of parameters r for which the $\mathscr{S}_r$-transform exists in the sense of [8].

Let

$$(\sin t)_+ = \begin{cases}\sin t, & t>0,\\ 0, & t\leqslant 0.\end{cases}$$

This is a regular distribution for which we shall find the S_r-transform given by Definition 3, first for $r=-q-1$, $q\in N_0$, and afterwards for $r<0$. We shall also show that for $r\leqslant -1$ the $\mathscr{S}_r$-transform of this distribution in the sense of [8] does not exist.

For $r=-q-1$, $q\in N_0$, we have

$$\widehat{(\sin t)}{}_+^{-q-1}(s) = \lim_{\omega\to 0^+}\int_0^\infty \exp(-\omega t)(s+t)^q \sin t\,dt$$
$$= \lim_{\omega\to 0^+}\sum_{k=0}^{q}\binom{q}{k}s^{q-k}\int_0^\infty \exp(-\omega t)\,t^k\sin t\,dt$$
$$= \sum_{j=0}^{[q/2]}(-1)^j\binom{q}{2j}s^{q-2j}\Gamma(2j+1), \qquad s\in C\setminus\bar{R}_-,\ \omega\in R_+.$$

The last equality follows from the relation ([9], p. 172)

$$\int_0^\infty \exp(-\omega t)\,t^k\sin t\,dt = \frac{1}{2i}\frac{\Gamma(k+1)}{(\omega^2+1)^{k+1}}[(\omega+i)^{k+1}-(\omega-i)^{k+1}]$$

where $k>-1$.

Now, let s be real and positive and suppose that $r<0$ and $a\neq 0$. If we put $-q=r+1$, we have

$$\int_0^\infty \exp(-\omega t)\exp(ait)(s+t)^q\,dt$$
$$= \exp\big(s(\omega-ai)\big)\left(\frac{\Gamma(q+1)}{(\omega-ai)^{q+1}} - \int_0^s \exp\big(-(\omega-ai)u\big)u^q\,du\right).$$

The limit of this expression as $\omega\to 0^+$ is

$$\exp(-sai)\left(\frac{\Gamma(q+1)}{(-ai)^{q+1}} - \int_0^s \exp(aiu)\,u^q\,du\right), \qquad q>-1.$$

Now, the S_r-transform of the distribution $(\sin t)_+$ can be obtained in the case $r<0$, $s\in R_+$ as

$$\widehat{(\sin t)}{}^r_+(s) = \int_0^s \sin(s-u)\,u^{-r-1}\,du - \Gamma(-r)\sin(s+\pi r/2).$$

Recall that θ denotes the Heaviside function and D the distributional derivative. We have

$$(\sin t)_+ = D\big(-(\cos t)_+ + \theta(t)\big), \qquad (\cos t)_+ = D(\sin t)_+.$$

Hence

$$(\sin t)_+ = D\big(-(\cos t)_+ + \theta(t)\big) = D^2\big(-(\sin t)_+ + (t)_+\big) = D^3\big((\cos t)_+ - \theta(t) + \tfrac{1}{2}(t^2)_+\big) = \ldots$$

This shows that $(\sin t)_+ = D^p F(t)$ where $F(t) \sim C_p t^{p-1}$, $t \to \infty$, and $p = 2, 3, \ldots$ Hence the integral

$$\int_1^\infty \frac{|F(t)|}{t^{r+p+1}}\,dt$$

exists only for $r > -1$ and, consequently, the Stieltjes transform in the sense of [8] of the distribution $(\sin t)_+$ exists only for $r > -1$. But we have shown that $\widehat{(\sin t)}{}^r_+$ exists for $r < -1$ too.

The following proposition gives a sufficient condition for the existence of the S_r-transform.

PROPOSITION 3. *Suppose that*

$$f(t)\,\eta(t)/(s+t)^{r+e}, \qquad s \in (C \setminus \bar{R})^n,\ \eta \in A(s),$$

has a quasiasymptotic at infinity of power $-n$, with $L(t) = 1$ (see [5] or [4]) *and with limit $g(s, t)$. Then f has the S_r-transform and*

$$\hat{f}^r(s) = \langle g(s, t), \exp(-e, t)\rangle, \qquad e = (1, \ldots, 1),\ s \in (C \setminus \bar{R}_-)^n.$$

Proof. We know that

$$\hat{f}^r(s) = \lim_{\omega \to 0^+} \left\langle f(t), \frac{\eta(t)}{(s+t)^{r+e}} \exp(-\omega, t)\right\rangle = \lim_{\omega \to 0^+} \left\langle f(t) \frac{\eta(t)}{(s+t)^{r+e}}, \exp(-\omega, t)\right\rangle, \qquad s \in (C \setminus \bar{R}_-)^n.$$

Using the Abelian theorem for the Laplace transform (see [5]) we have

$$\hat{f}^r(s) = \langle g(s, t), \exp(-e, t)\rangle, \qquad s \in (C \setminus \bar{R}_-)^n. \ \blacksquare$$

5. Initial value Abelian theorems. As we mentioned in the introduction, we shall give the initial value Abelian theorems for the Stieltjes transform of distributions using the concept of quasiasymptotic behaviour at 0^+.

For the proof of the main theorem we need the following lemma:

LEMMA 1. *Let L be a slowly varying function. Suppose that $r - b > 0$, $b + p + e > 0$ and $-r_i \notin N$, $i = 1, \ldots, n$. Then*

$$\lim_{k \to \infty} \frac{k^{-(r-b)}}{L(e/k)} (r+e)_p \int_{\bar{R}^n_+} \frac{f_{b+p+e}(t)\,L(t)}{(s/k+t)^{r+p+e}}\,dt = \prod_{i=1}^n \frac{\Gamma(r_i - b_i)}{\Gamma(r_i+1)} s^{-(r-b)}, \qquad s \in (C \setminus \bar{R}_-)^n.$$

Proof. Using theorems on integrals involving slowly varying functions [1] we have

$$\lim_{k \to \infty} \frac{k^{-(r-b)}}{L(e/k)} (r+e)_p \int_{\bar{R}^n_+} f_{b+p+e}(t) \frac{L(t)}{(s/k+t)^{r+p+e}}\,dt$$

$$= \lim_{k \to \infty} (r+e)_p \frac{1}{L(e/k) \prod_{i=1}^n \Gamma(b_i + p_i + 1)} \int_{\bar{R}^n_+} \frac{u^{b+p} L(u/k)}{(s+u)^{r+p+e}}\,du$$

$$= \frac{(r+e)_p}{\prod_{i=1}^n \Gamma(b_i + p_i + 1)} \int_{\bar{R}^n_+} \frac{u^{b+p}}{(s+u)^{r+p+e}}\,du$$

$$= \prod_{i=1}^n \frac{\Gamma(r_i - b_i)}{\Gamma(r_i+1)} s^{-(r-b)} \quad \text{(see [9], p. 6).} \ \blacksquare$$

THEOREM 3. *Suppose that $f \in J'(r_0)$ and that $f \overset{k}{\sim} g$, $t \to 0^+ (L, b)$. Then $g = Cf_{b+e}$ and for s belonging to the closed domain $\Omega \equiv \{s \in C^n\colon \arg s_i \in [-\pi + \varepsilon, \pi - \varepsilon],\ i = 1, \ldots, n\}$, $\varepsilon > 0$, we have*

$$\lim_{k \to \infty} \left(\frac{s}{k}\right)^{r-b} \frac{1}{L(e/k)} \hat{f}^r\left(\frac{s}{k}\right) = C \prod_{i=1}^n \frac{\Gamma(r_i - b_i)}{\Gamma(r_i+1)} \tag{8}$$

where $r_i \neq -1$, $r > b$ and $r \geq r_0$. If $n = 1$ and $L(x) \equiv 1$ the convergence is uniform in Ω.

Proof. By Definition 4, $f = S + D^q F$ where $\operatorname{supp} F \subset B = \{x \in R^n_+\colon \|x\| \geq \alpha,\ \alpha \in R_+\}$, $F(x)\,x^{-r-q-e}$ is summable and S is a distribution having support in the compact set $K = \overline{R^n_+ \setminus B}$.

By Proposition 2, $S \overset{k}{\sim} g$, $t \to 0^+$ (L, b). Hence $S = D^p G$, $G \in L^1_{\mathrm{loc}}$, $G \sim Cf_{b+p+e} L$ as $t \to 0^+$ (see Theorem 2).

Let K^ε be the ε-neighbourhood of the compact set K and let $\xi(t)$ be a C^∞-function such that $\xi(t) = 1$, $t \in K^\varepsilon$; $\xi(t) = 0$, $t \notin K^{3\varepsilon}$; $|D^\alpha \xi| \leq C_\alpha(\varepsilon)$. Using the Leibniz formula we have

$$S(t) = \xi(t) S(t) = \xi(t) D^p G(t) = D^p\big(\xi(t) S(t)\big) - \sum_{\substack{\alpha \leq p \\ \alpha \neq p}} D^\alpha\big(\xi_\alpha(t)\, G(t)\big)$$

where $\xi_\alpha(t)$ are functions with supports in $K^{3\varepsilon} \setminus K^\varepsilon$.

Let $s \in C^n$, $\arg s_i \neq \pi$, $i = 1, \ldots, n$. The Stieltjes transform of the distribution f is

$$\hat{f}^r(s) = \left\langle S(t)\,\xi(t),\ \frac{\eta(t)}{(s+t)^{r+e}}\right\rangle + (r+e)_q \int_B \frac{F(t)}{(s+t)^{r+q+e}}\,dt$$

$$= (r+e)_p \left[\int_K \frac{G(t)}{(s+t)^{r+p+e}}\,dt + \int_{K^{3\varepsilon}\setminus K} \frac{G(t)\,\xi(t)}{(s+t)^{r+p+e}}\,dt - \sum_{\substack{\alpha\leqslant p\\ \alpha\neq p}} \int_{K^{3\varepsilon}\setminus K} \frac{\xi_\alpha(t)\,G(t)}{(s+t)^{r+\alpha+e}}\,dt\right] + (r+e)_q \int_B \frac{F(t)}{(s+t)^{r+q+e}}\,dt.$$

Now by our Lemma 1

$$(9)\qquad \lim_{k\to\infty} \frac{k^{-(r-b)}}{L(e/k)} \hat{f}^r(s/k) - C\prod_{i=1}^{n} \frac{\Gamma(r_i-b_i)}{\Gamma(r_i+1)}\, s^{-(r-b)}$$

$$= \lim_{k\to\infty} \frac{k^{-(r-b)}}{L(e/k)} (r+e)_p \int_K \frac{G(t)-f_{b+p+e}(t)\,L(t)}{(s/k+t)^{r+p+e}}\,dt + \lim_{k\to\infty} \frac{k^{-(r-b)}}{L(e/k)} \int_{\bar{R}^n_+} H(s/k,\, t)\,dt$$

where $H(s/k,\, t)$ is a function with the properties $\lim_{k\to\infty} H(s/k,\, t) = H(0,\, t)$ and $H(0,\, t)\in L^1$. As $r > b$,

$$\lim_{k\to\infty} \frac{k^{-(r-b)}}{L(e/k)} \int_{\bar{R}^n_+} H(s/k,\, t)\,dt = 0.$$

It only remains to prove that the first term on the right-hand side of (9) tends to zero too.

In the set K we have

$$G(t) - Cf_{b+p+e}(t)\,L(t) = \delta(t)\,f_{b+p+e}(t)\,L(t)$$

where δ is a locally integrable function and $\delta(t)\to 0$ as $t\to 0^+$. Therefore we can choose k_0 in such a way that for any $\varepsilon > 0$, $|\delta(t/k)| < \varepsilon$ for $t\in K$ and $k \geqslant k_0$. Then

$$\left|\frac{k^{-(r-b)}}{L(e/k)} (r+e)_p \int_K \frac{G(t)-f_{b+p+e}(t)\,L(t)}{(s/k+t)^{r+p+e}}\,dt\right|$$

$$\leqslant \frac{k^{-(r-b)}}{L(e/k)} (r+e)_p \int_K \frac{\delta(t)\,f_{b+p+e}(t)\,L(t)}{|s/k+t|^{r+p+e}}\,dt$$

$$\leqslant \varepsilon \frac{(r+e)_p}{L(e/k)} \int_{\bar{R}^n_+} \frac{f_{b+p+e}(u)\,L(u/k)}{|s+u|^{r+p+e}}\,du.$$

In this way we have proved that the first term on the right-hand side of relation (9) tends to zero as $k\to\infty$. By the transformation $z\to 1/z$ and by the Montel theorem we can prove the uniformity of the limit process in Ω if $L(x)\equiv 1$ and $n = 1$. ∎

If in our Theorem 3 we only suppose that the distribution f has L-quasiasymptotic $f \overset{k}{\sim} g,\ t\to 0^+\ (L, b)$, without knowing for which r_0 it belongs to $J'(r_0)$, then we can only give a smaller set of r's satisfying relation (8):

Theorem 4. *Suppose that $f\in(S'_+)$ and $f \overset{k}{\sim} g,\ t\to 0^+\ (L, b)$. Then there exists $p\in \boldsymbol{R}^n_+$, $p+b+2e > 0$, such that $F = f_{p+e} * f$ and $F(t) \sim Cf_{b+p+e}(t)\,L(t)$, $t\to 0$. For $r > b$, $\sum_{i=1}^{n} r_i \geqslant \min\{p_i\colon i = 1, \ldots, n\}$, the statement of Theorem 3 is valid.*

The proof follows from Theorem 2 which says that there exists $m\in N_0$ such that $|F(t)| \leqslant C(1+\|t\|)^m$, $p_i > m$, $i = 1, \ldots, n$.

Sometimes a theorem which gives the asymptotical behaviour of $\hat{f}^r(s)$ knowing that of $\hat{f}^{r+e}(s)$ is very useful. For the proof of such a theorem we need the following lemma:

Lemma 2. *Let $f\in J'(r)$. Then for $x = (x_1, \ldots, x_n)\in \bar{\boldsymbol{R}}^n_+$ and $s = (s_1, \ldots, s_n) \in (\boldsymbol{C}\setminus \bar{\boldsymbol{R}}_-)^n$*

$$\hat{f}^r(sx) = \prod_{i=1}^{n} s_i(r_i+1) \int_{x_n}^{\infty} \ldots \int_{x_1}^{\infty} \hat{f}^{r+e}(st)\,dt_1 \ldots dt_n.$$

Proof. If $s\in(\boldsymbol{C}\setminus\bar{\boldsymbol{R}}_-)^n$, then $\hat{f}^r(sx)\to 0$ as $x\to\infty$ ($x_i\to\infty$, $i = 1, \ldots, n$). This follows from the definition of the space $J'(r)$ and the Lebesgue theorem. Hence

$$\hat{f}^r(sx) = (-1)^n \prod_{i=1}^{n} s_i \lim_{\omega\to\infty} \int_{x_n}^{\omega_n} \ldots \int_{x_1}^{\omega_1} (\hat{f}^r(st))^{(e)}\,dt_1 \ldots dt_n$$

$$= \prod_{i=1}^{n} s_i(r_i+1) \int_x^{\infty} \hat{f}^{r+e}(st)\,dt$$

where we put $\int_x^{\infty}$ instead of $\int_{x_n}^{\infty} \ldots \int_{x_1}^{\infty}$. ∎

Theorem 5. *If $f\in J'(r)$ and $\hat{f}^{r+e}(x) \sim cx^{-(r-b)-e}L(x)$, $x\to 0^+$, where $b < r$ ($b_i < r_i$, $i = 1, \ldots, n$) then*

$$\hat{f}^r(x) \sim c\prod_{i=1}^{n} \frac{r_i+1}{r_i-b_i}\, x^{-(r-b)}\,L(x), \qquad x\to 0^+.$$

Proof. Let $\varepsilon > 0$ be fixed. There exists $\delta = (\delta_1, \ldots, \delta_n)\in \boldsymbol{R}^n_+$ such that for $0 < x < \delta$

$$(4)\qquad \hat{f}^{r+e}(x) = c\big(1+\varepsilon(x)\big)\,x^{-(r-b)-e}\,L(x)$$

where $\varepsilon(x)$ is a locally integrable function in the δ-neighbourhood of zero such that $|c\varepsilon(x)| \leqslant \varepsilon/8$. With

$$M = c\prod_{i=1}^{n}(r_i-b_i)^{-1}, \qquad N = \prod_{i=1}^{n}(r_i+1)$$

we have from Lemma 2

$$\left|\frac{\hat{f}^r(x)}{NMx^{b-r}L(x)}-1\right| \leqslant \left|\frac{\int_x^\delta \hat{f}^{r+e}(t)\,dt}{Mx^{b-r}L(x)}-1\right| + \left|\frac{\int_\delta^\infty \hat{f}^{r+e}(t)\,dt}{Mx^{b-r}L(x)}\right|.$$

Now, there exists $\bar{\delta}$, $0<\bar{\delta}<\delta$, such that for $0<x<\bar{\delta}$

$$\frac{\left|\int_\delta^\infty \hat{f}^{r+e}(t)\,dt\right|}{Mx^{b-r}L(x)} \leqslant \frac{\varepsilon}{8}.$$

Thus for $0<x<\bar{\delta}$ we have

$$\left|\frac{\hat{f}^r(x)}{MNx^{b-r}L(x)}-1\right| \leqslant \left|\frac{\int_x^\delta (1+\varepsilon(t))\,ct^{b-r-e}L(t)\,dt}{Mx^{b-r}L(x)}-1\right| + \frac{\varepsilon}{8}$$

$$\leqslant \left|\frac{c\int_x^\delta t^{b-r-e}L(t)\,dt}{Mx^{b-r}L(x)}-1\right| + \frac{\varepsilon}{8}\,\frac{\left|\int_x^\delta t^{b-r-e}L(t)\,dt\right|}{Mx^{b-r}L(x)} + \frac{\varepsilon}{8}$$

$$\leqslant \left|\frac{c\int_e^{\delta/x} u^{b-r-e}L(ux)\,du}{ML(x)}-1\right| + \frac{\varepsilon}{8}\,\frac{\left|\int_e^{\delta/x} u^{b-r-e}L(ux)\,du\right|}{ML(x)} + \frac{\varepsilon}{8}$$

$$\leqslant \left|\frac{c\int_e^\infty u^{b-r-e}L(ux)\,du}{ML(x)}-1\right| + \frac{\left|c\int_\delta^\infty t^{b-r-e}L(t)\,dt\right|}{Mx^{b-r}L(x)}$$

$$+\frac{\varepsilon}{8}\,\frac{\left|\int_e^\infty u^{b-r-e}L(ux)\,du\right|}{ML(x)} + \frac{\varepsilon}{8} < \varepsilon.$$

We have made use of the relation (see [1])

$$\int_e^\infty u^{b-r-e}L(ux)\,du \sim L(x)\int_e^\infty u^{b-r-e}\,du, \qquad x \to 0^+. \ \blacksquare$$

Remarks. We shall compare the result of our Theorem 3 with the known initial value Abelian theorems. As far as we know, they are all given only in the case $n=1$. Examples 3 and 4 in the first part show that the

results of Lavoine and Misra [7], [8], relating to the point zero, with special regularly varying functions, can be obtained from Theorem 1. Also, Theorem 3 generalizes results from [10] (relating to the point zero) because in [10] the authors assume $b > -1$ and $x \to 0^+$ on the real line. Carmichael and Milton [2] obtained results relating to the point zero with $L(x) \equiv 1$ and s converging to zero and remaining in the right half-plane.

To show that the quasiasymptotic has some advantage over the classical asymptotic used in [3] in the investigations of integral transformations, we give the following example connected with the Laplace transform.

The regular distribution $(\sin t)_+$ has no usual asymptotic as $t \to \infty$. That is the reason why we cannot use the Abelian theorem for the classical Laplace transform. Also the generalized classical Abelian theorem for the Laplace transform gives no information on the asymptotic of $\mathscr{L}(\sin t)$ (see [3], p. 459):

$$\liminf_{t\to\infty} t^{-\alpha}\sin t \leqslant \liminf_{s\to 0^+}\frac{s^{\alpha+1}}{\Gamma(\alpha+1)}\mathscr{L}(\sin t)$$

$$\leqslant \limsup_{s\to 0^+}\frac{s^{\alpha+1}}{\Gamma(\alpha+1)}\mathscr{L}(\sin t) \leqslant \limsup_{t\to\infty}\sin t.$$

But we know that

$$\lim_{s\to 0^+}\int_0^\infty \exp(-st)\sin t\,dt = \lim_{s\to 0}\frac{1}{s^2+1} = 1.$$

The same result follows by using the fact that $(\sin t)_+$ has a quasiasymptotic at ∞ of power -1 ($L(t) \equiv 1$) and the Abelian theorem for the distributional Laplace transform (see [4]).

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Estimates in Sobolev norms $\|\cdot\|_p^s$ for harmonic and holomorphic functions and interpolation between Sobolev and Hölder spaces of harmonic functions

by

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Abstract. In the paper the duality theory is extended to the case of Sobolev spaces of harmonic functions whose derivatives are in $L^p(D)$. The behaviour of Bell's operators L_s^r on each space is studied. These operators together with the orthogonal projection P on harmonic functions are used to the study of interpolation between L^p, Sobolev and Hölder spaces of harmonic functions. It turns out that all these spaces form a double interpolation scale. If P maps $L^\infty(D)$ onto the space of Bloch harmonic functions, as in the case of the unit ball, then this last space is the vertex of this scale. No assumptions on the existence of traces on the boundary are needed in this approach. The possible use of above approach in the study of the regularity of the Bergman projection and of solutions of the $\bar\partial$-Neumann problem is discussed. The duality and interpolation theorems are also proved for the spaces of holomorphic functions on strictly pseudoconvex domains.

I. Introduction and the statement of results. The present paper is a continuation of [16]–[19]. We extend the duality theory for spaces of harmonic functions, originated by S. Bell [3], [4] and developed in [5], [10], [16]–[18], to the spaces $\mathrm{Harm}_p^s(D)$ of harmonic functions belonging to the Sobolev space $W_p^s(D)$, $1 < p < \infty$. In [16] we gave the detailed description of this duality for $p = 2$. Let us define the "negative Sobolev spaces" $W_p^{-s}(D)$, $1 < p < \infty$, s an integer, $s > 0$, as the spaces of distributions g on the domain D such that $g = \sum_{|\zeta|=s} D^\zeta g_\zeta + g_0$, where $g_0, g_\zeta \in L^p(D)$. The space $W_p^{-s}(D)$ is the adjoint space to $\mathring{W}_q^s(D)$ which is the closure of $C_0^\infty(D)$ in $W_q^s(D)$, $q = p/(p-1)$.

In fact, $W_p^{-s}(D)$ and $\mathring{W}_q^s(D)$ are mutually dual with respect to the L^2 scalar product $\langle\ ,\ \rangle$. We equip $W_p^{-s}(D)$ with the dual norm of $(\mathring{W}_q^s(D))^*$.

If s is not an integer, we define $W_p^s(D)$ as the value of the complex interpolation functor $[W_p^{[s]}(D), W_p^{[s]+1}(D)]_{[\theta]}$ for $\theta = s-[s]$, where $[s]$ is the integer part of s. If $s > 0$ then the "negative Sobolev space" $W_p^{-s}(D)$ represents the dual space to $\mathring{W}_q^s(D) = [\mathring{W}_q^{[s]}(D), \mathring{W}_q^{[s]+1}(D)]_{[\theta]}$, $\theta = s-[s]$, $q = p/(p-1)$. The space $\mathring{W}_q^s(D)$ is equal to the closure of $C_0^\infty(D)$ in $W_q^s(D)$ for $s \neq k+1/q$, $k = 0, 1, 2, \ldots$