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Improper integrals of distributions

by

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Abstract. We introduce a space of functions with bounded variation on $\mathbf{R}^n$ and call its dual space the space of improper integrable distributions. This definition turns out to be a generalization of the classical Schwartz definition of integrable distributions and of the definition of improper integrals for L^1_{loc} functions. We also define the improper convolution of distributions.

In this paper we define improper integrals for distributions. This is a slight modification of the definitions of Sikorski [15] and Musielak [9] and a generalization of the classical Schwartz definition. This modification allowed us to prove a representation theorem. The class of distributions having improper integrals turns out to be the dual space of a space which can be called a space of functions with bounded variation in $\mathbf{R}^n$.

We also define the convolution of two distributions using the notion of improper integral. This definition is more general than the classical Schwartz definition of convolution. We show that the exchange formula is still valid for the wider definition of convolution.

0. Notation, definitions and basic facts. We employ the usual notation of the theory of distributions. We denote by d_k, s_k for $k \in N_0 = \{0, 1, \ldots\}$ the seminorms in the spaces $\mathscr{D}(\mathbf{R}^n)$ and $\mathscr{S}(\mathbf{R}^n)$, i.e.

$$d_k(\varphi) = \sum_{|\alpha| \leqslant k} \sup |D^\alpha \varphi| \qquad \text{for } \varphi \in C_0^\infty(\mathbf{R}^n),$$

$$s_k(\sigma) = \sum_{|\alpha| \leqslant k} \sup_x \left|(1+|x|^2)^k D^\alpha \sigma(x)\right| \quad \text{for } \sigma \in \mathscr{S}(\mathbf{R}^n).$$

For every compact $K \subset \mathbf{R}^n$ and every $k \in N_0$, $\mathscr{D}_k(K)$ denotes the space $C_0^k(K)$ with topology given by the norm d_k; $\mathscr{S}_k(\mathbf{R}^n)$ is the space of all $\sigma \in C^k(\mathbf{R}^n)$ with $s_k(\sigma) < +\infty$, with topology given by s_k. $\mathscr{D}'_k(K)$, $\mathscr{S}'_k(\mathbf{R}^n)$ are their dual spaces.

The Fourier transform and inverse Fourier transform are denoted by "ˆ" and "ˇ", i.e.

$$\hat{\sigma}(\xi) = (2\pi)^{-n/2} \int \sigma(x) e^{-ix\xi} dx \qquad \text{for } \sigma \in \mathscr{S}(\mathbf{R}^n).$$

If f is a function on $\mathbf{R}^n$ we define $\tilde{f}$ as the function satisfying $\tilde{f}(x) =$

$f(-x)$ for $x \in \mathbf{R}^n$. For a distribution $T \in \mathscr{D}'(\mathbf{R}^n)$, $\tilde{T}$ is the distribution given by $\tilde{T}[\varphi] = T[\tilde{\varphi}]$ for $\varphi \in \mathscr{D}(\mathbf{R}^n)$.

A sequence η_ν is *convergent to unity in Schwartz sense* in $\mathbf{R}^n$ if

(i) $\eta_\nu \in C_0^\infty(\mathbf{R}^n)$ for $\nu = 1, 2, \ldots$

(ii) $d_k(\eta_\nu) \leqslant C_k < +\infty$ for $\nu = 1, 2, \ldots, k = 0, 1, \ldots$

(iii) For every compact $K \subset \mathbf{R}^n$ there exists $N > 0$ such that $\eta_\nu(x) = 1$ for $\nu \geqslant N$, $x \in K$.

For $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ and $\beta = (i_1, \ldots, i_n)$, $i_j = 0$ or 1, let $x_\beta \in \mathbf{R}^{|\beta|}$ denote the projection of x onto the space of those coordinates which correspond to $i_j = 1$. Obviously, for $\beta = \mathbf{1} = (1, \ldots, 1)$, $x_1 = x$. If $x_\beta = (x_{j_1}, \ldots, x_{j_{|\beta|}})$, then we write dx_β for $dx_{j_1} \ldots dx_{j_{|\beta|}}$. Sometimes instead of $\mathbf{R}^{|\beta|}$ we write $\mathbf{R}^{|\beta|}_{x_\beta}$.

An *open* (n-dimensional) *interval* is a set of the form

$$P = \{x \in \mathbf{R}^n\colon -\infty \leqslant a_i < x_i < b_i \leqslant \infty,\ i = 1, \ldots, n\}.$$

Every set Q such that $P \subset Q \subset \bar{P}$ for some open interval P will be called an *interval* in $\mathbf{R}^n$.

We now define the following spaces:

$$B = \{\psi \in C^\infty(\mathbf{R}^n)\colon \sup |D^\alpha \psi| < +\infty \text{ for } \alpha \in N_0^n\}$$

with topology given by the sequence of seminorms $(d_k)_k$;

$$\dot{B} = \{\psi \in C^\infty(\mathbf{R}^n)\colon D^\alpha \psi(x) \to 0 \text{ if } |x| \to +\infty \text{ for } \alpha \in N_0^n\}$$

equipped with the same sequence of seminorms;

$$\bar{B} = \{\psi \in C^\infty(\mathbf{R}^n)\colon \sup |D^\alpha \psi| < +\infty \text{ for } \alpha \in N_0^n\}$$

with the following topology: $\psi_\nu \to 0$ in $\bar{B}$ as $\nu \to \infty$ if

(i) $d_k(\psi_\nu) \leqslant C_k < +\infty$ for $k = 0, 1, \ldots, \nu = 1, 2, \ldots$

(ii) $D^\alpha \psi_\nu \to 0$ almost uniformly as $\nu \to \infty$ for $\alpha \in N_0^n$.

The space $\mathscr{D}$ is dense in $\dot{B}$ and in $\bar{B}$ but not in B. The spaces $\dot{B}$ and $\bar{B}$ have the same dual space. We denote it by $\mathscr{D}'_{L^1}$. Since $\mathscr{S} \subset \dot{B}$ and the inclusion is continuous, $\mathscr{D}'_{L^1}$ is a space of tempered distributions.

Definition 1. A distribution $T \in \mathscr{D}'(\mathbf{R}^n)$ is *integrable in Schwartz sense* (over $\mathbf{R}^n$) if for every sequence η_ν converging to unity in $\mathbf{R}^n$ in Schwartz sense the limit $\lim_\nu T[\eta_\nu]$ exists and is finite. Obviously the limit does not depend on the choice of the sequence η_ν. We denote it by $T[1]$ or $\int T$ and call the *Schwartz integral* of T over $\mathbf{R}^n$.

Theorem 1 (see [12], Chap. VI, § 8, Th. XXV, and [11]). *Let $T \in \mathscr{D}'$. The following conditions are equivalent:*

(i) $T \in \mathscr{D}'_{L^1}$.

(ii) *$T * \varphi \in L^1$ for $\varphi \in \mathscr{D}$.*

(iii) *$T = \sum_{|\alpha| \leqslant r} D^\alpha f_\alpha$ where $f_\alpha \in L^1$.*

(iv) *T is integrable in Schwartz sense.*

Theorem 2 (see [13]). *Let $U, V \in \mathscr{D}'(\mathbf{R}^n)$ $[U, V \in \mathscr{S}'(\mathbf{R}^n)]$. The following conditions are equivalent:*

(i) *$(U_x \otimes V_y)\varphi(x+y) \in \mathscr{D}'_{L^1}(\mathbf{R}^{2n})$ for $\varphi \in \mathscr{D}(\mathbf{R}^n)$ $[\varphi \in \mathscr{S}(\mathbf{R}^n)]$.*

(ii) *$U(\tilde{V} * \varphi) \in \mathscr{D}'_{L^1}(\mathbf{R}^n)$ for $\varphi \in \mathscr{D}(\mathbf{R}^n)$ $[\varphi \in \mathscr{S}(\mathbf{R}^n)]$.*

(iii) *$(\tilde{U} * \varphi) V \in \mathscr{D}'_{L^1}(\mathbf{R}^n)$ for $\varphi \in \mathscr{D}(\mathbf{R}^n)$ $[\varphi \in \mathscr{S}(\mathbf{R}^n)]$.*

(iv) *$(U * \varphi)(x)(\tilde{V} * \psi)(x) \in L^1(\mathbf{R}^n)$ for $\varphi, \psi \in \mathscr{D}(\mathbf{R}^n)$ $[\varphi, \psi \in \mathscr{S}(\mathbf{R}^n)]$.*

If one of these conditions holds then

$$(U_x \otimes V_y)\varphi(x+y)[1] = U(\tilde{V} * \varphi)[1] = (\tilde{U} * \varphi)V[1],$$
$$\int (U * \varphi)(\tilde{V} * \psi) = (U_x \otimes V_y)(\tilde{\varphi} * \psi)(x+y)[1]$$

for $\varphi, \psi \in \mathscr{D}(\mathbf{R}^n)$ $[\varphi, \psi \in \mathscr{S}(\mathbf{R}^n)]$.

Theorems 1 and 2 imply the following

Corollary 1. *Let $U, V \in \mathscr{D}'(\mathbf{R}^n)$ $[U, V \in \mathscr{S}'(\mathbf{R}^n)]$. The following conditions are equivalent:*

(i) *$(U_x \otimes V_y)\varphi(x+y) \in \mathscr{D}'_{L^1}(\mathbf{R}^{2n})$ for $\varphi \in \mathscr{D}(\mathbf{R}^n)$ $[\varphi \in \mathscr{S}(\mathbf{R}^n)]$.*

(ii) *The limit $\lim_\nu (U_x \otimes V_y)\varphi(x+y)[\eta_\nu(x, y)]$ exists and is finite for every sequence η_ν converging to unity in $\mathbf{R}^{2n}$ (in Schwartz sense) and $\varphi \in \mathscr{D}(\mathbf{R}^n)$ $[\varphi \in \mathscr{S}(\mathbf{R}^n)]$.*

(iii) *The limit $\lim_\nu (U_x \otimes V_y)[\varphi(x+y)\alpha_\nu(x)]$ exists and is finite for every sequence α_ν converging to unity in $\mathbf{R}^n_x$ and $\varphi \in \mathscr{D}(\mathbf{R}^n)$ $[\varphi \in \mathscr{S}(\mathbf{R}^n)]$.*

(iv) *The limit $\lim_\nu (U_x \otimes V_y)[\varphi(x+y)\beta_\nu(y)]$ exists and is finite for every sequence β_ν converging to unity in $\mathbf{R}^n_y$ and $\varphi \in \mathscr{D}(\mathbf{R}^n)$ $[\varphi \in \mathscr{S}(\mathbf{R}^n)]$.*

(v) *The limit $\lim_\nu (U_x \otimes V_y)\varphi(x+y)[\alpha_\nu(x)\beta_\nu(y)]$ exists and is finite for all sequences α_ν, β_ν converging to unity in $\mathbf{R}^n_x$, $\mathbf{R}^n_y$ respectively and $\varphi \in \mathscr{D}(\mathbf{R}^n)$ $[\varphi \in \mathscr{S}(\mathbf{R}^n)]$.*

If one of these conditions holds then all the limits equal $(U_x \otimes V_y)\varphi(x+y)[1]$.

Definition 2 (see [10], [11]). Let $U, V \in \mathscr{D}'(\mathbf{R}^n)$. The *convolution* $U * V$ of U, V is the distribution

$$(U * V)[\varphi] = \lim_\nu (U_x \otimes V_y)\varphi(x+y)[\eta_\nu(x, y)]$$

if the limit exists and is finite for every sequence η_ν converging to unity in $\mathbf{R}^{2n}$ and every function $\varphi \in \mathscr{D}(\mathbf{R}^n)$. (The Banach–Steinhaus theorem implies that $U * V$ is then a distribution.)

Remark 1. Convolution can also be defined as one of the limits (i)–(v) from Corollary 1, or as the value on unity of one of the distributions (ii)–(iv) from Theorem 2 (in the case $\mathcal{D}(\boldsymbol{R}^n)$). We can also equivalently define $(U*V)[\varphi*\psi]$ as $\int(U*\tilde{\varphi})(x)(\tilde{V}*\varphi)(x)\,dx$.

If $U, V\in\mathcal{S}'(\boldsymbol{R}^n)$, the existence of the convolution $U*V$ does not imply that $U*V$ belongs to $\mathcal{S}'(\boldsymbol{R}^n)$ (see [2]). Because of this the notion of $\mathcal{S}'$-convolution has been introduced (see e.g. [2], [13]).

Definition 3. Let $U, V\in\mathcal{S}'(\boldsymbol{R}^n)$. The *$\mathcal{S}'$-convolution* $U\circledast V$ of U, V is the tempered distribution

$$(U\circledast V)[\sigma] = \lim_{\nu}(U_x\otimes V_y)\,\sigma(x+y)\,[\eta_\nu(x, y)]$$

if the limit exists and is finite for every sequence η_ν converging to unity in $\boldsymbol{R}^{2n}$ and every $\sigma\in\mathcal{S}(\boldsymbol{R}^n)$. (Obviously the limit is a tempered distribution.)

Remark 2. $\mathcal{S}'$-convolution can be equivalently defined as described in Remark 1.

Definition 4 (see [5]). Let $T, S\in\mathcal{D}'(\boldsymbol{R}^n)$. We say that the *product* $T\circ S$ of T, S exists if the distributional limit $\lim_\nu(T*\varrho_\nu)S$ exists for every sequence ϱ_ν with the following properties:

(1) $\varrho_\nu\in\mathcal{D}(\boldsymbol{R}^n)$ for $\nu = 1, 2, \ldots$

(2) $\operatorname{supp}\varrho_\nu\to 0$ as $\nu\to\infty$.

(3) $\int\varrho_\nu = 1$ for $\nu = 1, 2, \ldots$

(4) $\varrho_\nu\geqslant 0$ for $\nu = 1, 2, \ldots$

It may be shown (see [5]) that this condition is equivalent to the existence of the distributional limit $\lim_\nu T(S*\varrho_\nu)$ or the limit $\lim_\nu(T*\varrho'_\nu)(S*\varrho''_\nu)$ for all sequences ϱ'_ν, ϱ''_ν satisfying conditions (1)–(4).

Theorem 3 (see [3], [5]). *If $U, V\in\mathcal{S}'(\boldsymbol{R}^n)$ and the $\mathcal{S}'$-convolution $U\circledast V$ exists then the product $\hat{U}\circ\hat{V}$ exists and*

$$(U\circledast V)\hat{} = (2\pi)^{n/2}\,\hat{U}\circ\hat{V}.$$

The same is true for the inverse Fourier transform.

Now we say a few words about the definitions of improper integrals in the sense of Sikorski [15] and Musielak [9].

For $\eta, \xi\in\boldsymbol{R}^n$, $\eta<\xi$, we write $\Omega_\eta^\xi = \{x\in\boldsymbol{R}^n\colon \eta\leqslant x\leqslant\xi\}$.

Definition 5 (Musielak). Let $T\in\mathcal{D}'(\boldsymbol{R}^n)$. The distribution $T*\tilde{\chi}_{\Omega_\eta^\xi}$ will be called the *integral* of T over Ω_η^ξ. We denote it by $\int_\eta^\xi T$. ($\chi_{\Omega_\eta^\xi}$ is the characteristic function of the set Ω_η^ξ.)

Definition 6 (Musielak). Let $T\in\mathcal{D}'(\boldsymbol{R}^n)$. The *improper integral* of T over $\boldsymbol{R}^n$ is the distributional limit

$$\Big(\int_{\boldsymbol{R}^n} T\Big)[\varphi] = \lim_{\substack{\eta\to-\infty\\ \xi\to+\infty}}\Big(\int_\eta^\xi T\Big)[\varphi]$$

if the limit exists and is finite for $\varphi\in\mathcal{D}(\boldsymbol{R}^n)$. ($\eta\to-\infty$, $\xi\to+\infty$ means: $\eta_1\to-\infty, \ldots, \eta_n\to-\infty$, $\xi_1\to+\infty, \ldots, \xi_n\to+\infty$.) The limit is then a constant distribution.

The Sikorski definition of improper integral of T over $\boldsymbol{R}^n$ is more difficult to describe. As mentioned in [9], in the case of $\boldsymbol{R}^n$ that definition is equivalent to Definition 6, so we will not quote it.

We notice that for all $\eta, \xi\in\boldsymbol{R}^n$, $\eta<\xi$, $\varphi\in\mathcal{D}(\boldsymbol{R}^n)$,

$$\Big(\int_\eta^\xi T\Big)[\varphi] = (T*\tilde{\chi}_{\Omega_\eta^\xi})[\varphi] = \int_{\Omega_\eta^\xi}(T*\tilde{\varphi})$$

and the last integral is the Lebesgue integral of the function $T*\tilde{\varphi}$.

Definition 7. Let P_ν be a sequence of compact intervals in $\boldsymbol{R}^n$. We say that the sequence is *absorbing* if for every compact $K\subset\boldsymbol{R}^n$ there exists $N\in N_0$ such that $K\subset P_\nu$ for $\nu\geqslant N$.

So we have

Remark 3. The existence of the Musielak improper integral of a distribution T is equivalent to the following condition: $\lim_\nu\int_{P_\nu}(T*\varphi)$ exists and is finite for every $\varphi\in\mathcal{D}(\boldsymbol{R}^n)$ and every absorbing sequence P_ν.

Remark 4. If $T\in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$ and the classical improper integral of the function T exists (i.e. $\int_\eta^\xi T(x_1, \ldots, x_n)\,dx_1\ldots dx_n$ has a limit as $\eta\to-\infty$, $\xi\to\infty$) then the Musielak improper integral of T exists and both integrals coincide.

1. Spaces of functions with bounded variation. We now define three subspaces of the space $C^\infty(\boldsymbol{R}^n)$. Their definitions are connected with the notion of function with bounded variation in the case $n = 1$. First we define, for every compact interval P and every $\lambda\in C_0^\infty(\boldsymbol{R}^n)$, the following sequence:

$$l_k^P(\lambda) = \sum_{|\alpha|\leqslant k}\sum_{\beta\leqslant 1}\sup_{x_{1-\beta}}\int_P|D^{\alpha+\beta}\lambda(x)|\,dx_\beta, \qquad k = 0, 1, \ldots,$$

with the obvious meaning of the terms corresponding to $\beta = 0 = (0, \ldots 0)$ and $\beta = \boldsymbol{1}$. We also define a sequence $l_k(\lambda)$ in the same way, replacing P by $\boldsymbol{R}^n$.

The sequence l_k is an increasing sequence of seminorms. We define the following topological vector spaces:

$$\mathcal{D}_L(\boldsymbol{R}^n) = \{\lambda\in C^\infty(\boldsymbol{R}^n)\colon l_k(\lambda) < +\infty \text{ for } k\in N_0\}$$

with topology given by the sequence of seminorms l_k;

$\mathcal{D}_{L_0}(\boldsymbol{R}^n) = \{\lambda\in C^\infty(\boldsymbol{R}^n)$: for every $\varepsilon>0$ and $k\in N_0$ there exists $N>0$ such that for every compact interval P such that $\langle -N, N\rangle^n\cap P = \emptyset$, $l_k^P(\lambda) < \varepsilon\}$

equipped with the same sequence of seminorms;

$$\mathcal{D}_{\bar{L}}(\boldsymbol{R}^n) = \{\lambda\in C^\infty(\boldsymbol{R}^n)\colon l_k(\lambda) < +\infty \text{ for } k\in N_0\}$$

with the following topology: $\lambda_\nu \to 0$ in $\mathcal{D}_{\bar{L}}(\boldsymbol{R}^n)$ as $\nu \to \infty$ if

(i) $l_k(\lambda_\nu) \leqslant C_k < +\infty$ for $k = 0, 1, \ldots, \nu = 1, 2, \ldots$

(ii) $D^\alpha \lambda_\nu \to 0$ almost uniformly as $\nu \to \infty$ for $\alpha \in N_0^n$.

The definitions imply that $\mathcal{D}_{L_0} \subset \mathcal{D}_L$ and the mappings $\mathcal{D}_{L_0} \hookrightarrow \mathcal{D}_L$ and id: $\mathcal{D}_L \to \mathcal{D}_{\bar{L}}$ are continuous.

Unity belongs to $\mathcal{D}_L$ and $\mathcal{D}_{\bar{L}}$ but not to $\mathcal{D}_{L_0}$.

We notice that for $\lambda \in \mathcal{D}_L$ $[\mathcal{D}_{L_0}, \mathcal{D}_{\bar{L}}]$

$$l_k(\lambda) = \sup_P l_k^P(\lambda)$$

(we take the supremum over all compact intervals).

It is not difficult to show that the spaces $\mathcal{D}_L$, $\mathcal{D}_{L_0}$, $\mathcal{D}_{\bar{L}}$ are complete.

Proposition 1. *If $\lambda \in \mathcal{D}_L$ $[\mathcal{D}_{L_0}, \mathcal{D}_{\bar{L}}]$, $x_0 \in \boldsymbol{R}^n$ then $\lambda(\cdot + x_0) \in \mathcal{D}_L$ $[\mathcal{D}_{L_0}, \mathcal{D}_{\bar{L}}]$. If $\lambda \in \mathcal{D}_L$ $[\mathcal{D}_{L_0}, \mathcal{D}_{\bar{L}}]$ then $D^\alpha \lambda \in \mathcal{D}_L$ $[\mathcal{D}_{L_0}, \mathcal{D}_{\bar{L}}]$ for $\alpha \in N_0^n$. Both mappings are continuous.*

Proposition 2. *If $\lambda \in \mathcal{D}_L$ $[\mathcal{D}_{L_0}, \mathcal{D}_{\bar{L}}]$, $\varphi \in \mathcal{D}$ then $\lambda * \varphi \in \mathcal{D}_L$ $[\mathcal{D}_{L_0}, \mathcal{D}_{\bar{L}}]$ and the mapping $\lambda \mapsto \lambda * \varphi$ is continuous. We have the following inequality:*

$$l_k(\lambda * \varphi) \leqslant \|\varphi\|_{L^1} l_k(\lambda) \quad \text{for } k = 0, 1, \ldots$$

Proposition 3. *If $\lambda_1 \in \mathcal{D}_L(\boldsymbol{R}^{n_1})$, $\lambda_2 \in \mathcal{D}_L(\boldsymbol{R}^{n_2})$ then the function $\lambda = \lambda_1 \otimes \lambda_2$ belongs to $\mathcal{D}_L(\boldsymbol{R}^{n_1+n_2})$ and $l_k(\lambda) \leqslant l_k(\lambda_1) l_k(\lambda_2)$ for $k = 0, 1, \ldots$ The mapping*

$$\mathcal{D}_L(\boldsymbol{R}^{n_1}) \times \mathcal{D}_L(\boldsymbol{R}^{n_2}) \ni (\lambda_1, \lambda_2) \mapsto \lambda_1 \otimes \lambda_2 \in \mathcal{D}_L(\boldsymbol{R}^{n_1+n_2})$$

is continuous.

The same is true for $\mathcal{D}_{L_0}$ and $\mathcal{D}_{\bar{L}}$.

Proposition 4. *$\mathcal{D} \subset \mathcal{D}_L$ and $\mathcal{S} \subset \mathcal{D}_L$ and both inclusions are continuous. The same is true for $\mathcal{D}_{L_0}$ and $\mathcal{D}_{\bar{L}}$.*

Definition 1. We say that a sequence η_ν is *L-convergent to unity* in $\boldsymbol{R}^n$ if

(i) $\eta_\nu \in C_0^\infty(\boldsymbol{R}^n)$ for $\nu = 1, 2, \ldots$

(ii) $l_k(\eta_\nu) \leqslant C_k < +\infty$ for $k = 0, 1, \ldots, \nu = 1, 2, \ldots$

(iii) For every compact $K \subset \boldsymbol{R}^n$ there exists $N \in N_0$ such that $\eta_\nu(x) = 1$ for $\nu > N$, $x \in K$.

Remark 1. If sequences α_ν, β_ν are L-convergent to unity in $\boldsymbol{R}^{n_1}$ and $\boldsymbol{R}^{n_2}$ respectively then the sequence $\eta_\nu = \alpha_\nu \otimes \beta_\nu$ is L-convergent to unity in $\boldsymbol{R}^{n_1+n_2}$.

Remark 2. Let $\varrho \in \mathcal{D}(\boldsymbol{R}^1)$, $\int \varrho = 1$. The sequence of functions $\eta_\nu = \chi_\nu * \varrho$ is L-convergent to unity in $\boldsymbol{R}^1$, where χ_ν is the characteristic function of the interval $\langle -\nu, \nu \rangle$. The sequence $\varkappa_\nu(x_1, \ldots, x_n) = \eta_\nu(x_1) \ldots \eta_\nu(x_n)$ is L-convergent to unity in $\boldsymbol{R}^n$.

Proposition 5. *$\mathcal{D}$ is dense in $\mathcal{D}_{L_0}$ and in $\mathcal{D}_{\bar{L}}$, but not in $\mathcal{D}_L$.*

Now we show the connection between the spaces just defined and the notion of function with bounded variation.

Let $f\colon \boldsymbol{R}^1 \to \boldsymbol{C}$, $a, b \in \boldsymbol{R}^1$, $a < b$. The *variation* of f on the interval $\langle a, b \rangle$ is the number (finite or infinite)

$$W_a^b(f) = \sup_{a = x_0 < \ldots < x_n = b} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|.$$

The variation of f on the real line is defined as $W(f) = \sup_{a<b} W_a^b(f)$. It is easily seen that $\mathcal{D}_L(\boldsymbol{R}^1)$ is the set of all functions $f \in C^\infty(\boldsymbol{R}^1)$ with all derivatives with bounded variation on $\boldsymbol{R}^1$. We have

$$l_0(\lambda) = \sup|\lambda| + W(\lambda) \quad \text{for } \lambda \in \mathcal{D}_L(\boldsymbol{R}^1) \quad [\mathcal{D}_{L_0}(\boldsymbol{R}^1), \mathcal{D}_{\bar{L}}(\boldsymbol{R}^1)].$$

2. Improper integrals. We now define improper integrals of functions and distributions.

Definition 1. A function $f \in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$ is *improper integrable* (over $\boldsymbol{R}^n$) if for every $\varepsilon > 0$ there exists $R > 0$ such that for every compact interval P with $P \cap \langle -R, R \rangle^n = \emptyset$, we have $\left|\int_P f\right| < \varepsilon$.

Remark 1. If a function f is improper integrable over $\boldsymbol{R}^n$ then the limit $\lim_\nu \int_{P_\nu} f$ exists for every absorbing sequence P_ν. The limit is finite and independent of the choice of such a sequence. We denote it by $\int_{\boldsymbol{R}^n} f$ or $\int f$. If P is an interval in $\boldsymbol{R}^n$ then the limit $\lim_\nu \int_{P \cap P_\nu} f$ exists, is finite and independent of the choice of P_ν. We write

$$\int_P f = \lim_\nu \int_{P \cap P_\nu} f.$$

Remark 2. Let $f \in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$, $n > 1$. The condition "For every interval P in $\boldsymbol{R}^n$ and for any absorbing sequence P_ν the limit $\lim_\nu \int_{P \cap P_\nu} f$ exists and is finite" does not imply that f is improper integrable:

Example 1.

$$f(x_1, \ldots, x_n) = a(x_1) \ldots a(x_{n-1}) b(x_n) + \sum_{k=1}^{\infty} a(2^{k/(n-1)} x_1) \ldots a(2^{k/(n-1)} x_{n-1}) b\Big(x_n/2^k - \sum_{l=0}^{k-1} 2^l\Big)$$

where $a(t) = -Y(t) + 2Y(t-1) - Y(t-2)$, $b(t) = Y(t) - Y(t-1)$ for $t \in \boldsymbol{R}^1$, and $Y(t)$ is the Heaviside function.

Remark 3. Let $f \in L^1_{\mathrm{loc}}(\boldsymbol{R}^1)$. If $\lim_\nu \int_{P_\nu} f$ exists and is finite for any absorbing sequence P_ν then f is improper integrable, i.e. in the case $n = 1$, Definition 1 and the classical definition of improper integral are equivalent.

Definition 2. Let $f \in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$. We say that f *has bounded improper integrals* if there exists $M > 0$ such that $\left|\int_P f\right| \leqslant M$ for every compact interval P in $\boldsymbol{R}^n$.

Remark 4. Every L^1 function is improper integrable. Every improper integrable function has bounded improper integrals. Both those inclusions between the three function classes are strict.

EXAMPLE 2.

$$f(x_1, \ldots, x_n) = (\sin x_1/x_1) \ldots (\sin x_n/x_n),$$
$$g(x_1, \ldots, x_n) = \sin x_1 \ldots \sin x_n.$$

The function f is improper integrable but it does not belong to L^1. The function g is not improper integrable and has bounded improper integrals.

Now we consider the dual spaces of $\mathcal{D}_{L_0}$ and $\mathcal{D}_L$. We denote them by $\mathcal{D}'_{L_0}$ and $\mathcal{D}'_L$. Since $\mathcal{D}$ is dense in $\mathcal{D}_{L_0}$ and in $\mathcal{D}_L$ both dual spaces are spaces of distributions. Proposition 1.4 implies that they are spaces of tempered distributions.

DEFINITION 3. We say that a distribution $T \in \mathcal{D}'(\boldsymbol{R}^n)$ is *improper integrable* if $\lim_\nu T[\eta_\nu]$ exists and is finite for any sequence η_ν L-convergent to unity in $\boldsymbol{R}^n$.

The definition implies that the limit does not depend on the choice of the sequence η_ν. We denote it by $T[1]$ or $\int T$.

Now we prove the main theorems of this part of the paper. First we cite the Banach–Steinhaus theorem in a form convenient for us.

THEOREM 1 (see [16], p. 13). *Let $\mathcal{M}$ be a vector space with convergence defined by an increasing sequence q_k of seminorms and suppose that $\mathcal{M}$ is complete. If $\{\Phi_t\}_{t\in T}$ is a family of continuous linear functionals on $\mathcal{M}$ such that $|\Phi_t[\xi]| < \varrho(\xi) < +\infty$ for $t \in T$, then there exist $C > 0$, $k \in N_0$ such that*

$$|\Phi_t[\xi]| \leqslant Cq_k(\xi) \quad \textit{for } \xi \in \mathcal{M},\ t \in T.$$

THEOREM 2. *Let $T \in \mathcal{D}'$. The following conditions are equivalent*:

(i) $T \in \mathcal{D}'_L$.

(ii) *$T * \varphi$ is an improper integrable function for $\varphi \in \mathcal{D}$.*

(iii) *$T = \sum_{|\alpha| \leqslant r} D^\alpha f_\alpha$, where f_α are improper integrable functions.*

(iv) *T is an improper integrable distribution.*

THEOREM 3. *Let $T \in \mathcal{D}'$. The following conditions are equivalent*:

(i) $T \in \mathcal{D}'_{L_0}$.

(ii) *$T * \varphi$ has bounded improper integrals for $\varphi \in \mathcal{D}$.*

(iii) *$T = \sum_{|\alpha| \leqslant r} D^\alpha f_\alpha$, where f_α are functions having bounded improper integrals.*

To prove both theorems we use the following

LEMMA 1. *If P is a compact interval and $\varphi \in \mathcal{D}(\boldsymbol{R}^n)$ then $\chi_P * \varphi$ belongs to $\mathcal{D}(\boldsymbol{R}^n)$ and*

$$l_k(\chi_P * \varphi) \leqslant C_k(\varphi) < +\infty \quad \textit{for } k = 0, 1, \ldots$$

where the constant $C_k(\varphi)$ does not depend on the interval P.

If $\sigma \in \mathcal{S}(\boldsymbol{R}^n)$ then $\chi_P * \sigma \in \mathcal{S}(\boldsymbol{R}^n)$ and the same inequality holds.

The proof of the lemma is left to the reader.

Proof of Theorem 2. (i)$\Rightarrow$(ii). It is enough to show that $\int_{P_\nu}(T * \varphi) \to 0$ as $\nu \to \infty$ for every sequence of compact intervals P_ν with $P_\nu \cap \langle -\nu, \nu\rangle^n = \emptyset$ for $\nu = 1, 2, \ldots$ We notice that

$$\int_{P_\nu} (T * \varphi) = T[\tilde{\varphi} * \chi_{P_\nu}].$$

By Lemma 1, $\tilde{\varphi} * \chi_{P_\nu} \to 0$ in $\mathcal{D}_L$ as $\nu \to \infty$. Since $T \in \mathcal{D}'_L$ the assertion follows.

(ii) $\Rightarrow$ (iii).

LEMMA 2. *If a distribution T satisfies condition* (ii) *of Theorem* 3 *then $T \in \mathcal{S}'(\boldsymbol{R}^n)$, so T is a distribution of finite order.*

Proof of Lemma 2. The assumption implies that for every function $\varphi \in \mathcal{D}$ the function

$$F(x) = \int_0^{x_1} \ldots \int_0^{x_n} (T * \varphi)(t_1, \ldots, t_n)\, dt_n \ldots dt_1 \quad \text{for } x \in \boldsymbol{R}^n$$

is bounded, so $T * \varphi \in \mathcal{S}'_n(\boldsymbol{R}^n) \subset \mathcal{S}'(\boldsymbol{R}^n)$. To finish the proof we use the following known fact: If $T \in \mathcal{D}'(\boldsymbol{R}^n)$ and $T * \varphi \in \mathcal{S}'(\boldsymbol{R}^n)$ for $\varphi \in \mathcal{D}(\boldsymbol{R}^n)$ then $T \in \mathcal{S}'(\boldsymbol{R}^n)$.

Now we show the implication (ii) $\Rightarrow$ (iii). We fix a compact $K = \overline{B(0, r)} \subset \boldsymbol{R}^n$. For every function $\varphi \in \mathcal{D}(K)$ there exists $\varrho(\varphi)$ such that

$$\Big|\int_P (T * \varphi)\Big| \leqslant \varrho(\varphi) < +\infty \quad \text{for every compact interval } P.$$

This is a consequence of condition (ii) and Remark 4. Moreover, $\int_P (T * \cdot) \in \mathcal{D}'(K)$ for every such P. So by the Banach–Steinhaus theorem there exist $C > 0$, $k \in N_0$ such that

$$\Big|\int_P (T * \varphi)\Big| \leqslant C d_k(\varphi) \quad \text{for } \varphi \in \mathcal{D}(K) \text{ and every } P.$$

Hence the functionals $\int_P (T * \cdot)$ on $\mathcal{D}(K)$ can be uniquely extended to functionals on $\mathcal{D}_k(K)$. Since T is a distribution of finite order, the number k can be chosen as large as is necessary to make the distribution $T * \gamma$ a continuous function for $\gamma \in \mathcal{D}_k(K)$. So we have

$$\Big|\int_P (T * \gamma)\Big| \leqslant C d_k(\gamma) \quad \text{for } \gamma \in \mathcal{D}_k(K) \text{ and every } P.$$

Now we show that $T*\gamma$ is an improper integrable function for $\gamma\in\mathcal{D}_k(K)$. Let φ_ν be a sequence of elements of $C_0^\infty(K)$ tending to γ in $\mathcal{D}_k(K)$. We fix $\varepsilon>0$. We notice that

$$\int_P (T*\gamma)=\int_P (T*\varphi_\nu)+\int_P \big(T*(\gamma-\varphi_\nu)\big) \quad \text{for } \nu=1,2,\ldots \text{ and every } P.$$

There exists $\nu_0\in N_0$ such that $\left|\int_P \big(T*(\gamma-\varphi_{\nu_0})\big)\right|<\varepsilon/2$ for every P. Condition (ii) implies the existence of $N>0$ such that for any compact interval P with $P\cap\langle -N,N\rangle^n=\emptyset$, $\left|\int_P (T*\varphi_{\nu_0})\right|<\varepsilon/2$. So $T*\gamma$ is an improper integrable function for $\gamma\in\mathcal{D}_k(K)$.

Now we take the function $\gamma=E^m\psi$, where E^m is a fundamental solution of Δ^m for m so large that $E^m\in C^k(\boldsymbol{R}^n)$ and $\psi\in\mathcal{D}(K)$, $\psi=1$ in a neighbourhood of the origin. So $T=\Delta^m f-T*\varrho$ for some improper integrable function f and some $\varrho\in C_0^\infty(K)$. Hence T satisfies condition (iii).

(iii) $\Rightarrow$ (i). Since $\mathcal{D}$ is dense in $\mathcal{D}_{\bar L}$, it is enough to show (see [16], Th. 4) that every distribution T satisfying condition (iii) has the following property: if a sequence ψ_ν of elements of $\mathcal{D}$ tends to 0 in $\mathcal{D}_{\bar L}$, then $T[\psi_\nu]\to 0$ as $\nu\to\infty$.

Let ψ_ν be such a sequence. Then

$$T[\psi_\nu]=\sum_{|\alpha|\leqslant r}(-1)^{|\alpha|}\int f_\alpha D^\alpha\psi_\nu.$$

We fix $\alpha\in N_0^n$ and write for simplicity $f=f_\alpha$, $\lambda_\nu=D^\alpha\psi_\nu$. Obviously $\lambda_\nu\in\mathcal{D}$ for $\nu=1,2,\ldots$ and $\lambda_\nu\to 0$ in $\mathcal{D}_{\bar L}$ as $\nu\to\infty$. We define

$$F(x)=\int_0^{x_1}\ldots\int_0^{x_n} f(t_1,\ldots,t_n)\,dt_n\ldots dt_1 \quad \text{for } x\in\boldsymbol{R}^n.$$

Since $f\in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$ we see that $D^1F=f$ in the distributional sense.

Let $\varepsilon>0$. We write for simplicity $Q_M=\langle -M,M\rangle^n$ for $M>0$ and fix $C>0$ satisfying the condition $\|D^1\lambda_\nu\|_{L^1}\leqslant C$ for $\nu=1,2,\ldots$ Since f is an improper integrable function, there exists $R>0$ such that $\left|\int_P f\right|<\varepsilon/(4nC)$ for every compact interval P with $P\cap Q_R=\emptyset$.

We define, for $x\in\boldsymbol{R}^n\setminus Q_R$, the following function H:

$$H(x)=H(x_1,\ldots,x_n)=\int_0^{\bar x_1}\ldots\int_0^{\bar x_n} f(t_1,\ldots,t_n)\,dt_n\ldots dt_1,$$

where $\bar x_i=\operatorname{sgn}x_i\cdot\min(R,|x_i|)$ for $i=1,\ldots,n$.

It is not difficult to see that $|F(x)-H(x)|<\varepsilon/(4C)$ for $x\in\boldsymbol{R}^n\setminus Q_R$. Since H locally depends only on at most $n-1$ variables, $D^1H=0$. So $D^1(F-H)=f$ in $\mathcal{D}'(\boldsymbol{R}^n\setminus Q_R)$.

We choose a function $\eta\in C_0^\infty(Q_{R+2})$, $|\eta|\leqslant 1$ and $\eta=1$ on Q_{R+1}. Then

$$\begin{aligned}\int f\lambda_\nu&=\int f\lambda_\nu\eta+\int f\lambda_\nu(1-\eta)\\&=\int f\lambda_\nu\eta+(-1)^n\int (F-H)D^1\big((1-\eta)\lambda_\nu\big)\\&=\int_{Q_{R+2}}\Big(f\lambda_\nu\eta+(-1)^n\sum_{0\leqslant\beta<1}(F-H)D^\beta\lambda_\nu D^{1-\beta}(1-\eta)\Big)\\&\quad+(-1)^n\int_{\boldsymbol{R}^n\setminus Q_R}(F-H)D^1\lambda_\nu(1-\eta).\end{aligned}$$

The first integral is not larger than $\varepsilon/2$ for ν large enough, because $\lambda_\nu\to 0$ in $\mathcal{D}_{\bar L}$ as $\nu\to\infty$, the set Q_{R+2} is bounded, f is locally integrable and $F-H$ is bounded. Since $|F(x)-H(x)|<\varepsilon/(4C)$ for $x\in\boldsymbol{R}^n\setminus Q_R$, the second integral can be estimated by $C\sup|1-\eta|\,\varepsilon/(4C)\leqslant\varepsilon/2$. Thus $\int f\lambda_\nu\to 0$ as $\nu\to\infty$ and the implication (iii) $\Rightarrow$ (i) is proved.

(i) $\Rightarrow$ (iv). Since every sequence which L-converges to unity in $\boldsymbol{R}^n$ tends to unity in $\mathcal{D}_{\bar L}$, this implication is trivial.

(iv) $\Rightarrow$ (i). Since $\mathcal{D}$ is dense in $\mathcal{D}_{\bar L}$ it is enough to show an extension of the distribution T to a linear continuous functional on $\mathcal{D}_{\bar L}$, or to prove, like in the proof of the implication (iii) $\Rightarrow$ (i), that if a sequence ψ_ν of elements of $\mathcal{D}$ tends to 0 in $\mathcal{D}_{\bar L}$ then $T[\psi_\nu]\to 0$ as $\nu\to\infty$. We leave the simple proof to the reader.

The proof of Theorem 2 is finished.

We omit the similar proof of Theorem 3.

Remark 5. If $T\in\mathcal{D}'_{\bar L}$ $[\mathcal{D}'_{L_0}]$ then $T*\sigma$ is an improper integrable function [a function with bounded improper integrals] for $\sigma\in\mathcal{S}$.

Remark 6. If T is an improper integrable function then it is an improper integrable distribution and the value $\int T$ in the sense of Definition 3 equals the value $\int T$ in the sense of Definition 1 and Remark 1.

COROLLARY 1. $\mathcal{D}'_{L^1}\subsetneq\mathcal{D}'_{\bar L}\subsetneq\mathcal{D}'_{L_0}$.

Proof. The inclusions $\mathcal{D}'_{L_1}\subset\mathcal{D}'_{\bar L}\subset\mathcal{D}'_{L_0}$ are simple consequences of condition (ii) of Theorems 0.1, 2 and 3 and Remark 4. The functions f, g from Example 2, considered as distributions, show that the inclusions are strict.

The spaces $\mathcal{D}_L$, $\mathcal{D}_{\bar L}$, $\mathcal{D}_{L_0}$ are in some way analogous to the spaces B, $\bar B$, $\dot B$ respectively. We obtain a representation theorem for the dual space $\mathcal{D}'_{\bar L}$ (Theorem 2) similar to the case of $\bar B$. There is an important difference in the case of $\mathcal{D}'_{L_0}$, because $\bar B'=\dot B'$ and $\mathcal{D}'_{\bar L}\neq\mathcal{D}'_{L_0}$. The reason is the following: the seminorms d_k have the property $d_k(\varphi+\psi)\leqslant\max\big(d_k(\varphi),d_k(\psi)\big)$ if $\operatorname{supp}\varphi\cap\operatorname{supp}\psi=\emptyset$. This is not true for the seminorms l_k (see [1], Property (III), p. 67).

Theorem 2 and Remarks 1 and 0.3 imply the following

Remark 7. If a distribution is improper integrable in the sense of Definition 1 then it is improper integrable in Musielak–Sikorski sense.

It is not difficult to see that in the case $n = 1$ both definitions are equivalent.

In the case $n > 1$ there exists a distribution T which is improper integrable in Musielak–Sikorski sense, but not in the sense of Definition 1:

EXAMPLE 3.

$$T(x_1, \ldots, x_n) = 1_{x_1} \otimes \ldots \otimes 1_{x_{n-1}} \otimes \big(Y(x_n+1) - 2Y(x_n) + Y(x_n - 1)\big).$$

Remark 8. It is not possible to prove a representation theorem for improper integrable distributions in Musielak–Sikorski sense, i.e. to prove that every such distribution is of the form $\sum_{|\alpha| \leq r} D^\alpha f_\alpha, f_\alpha \in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$, with additional conditions of integrability for f_α. In fact, an improper integrable distribution in Musielak–Sikorski sense need not be of finite order:

EXAMPLE 4 ($n = 2$).

$$T(x_1, x_2) = \big(Y(x_1+1) - 2Y(x_1) + Y(x_1 - 1)\big) \otimes \sum_{i=0}^{\infty} Y^{(i)}(x_2 - i).$$

3. Convolution of distributions. In this part of the paper we define the convolution of distributions, using the already defined notion of improper integral for distributions. This convolution will be called the improper convolution. It is a generalization of the classical convolution preserving all the properties which are required for this operation.

The following theorem analogous to Theorem 0.2 may be proved:

THEOREM 1. *Let $U, V \in \mathscr{D}'(\boldsymbol{R}^n)$. The following conditions are equivalent:*

(i) $(U_x \otimes V_y)\varphi(x+y) \in \mathscr{D}'_L(\boldsymbol{R}^{2n})$ *for* $\varphi \in \mathscr{D}(\boldsymbol{R}^n)$.

(ii) $U(\tilde{V} * \varphi) \in \mathscr{D}'_L(\boldsymbol{R}^n)$ *for* $\varphi \in \mathscr{D}(\boldsymbol{R}^n)$.

(iii) $(\tilde{U} * \varphi) V \in \mathscr{D}'_L(\boldsymbol{R}^n)$ *for* $\varphi \in \mathscr{D}(\boldsymbol{R}^n)$.

(iv) $(U * \varphi)(x)(\tilde{V} * \psi)(x)$ *is an improper integrable function for* $\varphi, \psi \in \mathscr{D}(\boldsymbol{R}^n)$.

If one of these conditions holds then

$$(U_x \otimes V_y)\varphi(x+y)[1] = U(\tilde{V} * \varphi)[1] = (\tilde{U} * \varphi) V[1],$$
$$\int (U * \varphi)(\tilde{V} * \psi) = (U_x \otimes V_y)(\tilde{\varphi} * \psi)(x+y)[1]$$

for $\varphi, \psi \in \mathscr{D}(\boldsymbol{R}^n)$.

Theorems 2.2 and 1 imply

COROLLARY 1. *Let $U, V \in \mathscr{D}'(\boldsymbol{R}^n)$. The following conditions are equivalent:*

(i) $(U_x \otimes V_y)\varphi(x+y) \in \mathscr{D}'_L(\boldsymbol{R}^{2n})$ *for* $\varphi \in \mathscr{D}(\boldsymbol{R}^n)$.

(ii) *The limit* $\lim_\nu (U_x \otimes V_y)\varphi(x+y)[\eta_\nu(x, y)]$ *exists and is finite for every sequence* η_ν *L-convergent to unity in* $\boldsymbol{R}^{2n}$ *and* $\varphi \in \mathscr{D}(\boldsymbol{R}^n)$.

(iii) *The limit* $\lim_\nu (U_x \otimes V_y)[\varphi(x+y)\alpha_\nu(x)]$ *exists and is finite for every sequence* α_ν *L-convergent to unity in* $\boldsymbol{R}^n_x$ *and* $\varphi \in \mathscr{D}(\boldsymbol{R}^n)$.

(iv) *The limit* $\lim_\nu (U_x \otimes V_y)[\varphi(x+y)\beta_\nu(y)]$ *exists and is finite for every sequence* β_ν *L-convergent to unity in* $\boldsymbol{R}^n_y$ *and* $\varphi \in \mathscr{D}(\boldsymbol{R}^n)$.

(v) *The limit* $\lim_\nu (U_x \otimes V_y)\varphi(x+y)[\alpha_\nu(x)\beta_\nu(y)]$ *exists and is finite for all sequences* α_ν, β_ν *L-convergent to unity in* $\boldsymbol{R}^n_x, \boldsymbol{R}^n_y$ *respectively and* $\varphi \in \mathscr{D}(\boldsymbol{R}^n)$.

If one of these conditions holds then all the limits equal $(U_x \otimes V_y)\varphi(x+y)[1]$.

DEFINITION 1. Let $U, V \in \mathscr{D}'(\boldsymbol{R}^n)$. The *improper convolution* $U * V$ of U, V is the distribution

$$(U * V)[\varphi] = \lim_\nu (U_x \otimes V_y)\varphi(x+y)[\eta_\nu(x, y)]$$

if the limit exists and is finite for every sequence η_ν L-convergent to unity in $\boldsymbol{R}^{2n}$ and every function $\varphi \in \mathscr{D}(\boldsymbol{R}^n)$.

Remark 1. Improper convolution can also be defined as described in Remark 0.1.

Remark 2. Definition 1 is a generalization of Schwartz' definition of convolution. There exist pairs of distributions U, V which are convolvable in improper sense, but not in Schwartz sense:

EXAMPLE 1.

$$U(x) \equiv 1, \quad V(x) = (\sin x_1/x_1) \ldots (\sin x_n/x_n) \quad \text{for } x \in \boldsymbol{R}^n.$$

EXAMPLE 2.

$$U(x, t) = E_1(x, t) = \frac{1}{2c} Y(ct - |x|) \quad \text{for some constant } c > 0 \text{ and } x, t \in \boldsymbol{R}^1,$$

$$V(x, t) = \sin t/t^2 \quad \text{for } x, t \in \boldsymbol{R}^1, |t| \geq 1, \quad V(x, t) = 0 \quad \text{for } |t| < 1.$$

We notice that E_1 is a fundamental solution of the wave operator $\Box_1 = \partial^2/\partial t^2 - c^2 \partial^2/\partial x^2$ (see [16], § 31). So the improper convolution $E_1 * V$ is a solution of the equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = V.$$

Remark 3. If $U, V \in \mathscr{D}'(\boldsymbol{R}^n)$ and the improper convolution $U * V$ exists then all improper convolutions which appear below exist and

(a) $U * V = V * U$.

(b) $U * \big(V(\cdot + t)\big) = \big(U(\cdot + t)\big) * V = (U * V)(\cdot + t)$ for $t \in \boldsymbol{R}^n$.

(c) $(U * V)^{\sim} = \tilde{U} * \tilde{V}$.

(d) $(D^\alpha U) * V = U * (D^\alpha V) = D^\alpha (U * V)$ for $\alpha \in N^n_0$.

If $V \in \mathscr{D}'(\boldsymbol{R}^n)$ then the operation $U \mapsto U * V$ is linear on the set of all distributions U such that the improper convolution $U * V$ exists.

For the same reason as in the classical case we introduce the improper $\mathscr{S}'$-convolution of tempered distributions. First we note that Theorem 1 and Corollary 1 are still true if we replace the functions $\varphi, \psi \in \mathscr{D}$ by functions from $\mathscr{S}$ and assume that $U, V \in \mathscr{S}'(\boldsymbol{R}^n)$.

DEFINITION 2. The *improper $\mathscr{S}'$-convolution* of tempered distributions U, V is a tempered distribution $U \circledast V$ defined as follows:

$$(U \circledast V)[\sigma] = \lim_{\nu} (U_x \otimes V_y)\,\sigma(x+y)[\eta_\nu(x, y)]$$

if the limit exists and is finite for every function $\sigma \in \mathscr{S}(\boldsymbol{R}^n)$ and every sequence η_ν L-convergent to unity in $\boldsymbol{R}^{2n}$.

$\mathscr{S}'$-convolution can also be defined as described in Remark 0.1.

Obviously the improper $\mathscr{S}'$-convolution is a generalization of the classical $\mathscr{S}'$-convolution. This is an essential generalization (see Examples 1 and 2).

4. Product of distributions and the exchange formula. In this paper we consider a more general definition of the product of distributions than Definition 0.4. It is connected with the notion of the value of a distribution at a point (in the sense of Łojasiewicz, see [6]–[8]). This definition of product can be found in [4], [8], [14]. Similarly we define the notion of the $\mathscr{S}'$-product of tempered distributions.

DEFINITION 1 (see [6], § 3, p. 15). We say that a distribution $T \in \mathscr{D}'(\boldsymbol{R}^n)$ *has a value* C at a point $x_0 \in \boldsymbol{R}^n$: $T(x_0) = C$, if the distributional limit $\lim_{\lambda\to 0+} T(x_0+\lambda x)$ exists and is the constant distribution C, i.e.

$$\lim_{\lambda\to 0+} T(x_0+\lambda\cdot)[\varphi] = C\int\varphi \quad \text{for } \varphi \in \mathscr{D}(\boldsymbol{R}^n),$$

where $T(x_0+\lambda\cdot)$ denotes the distribution defined as the composition of T with the function $f(x) = x_0+\lambda x$, so

$$T(x_0+\lambda\cdot)[\varphi] = T\left[\frac{1}{\lambda^n}\varphi\left(\frac{\cdot - x_0}{\lambda}\right)\right] \quad \text{for } \varphi \in \mathscr{D}(\boldsymbol{R}^n),\ \lambda \neq 0.$$

Remark 1. If T is an improper integrable distribution (in the sense of Definition 2.3) then $\hat{T}$ has a value at 0 and $\hat{T}(0) = (2\pi)^{-n/2}\,T[1]$.

Proof. Since

$$\left(\frac{1}{\lambda^n}\varphi\left(\frac{\cdot}{\lambda}\right)\right)^{\wedge} = \hat{\varphi}(\lambda\cdot)$$

we have

$$\hat{T}\left[\frac{1}{\lambda^n}\varphi\left(\frac{\cdot}{\lambda}\right)\right] = T[\hat{\varphi}(\lambda\cdot)].$$

We notice that $\hat{\varphi}(\lambda\cdot) \to (2\pi)^{-n/2}\int\varphi$ in $\mathscr{D}_{L}$ as $\lambda \to 0+$, so

$$\lim_{\lambda\to 0+} T[\hat{\varphi}(\lambda\cdot)] = T[1](2\pi)^{-n/2}\int\varphi.$$

Hence $\hat{T}(0)$ exists and $\hat{T}(0) = (2\pi)^{-n/2}\,T[1]$.

Improper integrable distributions in Musielak–Sikorski sense do not have this property. In fact, an improper integrable distribution in this meaning need not be a tempered distribution (see Example 2.4).

DEFINITION 2. We say that a sequence ϱ_ν *converges regularly to* δ (is a *regular δ-sequence*) if

(1) $\varrho_\nu \in \mathscr{D}(\boldsymbol{R}^n)$ for $\nu = 1, 2, \ldots$

(2) $\operatorname{supp}\varrho_\nu \to 0$ as $\nu \to \infty$.

(3) $\int\varrho_\nu = 1$ for $\nu = 1, 2, \ldots$

(4) For every $\alpha \in N_0^n$ there exists $M_\alpha > 0$ such that $\sup|D^\alpha \varrho_\nu| \leqslant M_\alpha/\lambda_\nu^{|\alpha|+n}$ for $\nu = 1, 2, \ldots$ where $\operatorname{supp}\varrho_\nu \subset \langle -\lambda_\nu, \lambda_\nu\rangle^n$.

The following theorem is true:

THEOREM 1. *Let $U, V \in \mathscr{D}'(\boldsymbol{R}^n)$ $[\mathscr{S}'(\boldsymbol{R}^n)]$. The following conditions are equivalent:*

(i) *The limit $\lim_\nu (U * \varrho_\nu)V$ exists in $\mathscr{D}'(\boldsymbol{R}^n)$ $[\mathscr{S}'(\boldsymbol{R}^n)]$ for every regular δ-sequence ϱ_ν.*

(ii) *The limit $\lim_\nu U(V * \varrho_\nu)$ exists in $\mathscr{D}'(\boldsymbol{R}^n)$ $[\mathscr{S}'(\boldsymbol{R}^n)]$ for every regular δ-sequence ϱ_ν.*

(iii) *The limit $\lim_\nu (U * \varrho'_\nu)(V * \varrho''_\nu)$ exists in $\mathscr{D}'(\boldsymbol{R}^n)$ $[\mathscr{S}'(\boldsymbol{R}^n)]$ for all regular δ-sequences ϱ'_ν, ϱ''_ν.*

(iv) *The distribution $\tilde{U} * (V\varphi)$ has a value at 0 for $\varphi \in \mathscr{D}(\boldsymbol{R}^n)$ $[\mathscr{S}(\boldsymbol{R}^n)]$.*

(v) *The distribution $(U\varphi) * \tilde{V}$ has a value at 0 for $\psi \in \mathscr{D}(\boldsymbol{R}^n)$ $[\mathscr{S}(\boldsymbol{R}^n)]$.*

(vi) *The distribution $(\tilde{U}\tilde{\psi}) * (V\varphi)$ has a value at 0 for $\varphi, \psi \in \mathscr{D}(\boldsymbol{R}^n)$ $[\mathscr{S}(\boldsymbol{R}^n)]$.*

If one of these conditions holds then the limits (i)–(iii) are equal and

$$\lim_\nu (U * \varrho'_\nu)(V * \varrho''_\nu)[\varphi] = (\tilde{U} * (V\varphi))(0) = ((U\varphi) * \tilde{V})(0),$$

$$\lim_\nu (U * \varrho'_\nu)(V * \varrho''_\nu)[\varphi\psi] = ((\tilde{U}\tilde{\psi}) * (V\varphi))(0)$$

for all regular δ-sequences ϱ'_ν, ϱ''_ν and $\varphi, \psi \in \mathscr{D}(\boldsymbol{R}^n)$ $[\varphi, \psi \in \mathscr{S}(\boldsymbol{R}^n)$ if $U, V \in \mathscr{S}'(\boldsymbol{R}^n)]$.

DEFINITION 3 (see [14]). If distributions U, V satisfy one of conditions (i)–(vi) of Theorem 1 (in the case of $\mathscr{D}'(\boldsymbol{R}^n)$) then we say that the *product* UV of these distributions exists, defined as one of the limits (i)–(iii).

Since, for $U, V \in \mathscr{S}'(\boldsymbol{R}^n)$, the existence of the product UV does not imply that $UV \in \mathcal{S}'(\mathbf{R}^n)$, we introduce the notion of *$\mathcal{S}'$-product*, denoted by $U \odot V$. It is one of the limits (i)–(iii), under the assumption that conditions (i)–(vi) hold in the case of $\mathcal{S}'(\mathbf{R}^n)$. So $U \odot V$ is a tempered distribution.

Now we give a generalization of Theorem 0.3 on the exchange formula:

THEOREM 2. *Let $U, V \in \mathcal{S}'(\mathbf{R}^n)$. If the improper $\mathcal{S}'$-convolution of U, V exists then the $\mathcal{S}'$-product of their Fourier transforms exists and*

$$(U \circledast V)^{\hat{}} = (2\pi)^{n/2}\, \hat{U} \odot \hat{V}.$$

The same is true for the inverse Fourier transform.

The theorem is a consequence of Theorem 1 and Corollary 3.1 in the case of $\mathcal{S}'(\mathbf{R}^n)$ and the definitions of improper $\mathcal{S}'$-convolution and $\mathcal{S}'$-product. We omit the simple proof.

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Nonlinear transformations on spaces of continuous functions

by

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Abstract. We study nonlinear transformations on spaces of continuous functions with values in a Banach space. The continuous functions are defined on an arbitrary topological space and have totally bounded range in a second Banach space. In particular, we consider transformations which satisfy the Hammerstein property of Batt [1973] and integral operators. Our results, which are obtained by a systematic use of the semivariation, generalize some of the results of Batt. Further, we point out some connections between our results and the theory of locally solid Riesz spaces and abstract integration theory.

0. Introduction. This paper deals with a generalization of the classical Riesz representation theorem to nonlinear transformations. In [VW2] we have given a new approach to the representation of continuous linear operators $T: \mathscr{C}(\Omega, E) \to F$, where Ω is an arbitrary topological space, E and F are Banach spaces and $\mathscr{C}(\Omega, E)$ denotes the space of all E-valued continuous functions on Ω with totally bounded range endowed with the topology of uniform convergence. J. Batt asked whether it is possible to obtain with our method [VW2] his integral representation theorem for certain nonlinear transformations [B2]. Batt investigated in [B2], for a compact Hausdorff space Ω, transformations $T: \mathscr{C}(\Omega, E) \to F$ which are uniformly continuous on bounded sets and satisfy $T0 = 0$ and the "Hammerstein property"

$$T(f+f_1+f_2)+T(f) = T(f+f_1)+T(f+f_2)$$

for all $f, f_1, f_2 \in \mathscr{C}(\Omega, E)$ with f_1 and f_2 having disjoint supports. For a short description of the relation with earlier research of Drewnowski–Orlicz, Mizel–Sundaresan, Chacon–Friedman, Friedman–Katz and Friedman–Tong, see [B1].

Our approach to the representation of nonlinear transformations $T: \mathscr{C}(\Omega, E) \to F$ is based on a systematic study of the semivariation of T (= modulus of continuity) in Section 1. The semivariation is used to obtain a continuous extension of T in Section 2. In Theorem (4.3)(a) we show the following result, which may be considered as the kernel of a representation theorem: Let Ω be an arbitrary (not necessarily compact) topological space.