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On the convergence of Laplace–Beltrami operators associated to quasiregular mappings*

by

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Abstract. We prove that if (f_h) is a sequence of K-quasiregular mappings converging to f in L^1_{loc} whose jacobians satisfy a weak integrability condition, then the solutions of the Laplace–Beltrami operators associated to each f_h, which are degenerate elliptic, converge to the solution of the Laplace–Beltrami operator associated to f. A class of weighted Sobolev spaces associated to quasiconformal mappings is also studied.

§ 0. Introduction. Let $\boldsymbol{R}^n$, $n \geqslant 2$, be the Euclidean space of n-tuples $x = (x_1, \ldots, x_n)$ of real numbers and let Ω_0 be an open set in $\boldsymbol{R}^n$.

For every $f = (f^1, \ldots, f^n)$ in $\left(H^{1,n}_{\mathrm{loc}}(\Omega_0)\right)^n$ we denote by $[D_x f(x)]$ the matrix $[D_{x_i} f^j(x)]$, $i, j = 1, \ldots, n$ (here $D_{x_i} f^j(x) = (\partial f^j/\partial x_i)(x)$) and set

$$J_x(x, f) = \text{determinant of } D_x f(x), \qquad |D_x(x, f)| = \Big(\sum_{i,j=1}^{n} \big(D_{x_i} f^j(x)\big)^2\Big)^{1/2}.$$

In the following, when clear from the context, we will write D instead of D_x and $\sum_{ij}$ instead of $\sum_{i,j=1}^{n}$.

If $f \in \left(H^{1,n}_{\mathrm{loc}}(\Omega_0)\right)^n$ with $J(x, f) \neq 0$ a.e. in Ω_0 it is possible to define a "Laplace–Beltrami" operator

$$\Delta_f = -\sum_{ij} D_i\big(a_{ij}(x, f)\, D_j\big) \tag{0.1}$$

whose coefficient matrix is symmetric in Ω_0 and given by

$$[a_{ij}(x, f)] = J(x, f)\,[Df(x)^t \cdot Df(x)]^{-1}. \tag{0.2}$$

If $f \in \left(H^{1,n}_{\mathrm{loc}}(\Omega_0)\right)^n$ and $K \geqslant 1$ we say that f is *K-quasiregular* on Ω_0 (see for a general exposition [BI]) if

$$|D(x, f)|^n \leqslant K n^{n/2} J(x, f) \qquad \text{a.e. in } \Omega_0. \tag{0.3}$$

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Observe that the inequality

$$n^{n/2} J(x, f) \leqslant |D(x, f)|^n \quad \text{a.e. in } \Omega_0 \tag{0.4}$$

holds for every $f \in (H^{1,n}_{\mathrm{loc}}(\Omega_0))^n$.

For every nonconstant K-quasiregular mapping f, $J(x,f)$ is different from zero a.e. (see [B I], Th. 7.2) and the coefficients of the associated Laplace–Beltrami operator Δ_f satisfy the inequality

$$c_K^{-1} J(x,f)^{1-2/n}|z|^2 \leqslant \sum_{ij} a_{ij}(x,f) z_i z_j \leqslant c_K J(x,f)^{1-2/n}|z|^2 \tag{0.5}$$

for a.e. x in Ω_0 and for every $z \in \boldsymbol{R}^n$,

where $c_K = (K+\sqrt{K^2-1})^{2(n-1)/n}$.

Let us consider a sequence of nonconstant K-quasiregular functions (f_h) converging in $L^1_{\mathrm{loc}}(\Omega_0)$ to a function f and bounded in $(H^{1,n}_{\mathrm{loc}}(\Omega_0))^n$. Observe that under these assumptions f is still K-quasiregular by a result of Reshetnyak ([R1], [R2]).

Assume that f is nonconstant. For $\Omega \Subset \Omega_0$ set

$$F_h(u, \Omega) = \int_\Omega \sum_{ij} a_{ij}(x, f_h) D_i u D_j u\, dx, \qquad F(u, \Omega) = \int_\Omega \sum_{ij} a_{ij}(x, f) D_i u D_j u\, dx.$$

In a paper of 1974 ([S2]) S. Spagnolo, assuming that for every compact set S in Ω_0

$$\begin{aligned} &0 < p_S^{-1} \leqslant \int_S J(x, f_h)\, dx \leqslant p_S \quad \text{for every } h \quad \text{if } n = 2,\\ &0 < p_S^{-1} \leqslant J(x, f_h)^{1-2/n} \leqslant p_S \quad \text{a.e. in } S \quad \text{for every } h \quad \text{if } n > 2, \end{aligned} \tag{0.6}$$

proved that Δ_{f_h} converge to Δ_f in Ω_0 in the sense that (see [S1], [DG S]) for every $\Omega \Subset \Omega_0$ and $g \in H^{-1}(\Omega)$ the minimum points of $F_h(u, \Omega) + \langle g, u\rangle$ in $H^1_0(\Omega)$ converge in $L^2(\Omega)$ to the minimum point in the same space of $F(u, \Omega) + \langle g, u\rangle$.

In dimension 2 the result of Spagnolo is exhaustive enough since the Laplace–Beltrami operators are uniformly elliptic with ellipticity constants independent of h.

In higher dimensions the Laplace–Beltrami operators degenerate in a natural way (even satisfying $J(x, f_h) \neq 0$ a.e. in Ω_0). Spagnolo also suggested in [S2] a deeper study of that case.

In 1978 C. Sbordone ([Sb]) extended the result of Spagnolo allowing to take such degeneracy into account in some cases. He proved that for every $\Omega \Subset \Omega_0$ the minimum points of $F_h(u, \Omega) + \langle g, u\rangle$ in a suitable weighted Sobolev space, with g in the dual of this space, converge in $L^1(\Omega)$ to the minimum point of the functional $F(u, \Omega) + \langle g, u\rangle$ under the assumption

$$0 < w(x) \leqslant J(x, f_h)^{1-2/n} \leqslant \Lambda w(x) \quad \text{a.e. in } \Omega_0, \quad h \in \boldsymbol{N}, \tag{0.7}$$

with $\|w\|_{L^p(\Omega)} + \|w^{-1}\|_{L^p(\Omega)} \leqslant Q(\Omega)$ for every $\Omega \Subset \Omega_0$ and $p > 2n-1$.

In this paper we prove, using techniques different from those of the above authors, that (see Th. 3.4) the same result about the convergence of minimum points still holds under the assumption (0.7) if simply w and w^{-1} are in $L^1_{\mathrm{loc}}(\Omega_0)$, f_h and f being nonconstant by (0.7).

By strengthening the hypotheses on (f_h) it is possible to get an analogous result under weaker hypotheses on w. More precisely, recalling that a K-quasiregular mapping f on Ω_0 is said to be *K-quasiconformal* on Ω_0 if it is a homeomorphism, it will be sufficient to assume (see Th. 3.5) that (f_h) is a sequence of K-quasiconformal mappings satisfying (0.7) with $w \in L^1_{\mathrm{loc}}(\Omega_0)$ and no assumption on w^{-1}, (f_h) converging to f in $L^1_{\mathrm{loc}}(\Omega_0)$. Observe that under these hypotheses it can be proved that f is K-quasiconformal (see [Ge1], [Cr]).

The above problem, with the hypothesis of K-quasiconformality of f_h, requires the study of a particular class of weighted Sobolev spaces; this study will be done in Section 2.

The techniques we use in this paper are techniques of Γ-convergence (see § 1 for the definition) and the results described above will be deduced in Section 3 from some theorems about the Γ-convergence of the functionals F_h. We point out anyway that for K-quasiconformal mappings the result could also be deduced more directly via the results of Section 2.

We refer to [B I K] and [I K] for stability problems for more general differential equations associated to quasiregular mappings, but considered from a different point of view.

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§ 1. Preliminaries.

I. We recall the following result of Reshetnyak ([R1], [R2]).

Proposition 1.1. *Given a function $\varphi \in C^0_0(\Omega)$ consider the following functionals on $(H^{1,n}(\Omega))^n$:*

$$\alpha: f \to \int_\Omega J(x, f)\varphi(x)\, dx, \qquad \beta: f \to \Big(\int_\Omega |D(x,f)|^n |\varphi(x)|\, dx\Big)^{1/n}.$$

Then α is weakly continuous and β is weakly lower semicontinuous and uniformly convex.

We now recall some properties of quasiregular and quasiconformal mappings.

Proposition 1.2. *Let f be a nonconstant K-quasiregular mapping on Ω_0. Then*:

(1) $J(x,f) \neq 0$ *a.e. in Ω_0* (see [B I], Th. 7.2).

(2) *f is differentiable a.e. in Ω_0 and the chain rule holds, i.e. for every $\varphi \in C^1(f(\Omega_0))$, $\varphi \circ f \in H^{1,n}_{\mathrm{loc}}(\Omega_0)$ and*

$$\frac{\partial(\varphi\circ f)}{\partial x_i}(x) = \sum_{k=1}^n \frac{\partial f^k}{\partial x_i}(x)\frac{\partial \varphi}{\partial y_k}(f(x)) \quad \text{(see [B I], Th. 5.3 and Lemma 9.6).}$$

(3) *f maps sets of measure zero into sets of measure zero* (see [B I], Th. 8.1).

If further f is K-quasiconformal on Ω_0, *then:*

(4) f^{-1} *is* K^{n-1}*-quasiconformal on* $f(\Omega_0)$ (see [B I], Th. 9.1).

(5) $[D_x(x,f)]^{-1} = [D_y(f(x), f^{-1})]$ *a.e. in* Ω_0.

(6) *For every* $u \in L^\infty(\mathbf{R}^n)$ *and* $\Omega \Subset \Omega_0$

$$\int_\Omega u(f(x)) J(x,f)\,dx = \int_{f(\Omega)} u(y)\,dy \quad \text{(see [B I], Th. 8.4).}$$

We now state the following result on the L^p-integrability for quasiregular mappings (see [B], [B I], [Ge2], [G M]).

THEOREM 1.3. *Let f be K-quasiregular on* Ω_0. *Then there exists* $p = p(n, K) > n$ *depending only on n and K such that f is in* $(H^{1,p}_{\rm loc}(\Omega_0))^n$. *Moreover, for every compact subset S of* $\Omega \Subset \Omega_0$ *the estimate*

$$\Big(\int_S |Df|^p\,dx\Big)^{1/p} \leqslant \frac{c(n, p, K)}{\operatorname{dist}(S, \partial\Omega)^{1-n/p}} \Big(\int_\Omega |Df|^n\,dx\Big)^{1/n}$$

holds with $c(n, p, K)$ *independent of f, S and* Ω.

Given a quasiregular mapping f denote by U_{si} the algebraic complement of $\partial f^s/\partial x_i$ in the jacobian matrix $D(x,f)$. Obviously we have

$$U_{si} = \sum_j a_{ij}(x,f) \frac{\partial f^s}{\partial x_j}.$$

Hence by Lemma 1.9 of [B I] we have (see also [Ci2])

$$-\sum_{ij} D_i\big(a_{ij}(x,f) D_j f^s\big) = 0 \quad \text{in } \Omega_0. \tag{1.1}$$

Given a sequence (f_h) of K-quasiconformal mappings on Ω_0, bounded in $(H^{1,n}_{\rm loc}(\Omega_0))^n$ and converging to a function f in $L^1_{\rm loc}(\Omega_0)$, from Theorem 1.3 and the Sobolev embedding theorem it follows that the convergence holds uniformly on compact subsets of Ω_0. Then by a result of Gehring (see [Ge1] and [Cr]) the following property holds.

PROPOSITION 1.4. *Let* (f_h) *be a sequence of K-quasiconformal mappings on* Ω_0, *bounded in* $(H^{1,n}_{\rm loc}(\Omega_0))^n$ *and converging to a function f in* $L^1_{\rm loc}(\Omega_0)$. *Then f is either a constant or a K-quasiconformal mapping.*

Remark 1.5. Under the assumptions of Proposition 1.4 if we assume that (0.7) holds then by Proposition 1.1 the limit function f cannot be a constant.

II. We now pass to the definition of Γ-convergence; we refer to [DG F] and [DG] for complete references.

Let (U, τ) be a topological space satisfying the first countability axiom and let F_h, F $(h \in \mathbf{N})$ be extended-valued functionals on U.

DEFINITION 1.6. We say that

$$F(u) = \Gamma^-(\tau) \lim_{\substack{h\to\infty \\ v\to u}} F_h(v) \quad \text{for every } u \in U$$

if and only if

(i) for every $u \in U$ and for every $v_h \xrightarrow{\tau} u$

$$F(u) \leqslant \liminf_{h\to\infty} F_h(v_h),$$

(ii) for every $u \in U$ there exists a sequence (u_h), $u_h \xrightarrow{\tau} u$, such that

$$F(u) = \lim_{h\to\infty} F_h(u_h).$$

The following theorem holds (see [DG F]).

THEOREM 1.7. *Let* (F_h) *be a sequence of equicoercive functionals defined on U, i.e. for every real number c there exists a compact* K_c *in U such that* $\{u \in U:\ F_h(u) \leqslant c\} \subseteq K_c$ *for every* $h \in \mathbf{N}$. *Assume further that*

$$F(u) = \Gamma^-(\tau) \lim_{\substack{h\to\infty \\ v\to u}} F_h(v), \qquad u \in U.$$

Then F has a minimum in U and

$$\operatorname*{Min}_{v\in U} F(v) = \lim_{h\to\infty} \operatorname*{Inf}_{v\in U} F_h(v).$$

Further, if (u_h) *is a sequence such that* $u_h \xrightarrow{\tau} u$ *and*

$$\lim_{h\to\infty} \big(F_h(u_h) - \operatorname*{Inf}_{v\in U} F_h(v)\big) = 0$$

then

$$F(u) = \operatorname*{Min}_{v\in U} F(v).$$

In the following we will be concerned with functionals of the type $\int_\Omega \sum_{ij} a_{ij}(x,f) D_i u D_j u\,dx$ where u is a locally Lipschitz function ($u \in \mathrm{Lip}_{\rm loc}$), f is a K-quasiregular mapping on Ω_0 and $\Omega \Subset \Omega_0$.

Let (f_h) be a sequence of K-quasiregular functions on Ω_0. If $\Omega_1 \Subset \Omega_0$ define

$$\hat{a}^h_{ij}(x) = \begin{cases} a_{ij}(x, f_h) & \text{if } x \in \Omega_1, \\ \delta_{ij} & \text{if } x \in \mathbf{R}^n - \Omega_1. \end{cases} \tag{1.2}$$

Then from (0.5) it follows that

$$c_K^{-1}\big(J(x,f_h)^{1-2/n} \chi_{\Omega_1}(x) + \chi_{\mathbf{R}^n-\Omega_1}(x)\big)|z|^2 \leqslant \sum_{ij} \hat{a}^h_{ij}(x) z_i z_j \leqslant c_K\big(J(x,f_h)^{1-2/n} \chi_{\Omega_1}(x) + \chi_{\mathbf{R}^n-\Omega_1}(x)\big)|z|^2 \quad \text{for a.e. } x \text{ in } \mathbf{R}^n,\ z \in \mathbf{R}^n. \tag{1.3}$$

The following result is a particular case of a theorem due to L. Carbone and C. Sbordone ([C Sb]).

THEOREM 1.8. *Given a sequence of nonconstant K-quasiregular mappings* (f_h) *on* Ω_0, *set* $\hat{a}_{ij}^h$ *as in* (1.2). *Then if*

$$J(x, f_h)^{1-2/n} \leqslant a_h(x) \quad \text{a.e. in } \Omega_0,$$

with

$$\lim_{h\to\infty} \int_\Omega a_h(x)\,dx = \int_\Omega a(x)\,dx \quad \text{for every } \Omega \Subset \Omega_0,$$

there exists a subsequence $(h_r)\uparrow\infty$ *and a symmetric matrix* $[a_{ij}]$ *satisfying*

$$(1.4)\qquad 0 \leqslant \sum_{ij} a_{ij}(x) z_i z_j \leqslant a(x)|z|^2 \quad x\text{-a.e. in } \boldsymbol{R}^n, \text{ for every } z \in \boldsymbol{R}^n,$$

such that

$$\int_\Omega \sum_{ij} a_{ij}(x) D_i u D_j u\,dx = \Gamma^-(M_0(\Omega)) \lim_{\substack{r\to\infty\\ v\to u}} \int_\Omega \sum_{ij} \hat{a}_{ij}^{h_r}(x) D_i v D_j v\,dx$$
$$= \Gamma^-(C_0^0(\Omega)) \lim_{\substack{r\to\infty\\ v\to u}} \int_\Omega \sum_{ij} \hat{a}_{ij}^{h_r}(x) D_i v D_j v\,dx$$

for every bounded open set Ω *in* $\boldsymbol{R}^n$, $u \in \mathrm{Lip}_{\mathrm{loc}}$.

Here we have denoted by $M_0(\Omega)$ the topology of convergence in measure and by $C_0^0(\Omega)$ the one induced by the extended-valued metric

$$\delta(u, v) = \begin{cases} \|u-v\|_{C^0(\bar\Omega)} & \text{if } \operatorname{supp}(u-v) \Subset \Omega,\\ +\infty & \text{otherwise.}\end{cases}$$

In particular, from Theorem 1.8 it follows that

$$(1.5)\qquad \int_\Omega \sum_{ij} a_{ij}(x) D_i u D_j u\,dx = \Gamma^-(M_0(\Omega)) \lim_{\substack{r\to\infty\\ v\to u}} \int_\Omega \sum_{ij} a_{ij}(x, f_{h_r}) D_i v D_j v\,dx$$
$$= \Gamma^-(C_0^0(\Omega)) \lim_{\substack{r\to\infty\\ v\to u}} \int_\Omega \sum_{ij} a_{ij}(x, f_{h_r}) D_i v D_j v\,dx$$

for every $\Omega \Subset \Omega_1$ and $u \in \mathrm{Lip}_{\mathrm{loc}}$.

§ 2. **A class of weighted Sobolev spaces.** Let Ω be a bounded open set whose closure is contained in Ω_0. For any positive function w in $L^1(\Omega)$ define $L^2(\Omega, w)$ as the space of all measurable functions such that

$$\|u\|_{L^2(\Omega,w)} = \Big(\int_\Omega u^2 w(x)\,dx\Big)^{1/2}$$

is finite.

Denote by $G_0(\Omega)$ the set of n-tuples of functions $v = (v_1, \ldots, v_n)$ such that there exists a function u in $C_0^1(\Omega)$ with $v = Du$, i.e. v is the gradient of u.

Define $H_0^1(\Omega, w)$ as the closure of $G_0(\Omega)$ in $(L^2(\Omega, w))^n$, denote by $H^{-1}(\Omega, w)$ its dual and by $\langle\cdot,\cdot\rangle$ the duality among them.

Since $w(x) > 0$ a.e. in Ω, a sequence of functions in $G_0(\Omega)$ which is a Cauchy sequence in $(L^2(\Omega, w))^n$ has a unique limit in $(L^2(\Omega, w))^n$.

Observe that, in general, $H_0^1(\Omega, w)$ is only a closed subspace of $(L^2(\Omega, w))^n$.

If w, w^{-1} are in $L^1(\Omega)$ it is known that $H_0^1(\Omega, w) \subset H_0^{1,1}(\Omega)$.

If f is K-quasiconformal on Ω_0, from (5) of Proposition 1.2 it follows that

$$(2.1)\qquad J_y(f(x), f^{-1}) = (J_x(x, f))^{-1} \quad \text{a.e. in } \Omega_0.$$

Further, it is easy to verify that for every $z \in \boldsymbol{R}^n$

$$(2.2)\qquad c^{-1}|J(x,f)|^{2/n}|z|^2 \leqslant |D(x,f)\cdot z|^2 \leqslant c|J(x,f)|^{2/n}|z|^2 \quad x\text{-a.e. in } \Omega_0$$

where c is independent of z and x.

From now on in this section, set $w(x) = J(x,f)^{1-2/n}$; in the case $n = 2$ this gives $w(x) = 1$; we will set in this case $w^{n/(n-2)}(x) = 1$.

The following Poincaré type inequality holds.

PROPOSITION 2.1. *Let f be a K-quasiconformal mapping on Ω_0 and let $\Omega \Subset \Omega_0$. Then there exists a constant $c = c(n, \Omega, K)$ such that for every u in $C_0^1(\Omega)$*

$$\int_\Omega u^2 w(x)^{n/(n-2)}\,dx \leqslant c\int_\Omega |Du|^2 w(x)\,dx.$$

Proof. Let $u \in C_0^1(\Omega)$ and set $U(y) = u(f^{-1}(y))$, $y \in f(\Omega)$. By (2.1), (2.2), Proposition 1.2 and the Poincaré inequality it follows that U is in $H_0^1(f(\Omega))$ and

$$\int_\Omega |D_x u(x)|^2 J_x(x,f)^{1-2/n}\,dx$$
$$= \int_{f(\Omega)} |D_x(f^{-1}(y), f)\cdot D_y U(y)|^2 J_x(f^{-1}(y), f)^{-2/n}\,dy$$
$$\geqslant c_1 \int_{f(\Omega)} |D_y U(y)|^2\,dy \geqslant c_2 \int_{f(\Omega)} U(y)^2\,dy = c_2\int_\Omega u(x)^2 J_x(x,f)\,dx. \ \blacksquare$$

From Proposition 2.1 it follows that if an n-tuple v is the limit in $(L^2(\Omega, w))^n$ of two different sequences $(D\varphi_h')$ and $(D\varphi_h'')$, then (φ_h') and (φ_h'') are converging sequences and they converge to the same measurable function in $(L^2(\Omega, w^{n/(n-2)}))^n$.

This implies that to each n-tuple v in $H_0^1(\Omega, w)$ it is possible to associate uniquely a function u for which v can be considered the "gradient".

In the following we will say that a measurable function u is in $H_0^1(\Omega, w)$ if there exists a sequence (φ_h) in $C_0^1(\Omega)$ such that (φ_h) converges to u in measure and $(D\varphi_h)$ is a Cauchy sequence in $(L^2(\Omega, w))^n$.

Hence the continuous embedding $H_0^1(\Omega, w) \subset L^2(\Omega, w^{n/(n-2)})$ holds.

The above considerations imply the following corollary.

COROLLARY 2.2. *Let f be a K-quasiconformal mapping on Ω_0 and let $\Omega \Subset \Omega_0$. Then there exists a constant $c = c(n, \Omega, K)$ such that for every u in $H_0^1(\Omega, w)$*

$$\int_\Omega u^2 w(x)^{n/(n-2)}\,dx \leqslant c \int_\Omega |Du|^2 w(x)\,dx.$$

It is known (see [BI], Sec. 9.2) that the operator defined by $(f_* v)(x) = v(f(x))$ is an isomorphism between $H_0^{1,n}(f(\Omega))$ and $H_0^{1,n}(\Omega)$ provided $f\colon \Omega \to f(\Omega)$ is quasiconformal and in $(H^{1,n}(\Omega))^n$.

In the same order of ideas it is possible to give a characterization of $H_0^1(f(\Omega))$ in terms of $H_0^1(\Omega, w)$.

PROPOSITION 2.3. *Given a K-quasiconformal mapping f on Ω_0 let $\Omega \Subset \Omega_0$. Then a function u is in $H_0^1(\Omega, w)$ if and only if the function $U = u \circ f^{-1}$ is in $H_0^1(f(\Omega))$. Moreover, there exists a constant c independent of u such that*

$$\text{(2.3)}\qquad \begin{aligned} c^{-1}\|u\|_{H_0^1(\Omega,w)} &\leqslant \|U\|_{H_0^1(f(\Omega))} \leqslant c\|u\|_{H_0^1(\Omega,w)},\\ c^{-1}\|u\|_{L^2(\Omega,w^{n/(n-2)})} &\leqslant \|U\|_{L^2(f(\Omega))} \leqslant c\|u\|_{L^2(\Omega,w^{n/(n-2)})}. \end{aligned}$$

Proof. First suppose that u is in $C_0^1(\Omega)$. By (2) of Proposition 1.2 it follows that $u \circ f^{-1}$ is in $H_0^1(f(\Omega))$; further, from the definition of quasiconformality, (2.2) and the chain rule it follows that there exists a positive constant c independent of u such that (2.3) holds for u in $C_0^1(\Omega)$.

Suppose now that u is in $H_0^1(\Omega, w)$ and let $(u_h) \subset C_0^1(\Omega)$ be a sequence converging to u in $H_0^1(\Omega, w)$. Define $U_h = u_h \circ f^{-1}$; then from the above considerations it follows that (U_h) is a sequence in $H_0^1(f(\Omega))$ which is a Cauchy sequence in this space and in $L^2(f(\Omega))$.

By (3) of Proposition 1.2 and the pointwise convergence of $u_h(x)$ to $u(x)$ a.e. in Ω we have $U_h(y) \to u(f^{-1}(y))$ a.e. in $f(\Omega)$. Therefore $u \circ f^{-1}$ is in $H_0^1(f(\Omega))$ and, by standard approximation arguments, (2.3) follows for every u in $H_0^1(\Omega, w)$.

Suppose now that U is in $C_0^1(f(\Omega))$. First of all observe that $\operatorname{supp}(U \circ f) \subseteq \Omega$. Let (ψ_h) be a sequence of Lipschitz functions converging to f in $(H^{1,n}(\Omega))^n$ and let $\varphi \in C_0^\infty(\Omega)$, $0 \leqslant \varphi \leqslant 1$, $\varphi \equiv 1$ in a neighbourhood I of $\operatorname{supp}(U \circ f)$ such that $I \Subset \Omega$.

By (2) of Proposition 1.2 we can write for every h

$$\text{(2.4)}\qquad \begin{aligned} &\int_\Omega |D(\varphi(U \circ \psi_h)) - D(U \circ f)|^2 w(x)\,dx\\ &\leqslant 2\Big\{\int_\Omega |(D\varphi)(U \circ \psi_h) - (D\varphi)(U \circ f)|^2 w(x)\,dx\\ &\quad + \int_\Omega |D(U \circ \psi_h) - D(U \circ f)|^2 w(x)\,dx + \int_\Omega |\varphi D(U \circ f) - D(U \circ f)|^2 w(x)\,dx\Big\}. \end{aligned}$$

From the chain rule and the K-quasiconformality of f it follows that

$$\text{(2.5)}\qquad \begin{aligned} &\int_\Omega |D(U \circ \psi_h) - D(U \circ f)|^2 w(x)\,dx\\ &\leqslant \|D_y U\|_{L^\infty(f(\Omega))} \Big(\int_\Omega |D_x \psi_h - D_x f|^n\,dx\Big)^{2/n} \Big(\int_\Omega w(x)^{n/(n-2)}\,dx\Big)^{1-2/n}\\ &\quad + \int_\Omega |D_y U(\psi_h(x)) - D_y U(f(x))|^2 J_x(x, f)\,dx. \end{aligned}$$

Therefore by (2.4) and (2.5) we have

$$\lim_{h\to\infty} \int_\Omega |D(\varphi(U \circ \psi_h)) - D(U \circ f)|^2 w(x)\,dx = 0.$$

Hence $U \circ f$ is in $H_0^1(\Omega, w)$ by the definition of $H_0^1(\Omega, w)$ and (2.3) holds for U in $C_0^1(f(\Omega))$. Again using an approximation argument we can prove that if U is in $H_0^1(f(\Omega))$ then $U \circ f$ is in $H_0^1(\Omega, w)$ and (2.3) holds for U in $H_0^1(f(\Omega))$. ∎

Denote by $M_0(\Omega)$ the space of measurable functions on Ω endowed with the usual topology. We have

COROLLARY 2.4. *Under the assumptions of Proposition 2.3 the embedding of $H_0^1(\Omega, w)$ in $M_0(\Omega)$ is compact.*

Proof. Let (u_h) be a bounded sequence in $H_0^1(\Omega, w)$. From Proposition 2.3 and Rellich's theorem it follows that there exist a subsequence (u_{h_r}) and a function U in $H_0^1(f(\Omega))$ such that $U_{h_r} = u_{h_r} \circ f^{-1}$ converge to U in $L^2(f(\Omega))$. Then, by (3) of Proposition 1.2, (u_{h_r}) converges in measure to $u = U \circ f$, which is in $H_0^1(\Omega, w)$ by Proposition 2.3. ∎

§ 3. The convergence results.

I. Let (f_h) be a sequence of nonconstant K-quasiregular functions on Ω_0 satisfying

$$\text{(3.1)}\qquad \begin{aligned} &\text{(i)}\quad f_h \to f \quad \text{in } L^1_{\text{loc}}(\Omega_0),\\ &\text{(ii)}\quad \|f_h\|_{(H^{1,n}(\Omega))^n} \leqslant Q(\Omega) \quad \text{for every } \Omega \Subset \Omega_0. \end{aligned}$$

Let Ω_1 be as in (1.2). From (3.1) (ii) and the K-quasiregularity of each f_h it follows that there exists a subsequence (f_{h_r}) of (f_h) such that

$$\text{(3.2)}\qquad J(x, f_{h_r})^{1-2/n} \rightharpoonup a(x)^{1-2/n} \quad \text{weakly in } L^{n/(n-2)}(\Omega_1).$$

Choosing

$$a_{h_r}(x) = J(x, f_{h_r})^{1-2/n} \chi_{\Omega_1}(x) + \chi_{\mathbf{R}^n - \Omega_1}(x)$$

by virtue of (3.2) we are under the hypotheses of Theorem 1.8. We get the existence of a subsequence of (h_r) (still denoted by (h_r)) and of a symmetric matrix $[a_{ij}(x)]$ such that

(3.3) $$\int_\Omega \sum_{ij} a_{ij}(x) D_i u D_j u\, dx = \Gamma^-\,(L^n(\Omega)) \lim_{\substack{r\to\infty\\ v\to u}} \int_\Omega \sum_{ij} a_{ij}(x, f_{h_r}) D_i v D_j v\, dx$$

for every u in $\mathrm{Lip}_{\mathrm{loc}}$, Ω open subset of Ω_1.

LEMMA 3.1. *Let (f_h) be a sequence of nonconstant K-quasiregular functions on Ω_0 satisfying* (3.1) *and let (f_{h_r}) be a subsequence for which* (3.3) *holds. Then*

(3.4) $$\int_{\Omega_1} \sum_{ij} a_{ij}(x) D_i f^s D_j f^s\, dx = \lim_{r\to\infty} \int_{\Omega_1} \sum_{ij} a_{ij}(x, f_{h_r}) D_i f^s_{h_r} D_j f^s_{h_r}\, dx$$

for $s = 1, \ldots, n$.

Proof. In order to prove (3.4) we first observe that from Hölder's inequality it follows that for every $\Omega \Subset \Omega_1$

$$\int_\Omega \sum_{ij} a_{ij}(x, f_h) D_i u D_j u\, dx \leqslant c_K \int_\Omega J(x, f_h)^{1-2/n} |Du|^2\, dx$$
$$\leqslant c_K \Big(\int_\Omega J(x, f_h)\, dx\Big)^{1-2/n} \Big(\int_\Omega |Du|^n\, dx\Big)^{2/n}.$$

Hence the functionals $(\int_\Omega \sum_{ij} a_{ij}(x, f_h) D_i u D_j u\, dx)^{1/2}$ are equilipschitzian in $H^{1,n}(\Omega)$. Therefore arguing as in [MSb] (Proposition 3.2) it follows that

(3.5) $$\int_\Omega \sum_{ij} a_{ij}(x) D_i u D_j u\, dx = \Gamma^-\,(L^n(\Omega)) \lim_{\substack{r\to\infty\\ v\to u}} \int_\Omega \sum_{ij} a_{ij}(x, f_{h_r}) D_i v D_j v\, dx$$

for every $\Omega \subset \Omega_1$, u in $H^{1,n}(\Omega)$.

For s fixed in $\{1, \ldots, n\}$ set for simplicity $u_r = f^s_{h_r}$, $u = f^s$ and $a^r_{ij}(x) = a_{ij}(x, f_{h_r})$. Then (1.1), written with u_r instead of f, holds. Further, $u_r \to u$ in $L^n_{\mathrm{loc}}(\Omega_1)$.

Let (v_r) be a sequence in $H^{1,n}(\Omega_1)$ such that $v_r \to u$ in $L^n(\Omega_1)$ and

$$\lim_{r\to\infty} \int_{\Omega_1} \sum_{ij} a^r_{ij}(x) D_i v_r D_j v_r\, dx = \int_{\Omega_1} \sum_{ij} a_{ij}(x) D_i u D_j u\, dx.$$

Let $\Omega' \Subset \Omega_1$ and $\varphi \in C_0^\infty(\Omega_1)$ be such that $0 \leqslant \varphi \leqslant 1$, $\varphi \equiv 1$ in Ω'; then for every $t \in]0, 1[$

$$\int_{\Omega_1} \sum_{ij} a^r_{ij}(x) D_i u_r D_j u_r\, dx$$
$$\leqslant \int_{\Omega_1} \sum_{ij} a^r_{ij}(x) D_i(\varphi v_r + (1-\varphi) u_r) D_j(\varphi v_r + (1-\varphi) u_r)\, dx$$
$$= \int_{\Omega_1} \sum_{ij} a^r_{ij}(x) \Big\{\frac{t}{t}(D_i \varphi)(v_r - u_r) + \frac{1-t}{1-t}(\varphi D_i v_r + (1-\varphi) D_i u_r)\Big\}$$
$$\cdot \Big\{\frac{t}{t}(D_j \varphi)(v_r - u_r) + \frac{1-t}{1-t}(\varphi D_j v_r + (1-\varphi) D_j u_r)\Big\}\, dx$$

$$\leqslant t \int_{\Omega_1} \sum_{ij} a^r_{ij}(x) \Big\{\frac{1}{t}(D_i \varphi)(v_r - u_r)\Big\} \Big\{\frac{1}{t}(D_j \varphi)(v_r - u_r)\Big\}\, dx$$
$$+ (1-t) \int_{\Omega_1} \sum_{ij} a^r_{ij}(x) \Big\{\frac{1}{1-t}(\varphi D_i v_r + (1-\varphi) D_i u_r)\Big\} \times$$
$$\times \Big\{\frac{1}{1-t}(\varphi D_j v_r + (1-\varphi) D_j u_r)\Big\}\, dx$$
$$\leqslant \frac{1}{t} \int_{\Omega_1} J(x, f_{h_r})^{1-2/n} |D\varphi|^2 |v_r - u_r|^2\, dx$$
$$+ \frac{1}{1-t} \int_{\Omega_1} \sum_{ij} a^r_{ij}(x) D_i v_r D_j v_r \varphi\, dx + \frac{1}{1-t} \int_{\Omega_1} \sum_{ij} a^r_{ij}(x) D_i u_r D_j u_r (1-\varphi)\, dx.$$

Then we have

$$(1-t) \int_{\Omega_1} \sum_{ij} a^r_{ij}(x) D_i u_r D_j u_r\, dx$$
$$\leqslant \frac{1-t}{t} \|D\varphi\|^2_{L^\infty(\Omega_1)} \|v_r - u_r\|^2_{L^n(\mathrm{supp}\,\varphi)} \Big(\int_{\Omega_1} J(x, f_{h_r})\, dx\Big)^{1-2/n}$$
$$+ \int_{\Omega_1} \sum_{ij} a^r_{ij}(x) D_i v_r D_j v_r \varphi\, dx + \int_{\Omega_1} \sum_{ij} a^r_{ij}(x) D_i u_r D_j u_r (1-\varphi)\, dx,$$

that is,

$$\int_{\Omega_1} \sum_{ij} a^r_{ij}(x) D_i v_r D_j v_r \varphi\, dx \geqslant \int_{\Omega_1} \sum_{ij} a^r_{ij}(x) D_i u_r D_j u_r (1 - t - 1 + \varphi)\, dx$$
$$- \frac{1-t}{t} \|D\varphi\|^2_{L^\infty(\Omega_1)} \|v_r - u_r\|^2_{L^n(\mathrm{supp}\,\varphi)} \Big(\int_{\Omega_1} J(x, f_{h_r})\, dx\Big)^{1-2/n}.$$

Hence, passing to the limit as $r \to \infty$,

$$\int_{\Omega_1} \sum_{ij} a_{ij}(x) D_i u D_j u\, dx \geqslant \limsup_{r\to\infty} \int_{\Omega_1} \sum_{ij} a^r_{ij}(x) D_i u_r D_j u_r (\varphi - t)\, dx.$$

Therefore if $t \to 0$

$$\int_{\Omega_1} \sum_{ij} a_{ij}(x) D_i u D_j u\, dx \geqslant \limsup_{r\to\infty} \int_{\Omega_1} \sum_{ij} a^r_{ij}(x) D_i u_r D_j u_r \varphi\, dx$$
$$\geqslant \liminf_{r\to\infty} \int_{\Omega'} \sum_{ij} a^r_{ij}(x) D_i u_r D_j u_r\, dx \geqslant \int_{\Omega'} \sum_{ij} a_{ij}(x) D_i u D_j u\, dx.$$

From these inequalities, since Ω' is an arbitrary open subset of Ω_1, (3.4) follows. ∎

THEOREM 3.2. *Let (f_h) be a sequence of nonconstant K-quasiregular* *functions on* Ω_0 *satisfying* (3.1). *Then* f *is* K*-quasiregular on* Ω_0 *and if* f *is nonconstant*

$$\int_\Omega \sum_{ij} a_{ij}(x,f) D_i u D_j u\, dx = \Gamma^-(L^1(\Omega)) \lim_{\substack{h\to\infty\\ v\to u}} \int_\Omega \sum_{ij} a_{ij}(x,f_h) D_i v D_j v\, dx,$$

whereas if f *is constant*

$$0 = \Gamma^-(L^1(\Omega)) \lim_{\substack{h\to\infty\\ v\to u}} \int_\Omega \sum_{ij} a_{ij}(x,f_h) D_i v D_j v\, dx$$

for every $\Omega \Subset \Omega_0$, $u \in \mathrm{Lip}_{\mathrm{loc}}$.

Proof. The K-quasiregularity of f on Ω_0 follows from Proposition 1.1.

Since we are going to identify the Γ-limit of the sequence $(\int_\Omega \sum_{ij} a_{ij}(x,f_h) D_i u D_j u\, dx)$ we can assume that in (3.2), (3.4) and (3.5) the convergence of the whole sequences holds.

As in the previous Lemma 3.1 set $u_h = f_h^s$, $u = f^s$, $s = 1, \ldots, n$, and $a_{ij}^h(x) = a_{ij}(x, f_h)$. As usal let $\Omega_1 \Subset \Omega_0$.

If

$$\varphi = \sum_{i=1}^{\nu} \lambda_i \chi_{A_i}, \quad A_i \text{ open, meas } (\Omega_1 - \bigcup_{i=1}^{\nu} A_i) = 0,\ \lambda_i \geqslant 0,$$

from (3.5) it follows that

$$\liminf_{h\to\infty} \int_{\Omega_1} \sum_{ij} a_{ij}^h(x) D_i u_h D_j u_h \varphi\, dx \geqslant \int_{\Omega_1} \sum_{ij} a_{ij}(x) D_i u D_j u \varphi\, dx. \tag{3.6}$$

Further, it is easy to show that (3.6) still holds if φ is in $C^0(\overline{\Omega_1})$, $\varphi \geqslant 0$, since such functions can be approximated in $C^0(\overline{\Omega_1})$ by functions of the type $\sum_{i=1}^{\nu} \lambda_i \chi_{A_i}$.

Actually, for every φ in $C^0(\overline{\Omega_1})$ equality holds in (3.6). In fact, for a subsequence (h_r)

$$\lim_{r\to\infty} \int_{\Omega_1} \sum_{ij} a_{ij}^{h_r}(x) D_i u_{h_r} D_j u_{h_r} \varphi\, dx = \int_{\Omega_1} \varphi(x)\, d\mu(x) \quad \forall \varphi \in C^0(\overline{\Omega_1}) \tag{3.7}$$

where, by Theorem 1.3, μ is a positive absolutely continuous measure. Therefore if $\varphi \geqslant 0$, from (3.6) it follows that

$$\int_{\Omega_1} \sum_{ij} a_{ij}(x) D_i u D_j u \varphi\, dx \leqslant \int_{\Omega_1} \varphi(x)\, d\mu(x). \tag{3.8}$$

Let S be a measurable subset of Ω_1 and let $(\varphi_k) \subset C^0(\overline{\Omega_1})$ be such that $\varphi_k(x) \to \chi_S(x)$ a.e. in Ω_1. Then from (3.8) and the Lebesgue theorem it follows that

$$\int_S \sum_{ij} a_{ij}(x) D_i u D_j u\, dx \leqslant \int_S d\mu(x). \tag{3.9}$$

On the other hand, from (3.7) and Lemma 3.1 we deduce that

$$\int_{\Omega_1} \sum_{ij} a_{ij}(x) D_i u D_j u\, dx = \int_{\Omega_1} d\mu(x). \tag{3.10}$$

Then from (3.9) and (3.10) it follows that equality holds in (3.9), hence the Radon–Nikodym derivative of μ equals $\sum_{ij} a_{ij}(x) D_i u(x) D_j u(x)$ a.e. in Ω_1. Therefore we have for the whole sequence

$$\lim_{h\to\infty} \int_{\Omega_1} \sum_{ij} a_{ij}^h(x) D_i u_h D_j u_h \varphi\, dx = \int_{\Omega_1} \sum_{ij} a_{ij}(x) D_i u D_j u \varphi\, dx \tag{3.11}$$

for every $\varphi \in C^0(\overline{\Omega_1})$.

Recall that in our hypotheses we have

$$\sum_{ij} a_{ij}(x,f_h) D_i f_h^r(x) D_j f_h^s(x) = J(x,f_h)\delta_{rs} \quad \text{a.e. in } \Omega_0,\ r, s = 1, \ldots, n. \tag{3.12}$$

From the symmetry of the matrix $[a_{ij}(x,f_h)]$, (3.11), (3.12) and Proposition 1.1 we get

$$\begin{aligned}\int_{\Omega_1} \sum_{ij} a_{ij}(x) D_i f^r D_j f^s \varphi\, dx &= \lim_{h\to\infty} \int_{\Omega_1} \sum_{ij} a_{ij}(x,f_h) D_i f_h^r D_j f_h^s \varphi\, dx \\ &= \lim_{h\to\infty} \int_{\Omega_1} J(x,f_h)\delta_{rs}\varphi\, dx = \int_{\Omega_1} J(x,f)\delta_{rs}\varphi\, dx\end{aligned} \tag{3.13}$$

where $\varphi \in C_0^\infty(\Omega_1)$, $r, s = 1, \ldots, n$.

Therefore if f is nonconstant, $J(x,f) \neq 0$ a.e. in Ω_0 and, passing to the Lebesgue points in (3.13), we have

$$a_{ij}(x) = a_{ij}(x,f) \quad \text{a.e. in } \Omega_1.$$

Since Ω_1 is arbitrary, the above equality holds a.e. in Ω_0.

If f is constant the assertion follows from (3.5), Lemma 3.1 and the nonnegativity of the functionals. ■

II. In this part we prove the main results of our paper. We will need the following lemma which can be proved by the same arguments as for Proposition 3.2 in [M Sb].

Lemma 3.3. *Assume that the hypotheses of Theorem* 3.2 *hold and suppose further that for a function* $w \in L^1_{\mathrm{loc}}(\Omega_0)$ *with* $w^{-1} \in L^1_{\mathrm{loc}}(\Omega_0)$

$$0 \leqslant w(x) \leqslant J(x,f_h)^{1-2/n} \leqslant \Lambda w(x) \quad \text{a.e. in } \Omega_0,\ \Lambda \geqslant 1,\ h \in N.$$

Then

$$\int_\Omega \sum_{ij} a_{ij}(x,f) D_i u D_j u\, dx = \Gamma^-(L^1(\Omega)) \lim_{\substack{h\to\infty\\ v\to u}} \int_\Omega \sum_{ij} a_{ij}(x,f_h) D_i v D_j v\, dx$$

for every $\Omega \Subset \Omega_0$ *and* u *in* $H_0^1(\Omega, w)$.

Theorem 3.4. *Let* (f_h) *be a sequence of* K*-quasiregular mappings on* Ω_0 *converging in* $L^1_{\mathrm{loc}}(\Omega_0)$ *to a function* f. *If* w *is a nonnegative function on* Ω_0 *such* *that* $w, w^{-1} \in L^1_{\mathrm{loc}}(\Omega_0)$ *and*

(3.14) $$0 \leqslant w(x) \leqslant J(x, f_h)^{1-2/n} \leqslant \Lambda w(x) \quad \text{a.e. in } \Omega_0,\ \Lambda \geqslant 1,\ h \in N,$$

then f *is* K*-quasiregular on* Ω_0 *and satisfies*

(3.15) $$0 \leqslant w(x) \leqslant J(x, f)^{1-2/n} \leqslant \Lambda w(x) \quad \text{a.e. in } \Omega_0.$$

Further, for every $\Omega \Subset \Omega_0$ *and for every* $g \in H^{-1}(\Omega, w)$ *the solutions of the problems*

(3.16) $$\min_{v \in H^1_0(\Omega, w)} \int_\Omega \sum_{ij} a_{ij}(x, f_h) D_i v D_j v\, dx + \langle g, v \rangle$$

converge in $L^1(\Omega)$ *to the solution of the problem*

(3.17) $$\min_{v \in H^1_0(\Omega, w)} \int_\Omega \sum_{ij} a_{ij}(x, f) D_i v D_j v\, dx + \langle g, v \rangle.$$

Proof. The K-quasiregularity of f and (3.15) follow from Proposition 1.1. Further, (3.15) implies that f is nonconstant. From Lemma 3.3 and the compact embedding of $H^1_0(\Omega, w)$ in $L^1(\Omega)$ it is easily verified that

(3.18) $$\int_\Omega \sum_{ij} a_{ij}(x, f) D_i u D_j u\, dx + \langle g, u \rangle = \Gamma^-(L^1(\Omega)) \lim_{\substack{h \to \infty \\ v \to u}} \int_\Omega \sum_{ij} a_{ij}(x, f_h) D_i v D_j v\, dx + \langle g, v \rangle$$

for every $\Omega \Subset \Omega_0$, u in $H^1_0(\Omega, w)$ and $g \in H^{-1}(\Omega, w)$. At this point we only have to observe that the functionals in (3.18) are equicoercive in the topology $L^1(\Omega)$. Hence the assertion follows from Theorem 1.7. ∎

Let us now prove that if the functions f_h are K-quasiconformal, the only hypothesis that w is in $L^1_{\mathrm{loc}}(\Omega_0)$ is sufficient to guarantee a convergence of the solutions of problems (3.16) to the solution of problem (3.17).

THEOREM 3.5. *Let* (f_h) *be a sequence of* K*-quasiconformal mappings on* Ω_0 *converging in* $L^1_{\mathrm{loc}}(\Omega_0)$ *to a function* f. *If* w *is a positive function on* Ω_0 *such that*

(3.19) $$0 < w(x) \leqslant J(x, f_h)^{1-2/n} \leqslant \Lambda w(x) \quad \text{a.e. in } \Omega_0,\ \Lambda \geqslant 1,\ h \in N,$$

then f *is* K*-quasiconformal on* Ω_0 *and satisfies*

(3.20) $$0 < w(x) \leqslant J(x, f)^{1-2/n} \leqslant \Lambda w(x) \quad \text{a.e. in } \Omega_0.$$

Further, for every $\Omega \Subset \Omega_0$ *and for every* g *in* $H^{-1}(\Omega, w)$ *the solutions of the problems*

$$\min_{v \in H^1_0(\Omega, w)} \int_\Omega \sum_{ij} a_{ij}(x, f_h) D_i v D_j v\, dx + \langle g, v \rangle$$

converge in $M_0(\Omega)$ *to the solution of the problem*

$$\min_{v \in H^1_0(\Omega, w)} \int_\Omega \sum_{ij} a_{ij}(x, f) D_i v D_j v\, dx + \langle g, v \rangle.$$

Proof. The K-quasiconformality of f and (3.20) follow from (3.19), Proposition 1.4, Remark 1.5 and Proposition 1.1. Hence it is not restrictive to assume that $w(x) = J(x, f)^{1-2/n}$.

As in Lemma 3.3 and in the first part of the proof of Theorem 3.4, by Corollary 2.4, it is easily verified that

(3.21) $$\int_\Omega \sum_{ij} a_{ij}(x, f) D_i u D_j u\, dx + \langle g, u \rangle = \Gamma^-(M_0(\Omega)) \lim_{\substack{h \to \infty \\ v \to u}} \int_\Omega \sum_{ij} a_{ij}(x, f_h) D_i v D_j v\, dx + \langle g, v \rangle$$

for every $\Omega \Subset \Omega_0$, $u \in H^1_0(\Omega, w)$, $g \in H^{-1}(\Omega, w)$. At this point we only have to observe that by Corollary 2.4 the functionals in (3.21) are equicoercive in the topology $M_0(\Omega)$. Hence the assertion follows from Theorem 1.7. ∎

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Improper integrals of distributions

by

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Abstract. We introduce a space of functions with bounded variation on $\boldsymbol{R}^n$ and call its dual space the space of improper integrable distributions. This definition turns out to be a generalization of the classical Schwartz definition of integrable distributions and of the definition of improper integrals for L^1_{loc} functions. We also define the improper convolution of distributions.

In this paper we define improper integrals for distributions. This is a slight modification of the definitions of Sikorski [15] and Musielak [9] and a generalization of the classical Schwartz definition. This modification allowed us to prove a representation theorem. The class of distributions having improper integrals turns out to be the dual space of a space which can be called a space of functions with bounded variation in $\boldsymbol{R}^n$.

We also define the convolution of two distributions using the notion of improper integral. This definition is more general than the classical Schwartz definition of convolution. We show that the exchange formula is still valid for the wider definition of convolution.

0. Notation, definitions and basic facts. We employ the usual notation of the theory of distributions. We denote by d_k, s_k for $k \in N_0 = \{0, 1, \ldots\}$ the seminorms in the spaces $\mathscr{D}(\boldsymbol{R}^n)$ and $\mathscr{S}(\boldsymbol{R}^n)$, i.e.

$$d_k(\varphi) = \sum_{|\alpha| \leqslant k} \sup |D^\alpha \varphi| \qquad \text{for } \varphi \in C_0^\infty(\boldsymbol{R}^n),$$

$$s_k(\sigma) = \sum_{|\alpha| \leqslant k} \sup_x \left|(1+|x|^2)^k D^\alpha \sigma(x)\right| \quad \text{for } \sigma \in \mathscr{S}(\boldsymbol{R}^n).$$

For every compact $K \subset \boldsymbol{R}^n$ and every $k \in N_0$, $\mathscr{D}_k(K)$ denotes the space $C_0^k(K)$ with topology given by the norm d_k; $\mathscr{S}_k(\boldsymbol{R}^n)$ is the space of all $\sigma \in C^k(\boldsymbol{R}^n)$ with $s_k(\sigma) < +\infty$, with topology given by s_k. $\mathscr{D}'_k(K)$, $\mathscr{S}'_k(\boldsymbol{R}^n)$ are their dual spaces.

The Fourier transform and inverse Fourier transform are denoted by "$\hat{\ }$" and "$\check{\ }$", i.e.

$$\hat{\sigma}(\xi) = (2\pi)^{-n/2} \int \sigma(x) e^{-ix\xi} dx \qquad \text{for } \sigma \in \mathscr{S}(\boldsymbol{R}^n).$$

If f is a function on $\boldsymbol{R}^n$ we define $\tilde{f}$ as the function satisfying $\tilde{f}(x) =$