$$\geqslant \frac{1}{|I_0|}\sum_n \sum_{\substack{I\in Q_{n,j(n)}\\ I\subset I_0}} |I|\,|\alpha_I|^2$$

$$\geqslant \delta^2 \frac{1}{|I_0|} \sum_{\substack{I\in Q_{n,j(n)}\\ I\subset I_0}} |I| \geqslant \delta^2 (1/M-\delta)^2 n_0.$$

On the other hand we get:

$$\Big\|\sum_{n=1}^{n_0} h_{n,j(n)}\Big\|_{\mathrm{BMO}} \leqslant 4.$$

Hence

$$\|P_{n_0}\| \geqslant \|\xi\|\,\|\xi^{-1}\| \geqslant \frac{1}{\delta} 4 \frac{\big\|\sum_n \xi h_{n,j(n)}\big\|}{\big\|\sum h_{n,j(n)}\big\|} \geqslant \frac{1}{\delta}\frac{\delta^2}{4} n_0^{1/2}.$$

Using the fact that i_{n_0} is an isometry and (*) we obtain the estimate $\delta \geqslant \frac{1}{3}$ and consequently $\|P_{n_0}\| \geqslant \frac{1}{12} n_0^{1/2}$.

Part (b) is a special case of Theorem 0. The estimate $\|P_{n_0}\| \leqslant 4$ follows from the calculations in this special case.

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Continuous factorizations of covariance operators and Gaussian processes

by

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Abstract. A bounded linear operator $Q\in L(E', E)$, defined on the dual E' of a Banach space E with values in E, is called a *covariance operator* if Q is positive, symmetric and compact. If E is separable, such an operator Q is always of the form $Q = T\circ T^*$ where T is a bounded linear operator from the Hilbert space l^2 into E. The following theorem is proved. Let $P_c(E)$ denote the set of all covariance operators. Then there is a universal map T from $P_c(E)$ into $L(l^2, E)$ such that $Q = T(Q)\circ T(Q)^*$ for all $Q\in P_c(E)$ and such that T is continuous, if $P_c(E)$ and $L(l^2, E)$ are equipped e.g. with the norm topology. Roughly speaking, it is always possible to make a continuous choice of "square roots" for a given continuous family of covariance operators. This pure functional analytic theorem has the following application to probability theory. If $(\varrho_s)_{s\in S}$ is a continuously indexed family of Gaussian measures on a separable Banach space E (continuous relative to the topology of weak convergence of probability measures), then there is always a Gaussian process $(X_s)_{s\in S}$ associated with the family $(\varrho_s)_{s\in S}$ which is e.g. mean square continuous.

1. Introduction. A (centered) Gaussian measure ϱ on a real separable Banach space E is usually defined as a probability measure on E such that all one-dimensional projections of ϱ are normal distributions with mean zero. It follows that the Fourier transform $\hat{\varrho}\colon E'\to \mathbf{C}$, defined on the dual E' of E, is given by

$$\hat{\varrho}(f) = \exp\Big(-\tfrac{1}{2}\int_E \langle x, f\rangle^2\, \varrho(dx)\Big)$$

for all $f\in E'$. Hence ϱ is uniquely determined by the bilinear form $\int_E x\otimes x\,\varrho(dx)$ on $E'\times E'$, defined by

$$\Big(\int_E x\otimes x\,\varrho(dx)\Big)(f, g) = \int_E \langle x, f\rangle\langle x, g\rangle\,\varrho(dx)$$

for all $f, g\in E'$. Since for a Gaussian measure we always have $\int \|x\|^2\varrho(dx) < \infty$, it follows that the bilinear form $\int x\otimes x\,\varrho(dx)$ is given by a continuous linear operator $Q\colon E'\to E$, where

$$\langle Qf, g\rangle = \int \langle x, f\rangle\langle x, g\rangle\,\varrho(dx) \qquad (f, g\in E').$$

Q is called the *covariance operator* of ϱ and has the following properties:

(i) Q is symmetric and positive (i.e. $\langle Qf, g\rangle = \langle Qg, f\rangle$ and $\langle Qf, f\rangle \geq 0$ for all $f, g \in E'$),

(ii) Q is compact

(see [1] and [4] as general reference for these and the following known results on Gaussian measures).

Every covariance operator Q has a natural factorization through a separable Hilbert space H, the so-called reproducing kernel Hilbert space, and there exists a compact operator T: $H \to E$ such that $Q = T \circ T^*$ (where T^* denotes the transpose of T).

This factorization of Q leads to the following important representation of ϱ as the distribution of a series of independent Gaussian random vectors all taking values in one-dimensional subspaces of E. Let $(e_k)_{k\geq 1}$ denote a complete orthonormal system of H, and let $(\xi_k)_{k\geq 1}$ be a sequence of independent identically distributed (i.i.d.) Gaussian random variables with mean zero and variance 1. Then $\sum_{k=1}^{\infty} T(e_k)\,\xi_k$ is a.s. convergent (also in the pth mean for any $p > 0$) to an E-valued Gaussian random vector X whose distribution is ϱ (see [1], p. 143, Theorem 6.8). The factorization of the covariance operator thus gives a natural method for constructing an associated Gaussian random vector of an especially nice simple structure.

Now suppose that we have not just a single Gaussian measure but a whole family $(\varrho_s)_{s\in S}$ of Gaussian measures, where S is – say – a metric index space. If we know that there exists a Gaussian process $(X_s)_{s\in S}$ (with $(\varrho_s)_{s\in S}$ as corresponding family of distributions) which is continuous in the pth mean for some $1 < p < \infty$, then it is easy to prove that the associated family $(Q_s)_{s\in S}$ of covariance operators is necessarily continuous in s relative to the operator norm.

Far more interesting is the following converse problem concerning the existence of Gaussian processes. Suppose that we are given a family $(Q_s)_{s\in S}$ of covariance operators such that $s \mapsto Q_s$ is continuous. Is it possible to construct a Gaussian process $(X_s)_{s\in S}$ from the given family $(Q_s)_{s\in S}$ such that $(X_s)_{s\in S}$ has nice continuity properties (e.g. mean square continuity, a.s. continuity)? A solution of this problem is known in the special case that S is an interval of the real line and (besides other additional assumptions) $s \mapsto \langle Q_s f, f\rangle$ $(f \in E')$ is continuously increasing (see [2]). In that case one gets the existence of a continuous Gaussian process with independent increments. But for a general index space S no answer seems to be known to this problem. As will be shown in a forthcoming paper (of E.D.), the solution of this problem has other interesting implications as well. Thus the problem also arises (and this was indeed the starting point) in connection with the construction of Banach space valued diffusion processes, where the only information we are given about the Gaussian random "perturbances" is a field $(Q_{t,x})_{t\geq 0, x\in E}$ of covariance operators. Even in the case when E is finite-dimensional or is a Hilbert space, this leads to a wider class of diffusion processes than the class obtained by the method of stochastic integral equations. This method is based on the assumption that the "random forces" driving the diffusion process are already given by a field $(X_{t,x}(\cdot))_{t\geq 0, x\in E}$ of homogeneous Gaussian processes with independent increments and that in addition this field has rather strong continuity properties (in t and x); at least local Lipschitz conditions have to be fulfilled. But even in the finite-dimensional case this is a severe restriction on the class of diffusions obtainable by this method (see [3] for a thorough discussion).

It turns out that the above-stated problem on the existence of Gaussian processes can be reduced to the following more general, purely analytic problem, which seems to be of interest in itself. Suppose that $(Q_s)_{s\in S}$ (S metric space) is a family of positive symmetric compact operators from E' into E such that $s \mapsto Q_s$ is continuous relative to a given operator topology (e.g. norm continuous or only strongly continuous). Does there exist a common separable Hilbert space H such that

(i) for every fixed $s \in S$ there is a compact operator from H into E such that $Q_s = T_s \circ T_s^*$, and

(ii) $s \mapsto T_s$ is continuous relative to a corresponding operator topology?

Our main result (see Theorem 1 below) shows that it is indeed always possible to make such a continuous choice of "square roots" – even for the whole cone of positive symmetric compact operators. Of course, only in case that E itself is a Hilbert space can one expect and, indeed, get a continuous choice of genuine square roots by an application of the spectral mapping theorem (see below).

These analytic results finally solve the probabilistic problem stated at the beginning. To any continuous family of covariance operators there always exists a corresponding Gaussian process, which is at least continuous in the pth mean for any $1 < p < \infty$ (see Theorem 2).

Before stating the main theorem let us make clear the notation and terminology we shall be using. Given Banach spaces E, F, let $L(E, F)$ be the usual space of bounded linear operators mapping E into F, and let $L_c(E, F)$ be those elements of $L(E, F)$ which are compact. We shall use $(L(E, F), \mathrm{n})$, $(L(E, F), \mathrm{s})$ and $(L(E, F), \mathrm{w})$ to denote $L(E, F)$ equipped with its norm, strong and weak-operator topologies respectively. Likewise, (E, n) and (E, w) will denote E with its norm and weak topologies, and (E', w^*) will denote E' with its weak* topology. Recall that if U is the unit ball of E' then (U, w^*) is always compact, and that, if E is separable, then (U, w^*) is metrizable, so that it is separable and sequentially compact.

For a Banach space E we denote by $P(E)$ the set of elements of $L(E', E)$

which are symmetric and positive, as defined above, and we denote by $P_c(E)$ those elements of $P(E)$ which are compact. A *factorization* of a subset $M \subseteq P(E)$ through a Hilbert space H is a map

$$Q \mapsto T_Q \quad (M \to L(H, E))$$

such that $Q = T_Q T_Q^*$, for all $Q \in M$. If $M \subseteq P_c(E)$ we demand further that $T_Q \in L_c(H, E)$, for all $Q \in M$. We shall say that a factorization is *norm continuous* if the map $Q \mapsto T_Q$ is continuous as a map from (M, n) to $(L(H, E), \mathrm{n})$, and that it is *strongly continuous* if the map $Q \mapsto T_Q$ is continuous as a map from (M, s) to $(L(H, E), \mathrm{s})$.

Our main theorem is as follows.

THEOREM 1. *Let E be a separable Banach space. Then there is a separable Hilbert space H and a factorization of $P_c(E)$ through H which is*

(i) *norm continuous,*

(ii) *strongly continuous on* n-*bounded subsets of* $P_c(E)$.

If E has finite dimension n, then so has H, and if E is infinite-dimensional, then $H \cong l^2$.

If E is reflexive, then in case (ii), *the continuity is uniform on* n-*bounded subsets of* $P_c(E)$.

There are two easy cases we can dispose of very quickly. First, suppose that E is a Hilbert space, so that $P(E)$ is just the set of all bounded positive operators on E. Every $Q \in P(E)$ has a unique square root $Q^{1/2} \in P(E)$, and it is easy for us to show that the map $Q \mapsto Q^{1/2}$ is both norm and strongly continuous on bounded sets. Indeed, suppose $R > 0$ and that M is the set of all $Q \in P(E)$ satisfying $\|Q\| \leqslant R$. Given $\varepsilon > 0$, we can choose a real polynomial p such that

$$\sup_{0 \leqslant \lambda \leqslant R} |p(\lambda) - \lambda^{1/2}| < \varepsilon/3.$$

Now the spectrum $\sigma(Q)$ of each $Q \in M$ is contained in the real interval $[0, R]$; so, by the spectral mapping theorem,

$$\|p(Q) - Q^{1/2}\| = \sup_{\lambda \in \sigma(Q)} |p(\lambda) - \lambda^{1/2}| < \varepsilon/3,$$

for all $Q \in M$. So, if $Q_1, Q_2 \in M$, we obviously have

$$\|Q_1^{1/2} - Q_2^{1/2}\| < \|p(Q_1) - p(Q_2)\| + 2\varepsilon/3.$$

It is easy to see that the map $Q \mapsto p(Q)$ is norm continuous, uniformly on M; so, clearly, $Q \to Q^{1/2}$ is uniformly norm continuous on M. The reader will be able to verify uniform strong continuity on M using the same idea. So, not surprisingly, we can prove Theorem 1 a fortiori for a Hilbert space E.

If Theorem 1 is true for Hilbert spaces, then it is evidently true for finite-dimensional Banach spaces. Indeed, if E is a Banach space with finite dimension n, we can choose an isomorphism J mapping E onto $\boldsymbol{R}^n$. If $Q \in P(E)$, then $JQJ^* \in P(\boldsymbol{R}^n)$, so we can take

$$T_Q = J^{-1}(JQJ^*)^{1/2}.$$

2. Proof of Theorem 1 for a separable infinite-dimensional Banach space.

2.1. *Preliminaries.* Let us begin by describing, in an informal way, the known construction of H and T_Q for a single operator $Q \in P(E)$. Let $K = \ker Q$. Then we can put an inner product $\langle\ ,\ \rangle_Q$ on E'/K, namely

$$\langle f+K, g+K\rangle_Q = \langle Qf, g\rangle \quad (f, g \in E'). \tag{2.1}$$

We let H_Q be the completion of E'/K with respect to $\langle\ ,\ \rangle_Q$. If $f \in E'$ we define $T_Q(f+K) = Qf$. This operator can be extended by continuity to the whole of H_Q, and we find that T_Q^* is just the natural homomorphism of E' into H_Q, and that $Q = T_Q T_Q^*$.

It is easy to verify the details of this construction (e.g. the consistency of the definition (2.1)) once we realize that an operator $Q \in P(E)$ has many of the familiar properties of positive operators on Hilbert spaces. Thus there is a Schwarz inequality:

$$\langle Qf, g\rangle^2 \leqslant \langle Qf, f\rangle \langle Qg, g\rangle \quad (f, g \in E'),$$

which is proved in the usual way. Two immediate corollaries follow:

$$\ker Q = \{f \in E';\ \langle Qf, f\rangle = 0\}, \tag{2.2}$$

$$\|Q\| = \sup_{\|f\|=1} \langle Qf, f\rangle. \tag{2.3}$$

Let us also note here (for future reference) that if H is a Hilbert space and $T \in L(H, E)$, then $TT^* \in P(E)$, whence it follows, by (2.3), that

$$\|TT^*\| = \|T^*\|^2 = \|T\|^2. \tag{2.4}$$

Suppose now that, for our fixed operator Q, H_Q is known to be separable and infinite-dimensional. Then we can find a sequence $(f_n) \subseteq E'$ such that $(f_n + K)$ is an orthonormal basis for H_Q. And we can identify H_Q with l^2 by mapping $f+K$ to its sequence $(\langle Qf_n, f\rangle)$ of Fourier coefficients.

It is a reasonable conjecture that if we allow Q to vary continuously in some suitably "thin" subset $M \subseteq P(E)$, then we can vary each f_n continuously so as to obtain a parametrized sequence $(f_n(Q)) \subseteq E'$ such that $(f_n(Q)+K)$ is an orthonormal basis for H_Q, for every $Q \in M$. Thus we may be able to factorize each $Q \in M$ through l^2. For a separable infinite-dimensional space E we are able to make precise this notion of "thinness" and prove a restricted version of Theorem 1 from which the full theorem can be deduced. So from now on let E definitely be separable and infinite-dimensional.

Given a subset $\mathscr{F} \subseteq E'$ let $L(\mathscr{F})$ denote the set of all finite linear combinations of elements of $\mathscr{F}$, let $\bar{L}(\mathscr{F})$ be the norm closure of $L(\mathscr{F})$ and let $L^*(\mathscr{F})$ be the weak* sequential closure of $L(\mathscr{F})$, i.e. $f \in L^*(\mathscr{F})$ if and only if there is a sequence $(f_n) \subseteq L(\mathscr{F})$ such that $\langle x, f_n\rangle \to \langle x, f\rangle$, for all $x \in E$. Notice that, because the unit ball U of E' is weak* separable, we can always find a sequence $(F_n) \subseteq E'$ which is linearly independent and for which $L^*(\{F_n\}) = E'$.

One property of $L^*(\mathscr{F})$ is worth noting here. Suppose that $A, B \in P_c(E)$ and that $Af = Bf$, for all f in some subset $\mathscr{F} \subseteq E'$; then we know very well that $Af = Bf$, for all $f \in \bar{L}(\mathscr{F})$. But, because A and B are compact and symmetric, we can say that $Af = Bf$, for all f in the larger set $L^*(\mathscr{F})$; for suppose $f \in L^*(\mathscr{F})$ and that $(f_n) \subseteq L(\mathscr{F})$ is a sequence converging weak* to f. By the uniform boundedness principle, (f_n) is bounded so, without loss of generality, there is a $y \in E$ such that $Af_n \to y$, in norm. Now $Af = y$ because, for all $h \in E'$,

$$\langle Af, h\rangle = \langle Ah, f\rangle = \lim_{n\to\infty} \langle Ah, f_n\rangle = \lim_{n\to\infty} \langle Af_n, h\rangle = \langle y, h\rangle.$$

But, equally, $Bf_n \to y$ and $Bf = y$; so $Af = Bf$.

2.2. *Factorization of a thin set.*

DEFINITION 1. Let E be a separable infinite-dimensional Banach space. A subset $M \subseteq P(E)$ is called *thin* if there is a sequence $(F_n) \subseteq E'$ such that

(i) (F_n) is linearly independent,

(ii) $L^*(\{F_n\}) = E'$,

(iii) $L(\{F_n\}) \cap \ker Q = (0)$, for all $Q \in M$.

Remark. Notice that the set of all injective elements of $P(E)$ is thin.

PROPOSITION 1. *Let E be a separable infinite-dimensional Banach space and let M be a thin subset of $P_c(E)$. Then there is a factorization of M through l^2 which is norm continuous and strongly continuous on bounded sets.*

Proof. Let (F_n) be as in Definition 1, and write $\mathscr{F}$ for the set $\{F_n\}_{n\geqslant 1}$. Because of condition (iii) and (2.2), the function $\langle\ ,\ \rangle_Q$ on $L(\mathscr{F}) \times L(\mathscr{F})$:

$$\langle f, g\rangle_Q = \langle Qf, g\rangle \quad (f, g \in L(\mathscr{F})) \tag{2.5}$$

is an inner product on $L(\mathscr{F})$ (and not just a nonnegative bilinear form). For each Q we can orthonormalize (F_n) using the Gram–Schmidt process so as to obtain a sequence $(f_n(Q)) \subseteq L(\mathscr{F})$ satisfying

$$\langle f_i(Q), f_j(Q)\rangle_Q = \langle Qf_i(Q), f_j(Q)\rangle = \delta_{ij}. \tag{2.6}$$

The $f_i(Q)$'s take the form

$$f_i(Q) = \sum_{j=1}^{i} a_{ij}(Q) F_j, \tag{2.7}$$

where $a_{ij}(Q) \in \boldsymbol{R}$, for $1 \leqslant j \leqslant i$. It is easy to verify, by induction, that each

map $Q \mapsto a_{ij}(Q)$ $(M \to \boldsymbol{R})$ is weak-operator continuous (hence also strongly and norm continuous). Notice also that, for $n \geqslant 1$ and $Q \in M$,

$$L(f_1(Q), f_2(Q), \ldots, f_n(Q)) = L(F_1, F_2, \ldots, F_n). \tag{2.8}$$

Let $H_Q = [L(\mathscr{F}), \langle\ ,\ \rangle_Q]^{\sim}$. Then, for every $Q \in M$, $(f_n(Q))$ is an orthonormal basis for H_Q; so for each $f \in L(\mathscr{F})$ we have Parseval's identity:

$$\sum_{i=1}^{\infty} \langle f, f_i(Q)\rangle_Q^2 = \|f\|_Q^2,$$

which means that

$$\sum_{i=1}^{\infty} \langle Qf_i(Q), f\rangle^2 = \langle Qf, f\rangle \quad (f \in L(\mathscr{F})). \tag{2.9}$$

We can extend equality (2.9) to the whole of E' using the following lemma.

LEMMA 1. *Let U be the unit ball of E' with its weak* topology and let $P_c(E)$ have its norm topology. Then the map*

$$(Q, f, g) \mapsto \langle Qf, g\rangle \quad (P_c(E) \times U \times U \to \boldsymbol{R})$$

is continuous.

Proof. It is sufficient to prove sequential continuity, because $P_c(E) \times U \times U$ is metrizable. So suppose $(Q_n) \subseteq P_c(E)$, $(f_n) \subseteq U$, $(g_n) \subseteq U$ are sequences, and that $Q_n \to Q$ in norm and $f_n \to f$, $g_n \to g$ weak*. For every n

$$\begin{aligned}|\langle Q_n f_n, g_n\rangle - \langle Qf, g\rangle| \leqslant\ & |\langle Q_n f_n, g_n\rangle - \langle Qf_n, g_n\rangle| \\ & + |\langle Qf_n, g_n\rangle - \langle Qf, g_n\rangle| \\ & + |\langle Qf, g_n\rangle - \langle Qf, g\rangle|.\end{aligned}$$

Because (f_n) and (g_n) are bounded, it is easy to see that the first and third terms on the right here tend to 0 as $n \to \infty$. To show that the second term tends to 0 it is sufficient to show that $\|Qf_n - Qf\| \to 0$. Now we certainly have $Qf_n \to Qf$ weakly, because, for $h \in E'$,

$$\langle Qf_n, h\rangle = \langle Qh, f_n\rangle \to \langle Qh, f\rangle = \langle Qf, h\rangle;$$

so it is sufficient to show that every subsequence of (Qf_n) has a subsequence which is norm convergent. But this is guaranteed by the compactness of Q: thus the lemma is proved. Obviously the same result is true if we replace $U \times U$ with $\lambda U \times \mu U$, where $\lambda, \mu > 0$.

We can now extend the identity (2.9) as promised. Given $f \in E'$ and $Q \in M$, let $J_Q f$ be the linear functional on $L(\mathscr{F})$:

$$J_Q f(g) = \langle Qg, f\rangle \quad (g \in L(\mathscr{F})). \tag{2.10}$$

Each $J_Q f$ is bounded, because, for all $g \in L(\mathscr{F})$,

$$|\langle Qg, f\rangle| \leqslant \langle Qg, g\rangle^{1/2} \langle Qf, f\rangle^{1/2} \leqslant (\|Q\|^{1/2} \|f\|) \|g\|_Q. \tag{2.11}$$

So we can identify $J_Q f$ with a unique element (also called $J_Q f$) of H_Q satisfying

$$\langle g, J_Q f\rangle_Q = \langle Qg, f\rangle. \tag{2.12}$$

Because of the uniqueness of $J_Q f$

$$J_Q f = f \quad (f \in L(\mathscr{F})).$$

Applying Parseval's identity to $J_Q f$ we have

$$\|J_Q f\|_Q^2 = \sum_{i=1}^{\infty} \langle f_i(Q), J_Q f\rangle_Q^2 = \sum_{i=1}^{\infty} \langle Qf_i(Q), f\rangle^2; \tag{2.13}$$

so we need to show that $\|J_Q f\|_Q^2 = \langle Qf, f\rangle$, for all $f \in E'$: we already know that this is true for all $f \in L(\mathscr{F})$, by (2.9).

So suppose $f \in E'$ and let $(g_n) \subset L(\mathscr{F})$ be a sequence which converges weak* to f: we shall show that $J_Q g_n \to J_Q f$ in norm. For every $h \in L(\mathscr{F})$,

$$\begin{aligned} |\langle h, J_Q g_n - J_Q f\rangle_Q| &= |\langle Qh, g_n - f\rangle| \\ &\leqslant \langle Qh, h\rangle^{1/2} \langle Q(g_n - f), g_n - f\rangle^{1/2} \\ &= \|h\|_Q \langle Q(g_n - f), g_n - f\rangle^{1/2}; \end{aligned}$$

therefore

$$\begin{aligned} \|J_Q g_n - J_Q f\|_Q &= \sup_{\substack{h \in L(\mathscr{F}) \\ \|h\|_Q = 1}} |\langle h, J_Q g_n - J_Q f\rangle_Q| \\ &\leqslant \langle Q(g_n - f), g_n - f\rangle^{1/2} \to 0, \end{aligned}$$

by Lemma 1, and hence

$$\|J_Q f\|_Q^2 = \lim_{n\to\infty} \|J_Q g_n\|_Q^2 = \lim_{n\to\infty} \langle Qg_n, g_n\rangle = \langle Qf, f\rangle,$$

again by Lemma 1. So, by (2.13), we now have

$$\sum_{i=1}^{\infty} \langle Qf_i(Q), f\rangle^2 = \langle Qf, f\rangle \quad (f \in E',\ Q \in M). \tag{2.14}$$

This identity will be crucial in what follows.

We shall define $T_Q\colon l^2 \to E$ by the equation

$$T_Q x = \sum_{i=1}^{\infty} x_i Qf_i(Q), \tag{2.15}$$

where $x = (x_1, x_2, \ldots, x_i, \ldots) \in l^2$. Let us consider first the convergence of the series in (2.15) for each fixed $Q \in M$.

For each $i \geqslant 1$, let $A_i(Q)\colon l^2 \to E$ be the rank-1 operator defined by the ith term of the series in (2.15):

$$A_i(Q)x = x_i Qf_i(Q) \quad (x \in l^2). \tag{2.16}$$

Clearly $A_i(Q) \in L_c(l^2, E)$. If $1 \leqslant m \leqslant n$,

$$\begin{aligned} \Big\|\sum_{i=m}^{n} A_i(Q)\Big\|^2 &= \sup_{\substack{x \in l^2 \\ \|x\| = 1}} \Big\|\sum_{i=m}^{n} x_i Qf_i(Q)\Big\|^2 \\ &= \sup_{\substack{x \in l^2 \\ \|x\| = 1}} \sup_{f \in U} \Big|\sum_{i=m}^{n} x_i \langle Qf_i(Q), f\rangle\Big|^2 \end{aligned}$$

where U is the unit ball of E'. Hence, by Schwarz' inequality,

$$\Big\|\sum_{i=m}^{n} A_i(Q)\Big\|^2 \leqslant \sup_{f \in U} \sum_{i=m}^{n} \langle Qf_i(Q), f\rangle^2. \tag{2.17}$$

Fix $Q \in M$. Then we see, by Lemma 1, that each term in the series in (2.14) is a continuous nonnegative function of f on (U, w^*); so also is the sum $\langle Qf, f\rangle$. Since (U, w^*) is compact, it follows from Dini's theorem that (2.14) holds *uniformly* over $f \in U$. Therefore, by (2.17), the series $\sum A_i(Q)$ converges in norm to an element $T_Q \in L_c(l^2, E)$ satisfying (2.15).

It is a simple matter to find T_Q^* and verify that $Q = T_Q T_Q^*$, for all $Q \in M$. If $x \in l^2$ and $f \in E'$ then, by (2.15),

$$\langle T_Q x, f\rangle = \sum_{i=1}^{\infty} x_i \langle Qf_i(Q), f\rangle,$$

so the ith component of $T_Q^* f$ is given by

$$(T_Q^* f)_i = \langle Qf_i(Q), f\rangle,$$

and hence

$$T_Q T_Q^* f = \sum_{i=1}^{\infty} \langle Qf_i(Q), f\rangle Qf_i(Q) \quad (f \in E',\ Q \in M).$$

By (2.6), $T_Q T_Q^* f = Qf$, for every $f_j(Q)$, and hence for every $F \in \mathscr{F}$, by (2.8). So, since $Q, T_Q T_Q^* \in P_c(E)$, we see that $T_Q T_Q^* f = Qf$ for all $f \in L^*(\mathscr{F}) = E'$. Thus $Q \mapsto T_Q$ is a factorization of M through l^2.

Now let us consider the continuity of the map $Q \mapsto T_Q$. First we shall show that each map $Q \mapsto A_i(Q)$ is norm continuous and strongly continuous. So fix $i \geqslant 1$ and let (Q_n) be a generalized sequence in M and let $Q \in M$.

Suppose first that $\|Q_n - Q\| \to 0$. Clearly, by (2.16),

$$\begin{aligned} \|A_i(Q_n) - A_i(Q)\| &= \|Q_n f_i(Q_n) - Qf_i(Q)\| \\ &= \Big\|\sum_{j=1}^{i} \big(a_{ij}(Q_n) Q_n F_j - a_{ij}(Q) QF_j\big)\Big\|, \end{aligned} \tag{2.18}$$

where a_{ij} is as in (2.7). We noted after (2.7) that each a_{ij} is weak-operator continuous, so, a fortiori, $a_{ij}(Q_n) \to a_{ij}(Q)$, for $1 \leq j \leq i$. Also $Q_n F_j \to QF_j$ in norm, so $\|A_i(Q_n) - A_i(Q)\| \to 0$. Thus $Q \mapsto A_i(Q)$ is norm continuous.

Now suppose $Q_n \to Q$ strongly. If $x \in l^2$, then by (2.16),

$$\|A_i(Q_n)x - A_i(Q)x\| = |x_i| \, \|Q_n f_i(Q_n) - Qf_i(Q)\|, \tag{2.19}$$

which tends to 0 as before. In fact, $Q_n \to Q$ strongly implies $A_i(Q_n) \to A_i(Q)$ in norm.

Next let us consider the norm continuity of the map $Q \mapsto T_Q$. Suppose that $(Q_k) \subseteq M$ is a sequence, that $Q \in M$ and that $\|Q_k - Q\| \to 0$. We know that, for each k,

$$T_{Q_k} = \sum_{i=1}^{\infty} A_i(Q_k), \tag{2.20}$$

in the norm topology of $L_c(l^2, E)$, To show that $\|T_{Q_k} - T_Q\| \to 0$ it is sufficient to show that the (norm) convergence in (2.20) is uniform over k. Now the set consisting of all the operators $(Q_k)_{k \geq 1}$ together with Q is norm compact; so, by (2.17), it is sufficient to show that, if $K \subseteq M$ is norm compact, then the series in (2.14) is uniformly convergent on $K \times U$. This can be done using Dini's theorem as before. Thus the right-hand side is certainly continuous on $K \times U$, by Lemma 1. Also by Lemma 1, each term on the left of (2.14) is continuous on $K \times U$, because it is an easy consequence of (2.7) et seq. that the map $Q \mapsto f_i(Q)$ $((K, \text{n}) \to (E', \text{w}^*))$ is continuous. Thus we have proved that $Q \mapsto T_Q$ is norm continuous.

Now let $M' \subseteq M$ be bounded – say $\|Q\| \leq R < \infty$, for all $Q \in M'$. To study strong continuity, let $x \in l^2$ be fixed but arbitrary. We know that $T_Q x = \sum A_i(Q)x$ in the norm topology of E; we shall show that the convergence is uniform over $Q \in M'$. Let $1 \leq m \leq n$; then for all $Q \in M'$,

$$\begin{aligned} \Big\|\sum_{i=m}^{n} A_i(Q)x\Big\|^2 = \Big\|\sum_{i=m}^{n} x_i Qf_i(Q)\Big\|^2 &= \sup_{f \in U} \Big|\sum_{i=m}^{n} x_i \langle Qf_i(Q), f\rangle\Big|^2 \\ &\leq \Big(\sum_{i=m}^{n} x_i^2\Big) \sup_{f \in U} \sum_{i=m}^{n} \langle Qf_i(Q), f\rangle^2 \\ &\leq \Big(\sum_{i=m}^{n} x_i^2\Big) \sup_{f \in U} \langle Qf, f\rangle \leq R\Big(\sum_{i=m}^{n} x_i^2\Big), \end{aligned}$$

by (2.14). Hence $T_Q x = \sum A_i(Q)x$, uniformly over $Q \in M'$, and so $Q \mapsto T_Q$ is strongly continuous on bounded subsets of M. So Proposition 1 is now proved.

2.3. *Factorization of* $P_c(E)$: *continuity*. There is a very natural way of factorizing $P_c(E)$ through l^2, given Proposition 1 and the following lemma.

LEMMA 2. *Let E be a separable infinite-dimensional Banach space, and let $(F_n) \subseteq E'$ be a sequence satisfying*

(i) *(F_n) is linearly independent,*

(ii) *$L^*(\{F_n\}) = E'$.*

Let $E_1 = l^2 \oplus E$. Then there is a sequence $(G_n) \subseteq E_1' = l^2 \oplus E'$ such that

(iii) *(G_n) is linearly independent,*

(iv) *$L^*(\{G_n\}) = E_1'$,*

(v) *$L(\{G_n\}) \cap E' = (0)$.*

Proof. The precise norm we put on $l^2 \oplus E$ is not important, but for definiteness let us define

$$\|(\xi, x)\| = \|\xi\| + \|x\| \qquad (\xi \in l^2,\ x \in E),$$

so that, for $(\eta, f) \in l^2 \oplus E' = E_1'$,

$$\|(\eta, f)\| = \max(\|\eta\|, \|f\|).$$

Notice that a sequence $((\eta_n, f_n)) \subseteq E_1'$ is weak* convergent to $(\eta, f) \in E'$ if and only if $\eta_n \to \eta$ weakly in l^2 and $f_n \to f$ weak* in E'.

For $n \geq 1$, let $h_n \in l^2$ be the real sequence

$$h_n = (0, 0, \ldots, 0, 1, 0, \ldots),$$

where the 1 appears in the nth place. Use $\mathscr{F}$ to denote the set $\{F_n\}_{n \geq 1}$ and $\mathscr{H}$ to denote the set $\{h_n\}_{n \geq 1}$, each set considered as a subset of E_1'. Clearly the union $\mathscr{F} \cup \mathscr{H}$ is linearly independent and $L^*(\mathscr{F} \cup \mathscr{H}) = E_1'$. We shall define (G_n) by perturbing the elements of $\mathscr{F}$.

Now $l^2/L(\mathscr{H})$ is infinite-dimensional, so there is a sequence $(k_n) \subseteq l^2$ such that the sequence $(k_n + L(\mathscr{H}))$ is linearly independent in $l^2/L(\mathscr{H})$, i.e.

$$\sum_{i=1}^{n} \lambda_i k_i \in L(\mathscr{H}) \quad \text{implies} \quad 0 = \lambda_1 = \lambda_2 = \ldots = \lambda_n.$$

It is easy to see, therefore, that the union $(h_n) \cup (k_n) \cup (F_n)$ of the three sequences is linearly independent in E_1'.

Let the sequence $(G_n) \subseteq E_1'$ be an ordering of the union $(h_n) \cup (F_n + k_n)$ of the two sequences (h_n), $(F_n + k_n)$. Clearly (G_n) is linearly independent, and if

$$\sum_{i=1}^{n} \big(\lambda_i h_i + \mu_i (F_i + k_i)\big) \in E'$$

then $\lambda_i = \mu_i = 0$, for $1 \leq i \leq n$. So (G_n) satisfies (iii) and (v).

To show that (G_n) satisfies (iv) note first that (G_n) contains an orthonormal basis (h_n) for l^2, so clearly

$$l^2 \subseteq L(\{G_n\}) \subseteq L^*(\{G_n\}).$$

For any fixed $N \geqslant 1$ we can write

$$k_N = \sum_{i=1}^{\infty} \xi_i h_i,$$

where $\xi_i = \langle k_N, h_i \rangle$ and the series converges in the norm of l^2; so we certainly have

$$F_N = (F_N + k_N) - \sum_{i=1}^{\infty} \xi_i h_i \in \bar{L}(\{G_n\}),$$

and hence $L(\mathscr{F}) \subseteq \bar{L}(\{G_n\})$. It follows that $E' \subseteq L^*(\{G_n\})$; for if $f \in E'$ we can find a sequence $(u_n) \subseteq L(\mathscr{F})$ such that $u_n \to f$ weak*, and we can find a sequence $(v_n) \subseteq L(\{G_n\})$ such that $\|u_n - v_n\| < 1/n$. It follows easily that $v_n \to f$ weak*. Now we see that (G_n) satisfies (iv), and hence the lemma is proved.

Now we can construct a factorization of the whole of $P_c(E)$ through l^2. Notice first that, since E is separable and infinite-dimensional, a sequence $(F_n) \subseteq E'$ satisfying (i) and (ii) of Lemma 2 certainly exists; so there is a sequence $(G_n) \subseteq E_1' = l^2 \oplus E'$ satisfying (iii), (iv) and (v).

Let K be a fixed compact positive injective operator on l^2. Given $Q \in P_c(E)$ define $\bar{Q} \in P_c(E_1)$ by

$$\bar{Q}(\xi, f) = (K\xi, Qf). \tag{2.21}$$

It is easy to verify that $\bar{Q}$ is indeed an element of $P_c(E_1)$, and also that

$$\ker \bar{Q} = \ker Q \subseteq E'.$$

It follows that the set $\bar{M}$ of all operators $\bar{Q}$ with $Q \in M$ is a thin subset of $P_c(E_1)$; so there is a factorization $\bar{Q} \mapsto T_{\bar{Q}}$ of $\bar{M}$ through l^2 which is norm continuous and strongly continuous on bounded sets.

Let P be the projection of E_1 onto E:

$$P(\xi, x) = x \quad (\xi \in l^2,\ x \in E).$$

Then P is bounded, and P^*: $E' \to E_1'$ is just the injection

$$P^* f = (0, f) \quad (f \in E').$$

Clearly $Q = P\bar{Q}P^*$, for all $Q \in P_c(E)$. So if we put $T_Q = PT_{\bar{Q}}$ we see that $T_Q \in L_c(l^2, E)$ and that $Q = T_Q T_Q^*$. So we have constructed a factorization of $P_c(E)$ through l^2, and, because P is bounded, this factorization is norm continuous and strongly continuous on bounded sets.

2.4. *Factorization of $P_c(E)$: uniform continuity.* We take the same approach as in Section 2.3: first we prove a lemma about thin sets, then we extend its scope using the map $Q \mapsto \bar{Q}$ and the projection P described above.

LEMMA 3. *Let E be a separable infinite-dimensional Banach space and let $M \subseteq P_c(E)$. Suppose that there is a thin subset $M' \subseteq P(E)$ such that $M \subseteq M'$ and suppose that M' is weak-operator compact. Then the factorization $Q \mapsto T_Q$ constructed in Proposition 1 is uniformly strongly continuous on M.*

Proof. Notice first that M is necessarily norm bounded.

At the beginning of the proof of Proposition 1 we orthonormalized the sequence (F_n) so as to obtain a sequence $(f_n(Q))$ satisfying (2.6) and (2.7), for all $Q \in M$. The compactness of the Q's was not used; so in the present context we can equally well obtain a sequence $(f_n(Q))$ satisfying (2.6) and (2.7), for all $Q \in M'$. Just as before, each map $Q \mapsto a_{ij}(Q)$ will be weak-operator continuous. But now, M' being compact, these maps will be uniformly weak-operator continuous and bounded on M', and hence, a fortiori, on M. We can now dispense with M' and consider the factorization of M as constructed in Proposition 1.

For each fixed $x \in l^2$, we have $T_Q x = \sum A_i(Q) x$ (see (2.15), (2.16)). The convergence of the series is in the norm topology of E, uniformly over $Q \in M$. So to prove Lemma 3 it is sufficient to show that, for every $x \in l^2$ and $i \geqslant 1$, the map $Q \mapsto A_i(Q) x$ $((M, \mathrm{s}) \to (E, \mathrm{n}))$ is uniformly continuous.

Given $Q_1, Q_2 \in M$, we have

$$\|A_i(Q_1)x - A_i(Q_2)x\| = |x_i| \Big\| \sum_{j=1}^{i} a_{ij}(Q_1) Q_1 F_j - \sum_{j=1}^{i} a_{ij}(Q_2) Q_2 F_j \Big\|,$$

by (2.16) and (2.7). Each map $Q \mapsto a_{ij}(Q)$ $((M, \mathrm{s}) \to \boldsymbol{R})$ is bounded and uniformly continuous: so also are the maps $Q \mapsto QF_j$ $((M, \mathrm{s}) \to (E, \mathrm{n}))$. So, clearly, each map $Q \mapsto A_i(Q) x$ $((M, \mathrm{s}) \to (E, \mathrm{n}))$ is uniformly continuous and the lemma is proved.

Finally, suppose that E is reflexive and that $M \subseteq P_c(E)$ is norm bounded — say $\|Q\| \leqslant R < \infty$, for all $Q \in M$. Let

$$M' = \{Q \in P(E);\ \|Q\| \leqslant R\};$$

then, because E is reflexive, M' is weak-operator compact. If $E_1 = l^2 \oplus E$ is as in Lemma 2, and if

$$\bar{M} = \{\bar{Q};\ Q \in M\}, \qquad \bar{M}' = \{\bar{Q};\ Q \in M'\},$$

where the extension $Q \mapsto \bar{Q}$ is defined as in (2.21), then it is easy to verify that E_1, $\bar{M}$, $\bar{M}'$ satisfy the hypotheses of Lemma 3. Therefore the map $\bar{Q} \mapsto T_{\bar{Q}}$ is uniformly strongly continuous on $\bar{M}$.

Since $T_Q = PT_{\bar{Q}}$, the map $Q \mapsto T_Q$ is the composition of three maps: first $Q \mapsto \bar{Q}$, then $\bar{Q} \mapsto T_{\bar{Q}}$ and finally multiplication on the left by P. Now all three maps are strongly uniformly continuous. This is obvious for the third map, and we have just dealt with the second. But in the case of the first map we need only notice that, for all $Q_1, Q_2 \in M$,

$$\|\bar{Q}_1(x, f) - \bar{Q}_2(x, f)\| = \|Q_1 f - Q_2 f\| \quad (x \in l^2, f \in E').$$

Thus the proof of Theorem 1 is complete.

Remarks. If we apply the method of Section 2.4 to study uniform norm continuity, we are only able to prove that $Q \mapsto T_Q$ is uniformly norm continuous on norm compact sets. But norm continuity already implies this.

The reader who is interested in functional analysis for its own sake will have no difficulty in proving that the factorization we have constructed is weak-operator continuous on bounded sets, and that, in the reflexive case, the continuity is uniform on bounded sets. A slightly more difficult exercise is to prove the following. "If E and E' are separable then there is a factorization of $P(E)$ through l^2 which is strongly continuous on bounded sets. If E is also reflexive then the continuity is uniform on bounded sets." This can be done by modifying our method. Because E' is separable there is a sequence $(F_n) \subseteq E'$ such that $\bar{L}(\{F_n\}) = E'$. The main modification to be made is to replace $L^*(\{F_n\})$ throughout with $\bar{L}(\{F_n\})$ (see for instance Definition 1). For the most part this involves a simplification of the argument.

3. Applications to the construction of Gaussian processes. Suppose that $(X_s)_{s\in S}$ (where S is a metric space) is an E-valued Gaussian process defined on some probability space $(\Omega, \mathscr{F}, P)$ such that $s \mapsto X_s$ is continuous in the pth mean for a $1 < p < \infty$. Then necessarily the family $(Q_s)_{s\in S}$ of covariance operators associated with $(X_s)_{s\in S}$ is norm continuous in s. This is a consequence of the fact that the continuity of $(X_s)_{s\in S}$ in the pth mean implies the weak continuity of the family $(\varrho_s)_{s\in S}$ of distributions of the process $(X_s)_{s\in S}$. But the weak continuity of $(\varrho_s)_{s\in S}$ implies the norm continuity of $(Q_s)_{s\in S}$, since $\lim_{n\to\infty} \varrho_{s_n} = \varrho_s$ weakly implies $\lim_{n\to\infty} \hat{\varrho}_{s_n} = \hat{\varrho}_s$ uniformly on the unit ball of E'. The main application of Theorem 1 is the following converse result.

THEOREM 2. *Let $(Q_s)_{s\in S}$ be a family of covariance operators of Gaussian measures ϱ_s such that $s \mapsto Q_s$ is strongly continuous. Then there exists a Gaussian process $(X_s)_{s\in S}$ such that $s \mapsto X_s$ is continuous in the p-th mean for any $1 \leqslant p < \infty$.*

Proof. By Theorem 1 there exists a family $(T_s)_{s\in S}$ in $L_c(l^2, E)$ which is strongly continuous on norm bounded subsets of $(Q_s)_{s\in S}$. Let $(s_n)_{n\geqslant 0}$ be an arbitrary sequence in S with $\lim_{n\to\infty} s_n = s_0$. The strong continuity of $(Q_s)_{s\in S}$ implies $\sup_{n\geqslant 0} \|Q_{s_n}\| < \infty$ by the uniform boundedness principle. Hence $\lim_{n\to\infty} T_{s_n} = T_{s_0}$ strongly and $\sup_{n\geqslant 0} \|T_{s_n}\| < \infty$ by (2.4). Now let $(e_k)_{k\geqslant 1}$ denote a complete orthonormal system of l^2, and let $(\xi_k)_{k\geqslant 1}$ be an i.i.d. sequence of Gaussian random variables with mean zero and variance 1, defined on some probability space $(\Omega, \mathscr{F}, P)$. Then for every fixed $s\in S$ the series

$$X_s = \sum_{k=1}^{\infty} T_s(e_k)\xi_k$$

is a.s. convergent and $X_s \in L^p(\Omega, \mathscr{F}, P; E)$. Moreover, $X_s(P) = \varrho_s$ ([1], p. 143, Theorem 6.8).

For every fixed $s\in S$ and $n\in N$

$$E\Big\|\sum_{k=n+1}^{\infty} T_s(e_k)\xi_k\Big\|^p \leqslant E\Big\|\sum_{k=n}^{\infty} T_s(e_k)\xi_k\Big\|^p, \tag{3.1}$$

since $X = \sum_{k=n+1}^{\infty} T_s(e_k)\xi_k$ and $Y = T_s(e_n)\xi_n$ are independent, mean zero random vectors (see [1], p. 103, Lemma 2.7). Since

$$\lim_{n\to\infty} E\Big\|\sum_{k=n}^{\infty} T_s(e_k)\xi_k\Big\|^p = 0,$$

we get

$$\lim_{n\to\infty} E\Big\|\sum_{k=n}^{\infty} T_s(e_k)\xi_k\Big\|^p = 0, \quad \textit{uniformly on } K = \{s_m\}_{m\geqslant 0}, \tag{3.2}$$

by Dini's theorem, since K is compact.

On the other hand, for every fixed $n\in N$ we have for every $\omega\in\Omega$

$$\lim_{m\to\infty} \Big\|\sum_{k=1}^{n} T_{s_m}(e_k)\xi_k(\omega) - \sum_{k=1}^{n} T_{s_0}(e_k)\xi_k(\omega)\Big\| = 0, \tag{3.3}$$

since $\lim_{m\to\infty} T_{s_m} = T_{s_0}$ strongly, and

$$\Big\|\sum_{k=1}^{n} T_{s_m}(e_k)\xi_k(\omega) - \sum_{k=1}^{n} T_{s_0}(e_k)\xi_k(\omega)\Big\| \leqslant 2\sup_{m\geqslant 0}\|T_{s_m}\|\,\Big\|\sum_{k=1}^{n} e_k\xi_k(\omega)\Big\|. \tag{3.4}$$

(3.3) and (3.4) show that we can apply Lebesgue's theorem to get

$$\lim_{m\to\infty} E\Big\|\sum_{k=1}^{n} T_{s_m}(e_k)\xi_k - \sum_{k=1}^{n} T_{s_0}(e_k)\xi_k\Big\|^p = 0, \tag{3.5}$$

for every fixed $n\in N$. (3.2) and (3.5) together now imply that

$$\lim_{m\to\infty} E\|X_{s_m} - X_{s_0}\|^p = 0,$$

i.e. $(X_s)_{s\in S}$ is continuous in the pth mean.

Remarks 1. Starting with a strongly continuous family $(Q_s)_{s\in S}$ of covariance operators, the continuity of the process $(X_s)_{s\in S}$ in Theorem 2 implies that $(Q_s)_{s\in S}$ must necessarily be norm continuous.

2. As mentioned above, the weak continuity of a family of Gaussian measures $(\varrho_s)_{s\in S}$ implies the norm continuity of the corresponding family $(Q_s)_{s\in S}$ of covariance operators. Hence Theorem 2 could be equivalently formulated as follows. To any weakly continuous family $(\varrho_s)_{s\in S}$ of Gaussian

measures, there exists a Gaussian process $(X_s)_{s\in S}$ such that the distribution of X_s is ϱ_s and $s\mapsto X_s$ is continuous in the pth mean for any $1\leqslant p<\infty$.

3. With similar arguments as in the proof of Theorem 2 one can also show that the process $(X_s)_{s\in S}$ of the theorem has the additional property that $\lim_{m\to\infty} X_{s_m}=X_{s_0}$ a.s. for any sequence $(s_m)_{m\geqslant 0}$ with $\lim_{m\to\infty} s_m=s_0$. But, of course, this does not necessarily imply that $(X_s)_{s\in S}$ is a.s. continuous.

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