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A note on Lie nilpotency in operator algebras

by

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Abstract. A Lie algebra A of operators is called ad-*nilpotent* if for each $a \in A$ there exists a positive integer $n(a)$ such that $(\operatorname{ad} a)^{n(a)} = 0$, and A is called ad-*quasi-nilpotent* if for each $a \in A$, $\|(\operatorname{ad} a)^n\|^{1/n} \underset{n}{\to} 0$. It is shown that Lie algebras which arise as Lie ideals of certain associative operator algebras and which are ad-quasi-nilpotent, are, in fact, central.

1. Introduction and notation. A *Banach–Lie* algebra A is a Banach space and a Lie algebra with a Lie product $[\ ,\ ]$: $A \times A \to A$ which is continuous in the norm topology on A. In [8] such an algebra was called *nilpotent* if for each $a \in A$ the operator $\operatorname{ad} a$: $A \to A$ defined by $(\operatorname{ad} a)(x) = [a, x]$ is quasi-nilpotent, i.e. $\|(\operatorname{ad} a)^n\|^{1/n} \underset{n}{\to} 0$. It was shown [8, Proposition 8] that if $K(X)$ is the collection of compact operators on a Banach space X considered as a Lie algebra with $[S, T] = ST - TS$, and L a Lie subalgebra consisting of quasi-nilpotent operators, then the uniform closure A of L is a nilpotent Banach–Lie subalgebra of $K(X)$ in the above sense. And conversely, it was shown [8, Theorem 1] that if A is a nilpotent Banach–Lie subalgebra of $K(X)$, then either X has a nonzero finite-dimensional A-invariant subspace or A consists of quasi-nilpotent operators.

If A is finite-dimensional, then to say that A is nilpotent in the above sense implies there is a positive integer n_0 such that $(\operatorname{ad} a)^{n_0} = 0$ for all $a \in A$. Engel's Theorem (cf. [4, p. 12]) states that this is equivalent to the condition that there exists a positive integer n, depending only on A, such that $(\operatorname{ad} a_1 \circ \operatorname{ad} a_2 \circ \ldots \circ \operatorname{ad} a_n)(x) = 0$ for all $a_i, x \in A$. This latter condition, phrased in terms of the termination of the descending central series for A is usually taken as the definition of a nilpotent Lie algebra (cf. [4, p. 11]).

We prefer to call a Lie algebra A of operators ad-*nilpotent* if for each $a \in A$ there exists a positive integer $n(a)$ such that $(\operatorname{ad} a)^{n(a)} = 0$. And we call such an algebra ad-*quasi-nilpotent* if $\|(\operatorname{ad} a)^n\|^{1/n} \underset{n}{\to} 0$ for each $a \in A$. In this

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note we study ad-nilpotent and ad-quasi-nilpotent Lie algebras of operators which arise as Lie ideals of certain associative operator algebras equipped with the Lie product as above. Our study is restricted to classes of algebras whose Lie ideal structure is well understood. It is shown that in these rather special cases, the lower central series terminates quickly.

If A is an associative complex algebra with multiplication $(x, y) \to xy$, then A becomes a complex Lie algebra with $[x, y] = xy - yx$. For subsets $B, C \subseteq A$ we define

$$[B, C] = \{\sum_{i=1}^{n} [b_i, c_i] \mid b_i \in B, c_i \in C\}.$$

A *Lie ideal* in A is a linear subspace $U \subseteq A$ for which $[A, U] \subseteq U$, and the *centre* of A is the Lie ideal $Z_A = \{z \in A \mid [z, A] = \{0\}\}$. We shall always assume that our algebras contain an identity 1. An algebra of operators A acting on a vector space X is called *n-fold transitive* if given independent vectors $x_1, \ldots, x_n$ and arbitrary vectors $y_1, \ldots, y_n$ in X there exists $a \in A$ such that $ax_i = y_i$. If X is a Banach space we denote the algebra of all bounded linear operators on X by $B(X)$.

2. Ad-nilpotent Lie ideals in $B(X)$.

LEMMA 1. *Let A be a 1-fold transitive algebra of operators acting on a Banach space X with $\dim X = \infty$, and let U be a Lie ideal in A. If $\operatorname{ad} a|_U$ is nilpotent for all $a \in A$, then $U \subseteq Z_A$.*

Proof. By [3, Lemma 1.3] either $U \subseteq Z_A$ or there exists a two-sided nonzero ideal $I \subseteq A$ such that $[A, I] \subseteq U$. In the latter case, I is also 1-fold transitive and so by [2, Lemma 2] I is $2m$-fold transitive for each m. If $x_1, \ldots, x_m$ are linearly independent and $y_1, \ldots, y_m$ arbitrary vectors in X, choose $j \in I$ such that

$$jx_i = \begin{cases} 0, & i = 1, \ldots, m, \\ -y_{i-m}, & i = m+1, \ldots, 2m, \end{cases}$$

and choose $a \in A$ such that

$$ax_i = \begin{cases} x_{m+i}, & i = 1, \ldots, m, \\ 0, & i = m+1, \ldots, 2m. \end{cases}$$

Then $[a, j](x_i) = y_i$ for $i = 1, \ldots, m$. This shows that $[A, I]$ is m-fold transitive, and hence U is m-fold transitive for each m.

We now proceed much as in [5, Lemma 3 and Theorem 3]. Suppose $a \in A$ and $(\operatorname{ad} a)^n|_U = 0$ where n will, in general, depend on a. We claim that for each $x \in X$ the subspace $W = \operatorname{Span}\{x, ax, \ldots, a^n x\}$ reduces a. For, if to the contrary $\{x, ax, \ldots, a^n x\}$ is independent, let $\lambda \in C$ and $v \in X$ be arbitrary and choose $u \in U$ such that $ua^{n-k}x = \lambda^{n-k} v$ for $k = 0, \ldots, n$ by the $m = n+1$-fold transitivity of U. Then

$$0 = ((\operatorname{ad} a)^n u)(x) = \sum_{k=0}^{n} \frac{1}{k!}(-1)^k a^k u a^{n-k} x$$

$$= \sum_{k=0}^{n} \frac{1}{k!}(-1)^k a^k \lambda^{n-k} v = (a-\lambda)^n v.$$

Hence $(a-\lambda)^n = 0$ for all $\lambda \in C$. The Hahn–Banach Theorem now implies a contradiction so that $\{x, ax, \ldots, a^n x\}$ is dependent for each $x \in X$. Thus, if $a \in A$ there is an eigenvalue $\lambda \in C$ and a corresponding eigenvector $v \neq 0$. If $b \in U$, then

$$0 = ((\operatorname{ad} a)^n b)(v) = (\operatorname{ad}(a-\lambda)^n b)(v) = \sum_{k=0}^{n} \frac{1}{k!}(-1)^k (a-\lambda)^{n-k} b (a-\lambda)^k v$$

$$= (a-\lambda)^n bv.$$

By the transitivity of U, $(a-\lambda)^n = 0$ so that the spectrum of a consists of a single point for each $a \in A$. This is impossible by the 2-fold transitivity of A.

THEOREM 1. *Let A be a semi-simple Banach algebra and $U \subseteq A$ a Lie ideal. If $\operatorname{ad} a|_U$ is nilpotent for all $a \in A$, then $U \subseteq Z_A$.*

Proof. Let I be a maximal modular left ideal in A and let T: $A \to B(A/I)$ be the left regular representation of A on A/I. Then $T(A)$ is a 1-fold transitive algebra acting on $X = A/I$ and $T(U)$ is a Lie ideal in $T(A)$. Moreover, if $a \in A$ then $\operatorname{ad} T(a)|_{T(U)}$ is a nilpotent operator for all $a \in A$. If X is finite-dimensional then, since $T(A)$ is irreducible, $T(A) = B(X)$ and so $T(U)$ is one of $\{0\}$, C, the trace zero elements of $B(X)$, or $B(X)$ itself by [3, Theorem 1.3]. If $\dim X > 1$ the classical Engel's Theorem rules out the last two possibilities for $T(U)$. Hence, if $\dim X < \infty$, $T(U) \subseteq C$. If $\dim X = \infty$, Lemma 1 implies $T(U) \subseteq C$. Hence in all cases, $[T(u), T(a)] = 0$ for $u \in U$, $a \in A$, or $[u, a] \subseteq \operatorname{kernel} T$. Since each primitive ideal arises as the kernel of such a T we have $[u, a] = 0$ by the semi-simplicity of A. Hence $U \subseteq Z_A$.

3. Ad-quasi-nilpotent Lie ideals in von Neumann algebras.

THEOREM 2. *Let U be a (not necessarily closed) Lie ideal in $B(H)$ where H is a separable Hilbert space. If $\operatorname{ad} u|_U$ is a quasi-nilpotent operator on U for all $u \in U$, then $U \subseteq C$.*

Proof. By the proof of Corollary 3 in [1] either $U \subseteq C$ or $\mathscr{F}_0 \subseteq U$ where $\mathscr{F}_0$ is the collection of finite rank operators of trace zero. In the latter case, $\|(\operatorname{ad} u)^n(x)\|^{1/n} \underset{n}{\to} 0$ for all $x \in \mathscr{F}_0$, $\|x\| = 1$, and all $u \in U$ since $\operatorname{ad} u$ is quasi-nilpotent. By [8, Lemma 13], this implies that the subspace $H_u^\lambda = \{v \in H \mid \|(u-\lambda)^n v\|^{1/n} \underset{n}{\to} 0\}$ is x-invariant for each $x \in \mathscr{F}_0$. Hence H_u^λ is x-invariant for each $x \in \mathscr{F}$ where $\mathscr{F}$ is the ideal of finite rank operators on H, and so H_u^λ is $\{0\}$ or H for each $\lambda \in C$, $u \in U$. Neither of these possibilities can occur since $\mathscr{F}_0 \subseteq U$.

LEMMA 2. *Let A be a von Neumann algebra and J a uniformly closed two-sided ideal in A. If $\operatorname{ad} j|_J$ is quasi-nilpotent for each $j \in J$, then J is abelian.*

Proof. If $a \in J^+$, the set of positive elements of J, and if $\{p_a^\lambda\}$ is the spectral resolution of a then $1 - p_a^\lambda = q_a \in J$ for $\lambda > 0$. Since each self-adjoint element is the difference of two positive elements in J and since J is uniformly closed, it suffices to show that if $q \in J$ and q is a projection, then $q \in Z_J$.

If $q = q^2 \in J$ then $(\operatorname{ad} q)^3(x) = (\operatorname{ad} q)(x)$ for all $x \in A$ so $(\operatorname{ad} q)^{2n+1} = \operatorname{ad} q$ for each positive integer n. Hence

$$\sup_{\substack{\|x\|=1\\ x\in J}} \|(\operatorname{ad} q)^{2n+1}(x)\| = \sup_{\substack{\|x\|=1\\ x\in J}} \|(\operatorname{ad} q)(x)\|.$$

Since $\operatorname{ad} q|_J$ is quasi-nilpotent,

$$\Big(\sup_{\substack{\|x\|=1\\ x\in J}} \|(\operatorname{ad} q)(x)\|\Big)^{1/(2n+1)} = \Big(\sup_{\substack{\|x\|=1\\ x\in J}} \|(\operatorname{ad} q)^{2n+1}(x)\|\Big)^{1/(2n+1)} \underset{n}{\to} 0$$

so that $\sup_{\|x\|=1} \|(\operatorname{ad} q)(x)\| = 0$. Consequently, $[q, x] = 0$ for all $x \in J$.

THEOREM 3. *Let A be a von Neumann algebra, and let U be a uniformly closed Lie ideal of A. If $\operatorname{ad} u|_U$ is quasi-nilpotent for each $u \in U$, then $U \subseteq Z_A$.*

Proof. By [6, Theorem 1] if A is infinite there exists a uniformly closed two-sided ideal J with $J \subseteq U \subseteq J + Z_A$. Hence in this case if $j \in J$ then $\operatorname{ad} j|_J$ is quasi-nilpotent. Similarly, if A is finite there exists a uniformly closed two-sided ideal J such that (1) $J \cap A_0 \subseteq U$ where $A_0 = \{a \in A \mid a^\# = 0\}$ and $a^\#$ is the centre-valued trace on A, and (2) $J \subseteq U + Z_A \subseteq J + Z_A$. Now if $j \in J$, $j - j^\# \in J \cap A_0 \subseteq U$ so that $\operatorname{ad}(j - j^\#)|_U = \operatorname{ad} j|_U$ is quasi-nilpotent. If $x \in J$, $\|x\| = 1$, then $x - x^\# \in J \cap A_0 \subseteq U$ and $\|x - x^\#\| \leqslant 2$ so that

$$\Big(\sup_{\substack{\|x\|=1\\ x\in J}} \|(\operatorname{ad} j)^n(x)\|\Big)^{1/n} = \Big(\sup_{\substack{\|x\|=1\\ x\in J}} \|(\operatorname{ad} j)^n(x - x^\#)\|\Big)^{1/n} \leqslant \Big(\sup_{\substack{\|u\|\leqslant 2\\ u\in U}} \|(\operatorname{ad} j)^n(u)\|\Big)^{1/n} \underset{n}{\to} 0.$$

Thus $\operatorname{ad} j|_J$ is quasi-nilpotent in the finite case also.

In both cases, by Lemma 2, J is abelian. This implies that the ultra-weak closure $J^{-\mathrm{uw}}$ of J is abelian and so $J^{-\mathrm{uw}} = Ac = \{ac \mid a \in A\}$ where c is a central abelian projection in A. This implies $J \subseteq J^{-\mathrm{uw}} = Ac = Z_{Ac} = (Z_A)c \subseteq Z_A$. Thus in both cases, $U \subseteq Z_A$.

In general, if A is any von Neumann algebra, then $A = Ac + A(1-c)$ where Ac is finite, $A(1-c)$ is infinite, and c is a central projection in A. The above considerations then apply to both summands.

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