STUDIA MATHEMATICA, T. LXXXV. (1987)

$\mathscr{P}$-Universally bounded unitary operators and the structure of locally convex vector spaces

by

J. VUKMAN (Maribor)

Abstract. Let X be a real or complex Hausdorff locally convex vector space, and let $L(X)$ be the algebra of all continuous linear operators of X into itself. Suppose there exists a $*$-algebra $B(X) \subset L(X)$ which contains the identity operator I and all continuous linear operators with finite-dimensional range. If for some calibration $\mathscr{P}$ which generates the topology on X, the relation $p_\alpha(Ux) \leqslant p_\alpha(x)$ holds for all $p_\alpha \in \mathscr{P}$ and all unitary operators $U \in B(X)$ (i.e. $U^* U = UU^* = I$), then an inner product $(\cdot, \cdot)$ can be introduced into X so that X equipped with this inner product is a Hilbert space and the topology induced by the inner product coincides with the given topology on X. For each $A \in B(X)$ the relation $(Ax, y) = (x, A^* y)$ holds for all pairs $x, y \in X$.

Let X be a real or complex locally convex vector space. A *calibration* is any family $\mathscr{P}$ of seminorms generating the topology of X, in the sense that the topology of X is the coarsest with respect to which all the seminorms in $\mathscr{P}$ are continuous. A calibration $\mathscr{P}$ is characterized by the property that the sets

$$\{x \in X;\ p_\alpha(x) \leqslant \varepsilon\}, \quad \varepsilon > 0,\ p_\alpha \in \mathscr{P},$$

constitute a neighbourhood sub-base at 0. Let $\mathscr{P}(X)$ denote the collection of all calibrations $\mathscr{P}$ on X which determine the topology of X.

A linear operator $A\colon X \to X$ will be called *$\mathscr{P}$-universally bounded* if for each $p_\alpha \in \mathscr{P}$, where $\mathscr{P}$ is a calibration, the relation $p_\alpha(Ax) \leqslant Cp_\alpha(x)$ holds for all $x \in X$ and some constant C. The set of all $\mathscr{P}$-universally bounded operators will be denoted by $B_{\mathscr{P}}(X)$. Obviously, each $\mathscr{P}$-universally bounded operator is continuous. We shall denote by $L(X)$ the algebra of all continuous linear operators of X into itself and by X^* the space of all continuous linear functionals acting on X. Let $f \in X^*$ be a fixed continuous linear functional and $x \in X$ a fixed vector. We shall write $x \otimes f$ for the

1980 *Mathematics Subject Classification*: 46A05, 46C99.

Key words and phrases: locally convex vector space, calibration, continuous linear operator, $\mathscr{P}$-universally bounded operator, involution, unitary operator, hermitian operator, inner product, Hilbert space.

This research was supported by the Research Council of Slovenia.

continuous linear operator defined by the relation $(x\otimes f)y=f(y)x$. By *involution* we mean a linear (in the complex case a conjugate-linear) mapping $A\mapsto A^*$ on $L(X)$ or on some subalgebra of $L(X)$ such that $(AB)^*=B^*A^*$ and $A^{**}=A$. We call a continuous linear operator U *unitary* if $U^*U=UU^*=I$, where I denotes the identity operator; a continuous linear operator H is called *hermitian* if $H^*=H$.

The $\mathscr{P}$-universally bounded operators, which were to our knowledge introduced by T. Moore [6], have been extensively studied (see [1], [3] where further references can be found). Let us point out that $B_{\mathscr{P}}(X)$ equipped with the norm

$$\|A\|_{\mathscr{P}}=\sup\{p_\alpha(Ax);\ p_\alpha(x)\leqslant 1 \text{ for all } x\in X \text{ and all } p_\alpha\in\mathscr{P}\}$$

is a unital normed algebra (see [6] for details). The main purpose of this paper is to prove the result below which characterizes Hilbert spaces among all Hausdorff locally convex vector spaces in terms of $\mathscr{P}$-universally bounded unitary operators.

THEOREM 1. *Let X be a real or complex Hausdorff locally convex vector space and let $B(X)$ be a subalgebra of $L(X)$ containing the identity operator I and all continuous linear operators with finite-dimensional range. Suppose further that there exists an involution $A\mapsto A^*$ on $B(X)$. If for some calibration $\mathscr{P}\in\mathscr{P}(X)$ the relation*

$$p_\alpha(Ux)\leqslant p_\alpha(x),\quad x\in X,$$

holds for all unitary operators $U\in B(X)$ and all $p_\alpha\in\mathscr{P}$, then an inner product $(\cdot,\cdot)$ can be introduced into X so that X equipped with this inner product is a Hilbert space and the topology induced by the inner product coincides with the given topology on X. For each $A\in B(X)$ the relation $(Ax, y)=(x, A^*y)$ holds for all pairs $x, y\in X$.

Remarks. It is somewhat surprising that in the result above we obtain the completeness of X without assuming the completeness of any kind. The involution is not assumed to be continuous. The assumptions are almost purely algebraic—the only condition in terms of seminorms is the condition concerning the unitary operators, but there is no assumption concerning the existence of nontrivial unitary operators. The question arises how can these, by our opinion, weak conditions imply that X is not only complete but even a Hilbert space. Speaking roughly, the completeness is a consequence of the fact that for each normed space X the dual space X^* equipped with the usual norm is complete. More precisely, at the end of the proof of Theorem 1 we shall use the following well-known result whose proof is almost obvious and will therefore be omitted.

LEMMA 2. *Let $(X,(\cdot,\cdot))$ be a real or complex pre-Hilbert space. If for each linear functional f on X which is continuous with respect to $(\cdot,\cdot)$, there exists $y\in X$ such that $f(x)=(x, y)$ for all $x\in X$, then $(X,(\cdot,\cdot))$ is a Hilbert space.*

Speaking about the existence of nontrivial unitary operators, let us point out that in the proof of Theorem 1 we shall construct these operators using continuous linear idempotents and it is obvious that at least continuous linear idempotents with finite-dimensional range exist in abundance since we require that $B(X)$ contains all continuous linear operators with finite-dimensional range.

Proof of Theorem 1. From the requirements of the theorem it follows immediately that the relation

$$p_\alpha(Ux)=p_\alpha(x),\quad x\in X, \tag{1}$$

holds for all unitary operators $U\in B(X)$ and all $p_\alpha\in\mathscr{P}$. We intend to prove that for an arbitrary idempotent $P\in B(X)$ the implication

$$P^*P=0\ \Rightarrow\ P=0 \tag{2}$$

holds. Suppose on the contrary that there exists a nonzero idempotent $P\in B(X)$ such that $P^*P=0$. Then a routine calculation shows that for any real number t the operator U_t defined by the relation

$$U_t=I+(\exp t-1)P+(\exp(-t)-1)P^*-\tfrac{1}{2}(\exp t+\exp(-t)-2)PP^*$$

is unitary. Let $e\in X$ be a nonzero vector such that $Pe=e$. Then $P^*P=0$ implies $P^*e=0$. Hence

$$U_te=(\exp t)e. \tag{3}$$

Since $e\neq 0$ there exists $p_\alpha\in\mathscr{P}$ such that $p_\alpha(e)\neq 0$ (recall that X is by assumption Hausdorff). Hence, according to (3) we have $p_\alpha(U_te)=(\exp t)p_\alpha(e)$ which is in contradiction with (1). The implication (2) is proved.

Now we are going to introduce an inner product into X. For this purpose let us choose a one-dimensional hermitian operator $H\in B(X)$. The existence of such an operator is ensured by the implication (2). Namely, for an arbitrary nonzero one-dimensional idempotent P we have a hermitian operator P^*P which is by (2) nonzero and it is obvious that the range of P^*P is one-dimensional. The operator H can be expressed in the form

$$H=e\otimes f_0 \tag{4}$$

where $e\in X$ is a fixed nonzero vector, and $f_0\in X^*$ a fixed nontrivial functional. It is easy to see that for each $A\in B(X)$ we have $HAH=f_0(Ae)H$. Therefore for each $A\in B(X)$ there exists a real (complex) number λ such that

$$HAH=\lambda H. \tag{5}$$

Let us denote by L the left ideal $B(X)H$. Let $A=A_1H$, $B=B_1H$ be from L. Using the fact that H is hermitian, we obtain $B^*A=(B_1H)^*(A_1H)=HB_1^*A_1H$. Hence, according to (5) for each pair $A, B\in L$ there exists a number λ such that $B^* A = \lambda H$. In other words, one can introduce a mapping $(\cdot, \cdot)$ from $L \times L$ into the real (complex) field as follows:

(6) $$(A, B) H = B^* A, \quad A, B \in L.$$

It is easy to see that the left ideal L contains exactly those operators which can be written in the form $x \otimes f_0$, where $x \in X$ is an arbitrary vector and f_0 the functional from (4). Using the isomorphism $x \mapsto x \otimes f_0$ one can introduce a mapping $(\cdot, \cdot)$ from $X \times X$ into the real (complex) field as follows:

(7) $$(x, y) = (x \otimes f_0, y \otimes f_0), \quad x, y \in X,$$

where $(x \otimes f_0, y \otimes f_0)$ means the same as in (6). It is obvious that $(\cdot, \cdot)$ is linear in the first argument for the fixed second. From the fact that H is hermitian, it follows immediately that $(x, y) = (y, x)$ (in the complex case $(x, y) = \overline{(y, x)}$) for all pairs $x, y \in X$. We do not know yet if the mapping $(\cdot, \cdot)$ is an inner product since we did not prove its positive definiteness. This problem will be considered later on; now we are going to prove that

(8) $$(Ax, y) = (x, A^* y)$$

for each $A \in B(X)$ and all pairs $x, y \in X$. Using the relation $(Ax) \otimes f_0 = A(x \otimes f_0)$, we obtain

$$\begin{aligned}(Ax, y) H &= ((Ax) \otimes f_0, y \otimes f_0) H \\ &= (A(x \otimes f_0), y \otimes f_0) H = (y \otimes f_0)^* A(x \otimes f_0).\end{aligned}$$

On the other hand

$$\begin{aligned}(x, A^* y) H &= (x \otimes f_0, (A^* y) \otimes f_0) H = (x \otimes f_0, A^*(y \otimes f_0)) H \\ &= (A^*(y \otimes f_0))^* (x \otimes f_0) = (y \otimes f_0)^* A(x \otimes f_0).\end{aligned}$$

Let us prove that all linear functionals of the form $f(x) = (x, y)$ are continuous. Let H, e and f_0 be from (4), and let us choose $u \in X$ such that $f_0(u) = 1$. Let $p_\alpha \in \mathscr{P}$ be such that $p_\alpha(e) \neq 0$. We have

$$\begin{aligned}|f(x)| p_\alpha(e) &= p_\alpha((x, y) Hu) = p_\alpha((y \otimes f_0)^* (x \otimes f_0) u) \\ &\leqslant q((x \otimes f_0) u) = q(x).\end{aligned}$$

Hence $|f(x)| \leqslant p_\alpha(e)^{-1} q(x)$, where q is some continuous seminorm, which proves the continuity of the functional.

Now we intend to prove that each continuous linear functional $f \in X^*$ can be written in the form $f(x) = (x, y)$ for some fixed $y \in X$. Let therefore $f \in X^*$ be given and let us choose z and y such that $(y, z) = 1$. According to (8) we have $f(x) = ((y \otimes f) x, z) = (x, (y \otimes f)^* z)$.

All is now prepared to prove the implication

(9) $$(x, x) = 0 \Rightarrow x = 0.$$

Assume on the contrary that there exists a nonzero vector $e \in X$ such that $(e, e) = 0$. Then the idempotent P defined by $P = e \otimes f$, $f \in X^*$, $f(e) = 1$, is nonzero. For all pairs $x, y \in X$ we have $(P^* Px, y) = (Px, Py) = (f(x) e, f(y) e) = 0$. Therefore $P^* P = 0$ which is in contradiction with $P \neq 0$ according to the implication (2).

Now we are going to prove that the mapping $(\cdot, \cdot)$ is positive or negative definite. We shall assume that X is a real space, the proof for the complex case is similar. Suppose there exist vectors $u, v \in X$ such that $(u, u) > 0$, $(v, v) < 0$. In this case there exists a real number t such that $t^2(u, u) + 2t(u, v) + (v, v) = 0$. In other words, for some real number t we have $(tu+v, tu+v) = 0$ which implies $tu+v = 0$ according to (9). Then $(v, v) = t^2(u, u)$ which is in contradiction with the assumption.

We may assume that $(\cdot, \cdot)$ is positive definite, since in case $(\cdot, \cdot)$ is negative definite one can introduce a positive definite mapping $(\cdot, \cdot)_0$ by $(x, y)_0 = -(x, y)$. Hence $(\cdot, \cdot)$ can be considered an inner product.

Now we intend to prove that the topology induced by the inner product $(\cdot, \cdot)$ coincides with the given topology on X. For this purpose let us prove that for each $p_\alpha \in \mathscr{P}$ the implication

(10) $$(e_1, e_1) = (e_2, e_2) = 1 \Rightarrow p_\alpha(e_1) = p_\alpha(e_2)$$

holds for all $e_1, e_2 \in X$. Let therefore $e_1, e_2 \in X$ be such that $(e_1, e_1) = (e_2, e_2) = 1$, and let us first assume that

(11) $$(e_1, e_2) = 0.$$

Let us define P_1 and P_2 by the relations $P_1 x = (x, e_1) e_1$, $P_2 x = (x, e_2) e_2$. It is not difficult to see that the idempotents P_1 and P_2 are hermitian. From (11) we obtain $P_1 P_2 = P_2 P_1 = 0$, whence it follows that the hermitian operator $P = P_1 + P_2$ is also an idempotent. According to (11), e_1 and e_2 are linearly independent. Denote by X_2 the two-dimensional subspace of X determined by e_1 and e_2, and let a subspace $X_c \subset X$ be such that $X = X_2 \oplus X_c$ is a decomposition of X made by the idempotent P. Then each $x \in X$ can be uniquely expressed in the form $x = \lambda_1 e_1 + \lambda_2 e_2 + x_c$, $x_c \in X_c$, which allows us to introduce a linear operator U by the relation $Ux = \lambda_2 e_1 + \lambda_1 e_2 + x_c$. It is not difficult to see that $U \in B(X)$ and that U is unitary. Therefore since $Ue_1 = e_2$ and since for each unitary operator the relation (1) holds, we have $p_\alpha(e_2) = p_\alpha(Ue_1) = p_\alpha(e_1)$, $p_\alpha \in \mathscr{P}$, which proves the implication (10) for the special case $(e_1, e_2) = 0$.

Let us prove the general case. Let therefore $e_1, e_2 \in X$ be vectors such that $(e_1, e_1) = (e_2, e_2) = 1$. There exists a nontrivial functional $f \in X^*$ such that $f(e_1) = f(e_2) = 0$. Since we have proved that each continuous linear functional can be represented by the inner product, it follows that there exists a nonzero vector $e \in X$ such that $(e_1, e) = (e_2, e) = 0$. We may assume that $(e, e) = 1$. Hence $p_\alpha(e_1) = p_\alpha(e_2)$, $p_\alpha \in \mathscr{P}$, which proves the implication

(10) in its full generality. From the implication (10) we deduce that for each $p_\alpha \in \mathscr{P}$ there exists a constant C_α such that

$$p_\alpha(x) = C_\alpha(x, x)^{1/2} \tag{12}$$

for all $x \in X$. From (12) we can conclude that each $p_\alpha \in \mathscr{P}$ is a norm and that all these norms are not only equivalent but in fact even equal. Therefore X equipped with the original topology can be considered a normed space, and according to (12) the topology induced by the inner product coincides with the original topology.

It remains to show that X equipped with the inner product is complete. Since we have just proved that the topology induced by the inner product coincides with the original topology, and since we know that linear functionals which are continuous with respect to the original topology are exactly those which can be represented by the inner product, all requirements of Lemma 2 are fulfilled and X is complete. The proof of the theorem is complete.

Remark. The proof of Theorem 1 is rather long, but it is elementary in the sense that we do not use any results and methods from B^*-algebra theory.

We conclude with the result below which also characterizes Hilbert spaces.

THEOREM 3. *Let X be a real or complex Hausdorff locally convex vector space and let $B(X)$ be a subalgebra of $L(X)$ containing the identity operator I and all continuous linear operators with finite-dimensional range. Suppose further that there exists an involution $A \mapsto A^*$ on $B(X)$. If the group of all unitary operators in $B(X)$ is equicontinuous, then an inner product $(\cdot, \cdot)$ can be introduced into X so that X equipped with this inner product is a Hilbert space and the topology induced by the inner product coincides with the given topology on X. For each $A \in B(X)$ the relation $(Ax, y) = (x, A^* y)$ holds for all pairs $x, y \in X$.*

Proof. By Theorem 4 in [6] there exists a calibration $\mathscr{P} \in \mathscr{P}(X)$ such that each unitary operator from $B(X)$ is contained in the unit ball of $B_{\mathscr{P}}(X)$ (so-called *recalibration*). Therefore all requirements of Theorem 1 are fulfilled and the proof is complete.

Concluding remarks. The history of the results presented in this paper began with the classical result of Kakutani and Mackey [4], [5] which characterizes real or complex Hilbert spaces among all Banach spaces in terms of involution on $L(X)$. A simple and elementary proof of the Kakutani–Mackey theorem can be found in J. Bognár's paper [2]. Some results in the sense of the Kakutani–Mackey theorem can be found in our earlier paper [7].

Acknowledgements. The author wishes to express his thanks to Professor S. J. Bhatt for his interest. He also thanks the members of the Seminar on Operator Theory, Department of Mathematics, University of Ljubljana, and in particular Professor M. Omladič, for their interest and fruitful discussions on the subject of this paper.

References

[1] S. J. Bhatt, *Quotient bounded elements in locally convex algebras*, J. Austral. Math. Soc. Ser. A 33 (1982), 102–113.
[2] J. Bognár, *Star-algebras induced by non-degenerate inner product*, Acta Sci. Math. (Szeged) 39 (1977), 15–26.
[3] G. A. Joseph, *Boundedness and completeness in locally convex spaces and algebras*, J. Austral. Math. Soc. Ser. A 24 (1977), 50–63.
[4] S. Kakutani and G. W. Mackey, *Two characterizations of real Hilbert space*, Ann. of Math. (2) 45 (1944), 50–58.
[5] –, –, *Ring and lattice characterizations of complex Hilbert space*, Bull. Amer. Math. Soc. 52 (1946), 727–733.
[6] R. T. Moore, *Banach algebras of operators on locally convex spaces*, ibid. 75 (1969), 68–73.
[7] J. Vukman, *Involution on $L(X)$*, Glasnik Mat. 17 (37) (1982), 65–72.

UNIVERSITY OF MARIBOR
VEKŠ
Razlagova 14, 62000 Maribor, Yugoslavia

Received July 18, 1985 (2068)