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Sharp local uncertainty inequalities

by

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Abstract. For each measurable $E \subseteq \boldsymbol{R}^d$ and $\alpha > d/2$, the local uncertainty principle inequality

$$\int_E |F(\omega)|^2 \, d\omega < \mathrm{const} \cdot m(E) \, \|f\|_2^{2-d/\alpha} \, \||t|^\alpha f\|_2^{d/\alpha}$$

is established for all $f \in L^2(\boldsymbol{R}^d)$ and the best constant found, where F denotes the Fourier transform of f. The constant only depends on d and α.

1. Introduction. Uncertainty principles in Fourier analysis assert that the more a function f is concentrated, the more its Fourier transform F will be spread out. The stronger concept that not only must the transform of a concentrated function be spread out, but that it cannot be "too" localized at any point is now referred to as the *local uncertainty principle.*

As an example, consider the following interpretation of certain Sobolev inequalities. Sobolev norms of a function F involve L^p norms of F and its derivatives, and derivatives of F are transforms of functions of the type $t^\alpha f(t)$, so that concentration of f implies small Sobolev norms. Now the basic Sobolev inequality gives bounds for $\|f\|_\infty$ in terms of its Sobolev norm. Hence concentration of f implies smallness of the L^∞ norm of its transform.

In this paper we develop a family of inequalities in their sharpest forms which more directly displays the principle of local uncertainty. Unless stated otherwise, all analysis will be over $\boldsymbol{R}^d$ and $\int$ will denote integration over this space. Given $f \in L^1 = L^1(\boldsymbol{R}^d)$, its *Fourier transform* F is defined by

$$F(\omega) = \hat{f}(\omega) = \int f(t) \exp(-2\pi i t \cdot \omega) \, dt, \qquad \omega \in \boldsymbol{R}^d,$$

where $t \cdot \omega = t_1 \omega_1 + \ldots + t_d \omega_d$. Throughout $K_1, K_2, \ldots$ will denote specific constants defined as the need arises.

The following is the main result where $\theta(d) = 2\pi^{d/2}/\Gamma(d/2)$ and $m(\cdot)$ denotes Lebesgue measure.

1.1. THEOREM. *Suppose $E \subseteq \boldsymbol{R}^d$ is measurable and $\alpha > d/2$. Then*

$$\int_E |F(\omega)|^2 \, d\omega < K_1 \, m(E) \, \|f\|_2^{2-d/\alpha} \, \||t|^\alpha f\|_2^{d/\alpha} \tag{1.1}$$

for all $f \in L^2$ *where*

$$K_1 = \frac{\theta(d)}{2\alpha}\Gamma\left(\frac{d}{2\alpha}\right)\Gamma\left(1-\frac{d}{2\alpha}\right)\left(\frac{2\alpha}{d}-1\right)^{d/2\alpha}\left(1-\frac{d}{2\alpha}\right)^{-1} \tag{1.2}$$

and $K_1 m(E)$ *is the smallest possible constant.*

1.2. Remarks. (i) The above family of inequalities is a generalization of

$$\int_E |F(\omega)|^2\, d\omega \leqslant 2\pi m(E)\, \|f\|_2\, \|tf\|_2$$

for $f \in L^1(\boldsymbol{R})$ and measurable $E \subseteq \boldsymbol{R}$, established in Faris [2, (3.2)]. (Notice that $K_1 = 2\pi$ when $d = 1 = \alpha$.) It is also related to and partially complements a family of inequalities developed in [5] which stem from [2] as well. This family includes:

$$\Big(\int_E |F(\omega)|^2\, d\omega\Big)^{1/2} \leqslant \mathrm{const}\cdot m(E)^{\alpha}\, \||t|^{\alpha} f\|_2$$

for all $f \in L^2$ and measurable $E \subseteq \boldsymbol{R}^d$ provided $0 \leqslant \alpha < d/2$.

(ii) The form of inequality (1.1) is unique in the following sense: given $\alpha, r, s \geqslant 0$, an inequality of the type

$$\int_E |F(\omega)|^2\, d\omega \leqslant \mathrm{const}\cdot m(E)\, \|f\|_2^r\, \||t|^{\alpha} f\|_2^s \tag{1.3}$$

is possible for all $f \in L^2$ and measurable $E \subseteq \boldsymbol{R}^d$ only if $\alpha > d/2$, $r = 2 - d/\alpha$ and $s = d/\alpha$.

To see this, first note that homogeneity requires $r + s = 2$. Next, replacement of f by its dilate f_a: $t \to f(at)$, $a > 0$, and E by aE converts the inequality to

$$a^{-d+s\alpha} \int_E |F(\omega)|^2\, d\omega \leqslant \mathrm{const}\cdot m(E)\, \|f\|_2^{2-s}\, \||t|^{\alpha} f\|_2^s.$$

(See the proof of Theorem 1.1 in § 3.) This requires $s = d/\alpha$ and hence $r = 2 - d/\alpha$. Finally, whenever $\alpha \leqslant d/2$, there exist functions $f \in L^2$ with $\||t|^{\alpha} f\|_2 < \infty$ so that F is infinitely differentiable everywhere except at $\omega = 0$ and as $\omega \to 0$, $F(\omega) \to \infty$ [8, Theorem 4]. By letting E range over a sequence of balls with radii tending to 0 and centres at 0, it is evident that (1.3) is contradicted.

1.3. *Constants.* The usual proofs of the general Sobolev inequalities give poor control of the constants. In linear partial differential equations this is of little consequence but can be more serious in the nonlinear situation. In quantum mechanics knowledge of the constants in Sobolev and related local uncertainty principle inequalities is frequently quite important since they can appear in estimates of thresholds for various physical phenomena. For example, such constants appear in certain descriptions of families of potentials which do not have negative energy bound states [2, 5]. They also appear in lower bounds for the energy per particle in bulk matter [4].

A number of authors, including Talenti [7] and Glaser, Martin, Grosse and Thirring [3], have obtained best constants for certain special, but useful, cases of the Sobolev inequality.

1.4. *Contents.* Throughout the paper care has been taken to obtain best constants in all cases and to describe the functions, if they exist, which give rise to them. Section 2 is concerned with a number of inequalities leading to the proof of the main result, Theorem 1.1, the proof being completed in Section 3.

By using a modified version of the usual Sobolev norm, the results of Section 2 are used in Section 4 to give a sharp form of a special case of the Sobolev inequality. The results of Section 5 give some precision to the piece of folklore that local uncertainty principles are strictly stronger than global ones. I am grateful to Henry Landau for many helpful remarks and conversations during all stages of this paper, from the initial vague ideas to the final typing.

2. Preliminary inequalities. This section opens with a pair of propositions which are just different versions of each other, one additive and one multiplicative. The process of passing from one to the other is the same as that employed in [1, Lemma 2.1]. Here we prove the additive version first and deduce the multiplicative from it. By using the calculus of variations it is possible to begin with a proof of the multiplicative version and then pass to the additive.

When it is stated that equality is achieved only for certain functions, it is meant that both sides of the inequality are to be equal and finite. $B(\ ,\ )$ denotes the Beta function.

2.1. Proposition. *Suppose* $p, q \in [1, \infty]$ *and* $\alpha > 0$ *satisfy* $p < q$ *and* $\alpha > d\beta$ *where* $\beta = 1/p - 1/q$. *Given* $\lambda, \mu > 0$,

$$\|f\|_p^q \leqslant K_2 \lambda^{-1} (\lambda/\mu)^{d\beta/\alpha} (\lambda \|f\|_q^q + \mu \||t|^{\alpha} f\|_q^q) \tag{2.1}$$

for all $f \in L^q$ *where*

$$K_2 = K_2(d, \alpha, p, q) = \big[(\theta(d)/\alpha q)\, B(d/\alpha q,\ 1/\beta q - d/\alpha q)\big]^{\beta q}. \tag{2.2}$$

There is equality (with both sides finite) if and only if f *satisfies*

$$|f| = \mathrm{const}\cdot(\lambda + \mu |t|^{\alpha q})^{-1/(q-p)}. \tag{2.3}$$

Proof. Assume the conditions of the theorem and let $r = q/p > 1$. Using Hölder's inequality with parameter r,

$$\begin{aligned}\|f\|_p^q &= \Big(\int (\lambda + \mu |t|^{\alpha q})^{1/r} |f|^p \cdot (\lambda + \mu |t|^{\alpha q})^{-1/r}\, dt\Big)^r \\ &\leqslant \|(\lambda + \mu |t|^{\alpha q})^{1/r} |f|^p\|_r^r\, \|(\lambda + \mu |t|^{\alpha q})^{-1/r}\|_{r'}^r.\end{aligned}$$

(As usual, r' is defined by $1/r+1/r' = 1$ with $1' = \infty$ and $\infty' = 1$.) The first norm is $\lambda\|f\|_q^q+\mu\||t|^\alpha f\|_q^q$ while the second is

$$\Big(\int(\lambda+\mu|t|^{\alpha q})^{-1/(r-1)}\,dt\Big)^{r-1} = \Big(\theta(d)\int_0^\infty(\lambda+\mu u^{\alpha q})^{-1/(r-1)}u^{d-1}\,du\Big)^{r-1}.$$

The substitution $v = (\mu/\lambda)u^{\alpha q}$ shows that this last integral is $K_2\lambda^{-1}(\lambda/\mu)^{d\beta/\alpha}$ which completes the demonstration of the inequality (2.1).

Equality can come about only if we have equality in Hölder's inequality which requires

$$[(\lambda+\mu|t|^{\alpha q})^{1/r}|f|^p]^r = \mathrm{const}\cdot[(\lambda+\mu|t|^{\alpha q})^{-1/r}]^{r'}.$$

Hence we have equality only if f is as described in (2.3). The proof is completed by noting that functions f of this type satisfy $\|f\|_q$, $\||t|^\alpha f\|_q < \infty$. ∎

2.1′. Proposition. *Suppose p, q and α are as in Proposition 2.1. Then*

$$\|f\|_p \leqslant K_3\|f\|_q^{(1-d\beta/\alpha)}\||t|^\alpha f\|_q^{d\beta/\alpha} \tag{2.4}$$

for all $f\in L^q$ where

$$K_3 = K_2^{1/q}(\alpha/d\beta-1)^{d\beta/q\alpha}(1-d\beta/\alpha)^{-1/q} \tag{2.5}$$

with equality if and only if f is of the form (2.3) *for some λ, $\mu>0$.*

Proof. For each $a>0$ define f_a by $f_a(t) = f(at)$, $t\in\mathbf{R}^d$. Then

$$\|f_a\|_p = a^{-d/p}\|f\|_p \quad\text{and}\quad \||t|^\alpha f_a\|_q = a^{-(\alpha+d/q)}\||t|^\alpha f\|_q.$$

Replacement of f by f_a in (2.1) gives

$$\|f\|_p^q \leqslant K_2\lambda^{-1}(\lambda/\mu)^{d\beta/\alpha}a^{d\beta q}(\lambda\|f\|_q^q+a^{-\alpha q}\mu\||t|^\alpha f\|_q^q). \tag{2.6}$$

The right side of (2.6) is minimized by choosing

$$a = (\alpha/d\beta-1)^{1/\alpha q}(\mu/\lambda)^{1/\alpha q}\|f\|_q^{-1/\alpha}\||t|^\alpha f\|_q^{1/\alpha}. \tag{2.7}$$

When this is done, (2.6) simplifies to the required (2.4).

It is not difficult (but needs care) to show that equality occurs in (2.4) when f is of the form (2.3) for some λ, $\mu>0$. To show that these are the only functions for which equality is achieved, begin by assuming we have equality in (2.4) for a particular function f. The argument in the preceding paragraph can be reversed to show that, for all λ, $\mu>0$, there is equality in (2.6) whenever a satisfies (2.7). But equality in (2.6) means precisely that

$$\|f_a\|_p^q = K_2\lambda^{-1}(\lambda/\mu)^{d\beta/\alpha}(\lambda\|f_a\|_q^q+\mu\||t|^\alpha f_a\|_q^q).$$

Proposition 2.1 now enters and confirms that f_a must be of the form

$$|f_a| = \mathrm{const}\cdot(\lambda+\mu|t|^{\alpha q})^{-1/(q-p)}.$$

Hence f must be of the form

$$|f| = \mathrm{const}\cdot(\lambda+\mu a^{-\alpha q}|t|^{\alpha q})^{-1/(q-p)},$$

which completes the proof since λ, $\mu>0$ are arbitrary. ∎

2.2. Corollary. *Suppose $q\in(1,\infty]$ and $\alpha>0$ satisfy $\alpha>d/q'$. Then*

$$\|F\|_\infty \leqslant K_4\|f\|_q^{1-d/\alpha q'}\||t|^\alpha f\|_q^{d/\alpha q'} \tag{2.8}$$

for all $f\in L^q$ where

$$K_4 = K_2^{1/q}(\alpha q'/d-1)^{d/qq'\alpha}(1-d/\alpha q')^{-1/q} \tag{2.9}$$

with equality if and only if f is of the form

$$f = c\exp(2\pi i\omega_0\cdot t)(\lambda+\mu|t|^{\alpha q})^{-1/(q-1)} \tag{2.10}$$

for some $c\in\mathbf{C}$, $\omega_0\in\mathbf{R}^d$ and λ, $\mu>0$.

Proof. When the right side of (2.8) is finite, $f\in L^1$ by 2.1′ so that F is defined and bounded. For such f,

$$\|F\|_\infty \leqslant \|f\|_1 \leqslant K_4\|f\|_q^{1-d/\alpha q'}\||t|^\alpha f\|_q^{d/\alpha q'} \tag{2.11}$$

by 2.1′, which establishes (2.8). Assume we have equality in (2.8). This means that the inequalities in (2.11) must be equalities. The second of these implies that f is of the form $f = \varphi(t)(\lambda+\mu|t|^{\alpha q})^{-1/(q-1)}$ for some λ, $\mu>0$ and measurable function φ with $|\varphi|$ a constant. Since $f\in L^1$, $F\in C_0$ so that there exists $\omega_0\in\mathbf{R}^d$ and $b\in\mathbf{C}$, $|b|=1$, with $\|F\|_\infty = bF(\omega_0)$. Equality in the first inequality of (2.11) now means that

$$b\int\varphi(t)(\lambda+\mu|t|^{\alpha q})^{-1/(q-1)}\exp(-2\pi i\omega_0\cdot t)\,dt = bF(\omega_0)$$
$$= \|F\|_\infty = \|f\|_1 = |\varphi|\int(\lambda+\mu|t|^{\alpha q})^{-1/(q-1)}\,dt.$$

This is possible only if $b\varphi(t)\exp(-2\pi i\omega_0\cdot t) = |\varphi|$, in other words, only if $\varphi(t) = c\exp(2\pi i\omega_0\cdot t)$ for some $c\in\mathbf{C}$, which completes the proof. ∎

3. Proof of Theorem 1.1. We are now in a position to prove Theorem 1.1. Choose measurable $E\subseteq\mathbf{R}^d$ with $0<m(E)<\infty$. Since

$$\int_E|F(\omega)|^2\,d\omega \leqslant m(E)\|F\|_\infty^2, \tag{3.1}$$

Corollary 2.2 with $\alpha>d/2$ and $q=2$ gives

$$\int_E|F(\omega)|^2\,d\omega \leqslant K_1 m(E)\|f\|_2^{2-d/\alpha}\||t|^\alpha f\|_2^{d/\alpha} \tag{3.2}$$

for all $f\in L^2$. What is now required is to show that equality is never attained and that $K_1 m(E)$ is the smallest possible constant.

Assume that (3.2) is valid for K instead of $K_1 m(E)$ where $K\leqslant K_1 m(E)$.

Given $a > 0$, define f_a by $f_a(t) = f(at)$. Replacement of f by f_a in the new (3.2) yields

$$m(a^{-1}E)^{-1} \int_{a^{-1}E} |F(\omega)|^2 d\omega \leqslant Km(E)^{-1} \|f\|_2^{2-d/\alpha} \||t|^\alpha f\|_2^{d/\alpha}, \tag{3.3}$$

where $a^{-1}E = \{a^{-1}t\colon t \in E\}$. (Use the identities at the commencement of the proof of 2.1′ along with $m(a^{-1}E) = a^{-d}m(E)$ and $\int_E |(f_a)\hat{}|^2 = a^{-d} \int_{a^{-1}E} |F|^2$.)

Without loss of generality assume that the right side of (3.2) is finite. Consequently $f \in L^1$ by Proposition 2.1′, whence F is continuous. With this the case, the limit as $a \to \infty$ of the left side of (3.3) is $|F(0)|^2$. (This result is routine when E is bounded. For the general result, commence by approximating E in measure by a bounded set.) Replacement of f with $\exp(2\pi i\omega\cdot t)f(t)$ for $\omega \in \boldsymbol{R}^d$ means that the left side in the new (3.3) will converge to $|F(\omega)|^2$. Hence

$$\|F\|_\infty^2 \leqslant Km(E)^{-1} \|f\|_2^{2-d/\alpha} \||t|^\alpha f\|_2^{d/\alpha}.$$

Comparison of this inequality with Corollary 2.2 shows that $Km(E)^{-1} \geqslant K_1$, so that $K = K_1 m(E)$, as required.

Finally, suppose that equality in (3.2) is achieved by some function f. This means that f must give equality in Corollary 2.2 with $q = 2$ and equality in (3.1). This first equality requires that f be of the form (2.10) with $q = 2$ while the second requires that its transform satisfy $F(\omega) = \|F\|_\infty$ a.e. for $\omega \in E$. These two requirements cannot be satisfied simultaneously since the maximum of $|F(\omega)|$ occurs at ω_0 and nowhere else when f is of the form (2.10). Hence there is never equality in (3.1). ∎

4. Sobolev inequalities. In this section we show our methods and results can be used to obtain some simple Sobolev inequalities. For any nonnegative integer k, the *Sobolev space* $L_k^2 = L_k^2(\boldsymbol{R}^d)$ is defined as the space of functions g with the property that $D^\theta g$ exists in the weak sense with $D^\theta g \in L^2$ for all $\theta = (\theta_1, \ldots, \theta_d)$ satisfying $|\theta| = \theta_1 + \ldots + \theta_d \leqslant k$. (Details are contained in Stein [6].) The standard norm on L_k^2 is

$$\|g\|_{L_k^2} = \sum_{|\theta| \leqslant k} \|D^\theta g\|_2,$$

under which it becomes a Banach space.

Since the Fourier transform provides an isometry on L^2 and since, formally, $(D^\theta g)\hat{} = (2\pi i t)^\theta \hat{g}$, it is clear that $g \in L_k^2$ if and only if $g \in L^2$ and $\mathscr{F}^{-1}(|t|^k \hat{g}) \in L^2$. Relaxing the restriction on k, for each $\alpha \geqslant 0$ we define the *Bessel potential space* $\mathscr{L}_\alpha^2$ as the space of functions g for which $g, \mathscr{F}^{-1}(|t|^\alpha \hat{g}) \in L^2$. For each $\lambda, \mu > 0$ define a norm on $\mathscr{L}_\alpha^2$ by

$$\|g\|_{2,\alpha} = \|\mathscr{F}^{-1}((\lambda + \mu|t|^{2\alpha})^{1/2} \hat{g})\|_2. \tag{4.1}$$

Evidently all these norms are equivalent for $\lambda, \mu > 0$ and make $\mathscr{L}_\alpha^2$ into a Banach space. (The usual norm on $\mathscr{L}_\alpha^2$ is slightly different [6].)

We can now give a sharp form of a Sobolev inequality.

4.1. Corollary. *Given $\lambda, \mu > 0$, suppose $\alpha > d/2$. Each $g \in \mathscr{L}_\alpha^2$ can be modified on a set of measure zero to become continuous and*

$$\|g\|_\infty \leqslant K_5 \lambda^{-1/2} (\lambda/\mu)^{d/4\alpha} \|g\|_{2,\alpha} \tag{4.2}$$

for all $g \in \mathscr{L}_\alpha^2$ where

$$K_5 = (\theta(d)/2\alpha)^{1/2} \Gamma(d/2\alpha)^{1/2} \Gamma(1 - d/2\alpha)^{1/2}. \tag{4.3}$$

There is equality if and only if g satisfies

$$\hat{g} = c\exp(2\pi i\omega_0\cdot t)(\lambda + \mu|t|^\alpha)^{-1} \tag{4.4}$$

for some $c \in \boldsymbol{C}$ and $\omega_0 \in \boldsymbol{R}^d$.

Proof. Given $g \in \mathscr{L}_\alpha^2$, choose f so that $f = \hat{g}$. Under the conditions of the corollary we have from Proposition 2.1 (taking $p = 1$ and $q = 2$):

$$\|g\|_\infty^2 \leqslant \|f\|_1^2 \leqslant K_5^2 \lambda^{-1} (\lambda/\mu)^{d/2\alpha} \{\lambda \|f\|_2^2 + \mu \||t|^\alpha f\|_2^2\}$$

which gives (4.2). Arguing as in the proof of 2.2 shows that only the functions of the form (4.4) give equality in (4.2).

The first part of the corollary is a consequence of C_c being dense in $\mathscr{L}_\alpha^2$ [6]. ∎

The preceding corollary can be used to obtain an explicit constant for the embedding described in 4.1 using the standard Sobolev norm introduced at the beginning of this section in place of the norm defined in (4.1). Given $k \in \boldsymbol{Z}^+$, define

$$C(d, k) = \max\{k!/\theta_1! \ldots \theta_d!\colon \theta_1 + \ldots + \theta_d = k\}.$$

It is easily seen that $C(d, k) = k!/(u!)^{d-v}/(u+1!)$ where $k = du + v$ with $u \geqslant 0$ and $0 \leqslant v < d$.

4.2. Corollary. *Suppose $k \in \boldsymbol{Z}^+$ satisfies $k > d/2$. For each $g \in L_k^2$,*

$$\|g\|_\infty \leqslant K_5 (2\pi)^{-d/2} C^{d/2k} \sum_{|\theta| \leqslant k} \|D^\theta g\|_2.$$

Proof. Let $\lambda = 1$ and $\mu = (2\pi)^{2k} C^{-2}$. Then

$$\begin{aligned}
\|g\|_{2,k} &= \left(\int (1 + \mu|t|^{2k}) |\hat{g}|^2 dt\right)^{1/2} \leqslant \|g\|_2 + \mu^{1/2} \||t|^k \hat{g}\|_2 \\
&\leqslant \|g\|_2 + \mu^{1/2} \|(|t_1| + \ldots + |t_d|)^k \hat{g}\|_2 \\
&\leqslant \|g\|_2 + \mu^{1/2} C \sum_{|\theta| = k} \|t^\theta \hat{g}\|_2 \\
&= \|g\|_2 + \mu^{1/2} C (2\pi)^{-k} \sum_{|\theta| = k} \|D^\theta g\|_2
\end{aligned}$$

$$\leqslant \sum_{|\theta|\leqslant k} \|D^\theta g\|_2,$$

with substitution in 4.2 completing the proof. ■

5. Uncertainty principle inequalities. Classical uncertainty principles assert that if f is concentrated, its transform F is spread out. However, they do not preclude the possibility that this spreading could be a result of F having a number of peaks far from the origin, whereas this is ruled out by local uncertainty principles. Thus, intuitively speaking, local uncertainty principle inequalities are strictly stronger than global ones.

One way to add precision to this is to notice that if f satisfies

$$\int_E |F(\omega)|^2\,d\omega \leqslant Cm(E)\,\|f\|_2^{2-d/\alpha}\,\||t|^\alpha f\|_2^{d/\alpha} \tag{5.1}$$

for all measurable $E \subseteq \boldsymbol{R}^d$ for some constant C and $\alpha > 0$, then

$$\|f\|_2^2 \leqslant C^{\alpha/d}\,\Omega_d^{\alpha/d}\left\{\left(\frac{2\alpha}{d}\right)^{d/(2\alpha+d)}+\left(\frac{d}{2\alpha}\right)^{2\alpha/(2\alpha+d)}\right\}^{(2\alpha+d)/2d}\||t|^\alpha f\|_2\,\||\omega|^\alpha F\|_2, \tag{5.2}$$

where $\Omega_d = \pi^{d/2}/\Gamma(1+d/2)$ is the volume of the unit ball in $\boldsymbol{R}^d$. (To see this, use (5.1) to establish

$$\|F\|_2^2 = \int_{|\omega|\leqslant b} |F(\omega)|^2\,d\omega + \int_{|\omega|>b} |F(\omega)|^2\,d\omega$$
$$\leqslant Cb^d\,\Omega_d\,\|f\|_2^{2-d/\alpha}\,\||t|^\alpha f\|_2^{d/\alpha} + b^{-2\alpha}\,\||\omega|^\alpha F\|_2^2;$$

substitution of $b = (2\alpha B/dA)^{1/(2\alpha+d)}$ where $A = C\Omega_d\|f\|_2^{2-d/\alpha}\||t|^\alpha f\|_2^{d/\alpha}$ and $B = \||\omega|^\alpha F\|_2^2$ completes the result.)

The significance of this is that smallness of the constant in (5.1) ensures smallness of the constant in (5.2). However, this is not possible in the opposite direction demonstrating the asymmetry between the families of inequalities. (Let ψ be a C^∞ positive function with support in the unit ball and define f_n by

$$F_n(\omega) = \psi(\omega-(n, 0, \ldots, 0)) + \psi(\omega+(n, 0, \ldots, 0)) \qquad (\omega \in \boldsymbol{R}^d,\ n \in \boldsymbol{Z}^+).$$

Suppose $\alpha > 0$. Since $\|f_n\|_2 = \|F_n\|_2$ is constant and $\||t|^\alpha f_n\|_2$ is uniformly bounded by $2\||t|^\alpha \mathscr{F}^{-1}\psi\|_2$, it is readily seen that

$$\|f_n\|_2^2/(\||t|^\alpha f_n\|_2\,\||\omega|^\alpha F_n\|_2) \to 0 \quad \text{as } n\to\infty$$

whereas

$$\sup\Big\{\int_E |F_n(\omega)|^2\,d\omega/m(E)\,\|f_n\|_2^{2-d/\alpha}\,\||t|^\alpha f_n\|_2^{d/\alpha}:\ \text{measurable } E \subseteq \boldsymbol{R}^d\Big\}$$

is bounded away from 0.)

Finally we point out that the methods leading from (5.1) to (5.2) couple with Theorem 1.1 to establish:

5.1. Corollary. *Suppose $\alpha > d/2$; then (5.2) is valid for all $f \in L^2$ with C replaced by K_1.*

Inequalities of this type are studied in detail in [1].

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