On basic Hahn–Banach extensions

by

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Abstract. A criterion is derived for the existence of a basic sequence of Hahn–Banach extensions of the coefficient functionals of a basic sequence of codimension one in a Banach space. Using this criterion and known results which are shown to characterize the usual basis for $c_0$, a negative answer is given to a question of Retherford concerning the existence of such extensions.

A problem of Retherford concerning the existence of norm-preserving extensions of coefficient functionals is the following (see [7, p. 66], or [6, p. 84]).

Given a basic sequence $\{x_n\}_{n=1}^{\infty}$ in a Banach space $X$ having coefficient functionals $\{x_n^*\}_{n=1}^{\infty}$ in $[x_n]^*$, does there exist a sequence of Hahn–Banach extensions of the functionals $\{x_n^*\}_{n=1}^{\infty}$ which is a basic sequence in $X^*$?

i.e. does there exist a sequence $\{g_n\}_{n=1}^{\infty}$ in $X^*$ for which $\|g_n\| = \|x_n^*\|$ for all $n$ and for which $\{x_n, g_n\}_{n=1}^{\infty}$ is a bi-basic system (see [2] and [6, p. 85])?

In a recent paper [7] Terenzi has given a partial answer to this question by showing that there always exists some basic sequence $\{g_n\}_{n=1}^{\infty}$ in $X^*$ which is biorthogonal to $\{x_n\}_{n=1}^{\infty}$, but it may not be that $\|g_n\| = \|x_n^*\|$ for all $n$. In fact, the proof does not even guarantee that $\sup_n \|x_n\| \|g_n\| < +\infty$.

In the purpose of this paper is to give a negative answer to the question of Retherford as an outgrowth of a general study of the problem of existence of basic Hahn–Banach extensions of coefficient functionals in the simplest possible case, where $\text{codim} [x_n]_{n=1}^{\infty} = 1$. This result is a consequence of a general existence criterion for such Hahn–Banach extensions (Theorem 1) and of related results which essentially show that the guaranteed existence of such extensions characterizes the unit vector basis $\{e_n\}_{n=1}^{\infty}$ for $c_0$ (Proposition 1 and Theorem 3). We begin with general discussion of Hahn–Banach extensions which will culminate in the first of these results.

Suppose $\{x_n\}_{n=1}^{\infty}$ is a basic sequence in $X$ for which $M = [x_n]_{n=1}^{\infty}$ is of codimension one in $X$. Then $\{x_n\}_{n=1}^{\infty}$ is a basis for $M$ having coefficient

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functionals \( \{ x_n^* \}_{n=1}^\infty \) in \( M^* \), and if \( x_0 \) is any vector not in \( M \) the sequence \( \{ x_n^* \}_{n=1}^\infty \) is a basis for \( X \) whose sequence of coefficient functionals \( \{ f_n \}_{n=1}^\infty \) forms a basic sequence in \( X^* \). Since \( \langle f_n, x_n^* \rangle = \alpha_{n0} \) for all \( n \) and \( m \) it must be that \( M^* = \{ f_0 \} \) and \( f_0^* = x_0^* \) for \( n \geq 1 \) if \( x_0 \) is any sequence of extensions of \( \{ x_n^* \}_{n=1}^\infty \) to \( X \) such that \( g_n^* = f_n^* = x_n^* \), from which it follows that \( g_n = f_n + \lambda_{n0} \lambda_n \) and hence that \( g_n = f_n - \lambda_n \lambda_{n0} \) for some scalars \( \alpha_{n0} \). Conversely, any sequence in \( X^* \) of the form \( \{ f_n - \lambda_n \lambda_{n0} \}_{n=1}^\infty \) is clearly a sequence of extensions of \( \{ x_n^* \}_{n=1}^\infty \) to \( X \). Therefore a sequence of \( X^* \) is a sequence of Hahn--Banach extensions of \( \{ x_n^* \}_{n=1}^\infty \) if it is of the form \( \{ f_n - \lambda_n \lambda_{n0} \}_{n=1}^\infty \), where \( \| f_n - \lambda_n \lambda_{n0} \| = \| x_n^* \| = \| f_0^* \| \).

But it is well known that if \( h \in X^* \) then

\[
\| h \| = \inf_{f \in M^*} \| h - f \| \quad [4, p. 121].
\]

so

\[
\| f_n - \lambda_n \lambda_{n0} \| = \| f_0^* \| \iff \| f_n - \lambda_n \lambda_{n0} \| = \inf_{f \in M^*} \| h - f \| = \text{dist}(f_n, f_0) = \text{dist}(f_n, f_0).
\]

That is, there exists a basic sequence of Hahn--Banach extensions of \( \{ x_n^* \}_{n=1}^\infty \) if there is a sequence of scalars \( \{ \alpha_{n0} \}_{n=1}^\infty \) for which \( f_n^* - \alpha_{n0} f_0^* \) is \text{dist}(f_n, f_0) \) and \( \{ f_n - \alpha_{n0} f_0 \}_{n=1}^\infty \) is a basic sequence in \( X^* \).

Now in a previous paper \[3\] we showed that the sequence of the form \( \{ f_n - \lambda_n \lambda_{n0} \}_{n=1}^\infty \) is a basic sequence in \( X^* \) if \( x_0 \in X \) and \( \{ \lambda_n \}_{n=1}^\infty \) is an \( \| \cdot \| \) in \( X^* \).

Hence \( \{ f_n - \lambda_n \lambda_{n0} \}_{n=1}^\infty \) is a basic sequence in \( X^* \) if \( G \) is a \( \| \cdot \| \) in \( X^* \). Therefore, \( \{ f_n - \lambda_n \lambda_{n0} \}_{n=1}^\infty \) is a basic sequence in \( X^* \).

Suppose \( \| f_n - \lambda_n \lambda_{n0} \| = \| f_0^* \| \) for all \( n \geq 1 \) and \( \| f_n \| = \| f_0 \| \) for all \( n \geq 1 \). Consequently, \( \{ f_n - \lambda_n \lambda_{n0} \}_{n=1}^\infty \) is a basic sequence in \( X^* \).  

satisfies \( \| f_n - \lambda_n \lambda_{n0} \| = \| f_0^* \| \) for all \( n \geq 1 \) and \( \| f_n \| = \| f_0 \| \) for all \( n \geq 1 \). Consequently, \( \{ f_n - \lambda_n \lambda_{n0} \}_{n=1}^\infty \) is a basic sequence in \( X^* \).  

Proof. Let \( \| \cdot \| \) denote both the original norm on \( X \) and the dual norm on \( X^* \). If \( x_0 \notin \{ x_n^* \}_{n=1}^\infty \) then \( \{ x_n^* \}_{n=1}^\infty \) is a basis for \( (X, \| \cdot \|) \) with coefficient functionals \( \{ f_n \}_{n=1}^\infty \) in \( X^* \). Since there is an equivalent norm on \( X \) under which \( \{ x_n^* \}_{n=1}^\infty \) is a normalized monotone basis \( [5, p. 250] \), we may assume \( \{ x_n^* \}_{n=1}^\infty \) is a monotone basis for \( (X, \| \cdot \|) \) with \( \| x_n \| = 1 \) for all \( n \geq 1 \).

Consequently we will have \( \{ f_n \}_{n=1}^\infty \) is a monotone bases in \( X \) of \( (X, \| \cdot \|) \) and \( \| \cdot \| \) is an isometry \( [5, p. 115] \). It follows that if we define an equivalent norm on \( \{ f_n \}_{n=1}^\infty \), say \( \| \cdot \| \), then the expression \( \| x_n \| = \sum_{n=1}^\infty \| f_n \| < 1 \) defines a norm on \( X \) equivalent to \( \| \cdot \| \).

Moreover, if the basic \( \{ f_n \}_{n=1}^\infty \) for \( \{ f_n \}_{n=1}^\infty \) is still monotone then for any \( f \in \{ f_n \}_{n=1}^\infty \) we have \( \| f \| = \sup \{ f(x), \| f \| < 1 \}, x \in X \) \{ f_n \}_{n=1}^\infty \) and \( \| \cdot \| \) will be embedded isometrically in \( (X, \| \cdot \|) \). That is, such a renorming \( \{ f_n \}_{n=1}^\infty \) induces an equivalent renorming of \( X \) with the property that the new dual norm on \( X^* \) agrees with the newly defined norm on \( \{ f_n \}_{n=1}^\infty \).

With this in mind we define on the space \( \{ f_n \}_{n=1}^\infty \) the norm

\[
\| f \| = \left( \sum_{n=0}^\infty c_n \| f_n \| \right)^{\frac{1}{2}} = \| c_0 f_0 \| + \left( \sum_{n=1}^\infty c_n \| f_n \| \right)^{\frac{1}{2}}.
\]

Obviously \( \| f \| \) is equivalent to \( \| f \| \) on \( \{ f_n \}_{n=1}^\infty \), and in this norm \( \{ f_n \}_{n=1}^\infty \) is still monotone. Therefore, by the above, if we define a new norm on \( X \) by \( \| f \| = \sup \{ f(x), \| f \| < 1 \}, x \in X \) \{ f_n \}_{n=1}^\infty \) then \( \| \cdot \| \) is equivalent to \( \| \cdot \| \) and \( \{ f_n \}_{n=1}^\infty \) is \( (X, \| \cdot \|) \) (isometrically). But for any \( n \geq 1 \),

\[
\inf_{f \in X^*} \| f_n - \lambda_n \lambda_{n0} \| = \inf_{f \in X^*} \| f - \lambda_n \lambda_{n0} \| = \| f_0^* \|.
\]
and this inf is attained only when \( \lambda = \lambda_n = 0 \). Hence by Theorem 1 it follows that the coefficient functionals \( \{x_n^*\}_{n=1}^{\infty} \) of the basic sequence \( \{x_n\}_{n=1}^{\infty} \) in \( (X, \|\cdot\|) \) have Hahn–Banach extensions which are a basic sequence in \( (X, \|\cdot\|) \), and the proof is complete.

Theorem 2 shows that one can (at least) equivalently renorm \( X \) to obtain a basic sequence of Hahn–Banach extensions for the coefficient functionals of a basic sequence of codimension one. Our next result shows that in the case of one particular type of basic sequence no renorming is necessary, even when \( \text{codim}(x^*) = +\infty \).

**Proposition 1.** Let \( \{x_n\}_{n=1}^{\infty} \) be a basic sequence in \( X \) which is equivalent to the usual basis \( \{e_n\}_{n=1}^{\infty} \) of \( c_0 \). Then there is a basic sequence in \( X^* \) of Hahn–Banach extensions of the coefficient functionals for \( \{x_n\}_{n=1}^{\infty} \).

**Proof.** If \( \{x_n\}_{n=1}^{\infty} \) is equivalent to the basis \( \{e_n\}_{n=1}^{\infty} \) for \( c_0 \) then, in particular, \( 0 < \delta = \inf \|x_n\| \leq \|x_0\| \leq M < +\infty \) for some \( \delta \) and \( M \).

Suppose \( \{x_n^*\}_{n=1}^{\infty} = [x^*] \) is biorthogonal to \( \{x_n\}_{n=1}^{\infty} \), and let \( \{f_n\}_{n=1}^{\infty} \subset X^* \) be any sequence of Hahn–Banach extensions of \( \{x_n^*\}_{n=1}^{\infty} \). Then \( \{f_n\}_{n=1}^{\infty} \) is biorthogonal to \( \{x_n\}_{n=1}^{\infty} \) and \( \|f_n\| = \|x_n^*\| \) for all \( n \), so \( \|f_n\| = \sup \|x_n^*\| < +\infty \), since \( \inf \|x_n\| = \delta > 0 \). Hence for any constants \( \lambda \) we have

\[
\sup_n \|f_n\| \left\| \sum_{n=1}^{N} \lambda_n c_n f_n \right\| \geq \sup_n \left\| \sum_{n=1}^{N} \lambda_n c_n x_n \right\| = \sup_n \left| \sum_{n=1}^{N} \lambda_n c_n \right| \|f_n\| = \sum_{n=1}^{N} \lambda_n c_n,
\]

But since \( \{x_n\}_{n=1}^{\infty} \) is equivalent to \( \{e_n\}_{n=1}^{\infty} \) in \( c_0 \) there is an \( \varepsilon > 0 \) (independent of \( \{c_n\} \)) for which this last is

\[
\varepsilon \geq \sup_n \left| \sum_{n=1}^{N} \lambda_n c_n \right| = \sum_{n=1}^{N} |c_n|.
\]

That is, the mapping \( T : \ell^1 \to X^* \) defined by \( T(e_n) = f_n \) is an isomorphism, implying that \( \{f_n\}_{n=1}^{\infty} \) is a basic sequence in \( X^* \) (which is, in fact, equivalent to the basis \( \{e_n\}_{n=1}^{\infty} \) for \( \ell^1 \)) and is therefore the desired sequence of extensions.

Now it follows from Proposition 1 that if \( \{x_n\}_{n=1}^{\infty} \) is a basic sequence in \( X \) which is equivalent to the basis \( \{e_n\}_{n=1}^{\infty} \) for \( c_0 \), then no matter how \( X \) is equivalently renormed there will still always exist a basic sequence of Hahn–Banach extensions for the coefficient functionals of \( \{x_n\}_{n=1}^{\infty} \). We now show that, at least for basic sequences of codimension one, this property characterizes the basis \( \{e_n\}_{n=1}^{\infty} \) for \( c_0 \), thereby not only providing numerous examples of basic sequences whose coefficient functionals do not admit basic Hahn–Banach extensions, but also completing the circle of ideas concerning the existence and stability of such basic sequences inherent in earlier parts of this paper.

**Theorem 3.** Let \( \{x_n\}_{n=1}^{\infty} \) be a bounded basic sequence in a Banach space \( X \) for which \( \text{codim}(x^*) = 1 \), and suppose \( \{x_n\}_{n=1}^{\infty} \) is not equivalent to the basis \( \{e_n\}_{n=1}^{\infty} \) for \( c_0 \). Then there is an equivalent norm on \( X \) for which no basic sequence of Hahn–Banach extensions of the coefficient functionals for \( \{x_n\}_{n=1}^{\infty} \) exists.

**Proof.** As in the proof of Theorem 2 let \( \|\cdot\| \) denote the original norm on \( X \) and let \( x_0 \notin [x_n^*]_{n=1}^{\infty} \) so that \( \{x_n\}_{n=1}^{\infty} \) is a basis for \( (X, \|\cdot\|) \) which is not equivalent to the basis \( \{e_n\}_{n=1}^{\infty} \) for \( c_0 \) and which may be assumed to be normalized and monotone. If \( \{f_n\}_{n=1}^{\infty} \subset X^* \) is biorthogonal to \( \{x_n\}_{n=1}^{\infty} \), then, just as in the proof of Theorem 2, defining an equivalent norm \( \|\cdot\| \) on \( [f_n]_{n=1}^{\infty} \) in which \( \{f_n\}_{n=1}^{\infty} \) is still monotone will result in the expression \( \|f_n\| = \sup \{\langle f_n, x \rangle : \|f_n\| \leq 1, f \in [f_n]_{n=1}^{\infty} \} \) defining an equivalent norm on \( X \) whose dual norm agrees with \( \|\cdot\| \) on \( [f_n]_{n=1}^{\infty} \). Consequently, to prove the theorem we need only (according to Theorem 1) define an equivalent norm \( \|\cdot\| \) on \( [f_n]_{n=1}^{\infty} \) in which \( \{f_n\}_{n=1}^{\infty} \) is still monotone and so that whenever \( \|f_n - \lambda_n f_0\| = \inf_{\lambda} \|f_n - \lambda f_0\| \) for all \( n \geq 1 \), then \( \sup \|\lambda_n c_n\| = +\infty \).

To define such a norm we first note that the assumption that \( \{x_n\}_{n=1}^{\infty} \) is not equivalent to \( \{e_n\}_{n=1}^{\infty} \) in \( c_0 \) implies \( \sum_{n=0}^{\infty} x_n \) is not weakly unconditionally Cauchy [1], and hence there is \( h_0 \in X^* \) for which \( \|h_0\| = 1 \) and \( \sum_{n=0}^{\infty} \langle h_0, x_n \rangle = +\infty \) [5, p. 434].

Now define a new norm \( \|\cdot\| \) on \( [f_n]_{n=1}^{\infty} \) by the expression

\[
\|f\| = \left\| \sum_{n=1}^{\infty} c_n f_n \right\| = \left\| c_0 f_0 \right\|^2 + \sum_{n=1}^{\infty} \left\| c_n f_n \right\|^2 + \sup_{n \neq 0} \|f_n\| + c_n \|f_n\|,
\]

where

\[
\varepsilon_n = \begin{cases} 1 & \text{if } \langle h_0, x_n \rangle \geq 0, \\ -1 & \text{if } \langle h_0, x_n \rangle < 0. \end{cases}
\]

It is trivial to check that \( \|\cdot\| \) is, indeed, a norm on \( [f_n]_{n=1}^{\infty} \) which is equivalent to \( \|\cdot\| \), and it is obvious that since \( \{f_n\}_{n=1}^{\infty} \) is monotone in the norm \( \|\cdot\| \) it will also be monotone in \( \|\cdot\| \). Moreover, for any \( n \geq 1 \) we have

\[
\inf_{\lambda} \|f_n - \lambda f_0\| = \inf_{\lambda} \|\lambda f_n - f_0\| = \inf_{\lambda} \left( \|\lambda f_n\|^2 + \|f_n\|^2 + \sup_{n \neq 0} \|f_n\|, |\lambda f_0|, -\|f_0\| + c_n \|f_n\| \right).
\]
and this inf will be attained only when \( \lambda \) has the sign of \( e_n \). Since \( \|x_n\| = 1 \) for all \( n \) implies \( \|f_n\| \geq 1 \), it now follows easily from elementary calculus that

\[
\inf_{\lambda} \|f_n - \lambda f\| \text{ is attained only when } \lambda = \lambda_n = \frac{\|f_n\|}{2\|f\|}.
\]

However, we then have

\[
\sup_N \left( \sum_{n=0}^N \lambda_n x_n \right) - \frac{1}{2} \sum_{n=0}^N \frac{\|f_n\|}{\|f\|} \geq \frac{1}{2} \|f\| \sup_N \left( \sum_{n=0}^N \lambda_n \langle h_0, x_n \rangle \right)
\]

(by the definition of \( e_n \) and the fact that \( \|f\| \geq 1 \)). But \( \sup_N \sum_{n=0}^N \langle h_0, x_n \rangle = +\infty \), so according to our previous remarks the proof is complete.

In particular, then, there is an equivalent norm \( \|\cdot\|_P \) on the space \( P \)

\( 1 \leq p \leq +\infty \) so that the coefficient functionals for the basic sequence \( \{e_n\}_{n=1}^\infty \) in \( P \) admit no basic Hahn–Banach extensions. Therefore the basic sequences \( \{e_n\}_{n=1}^\infty \) in the spaces \( (P, \|\cdot\|_P) \) provide an infinite number of (non-equivalent) counterexamples to the question mentioned at the beginning of the paper.

Remarks. Although we did not check the details it appears to be only a technical exercise to extend Theorems 2 and 3 to the case where \( \{x_n\}_{n=1}^\infty \) is of arbitrary finite codimension in \( X \). A more serious problem related to these results is:

**Problem 1.** Are Theorems 2 and 3 valid when \( \text{codim}(\{x_n\}_{n=1}^\infty) = +\infty \)?

We should also note the following problem to which we referred earlier, and which would seem to be fundamental in regard to questions of the sort treated in this paper:

**Problem 2.** If \( \{x_n\}_{n=1}^\infty \) is a normalized basic sequence in \( X \) for which \( \text{codim}(\{x_n\}_{n=1}^\infty) = +\infty \), does there exist a bounded basic sequence \( \{f_n\}_{n=1}^\infty \in X^* \) which is biorthogonal to \( \{x_n\}_{n=1}^\infty \)?

References


