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On functions whose improper Riemann integral is absolutely convergent

by

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Abstract. Absolutely convergent improper Darboux integrable functions on the compact support of a nonnegative Radon measure are introduced.

Introduction. S. Rolewicz deduced in [10] that a consistent definition of the Lebesgue integral is not possible for functions $f\colon [0, 1] \to X$ where X is a non-locally convex linear metric space. Hence, D. Przeworska-Rolewicz and S. Rolewicz [8] and independently B. Gramsch [1] introduced the Riemann integral for that situation. S. Rolewicz and the author [4] defined the Riemann integral for functions $f\colon K \to X$ where K is the compact support of a nonnegative Radon measure μ and where X is a topological linear space.

In [4], [5] of S. Rolewicz and the author, the translation of the classical result – i.e. a bounded function $f\colon [0, 1] \to \mathbf{R}$ is Riemann integrable iff f is continuous almost everywhere – was proved for Darboux integrable functions. These functions are characterized by Darboux lower and upper sums resp. by distance sums in the general case. Darboux integrable functions are Riemann integrable but the converse is false in general.

In this paper we characterize the Darboux integrability by a kind of fractional continuity. This allows us to obtain a definition of absolutely convergent improper Darboux integrability with respect to K and μ: Indeed, an unbounded function g is absolutely convergent improper Darboux integrable iff the following holds: (i) g fulfils the fractional continuity property and (ii) g is absolutely $\bar\mu$-Bochner integrable (or equivalently: the improper μ-Riemann integral of g is absolutely convergent).

There are also recent studies on Riemann integrable functions: R. Henstock [2], J. Kurzweil [6] and E. J. McShane [7] deduced that modifications of the Riemann integral on [0, 1] yield the Lebesgue and even the Perron–Ward integral. C. S. Hönig [3] found examples of Hilbert space valued functions on [0, 1] which are Riemann integrable but not measurable with respect to the complete Lebesgue measure. G. C. da Rocha Filho [9] analysed Riemann integration depending on the geometry of Banach spaces.

A. Pełczyński gave a definition of Riemann integrability for functions defined on an arbitrary measure space.

I am indebted to the referees for valuable remarks.

1. Preliminaries. Let K be a compact (Hausdorff) space. Let μ be a nonnegative Radon measure with $\mathrm{supp}(\mu) = K$. Here $\mathrm{supp}(\mu) := \{x \in K \mid \mu(U(x)) > 0$ for every open neighbourhood $U(x)$ of $x\}$ is the *support* of μ.

Let S be a subset of K. Then $\mathrm{cl}(S)$ resp. $\mathrm{int}(S)$ denotes the closure resp. the interior of S in K. If $\mathrm{cl\,int}(S) = S$ then S is *regular closed*. The *boundary* ∂S of S in K is $\partial S := \mathrm{cl}(S) \cap \mathrm{cl}(K \setminus S)$. If $\mu(\partial S) = 0$ then S is called a *μ-continuity set*. We denote by $\mathrm{rc}_0(K, \mu)$ the class of all regular closed, nonvoid μ-continuity subsets of K. Then $\mathrm{rc}_0(K, \mu)$ is a neighbourhood basis system of K (cf. [4] of S. Rolewicz and the author).

A *μ-partition* of K is a finite class $\mathscr{P} := \{P_i\}_{i=1}^n \subset \mathrm{rc}_0(K, \mu)$ such that $\bigcup_{i=1}^n P_i = K$ and such that $\mu(P_i \cap P_j) = 0$ for $i \neq j$. The intersection of finitely many μ-partitions yields a μ-partition of K. Therefore, if $S\mathscr{P}(\mu)$ denotes the class of all μ-partitions of K, then $S\mathscr{P}(\mu)$ is directed in a natural way: if $\mathscr{P} := \{P_i\}_{i=1}^n$ and $\mathscr{P}' := \{P'_j\}_{j=1}^m$ then $\mathscr{P}' \geqslant \mathscr{P}$ iff for every j there is an $i(j)$ with $P'_j \subset P_{i(j)}$ (cf. [4] for proofs and details).

A *μ-σ-partition* of K is a countable class $\mathscr{P} := \{P_i\}_{i=1}^\infty \subset \mathrm{rc}_0(K, \mu)$ such that the following holds: (i) $\mu(P_i \cap P_j) = 0$ for $i \neq j$ and (ii) $\mu(\mathrm{cl}(P_0)) = 0$ where $P_0 := K \setminus (\bigcup_{i=1}^\infty P_i)$. Instead of (ii) we can use the following equivalent condition: (iii) $\sum_{i=1}^\infty \mu(P_i) = \mu(K)$. We denote by σ-$S\mathscr{P}(\mu)$ the class of all μ-σ-partitions of K. The following lemma 1 shows that σ-$S\mathscr{P}(\mu)$ is analogously directed as $S\mathscr{P}(\mu)$:

LEMMA 1. *Let $\mathscr{P} := \{P_i\}_{i=1}^\infty$ and $\mathscr{P}' := \{P'_j\}_{j=1}^\infty$ be two μ-σ-partitions of K. Then $\mathscr{P} \cap \mathscr{P}' := \{\mathrm{cl\,int}(P_i \cap P'_j) \mid 1 \leqslant i, j < \infty\} \setminus \{\emptyset\}$ is a μ-σ-partition of K.*

Proof. $\mathscr{P} \cap \mathscr{P}'$ is a countable class of regular closed sets by its definition. Since $\partial\,\mathrm{cl\,int}(P_i \cap P'_j) \subset \partial(P_i \cap P'_j)$ it follows that $\mathscr{P} \cap \mathscr{P}'$ consists of μ-continuity sets. Let

$$P''_0 := K \setminus \Big(\bigcup_{1 \leqslant i,j < \infty} \mathrm{cl\,int}(P_i \cap P'_j)\Big).$$

If $x \in \mathrm{int}(P_i) \cap \mathrm{int}(P'_j)$ for $i, j \in N$ then $x \notin P''_0$. Since

$$\Big[\mathrm{cl}(P_0) \cup \Big(\bigcup_{i \in N} \partial P_i\Big)\Big] \cup \Big[\mathrm{cl}(P'_0) \cup \Big(\bigcup_{j \in N} \partial P'_j\Big)\Big]$$

is a closed μ-zero set containing P''_0 we obtain $\mu(\mathrm{cl}(P''_0)) = 0$. Hence $\mathscr{P} \cap \mathscr{P}'$ is a μ-σ-partition of K. ∎

Let B be a Banach space over $\boldsymbol{R}$ or $\boldsymbol{C}$. If $M \subset B$ then $\mathrm{dia}(M) := \sup\{\|x - y\| \mid x, y \in M\}$ denotes the *diameter* of M. We consider a function $g\colon K \to B$. Let $\mathscr{P} := \{P_i\}_{i \in I}$ be a μ-partition or a μ-σ-partition of K, i.e. $I = \{1, \ldots, n\}$ or $I = N$. The *μ-distance sum* of g with respect to $\mathscr{P}$ is

$$\mu\text{-dis}(g, \mathscr{P}) := \sum_{i \in I} \mathrm{dia}(g(P_i))\, \mu(P_i).$$

If $x_i \in P_i$ then $x := \sum_{i \in I} g(x_i)\, \mu(P_i)$ is a *$\mathscr{P}$-sum* of g. We denote

$$S(g, \mathscr{P}) := \{x \in B \mid x \text{ is a } \mathscr{P}\text{-sum of } g\}.$$

Moreover, $a := \sum_{i \in I} \|g(x_i)\|\, \mu(P_i)$ is an *absolute $\mathscr{P}$-sum* of g. We denote

$$AS(g, \mathscr{P}) := \{a \in \boldsymbol{R} \mid a \text{ is an absolute } \mathscr{P}\text{-sum of } g\}.$$

Remark that the above notations are meaningless if the existence of the related sums is not guaranteed.

Let $f\colon K \to B$ be a bounded function. Then f is called *μ-Darboux integrable* iff $\inf\{\mu\text{-dis}(f, \mathscr{P}) \mid \mathscr{P} \in S\mathscr{P}(\mu)\} = 0$. If the indexed class $\{S(f, \mathscr{P}) \mid \mathscr{P} \in S\mathscr{P}(\mu)\}$ is converging in B with respect to the directed class $S\mathscr{P}(\mu)$ then f is called *μ-Riemann integrable* and

$${}^{R}\!\int_K f\, d\mu := \lim_{\mathscr{P} \in S\mathscr{P}(\mu)} S(f, \mathscr{P})$$

is the *μ-Riemann integral* of f over K.

In [4] and [5], S. Rolewicz and the author characterized the bounded μ-Darboux integrable functions as follows: (i) If f is μ-Darboux integrable then f is μ-Riemann integrable (the converse is false in general). (ii) f is μ-Darboux integrable iff f is continuous μ-almost everywhere. (iii) If f is μ-Darboux integrable then f is $\bar{\mu}$-Bochner integrable and both integrals coincide. Here, $\bar{\mu}$ denotes the completion of μ.

2. μ-σ-partitionable functions. A – not necessarily bounded – function $g\colon K \to B$ is called *μ-σ-partitionable* iff for every $\varepsilon > 0$ there is a μ-σ-partition $\mathscr{P}(\varepsilon) := \{P_i(\varepsilon)\}_{i=1}^\infty$ of K with $\mathrm{dia}(g(P_i(\varepsilon))) < \varepsilon$ for $1 \leqslant i < \infty$.

If $x \in K$ and $a > 0$ then x is called an *a-discontinuity point* of g iff in every neighbourhood $U(x)$ of x there are $y, z \in U(x)$ with $\|g(y) - g(z)\| \geqslant a$. We denote by $\mathrm{DC}(g, a)$ the closed set of all a-discontinuity points of g.

PROPOSITION 2. *Let K be the compact support of a nonnegative Radon measure μ. Let $g\colon K \to B$ be a function where B is a Banach space. Then the following conditions are equivalent:*

(a) *g is μ-σ-partitionable.*

(b) *$\mu(\mathrm{DC}(g, a)) = 0$ for every $a > 0$.*

(c) $\inf\{\mu\text{-dis}(g, \mathscr{P}) \mid \mathscr{P} \in \sigma\text{-}S\mathscr{P}(\mu)\} = 0$.

Proof. The implications (a) $\Rightarrow$ (c), (a) $\Rightarrow$ (b) are obvious. We prove (c) $\Rightarrow$ (b). Assume that there is an $a > 0$ with $b := \mu(\mathrm{DC}(g, a)) > 0$. Then μ-$\mathrm{dis}(g, \mathscr{P}) \geqslant ab$ for every $\mathscr{P} \in \sigma$-$S\mathscr{P}(\mu)$. Indeed, this is a consequence of the equality

$$\mu\left(\mathrm{cl}(P_0) \cup \left(\bigcup_{i=1}^{\infty} \partial P_i\right)\right) = 0.$$

Hence $\mu(\mathrm{DC}(g, a)) = 0$ for every $a > 0$.

(b) $\Rightarrow$ (a). Let $\varepsilon > 0$. Then $\mathrm{DC}(g, \varepsilon/2)$ is a compact subset of K with $\mu(\mathrm{DC}(g, \varepsilon/2)) = 0$. Observe that μ is regular and that $\mathrm{rc}_0(K, \mu)$ is a neighbourhood basis system of K. This implies that there is a $U_1 \in \mathrm{rc}_0(K, \mu)$ with $\mathrm{DC}(g, \varepsilon/2) \subset \mathrm{int}(U_1)$ and $\mu(U_1) < \mu(K)/2^1$.

Let $V_1 := K \setminus \mathrm{int}(U_1)$. If $x \in V_1$ then $x \notin \mathrm{DC}(g, \varepsilon/2)$. Therefore there is a $U(x) \in \mathrm{rc}_0(K, \mu)$, $x \in \mathrm{int}(U(x))$, with $\|g(y) - g(z)\| < \varepsilon/2$ for all $y, z \in U(x)$. As V_1 is compact there are finitely many $x_j \in V_1$, $1 \leqslant j \leqslant n(1)$, such that $\{U(x_j) \mid 1 \leqslant j \leqslant n(1)\}$ is a covering of V_1. Let

$$\{\mathrm{clint}(U(x_r) \cap U(x_s)) \cap V_1 \mid 1 \leqslant r, s \leqslant n(1)\} \setminus \{\emptyset\} =: \{V_{11}, \ldots, V_{1m(1)}\}.$$

Then $\{V_{1j} \mid 1 \leqslant j \leqslant m(1)\}$ is a μ_1-partition of V_1 where μ_1 is the restriction of μ to V_1. Moreover, $\mathrm{dia}(g(V_{1j})) \leqslant \varepsilon/2 < \varepsilon$ for $1 \leqslant j \leqslant m(1)$.

By repeating successively the above construction we obtain a μ-σ-partition $\mathscr{P}(\varepsilon) := \{V_{ij} \mid 1 \leqslant i < \infty, 1 \leqslant j \leqslant m(i)\}$ of K with $\mathrm{dia}(g(V_{ij})) < \varepsilon$ for all i, j. ∎

Observe that condition (b) can be formulated as follows: (b′) g is continuous μ-almost everywhere. Therefore, if we restrict ourselves to bounded functions then Proposition 2 is a characterization of μ-Darboux integrable functions.

We denote by σ-$\mathscr{P}(K, \mu, B)$ the class of all μ-σ-partitionable B-valued functions (not necessarily bounded) on K. Then σ-$\mathscr{P}(K, \mu, B)$ is a linear space by Lemma 1. If B is an algebra then σ-$\mathscr{P}(K, \mu, B)$ is an algebra too.

For $g \in \sigma$-$\mathscr{P}(K, \mu, B)$ we define

$$\|g\|_{\mathrm{ess}} := \inf\{c > 0 \mid \bar{\mu}(\{x \in K \mid \|g(x)\| > c\}) = 0\}$$

provided that there exists a $c_0 \in \boldsymbol{R}$ with $\bar{\mu}(\{x \in K \mid \|g(x)\| > c_0\}) = 0$. Otherwise we define $\|g\|_{\mathrm{ess}} := \infty$. Then $\| \; \|_{\mathrm{ess}}$ is an $\boldsymbol{R}_+ \cup \{\infty\}$-valued mapping defined on σ-$\mathscr{P}(K, \mu, B)$. Let $\| \; \|'_{\mathrm{ess}}$: σ-$\mathscr{P}(K, \mu, B) \to [0, 1]$ be defined by $\|g\|'_{\mathrm{ess}} := \min\{1, \|g\|_{\mathrm{ess}}\}$. Then the mapping $\| \; \|'_{\mathrm{ess}}$ induces canonically on σ-$\mathscr{P}(K, \mu, B)$ by $(g, g') \mapsto \|g - g'\|'_{\mathrm{ess}}$ a pseudometric (i.e. $\|g\|'_{\mathrm{ess}} = 0 \Rightarrow g = 0$ is not always fulfilled) denoted by $\| \; \|'_{\mathrm{ess}}$ too.

We denote $Z\sigma$-$\mathscr{P}(K, \mu, B) := \{g \in \sigma$-$\mathscr{P}(K, \mu, B) \mid \|g\|'_{\mathrm{ess}} = 0\}$. Then $g \in Z\sigma$-$\mathscr{P}(K, \mu, B)$ iff g vanishes μ-almost everywhere. Hence, $Z\sigma$-$\mathscr{P}(K, \mu, B)$ is a closed subspace of σ-$\mathscr{P}(K, \mu, B)$ with respect to the pseudometric $\| \; \|'_{\mathrm{ess}}$. If B is a Banach algebra then $Z\sigma$-$\mathscr{P}(K, \mu, B)$ is a closed ideal.

We denote by $\sigma P(K, \mu, B) := \sigma$-$\mathscr{P}(K, \mu, B)/Z\sigma$-$\mathscr{P}(K, \mu, B)$ the quotient space (resp. the quotient algebra if B is a Banach algebra). Then $\| \; \|'_{\mathrm{ess}}$ induces canonically a metric on $\sigma P(K, \mu, B)$ which we denote by $\| \; \|'_{\mathrm{ess}}$ too. It follows that $(\sigma P(K, \mu, B), \| \; \|'_{\mathrm{ess}})$ is a metric space and a linear space. We do not obtain a metric linear space since the multiplication by scalars is discontinuous for unbounded functions.

We mention that $\| \; \|_{\mathrm{ess}}$ restricted to the μ-Darboux integrable (i.e. bounded) functions yields the usual ess sup norm. Moreover, $\| \; \|'_{\mathrm{ess}}$ restricted to the μ-Darboux integrable functions yields the same topology and the same uniformity as the ess sup norm. Observe that $\| \; \|_{\mathrm{ess}}$ is a modular function on $\sigma P(K, \mu, B)$ and that the μ-Darboux integrable functions are the related modular space (cf. S. Rolewicz [10], Chapter 1.2).

Proposition 3. *Let L be the compact support of a nonnegative Radon measure μ. Let B be a Banach space. Then $\sigma P(K, \mu, B)$ is a complete metric space with respect to the metric $\| \; \|'_{\mathrm{ess}}$.*

Proof. It suffices to prove that σ-$\mathscr{P}(K, \mu, B)$ is complete with respect to $\| \; \|'_{\mathrm{ess}}$. Let $\{g_n\}_{n \in \boldsymbol{N}}$ be a Cauchy sequence in σ-$\mathscr{P}(K, \mu, B)$ with respect to the pseudometric $\| \; \|'_{\mathrm{ess}}$. We can assume that $\|g_n - g_m\|'_{\mathrm{ess}} < 1/4$ for all $n, m \in \boldsymbol{N}$, i.e. that $\|g_n - g_m\|'_{\mathrm{ess}} = \|g_n - g_m\|_{\mathrm{ess}}$ for $n, m \in \boldsymbol{N}$.

We define $h_n := g_n - g_1$ for every $n \in \boldsymbol{N}$. It suffices to prove that $\{h_n\}_{n \in \boldsymbol{N}}$ is converging in σ-$\mathscr{P}(K, \mu, B)$. Observe that $\|h_n\|_{\mathrm{ess}} < 1/2$ for every $n \in \boldsymbol{N}$.

By Proposition 2 every h_n is continuous except on a μ-zero set Z_n. Then $Z := \bigcup_{n \in \boldsymbol{N}} Z_n$ is a μ-zero set. We denote $C := K \setminus Z$. If $x \in C$ then $h_n(x)$ is a Cauchy sequence in B. This is a consequence of the definition of $\| \; \|_{\mathrm{ess}}$ for x is a common continuity point of all h_n. We define h: $C \to B$ by $h(x) := \lim_{n \in \boldsymbol{N}} h_n(x)$.

It remains to define h on Z. Let $f = h_n$ or $f = h$. Then we define $\mathrm{dc}(f, a) := \{z \in Z \mid$ in every neighbourhood $U(z)$ of z there are $x, y \in C$ with $\|f(x) - f(y)\| \geqslant a\}$. If $z \in Z$ then we denote $r(f, z) := \sup\{a \mid z \in \mathrm{dc}(f, a)\}$. Since $\|f(x)\| \leqslant 1/2$ for every $x \in C$, we obtain $0 \leqslant r(f, z) \leqslant 1$ for $z \in Z$. Moreover, $r(h, z) = \limsup_{n \in \boldsymbol{N}} r(h_n, z)$. If $r(h, z) = 0$ then h can be extended continuously in z and we define

$$h(z) := \lim_{x \to z, x \in C} h|_C(x).$$

Suppose that $z \in Z$ and that $r(h, z) > 0$. Then there is an $A(z) \in \mathrm{rc}_0(K, \mu)$ with $\mathrm{dia}(h(A(z) \cap C)) < 2r(h, z)$. We choose a $y_0 \in h(A(z) \cap C)$ and we define $h(z) := y_0$. Then h is defined on K. Moreover, h is bounded on K.

We have to prove that $h \in \sigma$-$\mathscr{P}(K, \mu, B)$. By Proposition 2(b) it suffices to prove that $\mu(\mathrm{DC}(h, a)) = 0$ for every $a > 0$. Let $a > 0$ and let $x \in \mathrm{DC}(h, a)$.

We show that $x \notin C$. Indeed, assume that $x \in C$. Then there is a $k \in N$ with $\|h_n - h_m\|_{ess} < a/6$ for $n, m \geq k$. Let $A \in \mathrm{rc}_0(K, \mu)$, $x \in \mathrm{int}(A)$, with $\mathrm{dia}(h_k(A)) < a/6$. Then

$$\|h(z)-h(y)\| \leq \|h(z)-h_k(z)\|+\|h_k(z)-h_k(y)\|+\|h_k(y)-h(y)\| < a/2$$

if $z, y \in C \cap A$. We choose $B \subset \mathrm{int}(A)$, $x \in \mathrm{int}(B)$, $B \in \mathrm{rc}_0(K, \mu)$. Then $\mathrm{dia}(h(B)) < 2\,\mathrm{dia}(h(A \cap C)) = a$ by the definition of h on Z. This implies $x \notin \mathrm{DC}(h, a)$ which is a contradiction. Hence $x \notin C$ which implies $\mu(\mathrm{DC}(h, a)) = 0$. Therefore $h \in \sigma\text{-}\mathscr{P}(K, \mu, B)$.

We prove that $\|h_n - h\|_{ess} \to 0$ as $n \to \infty$. If $\varepsilon > 0$ and $k \in N$ with $\|h_n - h_m\|_{ess} < \varepsilon$ for $n, m \geq k$ then $\|h_n(x) - h(x)\| \leq \varepsilon$ for $n \geq k$ and $x \in C$. Since $\mathrm{supp}(\mu) = K$ and $\mu(Z) = 0$ we obtain $\|h_n - h\|_{ess} \leq \varepsilon$ for $n \geq k$. Hence $\{h_n\}_{n \in N}$ converges to h with respect to $\|\ \|_{ess}$. Therefore $\sigma\text{-}\mathscr{P}(K, \mu, B)$ is complete with respect to $\|\ \|'_{ess}$. ∎

The limit function h constructed in the above proof is bounded on K. Therefore, if we restrict ourselves to bounded μ-σ-partitionable functions – i.e. to μ-Darboux integrable functions – then it follows that these functions form a Banach space with respect to the usual ess sup norm (resp. a Banach algebra if B is one).

3. Absolutely convergent improper μ-Darboux integrable functions. We consider a function $g: K \to B$. Then g is called *absolutely convergent improper μ-Darboux integrable* if the following two conditions are satisfied:

(i) $\inf\{\mu\text{-dis}(g, \mathscr{P}) \mid \mathscr{P} \in \sigma\text{-}S\mathscr{P}(\mu)\} = 0$.

(ii) There is a $\mathscr{P}_0 \in \mu\text{-}S\mathscr{P}(\mu)$ such that $AS(g, \mathscr{P}_0)$ is bounded.

Suppose that $d: K \to B$ is an absolutely convergent improper μ-Darboux integrable function. Let $\mathscr{P}_0 \in \sigma\text{-}S\mathscr{P}(\mu)$ be such that $AS(d, \mathscr{P}_0)$ is bounded. Assume that the indexed class $\{S(d, \mathscr{P}) \mid \mathscr{P} \in \sigma\text{-}S\mathscr{P}(\mu),\ \mathscr{P} \geq \mathscr{P}_0\}$ is converging in B with respect to the naturally directed system $\sigma\text{-}S\mathscr{P}(\mu)$. Then d is called *improper μ-Riemann integrable* on K and

$$\lim_{\mathscr{P} \in \sigma\text{-}S\mathscr{P}(\mu)} S(d, \mathscr{P}) =: {}^{\mathrm{R}}\!\int_K d\, d\mu$$

is called the *improper μ-Riemann integral* of d on K.

Since the improper μ-Riemann integral is a limit, all operations compatible with limits are compatible with the above integral (provided that absolutely convergent improper μ-Darboux integrable functions are involved only). If $A \in \mathrm{rc}_0(K, \mu)$ then the improper μ-Riemann integral of d on A is the improper μ-Riemann integral of $\chi(A)d$ on K. Here $\chi(A)$ denotes the characteristic function of A. If $\mathscr{P} \in \sigma\text{-}S\mathscr{P}(\mu)$, $\mathscr{P} = \{P_i\}_{i=1}^{\infty}$, then we obtain

$${}^{\mathrm{R}}\!\int_K d\, d\mu = \sum_{i=1}^{\infty} {}^{\mathrm{R}}\!\int_{P_i} d\, d\mu.$$

The above definition of an improper μ-Riemann integral is not a complete generalization of the classical improper Riemann integral. Indeed, for such a purpose it does not suffice to consider the natural directed system $\sigma\text{-}S\mathscr{P}(\mu)$. It would be necessary to enumerate the μ-σ-partitions in an order in the definition. However, since we have restricted ourselves to the class of absolutely convergent improper μ-Darboux integrable functions, we will see that the above simplified definition is sufficient for our purposes. This is shown by the following

PROPOSITION 4. *Let K be the compact support of a nonnegative Radon measure μ. Let $d: K \to B$ be an absolutely convergent improper μ-Darboux integrable function where B is a Banach space. Then*

(a) *d is improper μ-Riemann integrable.*

(b) *d is absolutely $\bar{\mu}$-Bochner integrable.*

(c) *The improper μ-Riemann integral of d coincides with the $\bar{\mu}$-Bochner integral of d.*

Proof. (a) There is a μ-σ-partition $\mathscr{P}_0$ of K such that $AS(d, \mathscr{P}_0)$ is bounded. Therefore $S(d, \mathscr{P}_0)$ is bounded. Let $\mathscr{P}$ and $\mathscr{P}'$ be μ-σ-partitions with $\mathscr{P}_0 \leq \mathscr{P} \leq \mathscr{P}'$. Then an easy calculation shows that $\mathrm{dia}(S(d, \mathscr{P})) \geq \mathrm{dia}(S(d, \mathscr{P}'))$ and that $S(d, \mathscr{P}') \subset$ closed convex hull of $S(d, \mathscr{P})$. Hence d is improper μ-Riemann integrable iff there is a sequence $\mathscr{P}_i \geq \mathscr{P}_0$ of μ-σ-partitions of K with $\mathrm{dia}(S(d, \mathscr{P}_i)) \to 0$. The existence of such a sequence follows from condition (i) of the definition of the absolutely convergent improper μ-Darboux integrability.

(b) and (c) are easy consequences of the fact that $AS(d, \mathscr{P}_0)$ is bounded and that d is μ-σ-partitionable. ∎

We can formulate a characterization of absolutely convergent improper μ-Darboux integrable functions as follows:

Remark 5. Let $g: K \to B$ be a μ-σ-partitionable function. Then g is absolutely convergent improper μ-Darboux integrable iff g is absolutely $\bar{\mu}$-Bochner integrable.

We denote by $\mathscr{D}_\infty(K, \mu, B) := \{f: K \to B \mid f \in \sigma\text{-}\mathscr{P}(K, \mu, B)$ and f is bounded$\}$ the class of all μ-Darboux integrable functions. The ess sup norm $\|\ \|_{ess}$ is a seminorm on $\mathscr{D}_\infty(K, \mu, B)$. Then

$$Z\mathscr{D}_\infty(K, \mu, B) := \{f \in \mathscr{D}_\infty(K, \mu, B) \mid f \in Z\sigma\text{-}\mathscr{P}(K, \mu, B)\}$$

is a linear space – resp. an ideal if B is a Banach algebra – which is closed with respect to the seminorm $\|\ \|_{ess}$. We denote $D_\infty(K, \mu, B) := \mathscr{D}_\infty(K, \mu, B)/Z\mathscr{D}_\infty(K, \mu, B)$. From the proof of Proposition 3 it follows that $D_\infty(K, \mu, B)$ is a Banach space with respect to $\|\ \|_{ess}$, resp. a Banach algebra if B is one.

Let $0 < p < \infty$. We denote $\mathscr{D}_p(K, \mu, B) := \{g: K \to B \mid g \in \sigma\text{-}\mathscr{P}(K, \mu, B)$

and $\|g\|^p$ is absolutely convergent improper μ-Darboux integrable on $K\}$. Analogously as in the case of the L_p-spaces one deduces that $\mathscr{S}_p(K, \mu, B)$ is a linear space. Indeed, $\mathscr{D}_p(K, \mu, B)$ corresponds to those elements of $L_p(K, \mu, B)$ which possess a μ-σ-partitionable representative.

We consider the pseudometric $\| \ \|'_{\text{ess}}$ on $\mathscr{S}_p(K, \mu, B)$. Then $Z\sigma$-$\mathscr{P}(K, \mu, B)$ is a closed linear subspace of $\mathscr{D}_p(K, \mu, B)$. We denote $D_p(K, \mu, B) := \mathscr{D}_p(K, \mu, B)/Z\sigma$-$\mathscr{P}(K, \mu, B)$. Then $\| \ \|'_{\text{ess}}$ is a metric on $D_p(K, \mu, B)$. A consequence of Proposition 3 is the following

COROLLARY 6. *Let K be the compact support of a nonnegative Radon measure μ. Let B be a Banach space.*

(a) *If $0 < p < \infty$ then $D_p(K, \mu, B)$ is a linear space and $(D_p(K, \mu, B), \| \ \|'_{\text{ess}})$ is a complete metric space.*

(b) *$(D_\infty(K, \mu, B), \| \ \|'_{\text{ess}})$ is a Banach space, resp. a Banach algebra if B is one.*

We mention some points: (1) As in [4], the above results remain true if we consider a locally convex Fréchet space F instead of a Banach space B. (2) As in [5], we can consider $D_p(K, \mu, B)$ equivalently on dense subsets of K. (3) Invariance properties of $D_p(K, \mu, B)$ with respect to K and μ can be obtained as in [5]. (4) Locally compact spaces can be treated in the same way, i.e. Propositions 2 and 3 are not affected by that alteration of the assumptions. (5) Let M be a compact metric space. Let $\mathscr{D}(K, \mu, M) := \{f\colon K \to M \mid f$ fulfils condition (a) or (b) of Proposition 2$\}$. Then Proposition 3 can be applied analogously.

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