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Sequence space representations for (FN)-algebras
of entire functions modulo closed ideals

by
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Abstract. Let A^s denote the weighted (FN)-algebra of entire functions on C defined by an
appropriate weight system P. We prove that for every infinite-codimensional proper closed ideal
I in A^s the quotient A^s/I is isomorphic to a Köthe sequence space. In the interesting special
case that P is generated by a single weight function, A^s/I is even isomorphic to a power series
space of finite type. From the sequence space representation we deduce that in all relevant
equations P is not complemented in A^s. Furthermore, it follows that all proper closed infinite-
dimensional translation invariant subspaces of certain weighted (DFN)-spaces of entire functions
have a Schauder basis but are not complemented.

Let P = (\rho_k)_{k=1}^{\infty} be a decreasing sequence of radial subharmonic functions on C
which satisfy some mild technical conditions. Denote by A^s the vector space of all entire functions on C satisfying sup \{f(z) \exp(-\rho_k(z)) \} < \infty for all k \in \mathbb{N}. Under its natural locally convex topology A^s becomes a nuclear
Fréchet algebra. Algebras of this type have been studied since a long time.

They arise in complex analysis and functional analysis.

In the present article we use results and methods of Berenstein and Taylor [1] and Meise [7] to prove that for every proper closed infinite-codimensional ideal I of A^s the quotient space A^s/I is isomorphic to a nuclear Köthe sequence space. If the weight system P is of the special form
P = (k^{-1} \rho_k)_{k=1}^{\infty}, then we derive that A^s/I is isomorphic to a power series space of finite type.

This sequence space representation of A^s/I allows us to use the structure
theory of nuclear Fréchet spaces to investigate whether an ideal I is complemented in A^s.
It turns out that in all our examples no proper infinite-codimensional ideal I is complemented. This is essentially due to the fact that each continuous linear map from A^s/I into A^s is already
compact. Moreover, it explains the corresponding observation of Taylor [15] and gives results which cover a significantly larger class of examples.

By means of the Fourier-Borel transform, the information on the
structure of A^s/I implies that in certain weighted (DFN)-spaces of entire functions all translation invariant subspaces have a Schauder basis. As a
particular example we mention the following: For $s > 1$ denote by
\[ E^s := \{ f \in A(C) \mid \text{there exists } k \in \mathbb{N} \text{ with } \sup_{z \in C} |f(z)| \exp(-k |z|^s) < \infty \}. \]

Then every proper closed infinite-dimensional translation invariant subspace $W$ of $E^s$ is isomorphic to the strong dual of a nuclear power series space of finite type and is not complemented in $E^s$. This result should be compared with the results of Meise [7], Sect. 5, on the translation invariant subspaces of $A^2_{\phi}$, where $P = (k^{-1} |l|^s)_{k \in \mathbb{N}}$, $s > 1$. For applications of the results of the present article and for related work we refer to [8]-[12].

The article is divided into four sections. In the first one we introduce the weighted algebras $A^2_{\phi}$ and the sequence spaces which we need and give some examples. In section two, the representation theorem for $A^2_{\phi}$ is proved. The question of the complementation of the closed ideals in $A^2_{\phi}$ is treated in section three, and the results on the translation invariant subspaces are presented in section four.

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1. Weighted algebras, sequence spaces and examples. In this section we introduce the weighted algebras $A^2_{\phi}$ of entire functions on $C$ which will be treated in the sequel. Moreover, we give sequence space representations of these algebras.

1.1. Definition. A function $p: C \to [0, \infty[$ is called a weight function if it has the following properties:

(1) $p$ is continuous and subharmonic.
(2) $\log(1 + |z|^2) = O(p(z)).$
(3) There exists $C \geq 1$ such that for all $w \in C$
\[ \sup_{z \to w} \frac{p(z)}{p(z)-C} < \infty. \]

A weight function will be called radial if $p(z) = \mu |z|^s$ for all $z \in C$, $p$ will be called an inductive weight function if it satisfies (1), (3) and if $\log(1 + |z|^2) = O(p(z)).$

1.2. Definition. A sequence $P = (p_k)_{k \in \mathbb{N}}$ of weight functions is called a weight system if it has the following properties:

(1) For every $k \in \mathbb{N}$ there exists $M \geq 0$ with $p_{k+1} \leq p_k + M$.
(2) For every $k \in \mathbb{N}$ there exist $m \in \mathbb{N}$ and $L \geq 0$ with
\[ 2p_{\mu}(z) \leq p_k(z) + L \quad \text{for all } z \in C. \]

A weight system $P = (p_k)_{k \in \mathbb{N}}$ is called radial if $p_k$ is radial for all $k \in \mathbb{N}$.

For an open set $\Omega$ in $C$ let $A(\Omega)$ denote the algebra of all holomorphic functions on $\Omega$. If $P$ is a given weight system then we define the subalgebra $A^2_{\phi}$ of $A(\Omega)$ in the following way:

1.3. Definition. (a) For a weight function $p$ we put
\[ H^p_\phi := \{ f \in A(\Omega) : \|f\|_{p,\phi} := \sup_{|z| \leq 1} |f(z)| e^{-p(|z|)} < \infty \}. \]
\[ H^p_\phi := \{ f \in A(\Omega) : \|f\|_{p,\phi} := \left( \int_{C} |f(z)|^2 e^{-p(|z|)} dm(z) \right)^{1/2} < \infty \}, \]
where $m$ denotes the Lebesgue measure on $C = \mathbb{R}^2$.

(b) For a weight system $P$ we define
\[ A^2_{\phi} := \bigcap_{k \in \mathbb{N}} H^p_\phi \]
and endow this vector space with its natural projective limit topology. If $P = (k^{-1} p_{\lambda \in \mathbb{N}})$ then we write $A^2_{\phi}$ instead of $A^2_{\phi}$.

By standard arguments one proves:

1.4. Proposition. For every weight system $P$:

(a) $A^2_{\phi}$ is a locally convex algebra with unit under pointwise multiplication.
(b) $A^2_{\phi}$ is a nuclear Fréchet space.
(c) $A^2_{\phi} = \cap_{k \in \mathbb{N}} H^p_\phi = \text{proj}_{k \in \mathbb{N}} H^p_\phi$.

1.5. Examples. (1) Let $\phi : [0, \infty[ \to [0, \infty[$ be continuous, convex and increasing with $\lim \phi(t) = \infty$ and assume that there exists $D \geq 1$ with $\phi(2t) < D \phi(t) + D$ for all $t \in [0, \infty[$. Then it follows easily from Hörmander [2], Th. 1.6.7, that $\phi \circ \varphi$ is a weight function for every weight function $\varphi$.

(2) Let $\varphi : [0, \infty[ \to [0, \infty[$ be continuous with $\lim \varphi(t) = \infty$. Assume that $t \mapsto \varphi(t)$ is convex and increasing and that there exists $D \geq 1$ with $\varphi(2t) < D \varphi(t) + D$ for all $t \in [0, \infty[$. Then $p : z \mapsto \varphi(|z|^2)$ is a radial weight function for each $r > 0$.

Most of the following examples can be obtained from (1) or (2):

(3) $p(z) = |z|^s$, $r > 0$.
(4) $p(z) = (\log(1 + |z|^2))^s$, $s > 1$.
(5) $p(z) = |\Re z|^s + |\Im z|^s$, $r, s \geq 1$.
(6) $p(z) = \|z\|^s$, $r > 0$, $s \geq 1$.

1.6. Definition. (a) Let $A = (a_{j,k})_{j,k \in \mathbb{N}}$ be a matrix of nonnegative numbers $a_{j,k}$. $A$ is called a Köthe matrix if

(1) $a_{j,k} \leq a_{j,k+1}$ for all $j, k \in \mathbb{N}$.
(2) $a_{j,1} > 0$ for all $j \in \mathbb{N}$.
(b) Let $\mathcal{A}$ be a Köthe matrix and let $E = (E_j, \| \cdot \|_{j, \infty})$ be a sequence of Banach spaces. For $1 \leq p < \infty$ we define

$$
\lambda^{(p)}(A, E) = \{ x \in \prod_{j \in \mathbb{N}} E_j : \| x \| = \sup_{j \in \mathbb{N}} \| x_j \| \}
$$

and for $p = \infty$ we put

$$
\lambda^{(\infty)}(A, E) = \{ x \in \prod_{j \in \mathbb{N}} E_j : \| x \| = \sup_{j \in \mathbb{N}} \| x_j \| \}
$$

These spaces of vector-valued sequences are Fréchet spaces under their natural locally convex topology, induced by the norms $(\| \cdot \|_{j, \infty})_{j \in \mathbb{N}}$. If $E_j = (C_j(\cdot), \| \cdot \|_c)$ for all $j \in \mathbb{N}$, then we write $\lambda^{(p)}(A, E)$ instead of $\lambda^{(p)}(A, \mathbb{N})$. Instead of $\lambda^{(\infty)}(A, E)$ we sometimes write $\lambda(A)$.

1.7. Example. Let $x$ be an increasing unbounded sequence of positive real numbers (called an exponent sequence) and put

$$
a_{j,k} = \exp(k_x), \quad b_{j,k} = \exp(-\frac{1}{k_x} k).
$$

Then the corresponding space $\lambda^{(p)}(A)$ (resp. $\lambda^{(\infty)}(E)$) is denoted by $A_{\infty, a}(a)$ (resp. $A_{\infty, b}(b)$) and is called a power series space of infinite (resp. finite) type. Classical examples of power series spaces are the following: The space $C^\infty(S^1)$ of all $C^\infty$-functions on the unit circle $S^1$ is isomorphic to $A_{\infty, a}(\log(1+t))_{(0,\infty)}$. The space $A(C)$ is isomorphic to $A_{\infty, a}(\| \cdot \|_{C})_{(\mathbb{R}, \infty)}$. The space $\mathcal{L}(D)$, $D$ the open unit disk, is isomorphic to $A_{\infty, a}(\| \cdot \|_{C})_{(\mathbb{R}, \infty)}$.

Later in the applications we shall need sequence space representations of $A_{\infty, a}$. For radial weight functions such representations can be obtained by estimating the Taylor coefficients of the functions in $A_{\infty, a}$. Sufficiently precise estimates can be obtained by means of the Young conjugate of a convex function.

1.8. Definition. Let $\phi : [0, \infty) \to \mathbb{R}$ be an increasing convex function. Then its Young conjugate $\phi^* : [0, \infty) \to \mathbb{R}^*$ is defined by

$$
\phi^*(y) = \sup_{x \geq 0} \{ xy - \phi(x) \} = \begin{cases} \frac{y}{\phi(y)} & \text{if } y > 0, \\ 0 & \text{if } y = 0, \\ -\frac{1}{\phi'(0)} & \text{if } y < 0. \end{cases}
$$

1.9. Remark. The following facts are easy to check:

(a) $\phi^*$ is convex.
(b) If $\lim_{t \to a^+} \phi(t)/t = \infty$, then $\phi^*$ is strictly increasing on $[a, \infty)$, where $x = (\phi(a)/a)(0)$.
(c) If $\lim_{t \to a} \phi(t)/t = \infty$, then $(\phi^*)^* = \phi$.

The next result follows easily from Cauchy's inequality (see e.g. Taylor [14]).

1.10. Proposition. Let $q : [0, \infty) \to \mathbb{R}$ be an increasing function and put $\phi_x(x) = q(x)$. Assume that $q$ is constant on $[0, 1)$, that $\phi$ is convex and that $\lim_{x \to \infty} \phi(x)/x = \infty$. Then we have the following assertions for every entire function $f$:

(a) If $\sup_{x \in \mathbb{C}} \| f(x) \| \exp(-\phi(x)) < A$, then $|f(x)| \leq A \exp(-\phi^*(x))$ for all $x \in \mathbb{C}$.
(b) If $\sup_{x \in \mathbb{C}} \| f(x) \| \exp(-\phi^*(x)) < A$, then $\sup_{x \in \mathbb{C}} \| f(x) \| \exp(-\phi^*(x)) < A$.

Remark. It is easy to check that for a radial weight function $p$, the function $f_x : x \mapsto p(x)/x$ is an increasing convex function on $[0, \infty)$. We shall use this in the sequel. Moreover, we shall also use that we may assume w.l.o.g. that $p$ is constant on $[0, 1)$. Hence the following corollaries are immediate consequences of Proposition 1.10.

1.11. Corollary. Let $P = (\lambda_{j, \infty})_{j \in \mathbb{N}}$ be a radial weight system such that for each $k \in \mathbb{N}$ there exist $m \in \mathbb{N}$ and $D \geq 0$ with $P_{mk}(x) \leq P_k(x) + D$ for all $x \in C$. Then $A_{\infty, a}^p$ is isomorphic to $\lambda^{(1)}(A)$ for $A = (a_{j,k})_{j \in \mathbb{N}, k \in \mathbb{N}}$ with $a_{j,k} = \exp(q^*_k(j))$, where $\phi_k(x) = x^{-q^*_k(j)}$.

1.12. Corollary. Let $p$ be a radial weight function with $p(2) = O(p(c))$. Then $A_{\infty, a}^p$ is isomorphic to $\lambda^{(1)}(A)$ with $a_{j,k} = \exp(k^{-1} q^*_k(j))$, where $q^*_k(j)$ is defined by $q^*_k(j) = x^{-q^*_k(j)}$.

1.13. Example. (1) For $a > 0$ let $\phi : [0, \infty) \to \mathbb{R}$ be given by $\phi(x) = x^a$. Then it is easy to check that $\phi^*(y) = (y/a)/(log(y/a))$. Hence it follows from Corollary 1.12 that for $p : x \mapsto |x|^a$, $r > 0$, we have $A_{\infty, a}^p \simeq \lambda^{(1)}(A)$ with

$$
a_{j,k} = \exp\left(\frac{k}{k} j \log \frac{k}{k} \right) = \exp\left(\frac{j}{k} \log \frac{k}{k} \right).
$$

This shows that $A_{\infty, a}^p \simeq \lambda^{(1)}(A) \simeq A_{\infty, b}(\|^a_{\mathbb{R}})$.

(2) For $a > 1$ put $\beta = a(a-1)$ and $\phi : [0, \infty) \to \mathbb{R}$, $\phi(x) = (1/a)x^a$. Then it is easy to check that $\phi^*_k(j) = (1/\beta^k)$. Hence it follows from Corollary 1.12 that for $p(t) = (1+|t|^a)^a$, we have $A_{\infty, a}^p \simeq \lambda^{(1)}(A)$ with

$$
a_{j,k} = \exp\left(\frac{1}{k} \beta^k(j)^{\beta^k} \right) = \exp\left(\frac{1}{k} \beta^k(j)^{\beta^k} \right).
$$

This shows that $A_{\infty, a}^p \simeq A_{\infty, b}(\|^a_{\mathbb{R}})$.

1.14. Example. (1) Let $q : [0, \infty) \to [0, \infty]$ be continuous with $q(2) = O(q(t))$ and $\lim_{x \to \infty} q(x)/q(t) = 0$ for each $0 < r < 1$. Assume furthermore that $\psi : x \mapsto q(x)$ is increasing and convex. Let $(\lambda_{j, \infty})_{j \in \mathbb{N}}$ be a strictly decreasing sequence in $[0, \infty]$ and put $P = (\lambda_{j, \infty}^{-1} q(\cdot))_{j \in \mathbb{N}}$. Then $P$ is a weight system.
and we have
\[ A_{\varphi} = \begin{cases} A_1((\varphi x)z) & \text{if } \lim_{x \to 0} r_X > 0, \\ A_2(\varphi x) & \text{if } \lim_{x \to 0} r_X = 0. \end{cases} \]

To see this put \( p_k z \to z^{r_X^{-1}}q(|z|^2) \) for \( k \in \mathbb{N} \). Then \( p_k \) is a weight function by 1.5(2). Since \( \lim_{|z| \to 0} [p_{k+1}(z)/p_k(z)] = 0 \) by hypothesis, it follows that \( P \) is a weight system. For \( q_k(x) = x \to z^{r_X} \) we have
\[ q_k(x) = x \to z^{r_X} \]
\[
q_k(x) = x \to z^{r_X}.
\]
Hence it follows from Corollary 1.11 that \( A_{\varphi} \cong \lambda^1(A) \) with
\[ a_{\varphi} = \exp(\varphi (z)) = \exp \left( \frac{1}{r_X} \varphi (z) \right), \quad j \in \mathbb{N}_0, \quad k \in \mathbb{N}. \]

Obviously, this proves our claim.

(2) Let \( P := (\lambda^1)^{\mathbb{N}} \), where \((r_n)_{n=1}^{\infty} \) is a strictly decreasing sequence in \( (0, \infty) \). Then \( A_{\varphi} \cong A_1((\lambda^1)^{\mathbb{N}}) \) if \( \lim_{x \to 0} r_X > 0 \) and \( A_{\varphi} \cong A_2((\lambda^1)^{\mathbb{N}}) \) if \( \lim_{x \to 0} r_X = 0 \). This is an obvious consequence of (1) and the fact that for \( \psi : x \to z^y \) we have \( \psi(x) = x \to z^y \).

(3) Let \((r_n)_{n=1}^{\infty} \) be a strictly decreasing sequence in \( (1, \infty) \) and put \( P := (\lambda^1)^{\mathbb{N}} \). Then \( P \) is a weight system for which we have \( A_{\varphi} \cong \lambda^1(A) \) with
\[ a_{\varphi} = \exp \left( \frac{\log(1+|z|^2)}{r_X} \right). \]
This follows from Corollary 1.11 and the remark on the Young conjugate in 1.13(2). Note that for \( r_X = 1 + 1/k \) we have
\[ a_{\varphi} = \exp \left( \frac{1}{k+1} \varphi (z) \right). \]

(4) Let \((r_n)_{n=1}^{\infty} \) be a strictly decreasing sequence in \( (0, \infty) \). For \( s \in (0, \infty) \) put
\[ P := (\lambda^1 \exp ((\log(1+|z|^2))^{\mathbb{N}} \).
Then it follows from a remark in Meise [7], Example 2.13(5), that there is a weight system \( P \) with \( A_{\varphi} \cong A_2 \) and that \( A_{\varphi} \cong \lambda^1(A) \) with
\[ a_{\varphi} = \exp \left( -s (\log(1+|z|^2))^{\mathbb{N}} \right). \]

As in Meise [7], Proposition 2.8, one proves by means of Proposition 1.4 the following:

1.15. Proposition. Let \( P = (\nu)_{\mathbb{N}} \) be a radial weight system. Then
\[ A_{\varphi} \cong \lambda^1(B) = \lambda^1(B), \]
where \( B = (b_k) \) with \( b_k = \left[ 2^{2n+1} \exp(-s r_k(z)) \right]^{1/2}, \), \( j \in \mathbb{N}_0, \quad k \in \mathbb{N}. \)

1.16. Corollary. Let \( P \) be a radial weight function with the property
\[ (*) \quad \text{There exist } A > 1 \text{ and } B > 0 \text{ such that for all } z \in C \]
\[ p(z) \leq A p(z) \quad \text{and} \quad 2 p(z) \leq p(A z) + B. \]

Then \( A_{\varphi} \cong A_2((\lambda^1)^{\mathbb{N}}) \).

Proof. It is easy to check that because of (\( * \)) we have \( A_{\varphi} \cong A_{\varphi} \), where \( P \) is a weight function with \( b_k = \left[ 2^{2n+1} \exp(-s r_k(z)) \right]^{1/2} \).

we conclude from Proposition 1.15 that by a diagonal transformation we have \( A_{\varphi} \cong A_2((\lambda^1)^{\mathbb{N}}) \).

Remark. It is easy to see that the analogues of 1.15 and 1.16 hold for weight functions in several variables which are coordinatewise radial.

2. \( A_{\varphi} \) modulo localized ideals. In this section we derive a sequence space representation for \( A_{\varphi} \) modulo certain localized ideals. To do this we first have to establish an appropriate semi-local to global interpolation theorem.

2.1. Proposition. Let \( P \) be a radial weight system and put
\[ L_* := \left[ u \in L^1_1(C) \mid \int \left| u(z) \exp(-p(z)) \right|^{1/2} dm(z) < \infty \right], \]
Then, for every \( u \in L_* \), there exists \( u \in L_* \) with \( \partial u / \partial z = \tilde{v} \), where the derivative is taken in the sense of distributions. Moreover, if \( v \in C^o(C) \) then \( u \) is in \( C^o(C) \).

Proof. For \( k \in \mathbb{N} \) we define
\[ Z_k := \left[ f \in L^1_1(C) \mid \int \left| f(z) \exp(-p(z)) \right|^{1/2} dm(z) < \infty \right], \]
\[ Y_k := \left[ f \in L^1_1(C) \mid \int \left| f(z) \exp(-p(z) - \log(1+|z|^2)) \right|^{1/2} dm(z) < \infty \right], \]
and \( \partial f / \partial z \in Z_k \).

Then, for \( k \in \mathbb{N} \), \( \partial f / \partial z = \tilde{v} \), where \( \tilde{v} \) is a function of \( z \) in \( L_* \).

X_k := \left[ f \in Y_k \mid \partial f / \partial z = \tilde{v} \right] = H_k,
where \( \tilde{v} : z \to r_k(z) + \log(1+|z|^2) \). From Hörmander's \( L^2 \)-estimates for the \( \tilde{v} \)-operator, it then follows that the sequence
\[ 0 \to X_k \to Y_k \to Z_k \to 0 \]
is exact. See e.g. Berenstein and Taylor [1], Th. 1 or Hörmander [2],
Chap. 4. Now observe that $X_{k+1} = X_k$, $Y_{k+1} = Y_k$ and $Z_{k+1} = Z_k$, for all $k \leq N$. Since the weights $q_k$ are radial it follows easily that the polynomials are dense in $X_k$ (see Meise [7], 28, for the argument). This implies that the hypotheses of the Mittag-Leffler Lemma of Komatsu [4], Lemma 1.3, are satisfied. Hence the sequence

$$0 \to \text{proj} X_k \to \text{proj} Y_k \to \text{proj} Z_k \to 0$$

is exact. From the definition of weight functions and weight systems it follows easily that $\text{proj} Y_k = \text{proj} Z_k = F_k$. This proves the first assertion. The second assertion follows from regularity for the $\lambda$-operator (see Hörmander [23], 4.25).

22. Proposition. Let $P = (p_n)_{n \in \mathbb{N}}$ be a radial weight system, let $q$ be an inductive weight function with

$$\lim_{|z| \to +\infty} \frac{q(z)}{p_k(z)} = 0 \quad \text{for all } k \in \mathbb{N}$$

and let $F = (F_1, \ldots, F_n) \in (A(C))^n$ satisfy

$$\sup_{k \in \mathbb{N}} \sup_{|z| < e} \sup_{|c| < r} \exp(-Dq(z)) < \infty$$

for some $D > 0$. For $e > 0$ and $C > 0$ put

$$S_k(F; e, C) := \{z \in C : \left( \sum_{j=1}^{N} |F_j(z)|^2 \right)^{1/2} \leq e \exp(-Cq(z)) \}.$$ 

If $z \in S_k(F; e, C)$ satisfies for all $k \in \mathbb{N}$

$$\sup_{|z| < e} \exp(-p_k(z)) \in S_k(F; e, C) < \infty,$$

then there exist $\lambda \in A^*, \varepsilon_1, C_1$ with $0 < \varepsilon_1 < e$, $C_1 > C$ and $\gamma \in A(S_k(F; \varepsilon_1, C_1))$, $1 < j < N$, such that for all $z \in S_k(F; \varepsilon_1, C_1)$

$$\frac{\lambda(z)}{z} = \sum_{j=1}^{N} \frac{\gamma_j(z)}{z} F_j(z).$$

Proof. By Berenstein and Taylor [1], p. 120, there are $\varepsilon_1, C_1, A, B > 0$ and $\chi \in C^\infty(C)$ with $0 \leq \chi \leq 1$, $\text{Supp} (\chi) = S_k(F; \varepsilon_1, C_1)$, $\chi S_k(F; \varepsilon_1, C_1) \equiv 1$ and

$$\sup_{|z| < e} \exp(-Bq(z)) < \infty.$$ 

Then

$$\frac{\chi_j(z)}{z} = -F_j \left( \sum_{j=1}^{N} |F_j(z)|^{-1} \frac{\chi_j(z)}{z} \right), 1 < j < N,$$

is in $C^\infty(C) \cap L^2$. Hence Proposition 2.1 can be used to complete the proof similarly to the one of the semi-local interpolation theorem of Berenstein and Taylor [1], p. 110.

23. Lemma. Let $P$ be a radial weight system with the property that

$$\sup_{|z| < e} \frac{p_k(z)}{p_k(2z)} = C < \infty$$

for a suitable $R > 0$. Then for every $f \in A^*$ there exists a radial inductive weight function $q$ with the following properties:

1. $q(2z) = O(q(z))$;
2. $\sup_{|z| < e} |f(z)| \exp(-Aq(z)) < \infty$ for some $A > 0$;
3. $\lim_{|z| \to +\infty} \frac{q(z)}{p_k(z)} = 0$ for all $k \in \mathbb{N}$.

Proof. First we remark that the hypothesis on $P$ implies the existence of $a > 0$ with $\limsup_{|z| \to +\infty} [p_k(z)/a^k] < \infty$ for all $k \in \mathbb{N}$. Now let $f \in A^*$ be given. Without loss of generality we may assume $|f(0)| > e$ and $f$ not constant. Then we define $q_k : [0, \infty[ \to R$ by

$$q_k(r) := \log \left( \max \{|f(z)|: |z| = r \} \right).$$

Notice that $z \mapsto q_k(z)$ is subharmonic and continuous and that $\lim [q_k(z)/p_k(z)] = 0$ for all $k \in \mathbb{N}$. Next choose $b = \max(a + 2, \log 4C/2\log 2)$ and define

$$q_2(r) := \int_{1}^{r} q_1(t)(1+t)^{-b} \, dt.$$ 

Then we have

$$q_2(2r) = \int_{1}^{2r} q_1(2t)(1+t)^{-b} \, dt = \frac{1}{2} \int_{2}^{2r} q_1(t)(1+t/2)^{-b} \, dt$$

$$\leq 2^{b-1} \int_{1}^{2r} q_1(t)(1+t)^{-b} \, dt = 2^{b-1} q_2(r),$$

since $1 + (t/2) > (1 + t)$ for all $t \in [1, \infty[$.

Moreover, we have, with $L := \int_{1}^{(1+t)^{-b}} \, dt$, for all $r \in [0, \infty[$

$$L q_1(t) \leq \int_{1}^{t} q_1(t)(1+t)^{-b} \, dt = q_2(r)$$

and hence $q_1(1/L) q_2$. Notice that $z \mapsto q_2(z)$ is continuous. Hence it is subharmonic as a supremum of subharmonic functions. To see that $\lim_{x \to +\infty} [q_2(x)/p_k(x)] = 0$ for all $k \in \mathbb{N}$, let $k \in \mathbb{N}$ and $e > 0$ be given. Choose $m \in \mathbb{N}$ with $2^m > R$ such that $q_1(r) \leq 2^{-b-1} p_k(r)$ when $r \geq 2^m$. Then we have for
all \( r \geq 2^n \)
\[
q_2(r) = \sum_{i=1}^{n} q_i(r) + \sum_{m=1}^{n} \sum_{i=2}^{2^n} q_i(r)(1 + r)^{-n} dt \\
\leq I(m) + \sum_{m=1}^{n} \sum_{i=2}^{2^n} q_i(r)(1 + r)^{-n} dt \\
\leq I(m) + \sum_{m=1}^{n} \sum_{i=2}^{2^n} C_q(r) (2^n)^{-n} \\
\leq I(m) + \frac{2^n}{4} C_q(r) \sum_{m=1}^{n} \frac{2^n}{4} C_q(r). 
\]

Hence there exists \( r \) such that \( q_2(r) \leq \frac{2^n}{4} C_q(r) \) for all \( r \geq r \), which proves \( \lim_{r \to \infty} \frac{q_2(r)}{p_2(r)} = 0 \).

Concluding, we define \( q : z \mapsto q_2(|z|) + \log(1 + |z|^2) \). Then the preceding arguments show that \( q \) is a radial inductive weight function which has properties (1)-(3).

### 2.4. Definition
Let \( P \) be a weight system.

(a) For an arbitrary ideal \( I \) in the algebra \( A^p \), we define
\[
I_{\text{loc}} := \{ f \in A^p \mid f \in I_a \text{ for every } a \in C \},
\]
where \( I_a \) denotes the germ of \( f \) at \( a \) and \( I_a \) denotes the ideal generated by \( \{ f \mid f \in I \} \) in the ring \( C \), all germs of holomorphic functions at the point \( a \). \( I_{\text{loc}} \) is called the local ideal generated by \( I \) or just the localization of \( I \). We denote \( I_{\text{loc}} \) as \( I \).

(b) For \( F = (F_1, \ldots, F_n) \in (A^p)^n \), we denote by \( I(F) \) the ideal in \( A^p \) which is algebraically generated by the functions \( F_1, \ldots, F_n \). The localization of \( I(F) \) is denoted by \( I_{\text{loc}}(F) \).

### 2.5. Proposition
Let \( P \) be a radial weight system satisfying \( p_2(2k) = 0 \) for all \( k \in N \). Then for every closed ideal \( I \) in \( A^p \) there exist \( F_1, F_2 \in A^p \) with \( I = I_{\text{loc}}(F_1, F_2) \).

Proof: It is easy to check that \( A^p \) equals a space of type \( E(K) \) in the notation of Taylor [14], § 2, where \( K \) satisfies conditions K1 to K4. Hence every closed ideal in \( A^p \) is localized by Taylor [14], Th. 7.2. Knowing this, the "jiggling of zeros" argument, indicated in Berenstein and Taylor [1], p. 120, together with Lemma 2.3 shows that \( I = I_{\text{loc}}(F_1, F_2) \) for every closed ideal \( I \) in \( A^p \).

Next we want to determine the locally convex structure of \( A^p \). Let \( P \) be a radial weight system and let \( q \) be an inductive weight function with \( \lim_{z \to \infty} q(z/p_2(z)) = 0 \) for all \( k \in N \). Furthermore let \( F \) be given with
\[
= (F_1, \ldots, F_n) \in (A(C))^n \simeq \bigcap_{k \in N} (A(C))^k.
\]

for some \( B > 0 \) and assume that
\[
V(F) := \sup_{|z| \leq B} |F_j(z)| < \infty
\]
is an infinite set. Moreover, assume that \( F \) is slowly decreasing in the following sense:

(1) There exist \( \varepsilon > 0 \) and \( C > 0 \) such that for each \( k \in N \) there exist \( m \in N \) and \( D_k > 0 \) such that each component \( S \) of the set
\[
S_k(F; \varepsilon; C) := \{ |z| \leq B \mid \sum_{j=1}^{m} |F_j(z)|^2 \leq \varepsilon \exp(-Cq(z)) \}
\]
with \( S \cap V(F) = \emptyset \) is bounded and such that
\[
\sup_{z \in S} \rho_a(z) \leq \sup_{z \in S} \rho_a(z) + D_k.
\]

Since \( V(F) \) is infinite, it follows from (1) that there are infinitely many components \( S \) of \( S_k(F; \varepsilon; C) \) with \( S \cap V(F) = \emptyset \). We choose an enumeration \( (S_j)_{j \in N} \) of these components and choose \( z_j \in S_j \) for each \( j \in N \). Then we define the matrix \( A = (a_{jk})_{j,k \in N} \) by
\[
a_{jk} := \exp(-p_k(z_j)).
\]

Obviously (1) implies that for every \( k \in N \) there exist \( m \in N \) and \( D_k > 0 \) such that
\[
\sup_{z \in S_j} \rho_a(z) \leq \rho_a(z) + D_k
\]
and
\[
\rho_a(z) \leq \rho_a(z) + D_k.
\]

Next put \( I := I_{\text{loc}}(F) \) and \( V := V(F) \) and define for \( j \in N \)
\[
E_j := \prod_{a_{jk} \neq 0} \ell_a f_{j,k}.
\]

Then \( E_j \) is a finite-dimensional complex vector space. Let \( H^\infty(S_j) \) denote the Banach space of all bounded holomorphic functions on \( S_j \). Then it is easy to check that the map
\[
q_j : H^\infty(S_j) \to E_j, \quad q_j(f) = (f_{j,k})_{a_{jk} \neq 0}
\]
is linear and surjective. Hence we get a norm on \( E_j \) by letting
\[
\| f \| := \inf \| q_j(f) \| \quad \| q_j(f) \| := \sup_{|z| \leq B} |F_j(z)| < \infty.
\]

Now let \( E \) denote the sequence \( (E_j) \) of finite-dimensional normed spaces defined by (4) and (6). We want to show that for every \( f \in A^p \), we have
\[
\| q_j(f) \|, \quad \| q_j(f) \| \neq 0.
\]

Then, for every \( f \in A^p \), we have
\[
(q_j(f)(z)), \quad \| q_j(f)(z) \| \neq 0.
\]
To see this, let \( f \in A^2_{\mathbb{R}} \) be given. Then (3) implies that for each \( k \in \mathbb{N} \) there exists \( m \in \mathbb{N} \) such that for all \( j \in \mathbb{N} \) we have by (3)
\[
\|\varphi(jS)\| \leq \|f\|_{H^{a,j}} \leq \|f\|_{m} \exp(\sup_{x \in \mathbb{R}} p_\mu(x))
\]
\[
\leq \|f\|_{m} \exp(D_k + p_\mu(x_j)) = \|f\|_{m} \exp(D_k) \cdot (a_{k,j})^{-1}.
\]
This estimate shows that the linear map
\[
\varphi: A^2_{\mathbb{R}} \rightarrow \mathcal{L}^\infty(A, E), \quad \varphi(f) := (\varphi(jS))_{j \in \mathbb{N}}
\]
is continuous.

To show that \( \varphi \) is surjective, let \( \mu \in (\mu)_{j \in \mathbb{N}} \in \mathcal{L}^\infty(A, E) \) be given. Then we have
\[
\sup_{j \in \mathbb{N}} \|\mu\|_{j} a_{j,k} =: \|\mu\|_{k} < \infty \quad \text{for all} \quad k \in \mathbb{N}.
\]
Now remark that, by 1.2(2), for every \( k \in \mathbb{N} \) there exist \( n \in \mathbb{N} \) and \( L > 0 \) with
\[
a_{j,k} = \exp(-p_\mu(x_j) + p_\mu(x_j)) \leq L \exp(-\frac{1}{2} p_\mu(x_j)).
\]
Since \( \lim_{j \rightarrow \infty} |x_j| = \infty \) and since \( p_\mu \) satisfies 1.1(2) we have
\[
(9) \quad \text{For every} \quad k \in \mathbb{N} \quad \text{there exists} \quad n \in \mathbb{N} \quad \text{with} \quad \lim_{j \rightarrow \infty} a_{j,k} = 0.
\]
Hence (8) implies
\[
\lim_{j \rightarrow \infty} \|\mu\|_{j} a_{j,k} = 0 \quad \text{for all} \quad k \in \mathbb{N}.
\]
Now choose a strictly increasing sequence \( \langle j_k \rangle_{k \in \mathbb{N}} \) in \( \mathbb{N} \) with
\[
\|\mu\|_{j_k} \leq (a_{j_k})^{-1} \quad \text{for all} \quad j_k \in \mathbb{N}
\]
and choose \( \lambda_k \in \mathcal{L}^\infty(S_k) \) with \( \varphi(j_k) = \mu_k \) for all \( j \in \mathbb{N} \) and
\[
\|\lambda_k\|_{m} \exp \leq 2(a_{j_k})^{-1} \quad \text{for} \quad j_k \leq j < j_{k+1}.
\]
Next define \( \lambda: S_{\infty}(F; \varepsilon, C) \rightarrow C \) by
\[
\lambda(z) := \begin{cases} \lambda_{j_k}(z) & \text{if} \quad z \in S_{j_k}, \\ 0 & \text{if} \quad z \in S_j(F; \varepsilon, C) \setminus \bigcup_{j=1}^{j_k} S_{j_k}. \end{cases}
\]
Then \( \lambda \in A(S_{\infty}(F; \varepsilon, C)) \), and from (3) we deduce that for each \( k \in \mathbb{N} \) there exist \( m \in \mathbb{N} \) and \( D_k > 0 \) such that for all \( j > j_k \) and all \( z \in S_j \) we have
\[
|\lambda(z)| = |\lambda_j(z)| \leq 2(a_{j_k})^{-1} = 2 \exp(p_\mu(x_j)) \leq 2 e^{D_k} \exp(p_\mu(x_j)).
\]
This implies that \( \lambda \) satisfies the hypotheses of Proposition 2.2. Consequentially there exists \( \lambda \in A^2_{\mathbb{R}} \) with \( \varphi(\lambda) = (\varphi(jS))_{j \in \mathbb{N}} = \mu \). This shows that \( \varphi \) is surjective.

By this and (7), \( \varphi \) is open by the open mapping theorem. Since \( \ker \varphi = I_{\text{loc}}(F) \), we have proved
\[
(10) \quad A^2_{\mathbb{R}}/I_{\text{loc}}(F) \cong \mathcal{L}^\infty(A, E).
\]

Now remark that \( A^2_{\mathbb{R}} \) is nuclear by 1.4. Hence \( A^2_{\mathbb{R}}/I_{\text{loc}}(F) \) and consequently \( \mathcal{L}^\infty(A, E) \) is nuclear. From (9) we derive that \( \mathcal{L}^\infty(A, E) = \mathcal{L}^1(B, E) \), where
\[
\mathcal{L}^1(A, E) := \{ x \in \mathcal{L}^\infty(A, E) \mid \lim_{j \rightarrow \infty} \|x\|_{j} a_{j,k} = 0 \quad \text{for all} \quad k \in \mathbb{N} \}.
\]
Hence it follows from Meise [7], Remark a) after 1.3, that we have:

(11) For every \( k \in \mathbb{N} \) there exists \( l \in \mathbb{N} \) with \( \sum_{j=1}^{l} (\dim E_j) a_{j,k} < \infty \).

This implies that \( \mathcal{L}^1(A, E) = \mathcal{L}^1(A, E) \). Moreover, it follows from Meise [7], 1.4, that we have
\[
(12) \quad A^2_{\mathbb{R}}/I_{\text{loc}}(F) \cong \mathcal{L}^\infty(A, E) = \mathcal{L}^1(A, E) = \mathcal{L}^1(B, E),
\]
where \( B \) is obtained from \( A \) by repeating the \( j \)-th row of \( A \) \( (\dim E_j) \)-times.

All together we have proved:

2.6. Proposition. Let \( P \) be a radial weight system and let \( q \) be an inductive weight function with
\[
\lim_{j \rightarrow \infty} \frac{q(j)}{a_{j,k}} = 0 \quad \text{for all} \quad k \in \mathbb{N}.
\]
Furthermore let \( F = (F_1, \ldots, F_k) \in (A(C))^k \) be given with
\[
\sup_{1 \leq l \leq k} \sup_{x \in \mathbb{R}} \exp(-B q(x)) < \infty
\]
for some \( B > 0 \) and \( V(F) \) infinite.

If \( F \) is slowly decreasing in the sense of (1) above, then \( A^2_{\mathbb{R}}/I_{\text{loc}}(F) \) is isomorphic to the nuclear Fréchet space \( \mathcal{L}^1(B, E) \) where
\[
b_{j,k} := \exp(-p_\mu(w_j)), \quad j, k \in \mathbb{N},
\]
where \( (w_j)_{j \in \mathbb{N}} \) is an appropriate sequence in \( C \) with \( \lim |w_j| = \infty \).

2.7. Theorem. Let \( P = (p_k)_{k \in \mathbb{N}} \) be a radial weight system satisfying

(1) \quad \text{There exists} \quad R > 0 \quad \text{with} \quad \sup_{k \in \mathbb{N}} p_k(2z) < \infty.

Then for every proper closed infinite-codimensional ideal \( I \) in \( A^2_{\mathbb{R}} \) the quotient \( A^2_{\mathbb{R}}/I \) is isomorphic to the nuclear Fréchet space \( \mathcal{L}^1(B, E) \) with \( b_{j,k} := \exp(-p_\mu(w_j)), \quad j, k \in \mathbb{N}, \) where \( (w_j)_{j \in \mathbb{N}} \) is an appropriate sequence with \( \lim |w_j| = \infty \).
Proof. Let $I$ be an arbitrary proper closed ideal in $A^p_\omega$ which is of infinite codimension. By Proposition 2.5 we have $I = I_{m_\nu}(F_1, F_2)$, where we may assume $F_1 \neq 0$. By Lemma 2.3 there exists a radial inductive weight function $q$ with $q(2\alpha) = O(q(\alpha))$, $\lim_{|\alpha| \to \infty} q(\alpha)/p(\alpha) = 0$ for all $\alpha \in \mathbb{N}$ such that for an appropriate number $B > 0$ we have

$$\max_{j=1,2} \sup_{x \in C} \{|F_j(x)\exp(-Bq(x))| < \infty.$$

An application of the minimum modulus theorem implies (see Levin [5], p. 20, and the proof of Kelleher and Taylor [3], Prop. 5.2) that there exist $\varepsilon > 0, C > 0, n_0 \in \mathbb{N}$ and a sequence $(r_n)_{n \in \mathbb{N}}$ with $e^\varepsilon < r_n < e^{\varepsilon + 1}$ for all $n \geq n_0$ such that for $F = (F_1, F_2)$

$$S_\varepsilon(F; e, C) \cap \bigcup_{n \geq n_0} \{|\alpha| = r_n\} = \emptyset.$$

This shows that, up to finitely many exceptions, for each component $S$ of $S_\varepsilon(F; e, C)$ there exists $n \in \mathbb{N}$ with

$$S = R_n := \{\alpha \in C \mid \varepsilon < |\alpha| < e^{\varepsilon + 1}\}.$$

From (1) it follows that there exists $D > 0$ with

$$\sup_{x \in C} \max_{n \geq n_0} \max_{|\alpha| > R} p_n(\alpha) < D.$$

Now let $k \in \mathbb{N}$ be given. By 1.2(2) there exist $m \in \mathbb{N}$ and $D_k \geq 0$ with

$$D_{m_k}(\alpha) \leq p_k(\alpha) + D_k \quad \text{for all} \quad \alpha \in C.$$

Then for each component $S$ of $S_\varepsilon(F; e, C)$ with $S = R_n$ for $n \geq n_0$ and $S \subseteq \{\alpha \in C \mid |\alpha| > R\}$ we have

$$\sup_{x \in C} p_n(\alpha) \leq p_k(\alpha) + D_k \leq \inf_{x \in C} p_k(\alpha) + D_k.$$

This implies that $F = (F_1, F_2)$ is slowly decreasing in the sense of 2.6(1). Since $I$ has infinite codimension, the set $V(F)$ is infinite. Hence the result follows from Proposition 2.6.

Remark. From the proof of Theorem 2.7 and Proposition 2.6 it follows that the sequence $(w_k)_{k \in \mathbb{N}}$ in the assertion of Theorem 2.7 can be chosen in the following way:

Let $n_0$ and $r_0$ be as in the proof of Theorem 2.7 and put

$$M := \{n \in \mathbb{N} \mid n \geq n_0, \quad V(F) \cap R_n \neq \emptyset\},$$

where $R_{n_0} := \{\alpha \in C \mid |\alpha| < r_{n_0}\}$. Denote by $(n_k)_{k \in \mathbb{N}}$ the increasing arrangement of $M$ and denote by $n_0$ the number of the joint zeros of $F_1$ and $F_2$ in $R_n$ (counted with multiplicities). Then we can take as sequence $(n_k)_{k \in \mathbb{N}}$ the sequence which is obtained from the sequence $(\exp(n_k))_{k \in \mathbb{N}}$ by repeating $\exp(n_0)$ $v_0$-times.

2.8. Corollary. Let $p$ be a radial weight function with $p(2\alpha) = O(p(\alpha))$. Then for every proper closed infinite-dimensional ideal $I$ in $A^p_\omega$, $A^p_\omega/I$ is isomorphic to a nuclear power series type $A_\nu(x_0)$ of finite type.

Proof. By definition we have $A^p_\omega = A^p_{\omega'}$ for $P := (k^{-1} p_k)_{k \in \mathbb{N}}$. From the properties of $p$ it follows easily that $P$ satisfies condition (1) of Theorem 2.7. Hence we have $A^p_{\omega'}/I \approx \lambda^p(B)$, where

$$b_{\lambda k} = \exp \left( \frac{1}{k} p(w_k) \right), \quad j, k \in \mathbb{N}.$$

By the preceding remark we may assume that $\alpha := (p(w_k))_{k \in \mathbb{N}}$ is increasing, hence $\lambda^p(B) = A_\nu(x_0)$.

3. On the complementation of closed ideals in $A^p_\omega$. Now we use the information on the structure of $A^p_{\omega'}/\iota_\omega(F)$ which we have obtained in the previous section to decide whether $I_{m_\nu}(F)$ is complemented in $A^p_\omega$. This is done by means of certain linear topological invariants which have been introduced and investigated by Vogt [16], [17], [19], Vogt and Wagner [21] and Wagner [22]. We begin by recalling the definition of the invariants which we shall use later on.

3.1. Definition. Let $E$ be a metrizable locally convex space and let $(\|\cdot\|)_{k \in \mathbb{N}}$ be an (increasing) fundamental system of seminorms on $E$ generating the locally convex structure of $E$. For $k \in \mathbb{N}$ define $\|x\| := \inf \|y\| \|y\| < \|x\| \leq 1$. Then we say:

(a) $E$ has property (DN) if there exists $m \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ there exist $n \in \mathbb{N}$ and $C > 0$ with

$$\|x\| \leq C \|x\| \quad \|x\| \leq 1.$$

(b) $E$ has property (DN) if there exists $m \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ there exist $n \in \mathbb{N}$, $e > 0$ and $C > 0$ with

$$\|x\| < C \|x\| \quad \|x\| < 1.$$
(e) $E$ has property (Q) if for every $p \in \mathbb{N}$ there exists $q \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ and every $d > 0$ there exists $C > 0$ with
\[
\| q \|_p^{1+d} \leq C \| p \|_p \| q \|_p^d.
\]

3.2. Remark. (a) It is easy to check that properties (DN) and (DDN) are linear topological invariants which are inherited by topological linear subspaces. By Vogt [16], 1.7, a nuclear metrizable locally convex space $E$ has (DN) iff $E$ is isomorphic to a subspace of $s$. By Vogt [16], 2.4, a power series space $A_p(s)$ has (DN) iff $R = +\infty$. By Vogt [17], 3.3, a metrizable locally convex space $E$ is isomorphic to a subspace of a stable nuclear $A_1(s)$ iff $E$ has (DN) and is $A_1(s)$-nuclear.

(b) It is easy to check that properties (Q), (Q) and (Q) are linear topological invariants which are inherited by quotient spaces. By Vogt and Wagner [21], 1.8, a nuclear Fréchet space $E$ has (Q) iff $E$ is a quotient space of $s$. By Vogt [18], 2.8 and 7.3, a strongly nuclear Fréchet space $E$ has (Q) iff $E$ is a quotient of a nuclear power series space of finite type. By Vogt [19], 4.2, a Fréchet space $E$ has (Q) iff every continuous linear map $T : E \to A_1(s)$ is bounded for some (all) power series space $A_1(s)$ with $\sup_{x_{\infty}=1} \| x \|_s < \infty$. For other characterizations of Fréchet spaces satisfying (Q) see Vogt [20], Th. 4.2, and Meise and Vogt [13], Th. 3.3.

(c) From Vogt [18], 1.6, it follows that a nuclear Fréchet space which has properties (Q) and (DN) (resp. (Q) and (DDN)) is finite-dimensional.

3.3. Proposition. Let $P = (p_{hx})_{x<\omega}$ be a radial weight system which satisfies condition 2.7(1). Then for every proper closed ideal $I$ in $A^*_p$,

(a) $A^*_p/I$ has property (Q).

(b) If for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ with
\[
\lim_{x_{\infty}=m} p_n(x) = 0
\]
then $A^*_p/I$ has (Q).

Proof. If $I$ is of finite codimension, then $A^*_p/I$ has (Q) and hence (Q). Hence we may assume that $I$ is of infinite codimension. Then Theorem 2.7 implies that $A^*_p/I \cong \ell^1(A)$ with
\[
b_{jk} = \exp(-p_k(w_j)), \quad j, k \in \mathbb{N},
\]
where $(w_j)_{j<\omega}$ is a sequence in $C$ with $\lim |w_j| = \infty$.

To prove (a) let $n \in \mathbb{N}$ be given. By 1.2(2) there exist $m \in \mathbb{N}$ and $L > 0$ with
\[
2p_n(x) \leq p_m(x) + L \quad \text{for all } x \in C.
\]

Hence we have for each $k \in \mathbb{N}$ and all $l \in \mathbb{N}$
\[
-p_k(w_l) - p_k(w_l) \leq -2p_m(w_l) + L
\]
and hence
\[
b_{jk}b_{lm} \leq e^{b_{jk}}b_{lm}.
\]
By Wagner [22], 1.10, this implies that $\ell^1(A)$ has (Q). To prove (b) let $n \in \mathbb{N}$ be given and choose $m$ according to the hypothesis. Then for every $d > 0$ there exists $L \geq 0$ with
\[
p_m(z) \leq \frac{d}{1+d} p_m(z) + L \quad \text{for all } z \in C.
\]
Hence we have for each $k \in \mathbb{N}$ and each $l \in \mathbb{N}$
\[
-p_k(w_l) - dp_m(w_l) \leq -(1+d) p_m(w_l) + L(1+d)
\]
and consequently
\[
b_{jk}b_{lm} \leq Ch_{jm}^{1+d}.
\]
By standard arguments this implies that $\ell^1(A)$ has (Q).

3.4. Theorem. Let $P = (p_{hx})_{x<\omega}$ be a radial weight system which satisfies condition 2.7(1). Then no proper closed infinite-codimensional ideal $I$ in $A^*_p$ is complemented if one of the following conditions holds:

(a) $A^*_p$ has property (DN).

(b) $A^*_p$ has (DN) and for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ with
\[
\lim_{x_{\infty}=m} p_n(x) = 0.
\]

Proof. Let (a) be satisfied and assume that a proper closed ideal $I$ is complemented in $A^*_p$. Then $A^*_p/I$ has (Q) by Proposition 3.3(a) and has (DN), since $A^*_p/I$ is isomorphic to a topological linear subspace of $A^*_p$ and since (DN) is inherited by topological linear subspaces. Hence $A^*_p/I$ is finite-dimensional by 3.2(c).

If (b) is satisfied and if the proper closed ideal $I$ is complemented in $A^*_p$, then (b) and Proposition 3.3(b) imply by the same arguments as above that $A^*_p/I$ has (Q) and (DN). Hence $A^*_p/I$ is finite-dimensional by 3.2(c).

3.5. Corollary. Let $P$ be a radial weight function with $p(2z) = \Omega(p(z))$. If $A^*_p$ has (DN) then no proper closed infinite-codimensional ideal $I$ in $A^*_p$ is complemented.

3.6. Remark. By Corollary 1.12, $A^*_p$ is isomorphic to a KÖthe sequence space $\ell^1(A)$. Hence it can be characterized by Vogt [16], 2.3, when $A^*_p$ has property (DN). This characterization in terms of the conjugate function $\Omega^*$ of $\Omega : x \mapsto p(e^x)$ is given in [11], where also examples of algebras $A^*_p$ failing (DN) are given.
Here we restrict our attention to the discussion of the following examples.

3.7. Examples. Let \( P \) be any of the following weight systems. Then no proper closed infinite-codimensional ideal \( I \) in \( A^+_\mathbb{R} \) is complemented by Corollary 3.5.

1. \( p(z) = |z|^a, \quad a > 0. \) \( A^+_\mathbb{R} \) has (DN) by 1.13(1) or 1.16.

2. \( p(z) = (\log(1 + |z|^2))^\alpha, \quad \alpha > 1. \) \( A^+_\mathbb{R} \) has (DN) by 1.13(2).

3. Let \( (M_j)_{j=0}^\infty \) be a sequence of positive numbers with \( M_0 = 1 \) which has the following properties:

\[(M1) \quad M_j \leq M_{j-1} M_{j+1} \quad \text{for all} \quad j \in \mathbb{N}.\]

\[(M2) \quad \text{There exist} \quad A, H > 1 \quad \text{with} \quad M_n \leq AH^n \quad \text{for all} \quad n \in \mathbb{N}.\]

\[(\ast) \quad \text{There exists} \quad k \in \mathbb{N} \quad \text{with} \quad \lim \inf (M_{2^n}/M_n^{1/1}) > 1.\]

Then it has been remarked in Meise [7], 2.6(2), that the function

\[p_M: \quad z \mapsto \sup_{|z| = r} \log(|z|^2/M_z) \quad \text{for} \quad z \neq 0,
\]

\[p_M: \quad z \mapsto \log(M_z^z/M_z) \quad \text{for} \quad z = 0\]

is a weight function which satisfies condition 1.16(\( \ast \)). Hence \( A^+_\mathbb{R} \) has (DN) by Corollary 1.16.

3.8. Examples. Let \( P \) be any of the following weight systems. Then no proper closed infinite-codimensional ideal \( I \) in \( A^+_\mathbb{R} \) is complemented by Proposition 3.4.

1. \( P = (\varepsilon_{\mathbb{R}} \quad q(z))_{\mathbb{N}} \) as in Example 1.14(1). \( A^+_\mathbb{R} \) has (DN) by 1.14(1).

2. \( P = \varepsilon_{\mathbb{R}} h_{\mathbb{N}}, \) where \( h_{\mathbb{N}} \) is a strictly decreasing sequence in \( J_0, \): \( A^+_\mathbb{R} \) has (DN) by 1.14(2).

3. \( P = (r(z) \quad q(z))_{\mathbb{N}} \) By the sequence space representation given in Example 1.14(3) it is easy to check that \( A^+_\mathbb{R} \) has property (DN). It has even the stronger property (DN), introduced in Vogt [19], p. 190.

Remark. Let \( P \) (resp. \( \bar{P} \)) be as in Example 1.14(4). Then it is easy to check that \( A^+_\mathbb{R} \) has property (\( \bar{Q} \)). As a consequence of 3.2, \( A^+_\mathbb{R} \) does not have property (DN). Hence Theorem 3.4 cannot be used to decide whether closed ideals in \( A^+_\mathbb{R} \) are complemented. However, we can use other properties of the Fréchet spaces which are involved to decide this question.

3.9. Lemma. Let \( P = (p_{\mathbb{R}})_{\mathbb{N}} \) be a radial weight system satisfying condition 2.7(1). For \( k \in \mathbb{N} \) put \( \phi_k: x \mapsto p_k(e^x) \) and define the matrix \( B = (b_{jk}) \) by

\[b_{jk} := \exp(-\phi_j(\theta)) \quad \text{if} \quad \exp(-\phi_j(\theta)) \quad \text{is defined,} \]

then no proper closed infinite-codimensional ideal in \( A^+_\mathbb{R} \) is complemented.

Proof. Assume that \( I \) is a proper closed infinite-codimensional ideal in \( A^+_\mathbb{R} \) which is complemented. Then the quotient map \( \phi: A^+_\mathbb{R} \to A^+_\mathbb{R}/I \) has a continuous linear right inverse \( R: A^+_\mathbb{R}/I \to A^+_\mathbb{R} \), which is an injective topological homomorphism and hence noncompact. By Theorem 2.7 we have

\[A^+_\mathbb{R}/I \cong A^+_\mathbb{R}(C) \quad \text{where} \quad C = (c_{jk}) \quad \text{with} \quad c_{jk} = \exp(-\phi_j(\theta)), \quad \text{for an appropriate sequence} \quad (w_{jk})_{j=0}^\infty \quad \text{in} \quad C.\]

By the remark after Theorem 2.7, there exists a subsequence \( (m_{3n})_{n=0}^\infty \) of \( N \) such that for \( D = (d_{jk}) \) defined by

\[d_{jk} := \exp(-\phi_j(\theta)), \quad \lambda_1(D) \quad \text{is isomorphic to a complemented subspace of} \quad \lambda_1(C), \quad \text{and of} \lambda_1(B).\]

Denote a continuous linear projection from \( \lambda_1(C) \) onto \( \lambda_1(D) \). Then \( \lambda_1(D)_{\mathbb{N}} \) is a continuous linear map from \( \lambda_1(B) \) into \( \lambda_1(A) \) which is not compact. Hence the assumption that \( I \) is complemented leads to a contradiction to the hypothesis.

3.10. Proposition. Let \( P = (p_{\mathbb{R}})_{\mathbb{N}} \) be a radial weight system satisfying condition 2.7(1). For \( k \in \mathbb{N} \) put \( \phi_k: x \mapsto p_k(e^x). \) If the following holds:

*(\ast) \quad \text{For every} \quad (K(N))_{\mathbb{N}} \quad \text{there exists} \quad k \in \mathbb{N} \quad \text{such that for} \quad \text{each} \quad n \in \mathbb{N} \quad \text{there exists} \quad M \in \mathbb{N} \quad \text{and} \quad C > 0 \quad \text{such that for all} \quad v, j \in \mathbb{N},

\[\phi_k(v) + \phi_j(v) \leq \max_{1 \leq i \leq M} (\phi_k(v) + \phi_{kN}(v) + C),
\]

then no proper closed infinite-codimensional ideal in \( A^+_\mathbb{R} \) is complemented.

Proof. By Corollary 1.11 and Vogt [19], Satz 1.5, condition (\( \ast \)) is equivalent to the assertion that every continuous linear map from \( \lambda_1(B) \) into \( A^+_\mathbb{R} \) is compact, where \( B \) is the matrix defined in 3.9. Hence the result follows from Lemma 3.9.

Condition (\( \ast \)) looks rather complicated. However, it can be used to decide whether the ideals in \( A^+_\mathbb{R}, P \) as in Example 1.14(4), are complemented.

3.11. Example. For \( \epsilon > 0 \) define \( P = : a(z) \exp([\max(0, \log \log |z|^2)]^\epsilon))_{\mathbb{N}}, \) where \( (r_{jk})_{\mathbb{N}} \) is a strictly decreasing sequence in \( J_0, \) hence \( \epsilon \). Then no proper closed infinite-codimensional ideal in \( A^+_\mathbb{R} \) is complemented.

To show this we first remark that by Meise [7], Example 2.13(5), we have \( A^+_\mathbb{R} = A^+_{\mathbb{R}} \) where \( P \) satisfies condition 2.7(1). Hence we can apply Proposition 3.10. This is done essentially by the same arguments which have been used in Meise [7], Example 4.16.

Let \( (K(N))_{\mathbb{N}} \) be given. Without restriction we can assume that \( (K(N))_{\mathbb{N}} \) is strictly increasing. Choose \( k = K(1) + 1 \) and let \( n \in \mathbb{N} \) be given. Then choose \( M > n + 1 \) and \( \epsilon > 0 \) such that \( \phi_k - \phi_\epsilon \) and \( \phi_k - \phi_{\epsilon N} \) are strictly increasing on \( \mathbb{R}, \) where \( \phi_k - \phi_\epsilon \) Next fix \( s > \delta_0, \) where \( \delta_0 \) is large enough and can be determined from the following considerations. Define \( T(s) \) (resp. \( \tau(s) \)) as the solution of the following equation (7) (resp. (1)).

\[\phi_k^*(s) = \phi_k^*(s) \leq \phi_{kN}(t) - \phi_k(t),
\]

\[\phi_k^*(s) - \phi_k^*(s) = \phi_{kN}(t) - \phi_k(t).
\]
Assume for a moment that we can show:

(1) There exists \( t_0 \in [0, \infty[ \) with \( T(s) \leq \tau(s) \) for all \( s \geq t_0 \).

Then we have for all \( s \geq t_0 \)

\[
\phi(s) - \phi^*(t) \leq \phi(Kt) - \phi(t) \quad \text{for all } t \geq T(s),
\]

\[
\phi(s) - \phi^*(t) \geq \phi_0(t) - \phi_0(Kt) \quad \text{for all } t \in [\xi_0, \tau(s)],
\]

where \( \xi_0 \) is chosen appropriately. Hence

\[
\phi(s) + \phi_0(t) \leq \max(\phi(Kt) + \phi(s), \phi_0(t) + \phi_0^*(s))
\]

for all \( s \geq t_0 \) and all \( t \geq \xi_0 \). This implies the existence of \( J_0 \) and \( \eta_0 \) such that

\[
\phi(s) + \phi_0(t) \leq \max_{1 \leq k \leq m} (\phi(s) + \phi_0^* (j)) \quad \text{for all } j \geq j_0, \ n \geq n_0.
\]

Then it is easy to check that there exists \( C > 0 \) such that \( 3.10(\star) \) holds. Hence no proper closed infinite-codimensional ideal in \( A^*_D \) is complemented by Proposition 3.10 if we can show that (1) holds. To do this it suffices to show that for all large \( s \) we have

\[
\phi_0(T(s)) - \phi_0(KM^{-1}(T(s))) \leq \phi^*(s) - \phi_0^*(s).
\]

To do this, we note that by Meise [73, Example 2.13(5)], we have

\[
\phi^*(s) = \frac{2}{r} \log(s - \frac{s}{r} \log \log s) \quad \text{for large enough } s
\]

and that

\[
\phi_0(t) = \exp(\log r + \log \log r) \quad \text{for } t \text{ large enough.}
\]

To abbreviate we put \( f_1 : t \mapsto r^{-1}(\log \log s) \) and \( f_2 : t \mapsto (\log \log r) \). From the definition of \( T(s) \) we now get the identity

\[
s(f_1(s) - f_2(s)) = \exp(T(s)) \left[ \exp(\phi_0(T(s))) - \exp(\phi_1(T(s))) \right].
\]

Since \( \lim_{s \to \infty} T(s) = \infty \) and since \( k > K(1) \) we get

\[
\phi_0(T(s)) - \phi_0(KM^{-1}(T(s))) \leq \phi_0(T(s))
\]

\[
= s(f_1(s) - f_2(s)) \exp(\phi_0(T(s))) \left[ \exp(\phi_0(T(s))) - \exp(\phi_1(T(s))) \right]^{-1}
\]

\[
\leq \frac{s}{2} \exp(\phi_1(T(s)) - \phi_0(KM^{-1}(T(s))).
\]

From \( M > n + 1 \) we get for \( s \) large enough

\[
\phi(s) - \phi^*(s) = s(f_1(s) - f_2(s)) > \frac{s}{2} f_1(s).
\]

By \( \lim_{s \to \infty} T(s) = \infty \) and \( k > K(1) \) we get for large \( s \)

\[
\exp(\frac{1}{2} \gamma_{K1}(T(s))) \leq \exp(\gamma_{K1}(T(s)) - \phi_0(T(s))).
\]

Now (4), (5) and (6) show that (2) is implied by the inequality

\[
\frac{f_1(s)}{f_2(s)} \leq \frac{1}{2} \exp\left(\frac{1}{2} \gamma_{K1}(T(s))\right).
\]

To prove that (7) holds for large \( s \), one has to estimate \( T(s) \) from below. From the definition of \( T(s) \) we get

\[
rT(s) = \log s + \log f_1(s) - \log \left( \exp(\gamma_{K1}(T(s))) - \exp(\phi_0(T(s))) \right)
\]

This implies

\[
rT(s) \leq \log s + \log f_2(s) \leq 2 \log s
\]

for large \( s \) and hence by (9) and \( \lim_{s \to \infty} T(s) = \infty \)

\[
\exp(\frac{1}{2} \gamma_{K1}(T(s))) \geq \exp\left(\frac{1}{2} \log(\log s) - \log(2 \log s)\right) \geq \exp\left((r_1 - r_2) \log \log s\right)
\]

\[
= \frac{f_1(s)}{f_2(s)} \quad \text{for all } s \geq s_0.
\]

This shows that (7) holds for all \( s \) sufficiently large and hence completes the proof.

4. Translation invariant subspaces for some weighted \((DF)\)-spaces of entire functions. It was Martineau [6] who extended the classical work on convolution operators on Fréchet spaces of entire functions to convolution operators on \((DF)\)-spaces of entire functions. In this section we show that the results of Sections 2 and 3 can be used to determine the locally convex structure of the closed translation invariant subspaces of various \((DF)\)-spaces of entire functions, including those which were considered by Martineau [6].

We begin by introducing the \((DF)\)-spaces which we will work with.

4.1. Definition. Let \( Q = (q_0, q_{\infty}) \) be a sequence of radial weight functions with the following properties:

\( 1 \) For every \( k \in N \) there exists \( K \geq 0 \) with

\[
q_k(z) < q_{k+1}(z) + K \quad \text{for all } z \in C.
\]
(2) For every \( k \in \mathbb{N} \) there exist \( l \in \mathbb{N} \) and \( L \geq 0 \) with
\[
2q_k(z) \leq q_l(z) + L
\]
for all \( z \in C \).
(3) For every \( k \in \mathbb{N} \) there exist \( m \in \mathbb{N} \) and \( M \geq 0 \) with
\[
q_k(z) \leq q_m(z) + M
\]
for all \( z \in C \).
(4) \( q_k([0, \infty[) \) is convex and satisfies
\[
\lim_{x \to \infty} \frac{q_k(x)}{x} = \infty \quad \text{for all } k \in \mathbb{N}.
\]

Then we define
\[
A_k(C) := \{ f \in A(C) \mid \exists \text{ sup } |f(z)| \exp(-q_k(z)) < \infty \}
\]
and endow \( A_k(C) \) with its natural inductive limit topology. Because of (4) we can define \( \rho_k: z \mapsto (q_k([0, \infty[) \ast [0]) \ast [0]) \). We assume that \( A_k := (\rho_k)_{k \in \mathbb{N}} \) is a weight system which satisfies condition (1) of Theorem 2.7.

The following proposition is contained in Taylor [14], Th. 5.2. For the convenience of the reader we give the proof here too.

4.2. PROPOSITION. Let \( Q = (q_k)_{k \in \mathbb{N}} \) and \( P_0 = (\rho_k)_{k \in \mathbb{N}} \) be as in 4.1. Then the Fourier–Borel transform
\[
\mathcal{F}: A(Q)(C_0) \to A_k Q_0,
\]
\[
\mathcal{F}(T): \zeta \mapsto \sum_{j=0}^\infty \langle T_j, \exp(\zeta z) \rangle,
\]
is a linear topological isomorphism.

Proof. From the conditions on \( Q \) and Proposition 1.10 it follows that \( f: \sum_{j=0}^\infty a_j z^j \) is in \( A_k(Q)(C_0) \) iff there exist \( k \in \mathbb{N} \) and \( C > 0 \) with
\[
|a_j| \leq C \exp(-q_k(\exp(j))) \quad \text{for all } j \in \mathbb{N}.
\]

Since \( A_k(Q) \) is a (DFN)-space (see Meise [7], 2.4), this implies that a linear map \( T: A_k(Q)(C_0) \to C \) is continuous iff
(2) For every \( k \in \mathbb{N} \) there exists \( C_k \) such that for all \( j \in \mathbb{N} \)
\[
|T(z)| \leq C_k \exp((q_k \circ \exp) \ast (j))(j).
\]
By Taylor [14], Lemma 5.3, and Stirling's formula this implies that for every \( \mathcal{F}(T) \in A_k(Q) \) and every \( k \in \mathbb{N} \) there exists \( C_k \) such that for all \( j \in \mathbb{N} \)
\[
|T(z)| \leq C_k \exp((q_k \circ \exp) \ast (j))(j) \leq C_k \exp(-q_k(\exp(j))(j))
\]
By Proposition 1.10 this shows that
\[
\mathcal{F}(T): \zeta \mapsto \sum_{j=0}^\infty \langle T_j, \exp(\zeta z) \rangle,
\]
is in \( A_k Q_0 \). Hence \( \mathcal{F} \) maps \( A(Q)(C_0) \) into \( A_k Q_0 \) and is continuous because of (3).
To show that \( \mathcal{F} \) is surjective, let \( g: \zeta \mapsto \sum_{j=0}^\infty b_j \zeta^j \) be given. Then
\[
\tilde{g}: \zeta \mapsto \langle (\tilde{f} \circ \exp) \ast \rangle \sum_{j=0}^\infty (j+1)(j+2)b_j \zeta^j
\]
is in \( A_k Q_0 \). This implies by Proposition 1.10, Taylor [14], Lemma 5.3, and Stirling's formula that for each \( k \in \mathbb{N} \) there exists \( C_k > 0 \) such that for all \( j \in \mathbb{N} \)
\[
|j!b_j| \leq C_k \exp(-q_k(\exp(j))(j+1)(j+2)) \exp(j),
\]

By (1) this implies that \( T: f \mapsto \sum_{j=0}^\infty b_j \tilde{f}(0) \) is in \( A_k(C_0) \) and satisfies \( \mathcal{F}(T) = \tilde{g} \). Hence \( \mathcal{F} \) is a linear bijection. Since (4) implies the continuity of \( \mathcal{F}^{-1} \), the proof is complete.

4.3. DEFINITION. A linear subspace \( W \) of \( A(Q)(C_0) \) is called translation invariant if for every \( f \in W \) and every \( a \in C \) the function \( z \mapsto f(z+a) \) belongs to \( W \).

4.4. PROPOSITION. Let \( Q \) and \( P_0 \) be as in 4.1. Then a closed linear subspace \( W \) of \( A(Q)(C_0) \) is translation invariant if and only if \( \mathcal{F}(W) \) is an ideal in \( A_k Q_0 \).

This can be proved in the same way as Meise [7], Proposition 5.5.

4.5. THEOREM. Let \( Q \) and \( P_0 \) be as in 4.1. Then every closed linear translation invariant subspace \( W \) of \( A(Q)(C_0) \) has a Schauder basis.

Proof. By Proposition 4.2 and classical duality theory we have
\[
W = W_{\ast \ast} \cong \mathcal{F}(W) \cdot \mathcal{F}(W)'.
\]

If \( W \) is finite-dimensional or equal to \( A(Q)(C_0) \) then the result holds trivially. Hence we may assume that \( \mathcal{F}(W) = \mathcal{F}(W) \) is a proper infinite-codimensional ideal in \( A_k Q_0 \) because of Proposition 4.4. Consequently, Theorem 2.7 implies that \( W \cong A_k(C_0) \).

4.6. COROLLARY. Let \( q \) be a convex radial weight function which satisfies condition 1.1.6(\( \ast \)) and \( \lim_{x \to \infty} q(x)/x = \infty \). Assume that \( p: z \mapsto q(\exp(z)) \) satisfies condition 1.1.6(\( \ast \)) and put \( Q := (k)_{k \in \mathbb{N}} \). Then every proper closed linear infinite-dimensional translation invariant subspace \( W \) of \( A(Q)(C_0) \) is isomorphic to the strong dual of a nuclear power series space of finite type, and no such subspace is complemented in \( A(Q)(C_0) \).
Proof. Since \( q \) satisfies condition \((\ast)\) of 1.16, we have \( A_Q^e(C) = A_{Q^e}(C) \), where \( \tilde{Q} = \{q(k)\}_{k \in N} \). Hence we have \( P_{\tilde{Q}} = \{pq(k)\}_{k \in N} \). Since \( p \) satisfies condition \((\ast)\) of 1.16, we have \( A_{Q^e}^p = A_{Q^e}^p = A_{Q^e}^p \). By the proof of Theorem 4.5 we have
\[
W = (A_{Q^e}^p, \mathcal{F}(W^e))_n
\]
Hence the result follows from Corollary 2.8 and Theorem 3.4 in connection with Corollary 1.16.

47. EXAMPLES. (1) For \( s > 1 \) put
\[
E^e = \{ f \in A(C) \mid \exists k \in \mathbb{N} \text{ with } \mathbb{R} f(k) \exp(-k|z|^s) < \infty \}
\]
Then it follows easily from Corollary 4.6 that every proper closed infinite-dimensional translation invariant linear subspace \( W \) of \( E^e \) is isomorphic to a power series space of finite type, and no such subspace is complemented.

(2) Let \( \sigma = (\xi_k)_{k \in N} \) be a strictly increasing sequence in \( \mathbb{N} \), \( \sigma(1) > 1 \), and put
\[
E(\sigma) = \{ f \in A(C) \mid \exists k \in \mathbb{N} \text{ with } \mathbb{R} f(k) \exp(-|\xi_k|^s) < \infty \}
\]
Then every proper closed infinite-dimensional translation invariant linear subspace \( W \) of \( E(\sigma) \) has a Schauder basis, and no such subspace is complemented. Moreover, \( W \) has property \( (\tilde{Q}) \).

It is easy to check that \( E(\sigma) = A_{Q(\sigma)}(C) \), where \( Q(\sigma) = (\xi_k^{-1}|z|^\xi_k)_{k \in N} \). Since \( Q(\sigma) \) and \( P(\sigma) = (r_k^{-1}|z|^r_k)_{k \in N} \) with \( r_k = \sigma_k/(\sigma_k - 1) \) satisfy the condition of 4.1, Theorem 4.5 implies that \( W \) has a Schauder basis. By Proposition 3.3, \( W \) has property \( (\tilde{Q}) \), while Example 3.8(1) shows that \( \mathcal{F}(W^e) \) and hence \( W \) is not complemented.

References