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Sequence space representations for (FN)-algebras of entire functions modulo closed ideals

by

REINHOLD MEISE (Düsseldorf) and B. ALAN TAYLOR (Ann Arbor, Mich.)

Abstract. Let A_P^0 denote the weighted (FN)-algebra of entire functions on C defined by an appropriate weight system P. We prove that for every infinite-codimensional proper closed ideal I in A_P^0 the quotient A_P^0/I is isomorphic to a Köthe sequence space. In the interesting special case that P is generated by a single weight function, A_P^0/I is even isomorphic to a power series space of finite type. From the sequence space representation we deduce that in all relevant examples I is not complemented in A_P^0. Furthermore, it follows that all proper closed infinite-dimensional translation invariant subspaces of certain weighted (DFN)-spaces of entire functions have a Schauder basis but are not complemented.

Let $P = (p_k)_{k\in N}$ be a decreasing sequence of radial subharmonic functions on C which satisfy some mild technical conditions. Denote by A_P^0 the vector space of all entire functions on C satisfying $\sup_{z\in C} |f(z)| \exp(-p_k(z)) < \infty$ for all $k \in N$. Under its natural locally convex topology A_P^0 becomes a nuclear Fréchet algebra. Algebras of this type have been studied since a long time. They arise in complex analysis and functional analysis.

In the present article we use results and methods of Berenstein and Taylor [1] and Meise [7] to prove that for every proper closed infinite-codimensional ideal I of A_P^0 the quotient space A_P^0/I is isomorphic to a nuclear Köthe sequence space. If the weight system P is of the special form $P = (k^{-1} p)_{k\in N}$, then we derive that A_P^0/I is isomorphic to a nuclear power series space of finite type.

This sequence space representation of A_P^0/I allows to use the structure theory of nuclear Fréchet spaces to investigate whether an ideal I is complemented in A_P^0. It turns out that in all our examples no proper infinite-codimensional ideal I is complemented. This is essentially due to the fact that each continuous linear map from A_P^0/I into A_P^0 is already compact. Moreover, it explains the corresponding observation of Taylor [15] and gives results which cover a significantly larger class of examples.

By means of the Fourier–Borel transform, the information on the structure of A_P^0/I implies that in certain weighted (DFN)-spaces of entire functions all translation invariant subspaces have a Schauder basis. As a particular example we mention the following: For $s>1$ denote by

$$E^s := \{f \in A(C) \mid \text{there exists } k \in N \text{ with } \sup_{z\in C} |f(z)| \exp(-k|z|^s) < \infty\}.$$

Then every proper closed infinite-dimensional translation invariant subspace W of E^s is isomorphic to the strong dual of a nuclear power series space of finite type and is not complemented in E^s. This result should be compared with the results of Meise [7], Sect. 5, on the translation invariant subspaces of A_P^0, where $P=(k^{-1}|z|^s)_{k\in N}$, $s>1$. For applications of the results of the present article and for related work we refer to [8]–[12].

The article is divided into four sections. In the first one we introduce the weighted algebras A_P^0 and the sequence spaces which we need and give some examples. In section two, the representation theorem for A_P^0/I is proved. The question of the complementation of the closed ideals in A_P^0 is treated in section three, and the results on the translation invariant subspaces are presented in section four.

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1. Weighted algebras, sequence spaces and examples. In this section we introduce the weighted algebras A_P^0 of entire functions on C which will be treated in the sequel. Moreover, we give sequence space representations of these algebras.

1.1. Definition. A function $p\colon C\to [0,\infty[$ is called a *weight function* if it has the following properties:

(1) p is continuous and subharmonic.

(2) $\log(1+|z|^2) = o(p(z))$.

(3) There exists $C\geq 1$ such that for all $w\in C$

$$\sup_{|z-w|\leq 1} p(z) \leq C \inf_{|z-w|\leq 1} p(z) + C.$$

A weight function will be called *radial* if $p(z)=p(|z|)$ for all $z\in C$. p will be called an *inductive weight function* if it satisfies (1), (3) and if $\log(1+|z|^2) = O(p(z))$.

1.2. Definition. A sequence $P=(p_k)_{k\in N}$ of weight functions is called a *weight system* if it has the following properties:

(1) For every $k\in N$ there exists $M\geq 0$ with $p_{k+1}\leq p_k+M$.

(2) For every $k\in N$ there exist $m\in N$ and $L\geq 0$ with

$$2p_m(z)\leq p_k(z)+L \quad \text{for all } z\in C.$$

A weight system $P=(p_k)_{k\in N}$ is called *radial* if p_k is radial for all $k\in N$.

For an open set Ω in C let $A(\Omega)$ denote the algebra of all holomorphic functions on Ω. If P is a given weight system then we define the subalgebra A_P^0 of $A(C)$ in the following way:

1.3. Definition. (a) For a weight function p we put

$$H_p^\infty := \{f\in A(C) \mid \|f\|_{p,\infty} := \sup_{z\in C} |f(z)|\,e^{-p(z)} < \infty\}$$

$$H_p^2 := \{f\in A(C) \mid \|f\|_{p,2} := \Big(\int_C (|f(z)|\,e^{-p(z)})^2\, dm(z)\Big)^{1/2} < \infty\},$$

where m denotes the Lebesgue measure on $C=R^2$.

(b) For a weight system P we define

$$A_P^0 := \bigcap_{k\in N} H_{p_k}^\infty = \operatorname{proj}_{\leftarrow k} H_{p_k}^\infty$$

and endow this vector space with its natural projective limit topology. If $P=(k^{-1}p)_{k\in N}$ then we write A_p^0 instead of A_P^0.

By standard arguments one proves:

1.4. Proposition. *For every weight system* P:

(a) A_P^0 *is a locally convex algebra with unit under pointwise multiplication.*

(b) A_P^0 *is a nuclear Fréchet space.*

(c) $A_P^0 = \operatorname{proj}_{\leftarrow k} H_{p_k}^2 = \operatorname{proj}_{\leftarrow k} A_{p_k}^0$.

1.5. Examples. (1) Let $\varphi\colon [0,\infty[\,\to[0,\infty[$ be continuous, convex and increasing with $\lim_{t\to\infty}\varphi(t)=\infty$ and assume that there exists $D\geq 1$ with $\varphi(2t)\leq D\varphi(t)+D$ for all $t\in[0,\infty[$. Then it follows easily from Hörmander [2], Th. 1.6.7, that $\varphi\circ p$ is a weight function for every weight function p.

(2) Let $\varphi\colon [0,\infty[\,\to[0,\infty[$ be continuous with $\lim_{t\to\infty}\varphi(t)=\infty$. Assume that $t\mapsto\varphi(e^t)$ is convex and increasing and that there exists $D\geq 1$ with $\varphi(2t)\leq D\varphi(t)+D$ for all $t\in[0,\infty[$. Then $p\colon z\mapsto \varphi(|z|^r)$ is a radial weight function for each $r>0$.

Most of the following examples can be obtained from (1) or (2):

(3) $p(z)=|z|^r$, $r>0$.

(4) $p(z)=(\log(1+|z|^2))^s$, $s>1$.

(5) $p(z)=|\mathrm{Re}\,z|^r+|\mathrm{Im}\,z|^s$, $r,s\geq 1$.

(6) $p(z)=|z|^r+|\mathrm{Im}\,z|^s$, $r>0$, $s\geq 1$.

1.6. Definition. (a) Let $A=(a_{j,k})_{(j,k)\in N^2}$ be a matrix of nonnegative numbers $a_{j,k}$. A is called a *Köthe matrix* if

(1) $a_{j,k}\leq a_{j,k+1}$ for all $j,k\in N$.

(2) $a_{j,1}>0$ for all $j\in N$.

(b) Let A be a Köthe matrix and let $E=(E_j, \|\ \|_j)_{j\in N}$ be a sequence of Banach spaces. For $1\leqslant p<\infty$ we define

$$\lambda^p(A, E):=\{x\in\prod_{j\in N} E_j \mid \pi_{k,p}(x):=\Big(\sum_{j=1}^{\infty}(\|x_j\|_j a_{j,k})^p\Big)^{1/p}<\infty \quad \text{for all } k\in N\}$$

and for $p=\infty$ we put

$$\lambda^\infty(A, E):=\{x\in\prod_{j\in N} E_j \mid \pi_{k,\infty}(x):=\sup_{j\in N}\|x_j\|_j a_{j,k}<\infty \quad \text{for all } k\in N\}.$$

These spaces of vector-valued sequences are Fréchet spaces under their natural locally convex topology, induced by the norms $(\pi_{k,p})_{k\in N}$. If $E_j=(C, |\cdot|)$ for all $j\in N$, then we write $\lambda^p(A)$ instead of $\lambda^p(A, E)$. Instead of $\lambda^1(A)$ we sometimes write $\lambda(A)$.

1.7. Example. Let α be an increasing unbounded sequence of positive real numbers (called an *exponent sequence*) and put

$$a_{j,k}:=\exp(k\alpha_j) \quad \text{and} \quad b_{j,k}:=\exp\Big(-\frac{1}{k}\alpha_j\Big), \quad j, k\in N.$$

Then the corresponding space $\lambda^1(A)$ (resp. $\lambda^1(B)$) is denoted by $\Lambda_\infty(\alpha)$ (resp. $\Lambda_1(\alpha)$) and is called a *power series space of infinite* (resp. *of finite*) *type*. Classical examples of power series spaces are the following: The space $C^\infty(S^1)$ of all C^∞-functions on the unit circle S^1 is isomorphic to $\Lambda_\infty((\log(j+1))_{j\in N})$. The space $A(C)$ is isomorphic to $\Lambda_\infty((j)_{j\in N})$. The space $A(D)$, D the open unit disk, is isomorphic to $\Lambda_1((j)_{j\in N})$.

Later in the applications we shall need sequence space representations of A_P^0. For radial weight functions such representations can be obtained by estimating the Taylor coefficients of the functions in A_P^0. Sufficiently precise estimates can be obtained by means of the Young conjugate of a convex function.

1.8. Definition. Let φ: $[0, \infty[\to R$ be an increasing convex function. Then its *Young conjugate* φ^*: $[0, \infty[\to R\cup\{\infty\}$ is defined by

$$\varphi^*(y):=\sup\{xy-\varphi(x) \mid x\geqslant 0\}.$$

1.9. Remark. The following facts are easy to check:

(a) φ^* is convex.

(b) If $\lim_{t\to\infty}[\varphi(t)/t]=\infty$, then φ^* is strictly increasing on $[\alpha, \infty[$, where $\alpha=(d\varphi/dt)(0)$.

(c) If $\lim_{t\to\infty}[\varphi(t)/t]=\infty$, then $(\varphi^*)^*\doteq\varphi$.

The next result follows easily from Cauchy's inequality (see e.g. Taylor [14]).

1.10. Proposition. *Let q: $[0, \infty[\to R$ be an increasing function and put φ: $x\mapsto q(e^x)$. Assume that q is constant on $[0, 1]$, that φ is convex and that $\lim_{x\to\infty}[\varphi(x)/x]=\infty$. Then we have the following assertions for every entire function f: $z\mapsto\sum_{j=0}^{\infty}a_j z^j$:*

(a) *If $\sup_{z\in C}|f(z)|\exp(-q(|z|))=A$, then $|a_j|\leqslant A\exp(-\varphi^*(j))$ for all $j\in N_0$.*

(b) *If $\sup_{j\in N_0}|a_j|\exp(\varphi^*(j))=A$, then $\sup_{z\in C}|f(z)|\exp(-q(2|z|))\leqslant 2A$.*

Remark. It is easy to check that for a radial weight function p, the function φ: $x\mapsto p(e^x)$ is an increasing convex function on $[0, \infty[$. We shall use this in the sequel. Moreover, we shall also use that we may assume w.l.o.g. that p is constant on $[0, 1]$. Hence the following corollaries are immediate consequences of Proposition 1.10.

1.11. Corollary. *Let $P=(p_k)_{k\in N}$ be a radial weight system such that for each $k\in N$ there exist $m\in N$ and $D\geqslant 0$ with $p_m(2z)\leqslant p_k(z)+D$ for all $z\in C$. Then A_P^0 is isomorphic to $\lambda^1(A)$ for $A=(a_{j,k})_{j\in N_0, k\in N}$ with $a_{j,k}:=\exp(\varphi_k^*(j))$, where φ_k: $x\mapsto p_k(e^x)$.*

1.12. Corollary. *Let p be a radial weight function with $p(2z)=O(p(z))$. Then A_p^0 is isomorphic to $\lambda^1(A)$ with $a_{j,k}=\exp(k^{-1}\varphi^*(kj))$, where φ is defined by φ: $x\mapsto p(e^x)$.*

1.13. Example. (1) For $a>0$ let φ: $[0, \infty[\to R$ be given by $\varphi(x)=e^{ax}$. Then it is easy to check that φ^*: $y\mapsto(y/a)(\log(y/a)-1)$. Hence it follows from Corollary 1.12 that for p: $z\mapsto|z|^r$, $r>0$, we have $A_p^0\simeq\lambda^1(A)$ with

$$a_{j,k}=\exp\Big(\frac{1}{k}\frac{kj}{r}\Big(\log\frac{kj}{r}-1\Big)\Big)=\exp\Big(\frac{j}{r}\log\frac{j}{er}+\frac{j}{r}\log k\Big).$$

This shows that $A_p^0\simeq\lambda^1(A)\simeq\Lambda_\infty((j)_{j\in N})$.

(2) For $\alpha>1$ put $\beta:=\alpha/(\alpha-1)$ and φ: $[0, \infty[\to R$, $\varphi(x):=(1/\alpha)x^\alpha$. Then it is easy to check that $\varphi^*(y)=(1/\beta)y^\beta$. Hence it follows from Corollary 1.12 that for $p(z)=(\log(1+|z|^2))^\alpha$ we have $A_p^0\simeq\lambda^1(A)$ with

$$a_{j,k}=\exp\Big(\frac{1}{k}\frac{1}{\beta}(kj)^\beta\Big)=\exp\Big(\frac{1}{\beta}k^{\beta-1}j^\beta\Big).$$

This shows that $A_p^0\simeq\Lambda_\infty((j^\beta)_{j\in N})$.

1.14. Example. (1) Let q: $[0, \infty[\to[0, \infty[$ be continuous with $q(2t)=O(q(t))$ and $\lim_{t\to\infty}[q(t^r)/q(t)]=0$ for each $0<r<1$. Assume furthermore that ψ: $x\mapsto q(e^x)$ is increasing and convex. Let $(r_k)_{k\in N}$ be a strictly decreasing sequence in $]0, \infty[$ and put $P:=(r_k^{-1}q(|z|^{r_k}))_{k\in N}$. Then P is a weight system and we have

$$A_P^0 \simeq \begin{cases} \Lambda_1\left((\psi^*(j))_{j\in N_0}\right) & \text{if } \lim_{k\to\infty} r_k > 0, \\ \Lambda_\infty\left((\psi^*(j))_{j\in N_0}\right) & \text{if } \lim_{k\to\infty} r_k = 0. \end{cases}$$

To see this put $p_k\colon z \mapsto r_k^{-1} q(|z|^{r_k})$ for $k \in N$. Then p_k is a weight function by 1.5(2). Since $\lim_{|z|\to\infty} [p_{k+1}(z)/p_k(z)] = 0$ by hypothesis, it follows that P is a weight system. For $\varphi_k\colon x \mapsto p_k(e^x)$ we have

$$\varphi_k^*(y) = \sup_{x\geq 0}(xy - \varphi_k(x)) = \frac{1}{r_k}\sup_{x\geq 0}(r_k xy - \psi(r_k x)) = \frac{1}{r_k}\psi^*(y).$$

Hence it follows from Corollary 1.11 that $A_P^0 \simeq \lambda^1(A)$ with

$$a_{j,k} = \exp(\varphi_k^*(j)) = \exp\left(\frac{1}{r_k}\psi^*(j)\right), \quad j \in N_0,\ k \in N.$$

Obviously, this proves our claim.

(2) Let $P := (|z|^{r_k})_{k\in N}$, where $(r_k)_{k\in N}$ is a strictly decreasing sequence in $]0, \infty[$. Then $A_P^0 \simeq \Lambda_1((j)_{j\in N})$ if $\lim_{k\to\infty} r_k > 0$ and $A_P^0 \simeq \Lambda_\infty((j)_{j\in N})$ if $\lim_{k\to\infty} r_k = 0$. This is an obvious consequence of (1) and the fact that for $\psi\colon x \mapsto e^x$ we have $\psi^*\colon y \mapsto y(\log y - 1)$.

(3) Let $(r_k)_{k\in N}$ be a strictly decreasing sequence in $]1, \infty[$ and put $P := (r_k^{-1}[\log(1+|z|^2)]^{r_k})_{k\in N}$. Then P is a weight system for which we have $A_P^0 \simeq \lambda^1(A)$ with

$$a_{j,k} = \exp\left(\frac{r_k - 1}{r_k} j^{r_k/(r_k-1)}\right).$$

This follows from Corollary 1.11 and the remark on the Young conjugate in 1.13(2). Note that for $r_k = 1 + 1/k$ we have

$$a_{j,k} = \exp\left(\frac{1}{k+1} j^{k+1}\right).$$

(4) Let $(r_k)_{k\in N}$ be a strictly decreasing sequence in $]0, \infty[$. For $s \in]0, \infty[$ put

$$\tilde{P} := \left(|z|^{1/s} \exp([\log\log(|z|^{1/s} + e)]^{r_k})\right)_{k\in N}.$$

Then it follows from a remark in Meise [7], Example 2.13(5), that there is a weight system P with $A_P^0 = A_{\tilde{P}}^0$ and that $A_P^0 \simeq \lambda^1(A)$ with

$$a_{j,k} = \exp(-sj[\log\log(e+j)]^{r_k}).$$

As in Meise [7], Proposition 2.8, one proves by means of Proposition 1.4 the following:

1.15. Proposition. *Let $P = (p_k)_{k\in N}$ be a radial weight system. Then*

$$A_P^0 \simeq \lambda^2(B) = \lambda^1(B),$$

where $B = (b_{j,k})$ with $b_{j,k} = [2\pi \int_0^\infty r^{2j+1} \exp(-2p_k(r))\,dr]^{1/2}$, $j \in N_0$, $k \in N$.

1.16. Corollary. *Let p be a radial weight function with the property*

(*) *There exist $A \geq 1$ and $B \geq 0$ such that for all $z \in C$*

$$p(2z) \leq Ap(z) + B \quad \text{and} \quad 2p(z) \leq p(Az) + B.$$

Then $A_p^0 \simeq \Lambda_\infty((j)_{j\in N})$.

Proof. It is easy to check that because of (*) we have $A_p^0 = A_P^0$, where $P = (p_k)_{k\in N}$ with $p_k\colon z \mapsto p(z/k)$. Since

$$\int_0^\infty r^{2j+1} \exp(-2p(r/k))\,dr = k^{2j+2} \int_0^\infty s^{2j+1} \exp(-2p(s))\,ds$$

we conclude from Proposition 1.15 that by a diagonal transformation we have $A_P^0 \simeq \Lambda_\infty((j)_{j\in N_0}) \simeq \Lambda_\infty((j)_{j\in N})$.

Remark. It is easy to see that the analogues of 1.15 and 1.16 hold for weight functions in several variables which are coordinatewise radial.

2. A_P^0 modulo localized ideals. In this section we derive a sequence space representation for A_P^0 modulo certain localized ideals. To do this we first have to establish an appropriate semi-local to global interpolation theorem.

2.1. Proposition. *Let P be a radial weight system and put*

$$L_P^2 := \{u \in L_{\text{loc}}^2(C) \mid \int [|u(z)| \exp(-p_k(z))]^2\,dm(z) < \infty \text{ for all } k \in N\}.$$

Then, for every $v \in L_P^2$ there exists $u \in L_P^2$ with $\partial u/\partial\bar{z} = v$, where the derivative is taken in the sense of distributions. Moreover, if $v \in C^\infty(C)$ then u is in $C^\infty(C)$.

Proof. For $k \in N$ we define

$$Z_k := \{f \in L_{\text{loc}}^2(C) \mid \int_C [|f(z)| \exp(-p_k(z))]^2\,dm(z) < \infty\},$$

$$Y_k := \{f \in L_{\text{loc}}^2(C) \mid \int_C [|f(z)| \exp(-p_k(z) - \log(1+|z|^2))]^2\,dm(z) < \infty \text{ and } \partial f/\partial\bar{z} \in Z_k\},$$

$$X_k := \{f \in Y_k \mid \partial f/\partial\bar{z} = 0\} = H_{q_k}^2,$$

where $q_k\colon z \mapsto p_k(z) + \log(1+|z|^2)$. From Hörmander's L^2 estimates for the $\bar{\partial}$-operator, it then follows that the sequence

$$0 \to X_k \to Y_k \xrightarrow{\bar{\partial}} Z_k \to 0$$

is exact. See e.g. Berenstein and Taylor [1], Th. 1 or Hörmander [2], Chap. 4. Now observe that $X_{k+1} \subset X_k$, $Y_{k+1} \subset Y_k$ and $Z_{k+1} \subset Z_k$ for all $k \in N$. Since the weights q_k are radial it follows easily that the polynomials are dense in X_k (see Meise [7], 2.8, for the argument). This implies that the hypotheses of the Mittag–Leffler Lemma of Komatsu [4], Lemma 1.3, are satisfied. Hence the sequence

$$0 \to \operatorname*{proj}_{\leftarrow k} X_k \to \operatorname*{proj}_{\leftarrow k} Y_k \xrightarrow{\bar\partial} \operatorname*{proj}_{\leftarrow k} Z_k \to 0$$

is exact. From the definition of weight functions and weight systems it follows easily that $\operatorname*{proj}_{\leftarrow k} Y_k \subset \operatorname*{proj}_{\leftarrow k} Z_k = L^2_{\boldsymbol{P}}$. This proves the first assertion. The second assertion follows from regularity for the $\bar\partial$-operator (see Hörmander [2], 4.25).

2.2. PROPOSITION. *Let $\boldsymbol{P} = (p_k)_{k\in N}$ be a radial weight system, let q be an inductive weight function with*

$$\lim_{|z|\to\infty} \frac{q(z)}{p_k(z)} = 0 \quad \text{for all } k \in N$$

and let $F = (F_1, \ldots, F_N) \in (A(C))^N$ satisfy

$$\sup_{1\leqslant j\leqslant N} \sup_{z\in C} |F_j(z)| \exp(-Dq(z)) < \infty$$

for some $D > 0$. For $\varepsilon > 0$ and $C > 0$ put

$$S_q(F; \varepsilon, C) := \{z \in C \mid \Big(\sum_{j=1}^{N} |F_j(z)|^2\Big)^{1/2} < \varepsilon \exp(-Cq(z))\}.$$

If $\tilde\lambda \in A(S_q(F; \varepsilon, C))$ satisfies for all $k \in N$

$$\sup \{|\tilde\lambda(z)| \exp(-p_k(z)) \mid z \in S_q(F; \varepsilon, C)\} < \infty,$$

then there exist $\lambda \in A^0_{\boldsymbol{P}}$, ε_1, C_1 with $0 < \varepsilon_1 < \varepsilon$, $C_1 > C$ and $\alpha_j \in A(S_q(F; \varepsilon_1, C_1))$, $1 \leqslant j \leqslant N$, such that for all $z \in S_q(F; \varepsilon_1, C_1)$

$$\lambda(z) = \tilde\lambda(z) + \sum_{j=1}^{N} \alpha_j(z) F_j(z).$$

Proof. By Berenstein and Taylor [1], p. 120, there are $\varepsilon_1, C_1, A, B > 0$ and $\chi \in C^\infty(C)$ with $0 \leqslant \chi \leqslant 1$, $\operatorname{Supp}(\chi) \subset S_q(F; \varepsilon, C)$, $\chi | S_q(F; \varepsilon_1, C_1) \equiv 1$ and $\sup_{z\in C} |(\partial\chi/\partial\bar z)(z)| \exp(-Bq(z)) \leqslant A$. Then

$$v_j := -\bar F_j \Big(\sum_{j=1}^{N} |F_j|^2\Big)^{-1} \frac{\partial}{\partial \bar z}(\chi\tilde\lambda), \quad 1 \leqslant j \leqslant N,$$

is in $C^\infty(C) \cap L^2_{\boldsymbol{P}}$. Hence Proposition 2.1 can be used to complete the proof similarly to the one of the semi-local interpolation theorem of Berenstein and Taylor [1], p. 110.

2.3. LEMMA. *Let $\boldsymbol{P}$ be a radial weight system with the property that*

$$\sup_{k\in N} \sup_{|z|\geqslant R} \frac{p_k(2z)}{p_k(z)} = C < \infty$$

for a suitable $R > 0$. Then for every $f \in A^0_{\boldsymbol{P}}$ there exists a radial inductive weight function q with the following properties:

(1) $q(2z) = O(q(z))$.

(2) $\sup_{z\in C} |f(z)| \exp(-Aq(z)) < \infty$ *for some* $A > 0$.

(3) $\lim_{|z|\to\infty} \frac{q(z)}{p_k(z)} = 0$ *for all* $k \in N$.

Proof. First we remark that the hypothesis on $\boldsymbol{P}$ implies the existence of $a > 0$ with $\limsup_{x\to\infty} [p_k(x)/x^a] < \infty$ for all $k \in N$. Now let $f \in A^0_{\boldsymbol{P}}$ be given. Without loss of generality we may assume $|f(0)| > e$ and f not constant. Then we define $q_1\colon [0, \infty[\to \boldsymbol{R}$ by

$$q_1(r) := \log\big(\max_{|z|=r} |f(z)|\big).$$

Notice that $z \mapsto q_1(|z|)$ is subharmonic and continuous and that $\lim_{x\to\infty} [q_1(x)/p_k(x)] = 0$ for all $k \in N$. Next choose $b = \max(a+2, \log 4C/\log 2)$ and define

$$q_2(r) := \int_1^\infty q_1(tr)(1+t)^{-b}\, dt.$$

Then we have

$$q_2(2r) = \int_1^\infty q_1(2tr)(1+t)^{-b}\, dt = \tfrac{1}{2} \int_2^\infty q_1(tr)(1+t/2)^{-b}\, dt$$
$$\leqslant 2^{b-1} \int_1^\infty q_1(tr)(1+t)^{-b}\, dt = 2^{b-1} q_2(r),$$

since $1 + t/2 \geqslant \frac{1}{2}(1+t)$ for all $t \in [1, \infty[$.

Moreover, we have, with $L := \int_1^\infty (1+t)^{-b}\, dt$, for all $r \in [0, \infty[$

$$Lq_1(r) \leqslant \int_1^\infty q_1(tr)(1+t)^{-b}\, dt = q_2(r)$$

and hence $q_1 \leqslant (1/L) q_2$. Notice that $z \mapsto q_2(|z|)$ is continuous. Hence it is subharmonic as a supremum of subharmonic functions. To see that $\lim_{x\to\infty} [q_2(x)/p_k(x)] = 0$ for all $k \in N$, let $k \in N$ and $\varepsilon > 0$ be given. Choose $m \in N$ with $2^m \geqslant R$ such that $q_1(r) \leqslant \varepsilon 2^{-b-1} p_k(r)$ when $r \geqslant 2^m$. Then we have for all $r \geqslant 2^m$

$$q_2(r) = \int_1^{2^m} q_1(tr)(1+t)^{-b}\,dt + \sum_{n=m+1}^{\infty} \int_{2^{n-1}}^{2^n} q_1(tr)(1+t)^{-b}\,dt$$

$$\leqslant I(m) + \sum_{n=m+1}^{\infty} \frac{\varepsilon}{2^{b+1}} \int_{2^{n-1}}^{2^n} p_k(2^n r)(1+t)^{-b}\,dt$$

$$\leqslant I(m) + \sum_{n=m+1}^{\infty} \frac{\varepsilon}{2^{b+1}} C^n p_k(r) 2^{-(n-1)(b-1)}$$

$$\leqslant I(m) + \frac{\varepsilon}{4} p_k(r) \sum_{n=m+1}^{\infty} \left(\frac{2C}{2^b}\right)^n \leqslant I(m) + \frac{\varepsilon}{4} p_k(r).$$

Hence there exists r_ε such that $q_2(r) \leqslant \varepsilon p_k(r)$ for all $r \geqslant r_\varepsilon$, which proves $\lim_{x\to\infty} [q_2(x)/p_k(x)] = 0$.

Concluding, we define $q\colon z \mapsto q_2(|z|) + \log(1+|z|^2)$. Then the preceding arguments show that q is a radial inductive weight function which has properties (1)–(3).

2.4. Definition. Let $\boldsymbol{P}$ be a weight system.

(a) For an arbitrary ideal I in the algebra $A_{\boldsymbol{P}}^0$ we define

$$I_{\mathrm{loc}} := \{f \in A_{\boldsymbol{P}}^0 \mid [f]_a \in I_a \text{ for every } a \in C\},$$

where $[f]_a$ denotes the germ of f at a and I_a denotes the ideal generated by $\{[f]_a \mid f \in I\}$ in the ring $\mathcal{O}_a$ of all germs of holomorphic functions at the point a. I_{loc} is called the *local ideal generated by* I or just the *localization* of I. I is called *localized* if $I = I_{\mathrm{loc}}$.

(b) For $F = (F_1, \ldots, F_N) \in (A_{\boldsymbol{P}}^0)^N$ we denote by $I(F)$ the ideal in $A_{\boldsymbol{P}}^0$ which is algebraically generated by the functions $F_1, \ldots, F_N$. The localization of $I(F)$ is denoted by $I_{\mathrm{loc}}(F)$.

2.5. Proposition. *Let $\boldsymbol{P}$ be a radial weight system satisfying $p_k(2z) = O(p_k(z))$ for all $k \in N$. Then for every closed ideal I in $A_{\boldsymbol{P}}^0$ there exist $F_1, F_2 \in A_{\boldsymbol{P}}^0$ with $I = I_{\mathrm{loc}}(F_1, F_2)$.*

Proof. It is easy to check that $A_{\boldsymbol{P}}^0$ equals a space of type $E(K)$ in the notation of Taylor [14], § 2, where K satisfies conditions K1 to K4. Hence every closed ideal in $A_{\boldsymbol{P}}^0$ is localized by Taylor [14], Th. 7.2. Knowing this, the "jiggling of zeros" argument, indicated in Berenstein and Taylor [1], p. 120, together with Lemma 2.3 shows that $I = I_{\mathrm{loc}}(F_1, F_2)$ for every closed ideal I in $A_{\boldsymbol{P}}^0$.

Next we want to determine the locally convex structure of $A_{\boldsymbol{P}}^0/I_{\mathrm{loc}}(F_1, \ldots, F_N)$. Let $\boldsymbol{P}$ be a radial weight system and let q be an inductive weight function with $\lim_{|z|\to\infty} [q(z)/p_k(z)] = 0$ for all $k \in N$. Furthermore let $F = (F_1, \ldots, F_N) \in (A(C))^N$ be given with

$$\sup_{1 \leqslant j \leqslant N} \sup_{z \in C} |F_j(z)| \exp(-Bq(z)) < \infty$$

for some $B > 0$ and assume that

$$V(F) := \{z \in C \mid F_j(z) = 0 \text{ for } 1 \leqslant j \leqslant N\}$$

is an infinite set. Moreover, assume that F is slowly decreasing in the following sense:

(1) There exist $\varepsilon > 0$ and $C > 0$ such that for each $k \in N$ there exist $m \in N$ and $D_k \geqslant 0$ such that each component S of the set

$$S_q(F; \varepsilon, C) := \{z \in C \mid (\sum_{j=1}^{N} |F_j(z)|^2)^{1/2} < \varepsilon \exp(-Cq(z))\}$$

with $S \cap V(F) \neq \emptyset$ is bounded and such that

$$\sup_{z \in S} p_m(z) \leqslant \inf_{z \in S} p_k(z) + D_k.$$

Since $V(F)$ is infinite, it follows from (1) that there are infinitely many components S of $S_q(F; \varepsilon, C)$ with $S \cap V(F) \neq \emptyset$. We choose an enumeration $(S_j)_{j\in N}$ of these components and we choose $z_j \in S_j$ for each $j \in N$. Then we define the matrix $A = (a_{j,k})_{j,k\in N}$ by

(2) $$a_{j,k} := \exp(-p_k(z_j)).$$

Obviously (1) implies that for every $k \in N$ there exist $m \in N$ and $D_k \geqslant 0$ such that for all $j \in N$

(3) $$\sup_{z\in S_j} p_m(z) \leqslant p_k(z_j) + D_k \quad \text{and} \quad p_m(z_j) \leqslant \inf_{z \in S_j} p_k(z) + D_k.$$

Next put $I := I_{\mathrm{loc}}(F)$ and $V := V(F)$ and define for $j \in N$

(4) $$E_j := \prod_{a \in S_j \cap V} \mathcal{O}_a/I_a.$$

Then E_j is a finite-dimensional complex vector space. Let $H^\infty(S_j)$ denote the Banach space of all bounded holomorphic functions on S_j. Then it is easy to check that the map

(5) $$\varrho_j\colon H^\infty(S_j) \to E_j, \quad \varrho_j(f) = ([f]_a + I_a)_{a \in S_j \cap V}$$

is linear and surjective. Hence we get a norm on E_j by letting

(6) $$\|\ \|_j\colon E_j \to \boldsymbol{R}, \quad \|\varphi\|_j := \inf\{\|g\|_{H^\infty(S_j)} \mid g \in H^\infty(S_j),\ \varrho_j(g) = \varphi\}.$$

Now let E denote the sequence $(E_j, \|\ \|_j)_{j\in N}$ of finite-dimensional normed spaces defined by (4) and (6). We want to show that for every $f \in A_{\boldsymbol{P}}^0$ we have $(\varrho_j(f|S_j))_{j\in N} \in \lambda^\infty(A, E)$.

To see this, let $f \in A_P^0$ be given. Then (3) implies that for each $k \in N$ there exists $m \in N$ such that for all $j \in N$ we have by (3)

$$\|\varrho_j(f|S_j)\|_j \leqslant \|f|S_j\|_{H^\infty(S_j)} \leqslant \|f\|_{p_m,\infty} \exp\big(\sup_{z \in S_j} p_m(z)\big)$$

$$\leqslant \|f\|_{p_m,\infty} \exp\big(D_k + p_k(z_j)\big) = \|f\|_{p_m,\infty} \exp(D_k) \cdot (a_{j,k})^{-1}.$$

This estimate shows that the linear map

(7) $\varrho\colon A_P^0 \to \lambda^\infty(A, E)$, $\varrho(f) := \big(\varrho_j(f|S_j)\big)_{j \in N}$ is continuous.

To show that ϱ is surjective, let $\mu = (\mu_j)_{j \in N} \in \lambda^\infty(A, E)$ be given. Then we have

(8) $$\sup_{j \in N} \|\mu_j\|_j a_{j,k} =: |||\mu|||_k < \infty \quad \text{for all } k \in N.$$

Now remark that, by 1.2(2), for every $k \in N$ there exist $n \in N$ and $L > 0$ with

$$\frac{a_{j,k}}{a_{j,n}} = \exp\big(-p_k(z_j) + p_n(z_j)\big) \leqslant L\exp\big(-\tfrac{1}{2} p_k(z_j)\big).$$

Since $\lim_{j \to \infty} |z_j| = \infty$ and since p_k satisfies 1.1(2) we have

(9) For every $k \in N$ there exists $n \in N$ with $\lim_{j \to \infty} \frac{a_{j,k}}{a_{j,n}} = 0$.

Hence (8) implies

$$\lim_{j \to \infty} \|\mu_j\|_j a_{j,k} = 0 \quad \text{for all } k \in N.$$

Now choose a strictly increasing sequence $(j_k)_{k \in N}$ in N with

$$\|\mu_j\|_j \leqslant (a_{j,k})^{-1} \quad \text{for all } j \geqslant j_k$$

and choose $\lambda_j \in H^\infty(S_j)$ with $\varrho_j(\lambda_j) = \mu_j$ for all $j \in N$ and

$$\|\lambda_j\|_{H^\infty(S_j)} \leqslant 2(a_{j,k})^{-1} \quad \text{for } j_k \leqslant j < j_{k+1}.$$

Next define $\tilde{\lambda}\colon S_q(F; \varepsilon, C) \to C$ by

$$\tilde{\lambda}(z) := \begin{cases} \lambda_j(z) & \text{if } z \in S_j, \\ 0 & \text{if } z \in S_q(F; \varepsilon, C) \setminus \bigcup_{j \in N} S_j. \end{cases}$$

Then $\tilde{\lambda} \in A\big(S_q(F; \varepsilon, C)\big)$, and from (3) we deduce that for each $k \in N$ there exist $m \in N$ and $D_k > 0$ such that for all $j \geqslant j_m$ and all $z \in S_j$ we have

$$|\tilde{\lambda}(z)| = |\lambda_j(z)| \leqslant 2(a_{j,m})^{-1} = 2\exp\big(p_m(z_j)\big)$$

$$\leqslant 2e^{D_k} \exp\big(p_k(z)\big).$$

This implies that $\tilde{\lambda}$ satisfies the hypotheses of Proposition 2.2. Consequently there exists $\lambda \in A_P^0$ with $\varrho(\lambda) = \big(\varrho_j(\lambda_j)\big)_{j \in N} = \mu$. This shows that ϱ is surjective.

By this and (7), ϱ is open by the open mapping theorem. Since $\ker \varrho = I_{\text{loc}}(F)$, we have proved

(10) $$A_P^0/I_{\text{loc}}(F) \simeq \lambda^\infty(A, E).$$

Now remark that A_P^0 is nuclear by 1.4. Hence $A_P^0/I_{\text{loc}}(F)$ and consequently $\lambda^\infty(A, E)$ is nuclear. From (9) we derive that $\lambda^\infty(A, E) = \lambda^0(A, E)$, where

$$\lambda^0(A, E) := \{x \in \lambda^\infty(A, E) \mid \lim_{j \to \infty} \|x_j\|_j a_{j,k} = 0 \text{ for all } k \in N\}.$$

Hence it follows from Meise [7], Remark a) after 1.3, that we have:

(11) For every $k \in N$ there exists $l \in N$ with $\sum_{j=1}^{\infty} (\dim E_j) \frac{a_{j,k}}{a_{j,l}} < \infty$.

This implies that $\lambda^0(A, E) = \lambda^1(A, E)$. Moreover, it follows from Meise [7], 1.4, that we have

(12) $$A_P^0/I_{\text{loc}}(F) \simeq \lambda^\infty(A, E) = \lambda^1(A, E) \simeq \lambda^1(B),$$

where B is obtained from A by repeating the jth row of A $(\dim E_j)$-times.

All together we have proved:

2.6. Proposition. *Let P be a radial weight system and let q be an inductive weight function with*

$$\lim_{|z| \to \infty} \frac{q(z)}{p_k(z)} = 0 \quad \textit{for all } k \in N.$$

Furthermore let $F = (F_1, \ldots, F_N) \in (A(C))^N$ be given with

$$\sup_{1 \leqslant j \leqslant N} \sup_{z \in C} |F_j(z)| \exp\big(-Bq(z)\big) < \infty$$

for some $B > 0$ and $V(F)$ infinite.

If F is slowly decreasing in the sense of (1) above, then $A_P^0/I_{\text{loc}}(F)$ is isomorphic to the nuclear Fréchet space $\lambda^1(B)$ where

$$b_{j,k} = \exp\big(-p_k(w_j)\big), \quad j, k \in N,$$

where $(w_j)_{j \in N}$ is an appropriate sequence in C with $\lim_{j \to \infty} |w_j| = \infty$.

2.7. Theorem. *Let $P = (p_k)_{k \in N}$ be a radial weight system satisfying*

(1) *There exists $R > 0$ with* $\sup_{k \in N} \sup_{|z| \geqslant R} \frac{p_k(2z)}{p_k(z)} < \infty$.

Then for every proper closed infinite-codimensional ideal I in A_P^0 the quotient A_P^0/I is isomorphic to the nuclear Fréchet space $\lambda^1(B)$ with $b_{j,k} = \exp\big(-p_k(w_j)\big)$, $j, k \in N$, where $(w_j)_{j \in N}$ is an appropriate sequence with $\lim_{j \to \infty} |w_j| = \infty$.

Proof. Let I be an arbitrary proper closed ideal in A_P^0 which is of infinite codimension. By Proposition 2.5 we have $I = I_{loc}(F_1, F_2)$, where we may assume $F_1 \neq 0$. By Lemma 2.3 there exists a radial inductive weight function q with $q(2z) = O(q(z))$, $\lim_{|z|\to\infty} [q(z)/p_k(z)] = 0$ for all $k \in N$ such that for an appropriate number $B > 0$ we have

$$\max_{j=1,2} \sup_{z\in C} |F_j(z)| \exp(-Bq(z)) < \infty.$$

An application of the minimum modulus theorem implies (see Levin [5], p. 20, and the proof of Kelleher and Taylor [3], Prop. 5.2) that there exist $\varepsilon > 0$, $C > 0$, $n_0 \in N$ and a sequence $(r_n)_{n\in N}$ with $e^n < r_n < e^{n+1}$ for all $n \geqslant n_0$ such that for $F = (F_1, F_2)$

$$S_q(F; \varepsilon, C) \cap \Big(\bigcup_{n\geqslant n_0} \{z \in C \mid |z| = r_n\}\Big) = \emptyset.$$

This shows that, up to finitely many exceptions, for each component S of $S_q(F; \varepsilon, C)$ there exists $n \in N$ with

$$S \subset R_n := \{z \in C \mid r_n < |z| < r_{n+1}\}.$$

From (1) it follows that there exists $D \geqslant 1$ with

$$\sup_{k\in N} \sup_{|z|\geqslant R} \frac{p_k(e^2 z)}{p_k(z)} \leqslant D.$$

Now let $k \in N$ be given. By 1.2(2) there exist $m \in N$ and $D_k \geqslant 0$ with

$$Dp_m(z) \leqslant p_k(z) + D_k \quad \text{for all } z \in C.$$

Then for each component S of $S_q(F; \varepsilon, C)$ with $S \subset R_n$ for $n \geqslant n_0$ and $S \subset \{z \in C \mid |z| \geqslant R\}$ we have

$$\sup_{z\in S} p_m(z) \leqslant p_m(e^{n+2}) \leqslant Dp_m(e^n) \leqslant p_k(e^n) + D_k \leqslant \inf_{z\in S} p_k(z) + D_k.$$

This implies that $F = (F_1, F_2)$ is slowly decreasing in the sense of 2.6(1). Since I has infinite codimension, the set $V(F)$ is infinite. Hence the result follows from Proposition 2.6.

Remark. From the proof of Theorem 2.7 and Proposition 2.6 it follows that the sequence $(w_j)_{j\in N}$ in the assertion of Theorem 2.7 can be chosen in the following way:

Let n_0 and R_n be as in the proof of Theorem 2.7 and put

$$M := \{n \in N_0 \mid n \geqslant n_0 - 1 \text{ and } V(F) \cap R_n \neq \emptyset\},$$

where $R_{n_0-1} := \{z \in C \mid |z| < r_{n_0}\}$. Denote by $(n_l)_{l\in N}$ the increasing arrangement of M and denote by v_l the number of the joint zeros of F_1 and F_2 in R_{n_l} (counted with multiplicities). Then we can take as sequence $(w_j)_{j\in N}$ the sequence which is obtained from the sequence $(\exp(n_l))_{l\in N}$ by repeating $\exp(n_l)$ v_l-times.

2.8. Corollary. *Let p be a radial weight function with $p(2z) = O(p(z))$. Then for every proper closed infinite-codimensional ideal I in A_p^0, A_p^0/I is isomorphic to a nuclear power series space $\Lambda_1(\alpha)$ of finite type.*

Proof. By definition we have $A_p^0 = A_P^0$ for $P := (k^{-1}p)_{k\in N}$. From the properties of p it follows easily that P satisfies condition (1) of Theorem 2.7. Hence we have $A_p^0/I \simeq \lambda^1(B)$, where

$$b_{j,k} = \exp\Big(-\frac{1}{k} p(w_j)\Big), \quad j, k \in N.$$

By the preceding remark we may assume that $\alpha := (p(w_j))_{j\in N}$ is increasing, hence $\lambda^1(B) = \Lambda_1(\alpha)$.

3. On the complementation of closed ideals in A_P^0. Now we use the information on the structure of $A_P^0/I_{loc}(F)$ which we have obtained in the previous section to decide whether $I_{loc}(F)$ is complemented in A_P^0. This is done by means of certain linear topological invariants which have been introduced and investigated by Vogt [16], [17], [19], Vogt and Wagner [21] and Wagner [22]. We begin by recalling the definition of the invariants which we shall use later on.

3.1. Definition. Let E be a metrizable locally convex space and let $(\| \ \|_k)_{k\in N}$ be an (increasing) fundamental system of seminorms on E generating the locally convex structure of E. For $k \in N$ define $\| \ \|_k^*: E' \to [0, \infty]$ by $\|y\|_k^* = \sup\{|y(x)| \mid \|x\|_k \leqslant 1\}$. Then we say:

(a) E has *property* (DN) if there exists $m \in N$ such that for every $k \in N$ there exist $n \in N$ and $C > 0$ with

$$\| \ \|_k^2 \leqslant C \| \ \|_m \| \ \|_n.$$

(b) E has *property* ($\underline{\mathrm{DN}}$) if there exists $m \in N$ such that for every $k \in N$ there exist $n \in N$, $\varepsilon > 0$ and $C > 0$ with

$$\| \ \|_k^{1+\varepsilon} \leqslant C \| \ \|_m^{\varepsilon} \| \ \|_n.$$

(c) E has *property* (Ω) if for every $p \in N$ there exists $q \in N$ such that for every $k \in N$ there exist $d > 0$ and $C > 0$ with

$$\| \ \|_q^{*1+d} \leqslant C \| \ \|_k^* \| \ \|_p^{*d}.$$

(d) E has *property* ($\bar{\Omega}$) if there exists $d > 0$ such that for every $p \in N$ there exists $q \in N$ such that for every $k \in N$ there exists $C > 0$ with

$$\| \ \|_q^{*1+d} \leqslant C \| \ \|_k^* \| \ \|_p^{*d}.$$

(e) E has *property* $(\bar{\bar{\Omega}})$ if for every $p \in N$ there exists $q \in N$ such that for every $k \in N$ and every $d > 0$ there exists $C > 0$ with

$$\|\ \|_q^{*1+d} \leqslant C \|\ \|_k^* \|\ \|_p^{*d}.$$

3.2. Remark. (a) It is easy to check that properties (DN) and $(\underline{\mathrm{DN}})$ are linear topological invariants which are inherited by topological linear subspaces. By Vogt [16], 1.7, a nuclear metrizable locally convex space E has (DN) iff E is isomorphic to a subspace of s. By Vogt [16], 2.4, a power series space $\Lambda_R(\alpha)$ has (DN) iff $R = +\infty$. By Vogt [17], 3.3, a metrizable locally convex space E is isomorphic to a subspace of a stable nuclear $\Lambda_1(\alpha)$ iff E has $(\underline{\mathrm{DN}})$ and is $\Lambda_1(\alpha)$-nuclear.

(b) It is easy to check that properties (Ω), $(\bar{\Omega})$ and $(\bar{\bar{\Omega}})$ are linear topological invariants which are inherited by quotient spaces. By Vogt and Wagner [21], 1.8, a nuclear Fréchet space E has (Ω) iff E is a quotient space of s. By Vogt [18], 2.8 and 7.3, a strongly nuclear Fréchet space E has $(\bar{\Omega})$ iff E is a quotient of a nuclear power series space of finite type. By Vogt [19], 4.2, a Fréchet space E has $(\bar{\bar{\Omega}})$ iff every continuous linear map $T\colon E \to \Lambda_1(\alpha)$ is bounded for some (all) power series space $\Lambda_1(\alpha)$ with $\sup_{n \in N}(\alpha_{n+1}/\alpha_n) < \infty$. For other characterizations of Fréchet spaces satisfying $(\bar{\bar{\Omega}})$ see Vogt [20], Th. 4.2, and Meise and Vogt [13], Th. 3.3.

(c) From Vogt [18], 1.6, it follows that a nuclear Fréchet space which has properties $(\bar{\Omega})$ and (DN) (resp. $(\bar{\bar{\Omega}})$ and $(\underline{\mathrm{DN}})$) is finite-dimensional.

3.3 Proposition. *Let $P = (p_k)_{k \in N}$ be a radial weight system which satisfies condition* 2.7(1). *Then for every proper closed ideal I in A_P^0:*

(a) *A_P^0/I has property* $(\bar{\Omega})$.

(b) *If for every $n \in N$ there exists $m \in N$ with*

$$\lim_{|z| \to \infty} \frac{p_m(z)}{p_n(z)} = 0$$

then A_P^0/I has $(\bar{\bar{\Omega}})$.

Proof. If I is of finite codimension, then A_P^0/I has $(\bar{\bar{\Omega}})$ and hence $(\bar{\Omega})$. Hence we may assume that I is of infinite codimension. Then Theorem 2.7 implies that $A_P^0/I \simeq \lambda^1(B)$ with

$$b_{j,k} = \exp(-p_k(w_j)), \quad j, k \in N,$$

where $(w_j)_{j \in N}$ is a sequence in C with $\lim_{j \to \infty} |w_j| = \infty$.

To prove (a) let $n \in N$ be given. By 1.2(2) there exist $m \in N$ and $L \geqslant 0$ with

$$2p_m(z) \leqslant p_n(z) + L \quad \text{for all } z \in C.$$

Hence we have for each $k \in N$ and all $j \in N$

$$-p_k(w_j) - p_n(w_j) \leqslant -2p_m(w_j) + L$$

and hence

$$b_{j,k} b_{j,n} \leqslant e^L b_{j,m}^2.$$

By Wagner [22], 1.10, this implies that $\lambda^1(B)$ has $(\bar{\Omega})$.

To prove (b) let $n \in N$ be given and choose m according to the hypothesis. Then for every $d > 0$ there exists $L \geqslant 0$ with

$$p_m(z) \leqslant \frac{d}{1+d} p_n(z) + L \quad \text{for all } z \in C.$$

Hence we have for each $k \in N$ and each $j \in N$

$$-p_k(w_j) - d p_n(w_j) \leqslant -(1+d) p_m(w_j) + L(1+d)$$

and consequently

$$b_{j,k} b_{j,n}^d \leqslant C b_{j,m}^{1+d}.$$

By standard arguments this implies that $\lambda^1(B)$ has $(\bar{\bar{\Omega}})$.

3.4. Theorem. *Let $P = (p_k)_{k \in N}$ be a radial weight system which satisfies condition* 2.7(1). *Then no proper closed infinite-codimensional ideal I in A_P^0 is complemented if one of the following conditions holds:*

(a) *A_P^0 has property* (DN).

(b) *A_P^0 has* $(\underline{\mathrm{DN}})$ *and for every $n \in N$ there exists $m \in N$ with*

$$\lim_{|z| \to \infty} \frac{p_m(z)}{p_n(z)} = 0.$$

Proof. Let (a) be satisfied and assume that a proper closed ideal I is complemented in A_P^0. Then A_P^0/I has $(\bar{\Omega})$ by Proposition 3.3(a) and has (DN), since A_P^0/I is isomorphic to a topological linear subspace of A_P^0 and since (DN) is inherited by topological linear subspaces. Hence A_P^0/I is finite-dimensional by 3.2(c).

If (b) is satisfied and if the proper closed ideal I is complemented in A_P^0 then (b) and Proposition 3.3(b) imply by the same arguments as above that A_P^0/I has $(\bar{\bar{\Omega}})$ and $(\underline{\mathrm{DN}})$. Hence A_P^0/I is finite-dimensional by 3.2(c).

3.5. Corollary. *Let p be a radial weight function with $p(2z) = O(p(z))$. If A_p^0 has* (DN) *then no proper closed infinite-codimensional ideal I in A_p^0 is complemented.*

3.6. Remark. By Corollary 1.12, A_p^0 is isomorphic to a Köthe sequence space $\lambda^1(A)$. Hence it can be characterized by Vogt [16], 2.3, when A_p^0 has property (DN). This characterization in terms of the conjugate function φ^* of $\varphi\colon x \mapsto p(e^x)$ is given in [11], where also examples of algebras A_p^0 failing (DN) are given.

Here we restrict our attention to the discussion of the following examples.

3.7. EXAMPLES. Let p be any of the following weight functions. Then no proper closed infinite-codimensional ideal I in A_p^0 is complemented by Corollary 3.5.

(1) $p(z) = |z|^a$, $a > 0$. A_p^0 has (DN) by 1.13(1) or 1.16.

(2) $p(z) = (\log(1+|z|^2))^\alpha$, $\alpha > 1$. A_p^0 has (DN) by 1.13(2).

(3) Let $(M_j)_{j\in N_0}$ be a sequence of positive numbers with $M_0 = 1$ which has the following properties:

(M1) $M_j^2 \leqslant M_{j-1} M_{j+1}$ for all $j \in N$.

(M2) There exist $A, H \geqslant 1$ with $M_n \leqslant AH^n \min\limits_{0\leqslant j\leqslant n} M_j M_{n-j}$ for all $n\in N$.

(*) There exists $k\in N$ with $\liminf\limits_{j\to\infty}(M_{jk}/M_j^k)^{1/j} > 1$.

Then it has been remarked in Meise [7], 2.6(2), that the function

$$p_M\colon z\mapsto \begin{cases} \sup\limits_{j\in N_0} \log(|z|^j/M_j) & \text{for } z\neq 0,\\ 0 & \text{for } z=0\end{cases}$$

is a weight function which satisfies condition 1.16(*). Hence $A_{p_M}^0$ has (DN) by Corollary 1.16.

3.8. EXAMPLES. Let P be any of the following weight systems. Then no proper closed infinite-codimensional ideal I in A_P^0 is complemented by Proposition 3.4.

(1) $P = (r_k^{-1} q(|z|^{r_k}))_{k\in N}$ as in Example 1.14(1). A_P^0 has $\underline{\text{(DN)}}$ by 1.14(1).

(2) $P = (|z|^{r_k})_{k\in N}$, where $(r_k)_{k\in N}$ is a strictly decreasing sequence in $]0, \infty[$. A_P^0 has $\underline{\text{(DN)}}$ by 1.14(2).

(3) $P = (r_k^{-1}[\log(1+|z|^2)]^{r_k})_{k\in N}$. By the sequence space representation given in Example 1.14(3) it is easy to check that A_P^0 has property (DN). It has even the stronger property $(\overline{\text{DN}})$, introduced in Vogt [19], p. 190.

Remark. Let P (resp. $\tilde{P}$) be as in Example 1.14(4). Then it is easy to check that A_P^0 has property $(\tilde{\Omega})$. As a consequence of 3.2(c), A_P^0 does not have property $\underline{\text{(DN)}}$. Hence Theorem 3.4 cannot be used to decide whether closed ideals in A_P^0 are complemented. However, we can use other properties of the Fréchet spaces which are involved to decide this question.

3.9. LEMMA. *Let $P = (p_k)_{k\in N}$ be a radial weight system satisfying condition 2.7(1). For $k\in N$ put $\varphi_k\colon x\mapsto p_k(e^x)$ and define the matrix $B = (b_{j,k})$ by $b_{j,k} := \exp(-\varphi_k(j))$. If every continuous linear map from $\lambda^1(B)$ into A_P^0 is compact, then no proper closed infinite-codimensional ideal in A_P^0 is complemented.*

Proof. Assume that I is a proper closed infinite-codimensional ideal in A_P^0 which is complemented. Then the quotient map $\varrho\colon A_P^0 \to A_P^0/I$ has a continuous linear right inverse $R\colon A_P^0/I\to A_P^0$, which is an injective topological homomorphism and hence noncompact. By Theorem 2.7 we have $A_P^0/I \simeq \lambda^1(C)$, where $C = (c_{j,k})$ with $c_{j,k} = \exp(-p_k(w_j))$ for an appropriate sequence $(w_j)_{j\in N}$ in C. By the remark after Theorem 2.7, there exists a subsequence $(m_l)_{l\in N}$ of N such that for $D = (d_{j,k})$ defined by $d_{j,k} = \exp(-p_k(\exp(m_j)))$, $\lambda^1(D)$ is isomorphic to a complemented subspace of $\lambda^1(C)$, and of $\lambda^1(B)$. Let π denote a continuous linear projection of $\lambda^1(B)$ onto $\lambda^1(D)$. Then $(R|\lambda^1(D))\circ\pi$ is a continuous linear map from $\lambda^1(B)$ into $\lambda^1(A)$ which is not compact. Hence the assumption that I is complemented leads to a contradiction to the hypothesis.

3.10. PROPOSITION. *Let $P = (p_k)_{k\in N}$ be a radial weight system satisfying condition 2.7(1). For $k\in N$ put $\varphi_k\colon x\mapsto p_k(e^x)$. If the following holds:*

(*) *For every $(K(N))_{N\in N}\in N^N$ there exists $k\in N$ such that for each $n\in N$ there exist $M\in N$ and $C\geqslant 0$ such that for all $v, j\in N$*

$$\varphi_n^*(v) + \varphi_k(j) \leqslant \max_{1\leqslant N\leqslant M} (\varphi_N^*(v) + \varphi_{K(N)}(j)) + C,$$

then no proper closed infinite-codimensional ideal in A_P^0 is complemented.

Proof. By Corollary 1.11 and Vogt [19], Satz 1.5, condition (*) is equivalent to the assertion that every continuous linear map from $\lambda^1(B)$ into A_P^0 is compact, where B is the matrix defined in 3.9. Hence the result follows from Lemma 3.9.

Condition (*) looks rather complicated. However, it can be used to decide whether the ideals in A_P^0, P as in Example 1.14(4), are complemented.

3.11. EXAMPLE. For $r > 0$ define $P := (|z|^r \exp([\max(0, \log\log|z|^r)]^{r_k}))_{k\in N}$, where $(r_k)_{k\in N}$ is a strictly decreasing sequence in $]0, \infty[$. Then no proper closed infinite-codimensional ideal in A_P^0 is complemented.

To show this we first remark that by Meise [7], Example 2.13(5), we have $A_P^0 = A_{\tilde{P}}^0$ where $\tilde{P}$ satisfies condition 2.7(1). Hence we can apply Proposition 3.10. This is done essentially by the same arguments which have been used in Meise [7], Example 4.16.

Let $(K(N))_{N\in N}$ be given. Without restriction we can assume that $(K(N))_{N\in N}$ is strictly increasing. Choose $k = K(1)+1$ and let $n\in N$ be given. Then choose $M > n+1$ and $\xi\in[0, \infty[$ such that $\varphi_M - \varphi_n$ and $\varphi_n - \varphi_1$ are strictly increasing on $[\xi, \infty[$, where $\varphi_k\colon x\mapsto p_k(e^x)$. Next fix $s\geqslant s_0$, where s_0 is large enough and can be determined from the following considerations. Define $T(s)$ (resp. $\tau(s)$) as the solution of the following equation (T) (resp. (τ)):

$$(T)\qquad \varphi_n^*(s) - \varphi_1^*(s) = \varphi_{K(1)}(t) - \varphi_k(t),$$

$$(\tau)\qquad \varphi_M^*(s) - \varphi_n^*(s) = \varphi_k(t) - \varphi_{K(M)}(t).$$

Assume for a moment that we can show:

(1) There exists $s_0 \in [0, \infty[$ with $T(s) \leqslant \tau(s)$ for all $s \geqslant s_0$.

Then we have for all $s \geqslant s_0$

(T') $\qquad \varphi_n^*(s) - \varphi_1^*(s) \leqslant \varphi_{K(1)}(t) - \varphi_k(t)$ for all $t \geqslant T(s)$,

(τ') $\qquad \varphi_M^*(s) - \varphi_n^*(s) \geqslant \varphi_k(t) - \varphi_{K(M)}(t)$ for all $t \in [\xi_0, \tau(s)]$,

where ξ_0 is chosen appropriately. Hence

$$\varphi_n^*(s) + \varphi_k(t) \leqslant \max\big(\varphi_{K(1)}(t) + \varphi_1^*(s),\ \varphi_{K(M)}(t) + \varphi_M^*(s)\big)$$

for all $s \geqslant s_0$ and all $t \geqslant \xi_0$. This implies the existence of j_0 and v_0 such that

$$\varphi_n^*(v) + \varphi_k(j) \leqslant \max_{1 \leqslant N \leqslant M} \big(\varphi_N^*(v) + \varphi_{K(N)}(j)\big) \quad \text{for all } j \geqslant j_0,\ v \geqslant v_0.$$

Then it is easy to check that there exists $C > 0$ such that 3.10(*) holds. Hence no proper closed infinite-codimensional ideal in A_P^0 is complemented by Proposition 3.10 if we can show that (1) holds. To do this it suffices to show that for all large s we have

$$(2) \qquad \varphi_k(T(s)) - \varphi_{K(M)}(T(s)) \leqslant \varphi_M^*(s) - \varphi_n^*(s).$$

To do this, we note that by Meise [7], Example 2.13(5), we have

$$\varphi_l^*(s) = \frac{s}{r}\log s - \frac{s}{r}(\log\log s)^{r_l} \quad \text{for } s \text{ large enough}$$

and that

$$\varphi_l(t) = \exp\big(rt + (\log rt)^{r_l}\big) \quad \text{for } t \text{ large enough.}$$

To abbreviate we put $f_l\colon s \mapsto r^{-1}(\log\log s)^{r_l}$ and $g_l\colon t \mapsto (\log rt)^{r_l}$. From the definition of $T(s)$ we now get the identity

$$(3) \qquad s\big(f_1(s) - f_n(s)\big) = \exp\big(rT(s)\big)\big[\exp\big(g_{K(1)}(T(s))\big) - \exp\big(g_k(T(s))\big)\big].$$

Since $\lim_{s\to\infty} T(s) = \infty$ and since $k > K(1)$ we get

$$\begin{aligned}(4) \quad \varphi_k(T(s)) - \varphi_{K(M)}(T(s)) &\leqslant \varphi_k(T(s)) \\ &= s\big(f_1(s) - f_n(s)\big)\exp\big(g_k(T(s))\big)\big[\exp\big(g_{K(1)}(T(s))\big) - \exp\big(g_k(T(s))\big)\big]^{-1} \\ &\leqslant s f_1(s)\, 2\exp\big(g_k(T(s)) - g_{K(1)}(T(s))\big).\end{aligned}$$

From $M > n+1$ we get for s large enough

$$(5) \qquad \varphi_M^*(s) - \varphi_n^*(s) = s\big(f_n(s) - f_M(s)\big) \geqslant \frac{s}{2} f_n(s).$$

By $\lim_{s\to\infty} T(s) = \infty$ and $k > K(1)$ we get for large s

$$(6) \qquad \exp\big(\tfrac{1}{2} g_{K(1)}(T(s))\big) \leqslant \exp\big(g_{K(1)}(T(s)) - g_k(T(s))\big).$$

Now (4), (5) and (6) show that (2) is implied by the inequality

$$(7) \qquad \frac{f_1(s)}{f_n(s)} \leqslant \tfrac{1}{4}\exp\big(\tfrac{1}{2} g_{K(1)}(T(s))\big).$$

To prove that (7) holds for large s, one has to estimate $T(s)$ from below. From the definition of $T(s)$ we get

$$(8) \quad rT(s) = \log s + \log\big(f_1(s) - f_n(s)\big) - \log\big(\exp\big(g_{K(1)}(T(s))\big) - \exp\big(g_k(T(s))\big)\big).$$

This implies

$$(9) \qquad rT(s) \leqslant \log s + \log f_1(s) \leqslant 2\log s \quad \text{for large } s$$

and hence by (9) and $\lim_{s\to\infty} T(s) = \infty$

$$(10) \qquad rT(s) \geqslant \log s - g_{K(1)}(T(s)) \geqslant \log s - g_{K(1)}\Big(\frac{2}{r}\log s\Big)$$

for large s. Then we get

$$\begin{aligned}\tfrac{1}{4}\exp\big(\tfrac{1}{2} g_{K(1)}(T(s))\big) &\geqslant \tfrac{1}{4}\exp\big(\tfrac{1}{2}\big[\log\big(\log s - [\log(2\log s)]^{r_{K(1)}}\big)\big]^{r_{K(1)}}\big) \\ &\geqslant \tfrac{1}{4}\exp\big(\tfrac{1}{2}\big[\log\big(\tfrac{1}{2}\log s\big)\big]^{r_{K(1)}}\big) \geqslant \exp\big((r_1 - r_n)\log\log\log s\big) \\ &= \frac{f_1(s)}{f_n(s)} \quad \text{for all } s \geqslant s_0.\end{aligned}$$

This shows that (7) holds for all s sufficiently large and hence completes the proof.

4. Translation invariant subspaces for some weighted (DF)-spaces of entire functions. It was Martineau [6] who extended the classical work on convolution operators on Fréchet spaces of entire functions to convolution operators on (DF)-spaces of entire functions. In this section we show that the results of Sections 2 and 3 can be used to determine the locally convex structure of the closed translation invariant subspaces of various (DF)-spaces of entire functions, including those which were considered by Martineau [6].

We begin by introducing the (DF)-spaces which we will work with.

4.1. Definition. Let $Q = (q_k)_{k\in \boldsymbol{N}}$ be a sequence of radial weight functions with the following properties:

(1) For every $k \in \boldsymbol{N}$ there exists $K \geqslant 0$ with

$$q_k(z) \leqslant q_{k+1}(z) + K \quad \text{for all } z \in \boldsymbol{C}.$$

(2) For every $k \in N$ there exist $l \in N$ and $L \geq 0$ with

$$2q_k(z) \leq q_l(z) + L \quad \text{for all } z \in C.$$

(3) For every $k \in N$ there exist $m \in N$ and $M \geq 0$ with

$$q_k(2z) \leq q_m(z) + M \quad \text{for all } z \in C.$$

(4) $q_k|[0, \infty[$ is convex and satisfies

$$\lim_{x \to \infty} \frac{q_k(x)}{x} = \infty \quad \text{for all } k \in N.$$

Then we define

$$A_Q(C) := \{f \in A(C) | \text{ there exists } k \in N \text{ with } \sup_{z \in C} |f(z)| \exp(-q_k(z)) < \infty\}$$

and endow $A_Q(C)$ with its natural inductive limit topology. Because of (4) we can define $p_k: z \mapsto (q_k|[0, \infty[)^*(|z|)$. We assume that $P_Q := (p_k)_{k \in N}$ is a weight system which satisfies condition (1) of Theorem 2.7.

The following proposition is contained in Taylor [14], Th. 5.2. For the convenience of the reader we give the proof here too.

4.2. Proposition. *Let $Q = (q_k)_{k \in N}$ and $P_Q = (p_k)_{k \in N}$ be as in 4.1. Then the Fourier–Borel transform*

$$\mathscr{F}: A_Q(C)'_b \to A^0_{P_Q}, \quad \mathscr{F}(T): \zeta \mapsto \langle T_z, \exp(z\zeta) \rangle,$$

is a linear topological isomorphism.

Proof. From the conditions on Q and Proposition 1.10 it follows that $f: z \mapsto \sum_{j=0}^{\infty} a_j z^j$ is in $A_Q(C)$ iff there exist $k \in N$ and $C > 0$ with

(1) $$|a_j| \leq C \exp(-(q_k \circ \exp)^*(j)) \quad \text{for all } j \in N_0.$$

Since $A_Q(C)$ is a (DFN)-space (see Meise [7], 2.4), this implies that a linear map $T: A_Q(C) \to C$ is continuous iff

(2) For every $k \in N$ there exists C_k such that for all $j \in N_0$

$$|T(z^j)| \leq C_k \exp((q_k \circ \exp)^*(j)).$$

By Taylor [14], Lemma 5.3, and Stirling's formula this implies that for every $T \in A_Q(C)'$ and every $k \in N$ there exists C'_k such that for all $j \in N$

(3) $$|T(z^j/j!)| \leq C'_k \exp((q_k \circ \exp)^*(j) - j \log j + j) \leq C'_k \exp(-(q_k^* \circ \exp)^*(j)).$$

By Proposition 1.10 this shows that

$$\mathscr{F}(T): \zeta \mapsto \sum_{j=0}^{\infty} \langle T_z, (z\zeta)^j/j! \rangle$$

is in $A^0_{P_Q}$. Hence $\mathscr{F}$ maps $A_Q(C)'$ into $A^0_{P_Q}$ and is continuous because of (3). To show that $\mathscr{F}$ is surjective, let $g: \zeta \mapsto \sum_{j=0}^{\infty} b_j \zeta^j$ be given. Then

$$\tilde{g}: \zeta \mapsto (\zeta^2 g(\zeta))'' = \sum_{j=0}^{\infty} (j+1)(j+2) b_j \zeta^j$$

is in $A^0_{P_Q}$. This implies by Proposition 1.10, Taylor [14], Lemma 5.3, and Stirling's formula that for each $k \in N$ there exists $C_k > 0$ such that for all $j \in N$

(4) $$j!\,|b_j| \leq C_k \exp(-(q_k^* \circ \exp)^*(j) + j \log j - j) j^{-3/2} \leq C_k \exp((q_k \circ \exp)^*(j)) j^{-3/2}.$$

By (1) this implies that $T: f \mapsto \sum_{j=0}^{\infty} b_j f^{(j)}(0)$ is in $A_Q(C)'_b$ and satisfies $\mathscr{F}(T) = g$. Hence $\mathscr{F}$ is a linear bijection. Since (4) implies the continuity of $\mathscr{F}^{-1}$, the proof is complete.

4.3. Definition. A linear subspace W of $A_Q(C)$ is called *translation invariant* if for every $f \in W$ and every $a \in C$ the function $z \mapsto f(z+a)$ belongs to W.

4.4. Proposition. *Let Q and P_Q be as in 4.1. Then a closed linear subspace W of $A_Q(C)$ is translation invariant if and only if $\mathscr{F}(W^\perp)$ is an ideal in $A^0_{P_Q}$.*

This can be proved in the same way as Meise [7], Proposition 5.5.

4.5. Theorem. *Let Q and P_Q be as in 4.1. Then every closed linear translation invariant subspace W of $A_Q(C)$ has a Schauder basis.*

Proof. By Proposition 4.2 and classical duality theory we have

$$W = W^{\perp\perp} \simeq (A^0_{P_Q}/\mathscr{F}(W^\perp))'_b.$$

If W is finite-dimensional or equal to $A_Q(C)$ then the result holds trivially. Hence we may assume that $\mathscr{F}(W^\perp)$ is a proper infinite-codimensional ideal in $A^0_{P_Q}$ because of Proposition 4.4. Consequently, Theorem 2.7 implies that $W \simeq \lambda(B)'_b$.

4.6. Corollary. *Let q be a convex radial weight function which satisfies condition 1.16(*) and $\lim_{x \to \infty} [q(x)/x] = \infty$. Assume that $p: z \mapsto (q|[0, \infty[)^*(|z|)$ satisfies condition 1.16(*) and put $Q := (kp)_{k \in N}$. Then every proper closed linear infinite-dimensional translation invariant subspace W of $A_Q(C)$ is isomorphic to the strong dual of a nuclear power series space of finite type, and no such subspace is complemented in $A_Q(C)$.*

Proof. Since q satisfies condition $(*)$ of 1.16, we have $A_Q(C) = A_{\tilde{Q}}(C)$, where $\tilde{Q} := (q(kz))_{k\in N}$. Hence we have $P_{\tilde{Q}} = (p(z/k))_{k\in N}$. Since p satisfies condition $(*)$ of 1.16, we have $A^0_{P_Q} = A^0_{P_{\tilde{Q}}} = A^0_p$. By the proof of Theorem 4.5 we have

$$W \simeq (A^0_p/\mathcal{F}(W^\perp))'_b.$$

Hence the result follows from Corollary 2.8 and Theorem 3.4 in connection with Corollary 1.16.

4.7. EXAMPLES. (1) For $s > 1$ put

$$E^s := \{f \in A(C) \mid \text{there exists } k \in N \text{ with } \sup_{z\in C}|f(z)|\exp(-k|z|^s) < \infty\}.$$

Then it follows easily from Corollary 4.6 that every proper closed infinite-dimensional translation invariant linear subspace W of E^s is isomorphic to a power series space of finite type, and no such subspace is complemented.

(2) Let $\sigma = (s_k)_{k\in N}$ be a strictly increasing sequence in $]1, \infty[$ and put

$$E(\sigma) := \{f \in A(C) \mid \text{there exists } k \in N \text{ with } \sup_{z\in C}|f(z)|\exp(-|z|^{s_k}) < \infty\}.$$

Then every proper closed infinite-dimensional translation invariant linear subspace W of $E(\sigma)$ has a Schauder basis, and no such subspace is complemented. Moreover, W'_b has property $(\bar{\Omega})$.

It is easy to check that $E(\sigma) = A_{Q(\sigma)}(C)$, where $Q(\sigma) = (s_k^{-1}|z|^{s_k})_{k\in N}$. Since $Q(\sigma)$ and $P(\sigma) = (r_k^{-1}|z|^{r_k})_{k\in N}$ with $r_k = s_k/(s_k - 1)$ satisfy the condition of 4.1, Theorem 4.5 implies that W has a Schauder basis. By Proposition 3.3, $W'_b \simeq A^0_{P_{Q(\sigma)}}/\mathcal{F}(W^\perp)$ has $(\bar{\Omega})$, while Example 3.8(1) shows that $\mathcal{F}(W^\perp)$ and hence W is not complemented.

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MATHEMATISCHES INSTITUT DER UNIVERSITÄT DÜSSELDORF
Universitätsstr. 1, D-4000 Düsseldorf 1, Federal Republic of Germany
and
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN
Ann Arbor, Michigan 48109, U.S.A.

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