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Extension of characters in commutative Banach algebras

by

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Abstract. We characterize those continuous homomorphisms $f\colon A \to B$ whose transpose maps are surjective. This provides a necessary and sufficient condition for all characters of a closed subalgebra A of a commutative Banach algebra B to have an extension to B.

Let $f\colon A \to B$ be a homomorphism of commutative complex unital Banach algebras. In this paper we study the image of the transpose of f, f^*, which assigns to every character h of B the character $h \circ f$ of A (by a *character* of a Banach algebra A we mean a nonzero multiplicative linear functional of A; we denote by $M(A)$ the space of all characters of A with the weak* topology; see [1], [6], [7]). More precisely, we find necessary and sufficient conditions for $h \in M(A)$ to belong to $f^*(M(B))$. In particular, we show that f^* is surjective if and only if the spectral equality

$$\sigma_A(a) = \sigma_B(f(a)) \tag{1}$$

holds for every $a \in A^n$. This fact seems to be relevant when A is a closed subalgebra of B, for it gives conditions for every character of A to have an extension to B. By applying the corona theorem of Carleson [2] we get a continuous surjection of the Čech compactification $\beta\Delta$ of the open unit disk Δ onto the character space of H^∞, the algebra of all bounded holomorphic functions on Δ.

Let A be a complex unital algebra. For every $n \geq 1$, let $U_n(A) = \{a = (a_1, \ldots, a_n) \in A^n$: there exists $b \in A^n$ such that $\sum b_i a_i = 1\}$. The elements of $U_n(A)$ are called (*left*) *unimodular*. They are characterized by the condition $0 = (0, \ldots, 0) \in C^n \setminus \sigma(a)$, where $\sigma(a)$ is the (left) *joint spectrum* of a:

$$\sigma_A(a) = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in C^n\colon \sum_{i=1}^{n} A(a_i - \lambda_i) \text{ is a proper left ideal of } A\}.$$

It is well known that $\sigma(a)$ is a, possibly empty, compact subset of C^n [5] when A is a Banach algebra. Observe that every (not necessarily continuous) homomorphism $f\colon A \to B$ of Banach algebras induces a mapping from $U_n(A)$ into $U_n(B)$; we denote it again by f.

PROPOSITION 1. *Let $f\colon A\to B$ be a (not necessarily continuous) homomorphism of unital Banach algebras. Then the following conditions are equivalent*:

(i) $\sigma_A(a)=\sigma_B(f(a))$ *for every* $a\in A^n$;

(ii) $0\notin\sigma_B(f(a))\Rightarrow 0\notin\sigma_A(a)$ *for every* $a\in A^n$;

(iii) $f^{-1}(U_n(B))=U_n(A)$.

Proof. Observe, first, that the inclusions $U_n(A)\subset f^{-1}(U_n(B))$ and $\sigma_B(f(a))\subset\sigma_A(a)$ hold in general.

(i) $\Rightarrow$ (ii). Obvious.

(ii) $\Rightarrow$ (iii). If $a\in A^n$ and $f(a)\in U_n(B)$, then $0\notin\sigma_B(f(a))$ so, by hypothesis, $0\notin\sigma_A(a)$, which means that $a\in U_n(A)$.

(iii) $\Rightarrow$ (i). If $\lambda\in C^n\setminus\sigma_B(f(a))$, then $f(a-\lambda)=f(a)-\lambda\in U_n(B)$; by (iii), $a-\lambda\in U_n(A)$ and then $\lambda\in C^n\setminus\sigma_A(a)$. ∎

From now on, by a *Banach algebra* we mean a commutative complex Banach algebra with unit. If A is a Banach algebra then an element a of A^n is unimodular if and only if for every character $h\in M(A)$ there exists a coordinate a_i of a such that $h(a_i)\neq 0$. In other terms, define for an element $c\in A$, $Z_c=\{h\in M(A)\colon h(c)=0\}$; then $a\in U_n(A)\Leftrightarrow\bigcap_{i=1}^{n}Z_{a_i}=\emptyset$. In fact, $\bigcap_{i=1}^{n}Z_{a_i}=\emptyset$ means that the Gelfand transforms $\hat a_1,\dots,\hat a_n$ have no common zero in $M(A)$ and, then, $(\hat a_1,\dots,\hat a_n)\in U_n(C(M(A)))$, where $C(M(A))$ is the algebra of all continuous complex maps on $M(A)$. By Proposition 1, $a\in U_n(A)$ because $\sigma_A(a)=\sigma_{C(M(A))}(\hat a)$. Observe that Z_c is the hull of the ideal generated by c [7, 11.3].

PROPOSITION 2. *Let $f\colon A\to B$ be a continuous homomorphism of Banach algebras. For a character $h\in M(A)$ the following conditions are equivalent*:

(i) $h\in M(A)\setminus f^*(M(B))$;

(ii) *there exist* $n\in N$ *and* $a\in A^n$ *such that* $f(a)\in U_n(B)$ *and* $h\in\bigcap_{i=1}^{n}Z_{a_i}$.

Proof. (i) $\Rightarrow$ (ii). If $h\in M(A)\setminus f^*(M(B))$, for every $\psi\in M(B)$ there exists $a_\psi\in A$ such that $h(a_\psi)=0$ and $\psi(f(a_\psi))\neq 0$. Then there is an open neighborhood U_ψ of ψ in $M(B)$ such that $\varphi(f(a_\psi))\neq 0$ for every $\varphi\in U_\psi$. Now, the compact space $M(B)$ is covered by its open subsets U_ψ $(\psi\in M(B))$ so there exist $a_1,\dots,a_n\in A$ such that $M(B)=\bigcup_{i=1}^{n}U_{\psi_i}$, $h(a_1)=h(a_2)=\dots=h(a_n)=0$ and $\varphi(f(a_i))\neq 0$ for every $\varphi\in U_{\psi_i}$. This shows that $f(a)=(f(a_1),\dots,f(a_n))\in U_n(B)$ and $h\in\bigcap_{i=1}^{n}Z_{a_i}$, as claimed.

(ii) $\Rightarrow$ (i). If $h\in f^*(M(B))$ and $h(a_i)=0$ for $i=1,2,\dots,n$, then there exists $\psi\in M(B)$ such that $h=\psi\circ f$ and $\psi(f(a_i))=0$ for $i=1,\dots,n$; thus $f(a)\notin U_n(B)$, by the remarks preceding Proposition 2. ∎

THEOREM 3. *Under the same hypothesis, the following conditions are equivalent*:

(i) $f^*(M(B))=M(A)$;

(ii) $f^{-1}(U_n(B))=U_n(A)$ *for every* $n\geqslant 1$;

(iii) $\sigma_A(a)=\sigma_B(f(a))$ *for every* $n\geqslant 1$ *and* $a\in A^n$.

Proof. By Proposition 1 it suffices to prove that (i) $\Leftrightarrow$ (ii). Suppose that (i) holds and take $a\in A^n$ such that $f(a)\in U_n(B)$; given $h\in M(A)$ we find $\psi\in M(B)$ such that $h=\psi\circ f$; then there is an $i\in\{1,2,\dots,n\}$ with $h(a_i)=\psi(f(a_i))\neq 0$ and, h being arbitrary, we get $a\in U_n(A)$, as desired. Conversely, suppose that there exists $h\in M(A)\setminus f^*(M(B))$. Then we can find $a\in f^{-1}(U_n(B))$ such that $h\in\bigcap_{i=1}^{n}Z_{a_i}$; but this implies that $a\notin U_n(A)$, because $a\in U_n(A)\Leftrightarrow\bigcap_{i=1}^{n}Z_{a_i}=\emptyset$. ∎

COROLLARY 1. *Let $f\colon A\to B$ be an injective continuous homomorphism of Banach algebras. Then $f^*(M(B))=M(A)$ if and only if $f(U_n(A))=U_n(B)\cap f(A^n)$, for every $n\geqslant 1$.*

COROLLARY 2. *If A is a closed subalgebra of a Banach algebra B, then every character of A extends to a character of B if and only if $U_n(A)=U_n(B)\cap A^n$ for every $n\geqslant 1$.*

COROLLARY 3. *Let H^∞ be the algebra of all bounded holomorphic functions on the open unit disk Δ and let $\beta\Delta$ be the Čech compactification of Δ. Then the transpose of the inclusion of H^∞ into the algebra $BC(\Delta)$ of all bounded continuous functions on Δ gives a surjective continuous map $\beta\Delta\to M(H^\infty)$.*

Proof. In fact, the deep result known as the *corona theorem* [2] states that if $f_1,\dots,f_n\in H^\infty$ are such that $|f_1|+\dots+|f_n|\geqslant r$ in Δ for some $r>0$, then there exist $g_1,\dots,g_n\in H^\infty$ such that $f_1g_1+\dots+f_ng_n=1$. With our notation, Carleson proved that for every $n\geqslant 1$, $U_n(BC(\Delta))\cap(H^\infty)^n\subset U_n(H^\infty)$ (the other inclusion is trivial). Thus the statement follows by the equivalence of (i) and (ii) of Theorem 3. ∎

Recall [7, 15.8] that the *cortex* of a Banach algebra A is the set $\operatorname{cor}(A)$ of all $h\in M(A)$ which admit an extension to characters on any superalgebra of A: $\operatorname{cor}(A)=\bigcap_B i^*(M(B))$, where B runs through all (commutative) superalgebras of A.

COROLLARY 4. *Let $h\in M(A)$. Then $h\in\operatorname{cor}(A)$ if and only if for every superalgebra B of A and $a\in U_n(B)\cap A^n$ we have $h(a)\neq 0\in C^n$.*

Recall that for every Banach algebra A there exists a unique minimal closed subset of $M(A)$, called the (*Shilov*) *boundary* of A and denoted by $\Gamma(A)$, such that for every $a\in A$

$$\sup_{h\in M(A)}|h(a)|=\sup_{h\in\Gamma(A)}|h(a)|$$

[7, 15.2]. One of the relevant properties of $\Gamma(A)$ is that each $h \in \Gamma(A)$ admits an extension to a character of any superalgebra B of A; in other terms, $\Gamma(A)$ is contained in the cortex of A. In particular, we have the following

COROLLARY 5. *Let $h \in \Gamma(A)$ and let B be a superalgebra of A. Then for every $a \in A^n$ such that $a \in U_n(B)$ we have $h(a) \neq 0 \in C^n$.*

Remarks. Condition (ii) of Theorem 3 is related to the notion of full subalgebra (in French "sous-algèbre pleine"): A subalgebra A of B is *full* if every element of A which is invertible in B must be invertible in A. Naimark [6] then says that $A \subset B$ is a *Wiener pair*. Bourbaki [1, Ch. I, § 3, Proposition 7] proves that if every character of a closed subalgebra A of B can be extended to a character of B then A must be full, but the converse does not hold in general [1, Ch. I, § 3, Ex. 14]. Precisely, the interest of our Corollary 3 is that it provides a suitable converse.

It should be noted that we have made no use of the topology of $U_n(A)$. However, the connectedness properties of $U_n(A)$ are strongly related to an algebraic invariant of A called the *Bass stable rank* (see [3] and [4] for a precise formulation of this remark).

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