isometric. Hence, by Theorem 1, there exists a homeomorphism $k$ of $\Omega_1$ onto $\Omega_2$.

Next, for Borel sets $B \subseteq \Omega_2$, we define $\lambda(B) = \mu_1[k^{-1}(B)]$. If then $A$ is a Borel subset of $\Omega_1$ we have $\mu_1(A) = \lambda(k(A)) = \int_{k(A)} d\lambda$ so that the map

$$\sum_{j=1}^{n} e_j \chi_{A_j} \to \sum_{j=1}^{n} e_j \chi_{k(A_j)}$$

carries the dense subspace of simple functions in $L^1(\Omega_1, \Sigma_1, \mu_1, E)$ isometrically onto the corresponding subspace of $L^1(\Omega_2, \Sigma_2, \lambda, E)$ and can thus be extended to an isometry of $L^1(\Omega_1, \Sigma_1, \mu_1, E)$ onto $L^1(\Omega_2, \Sigma_2, \lambda, E)$. Then multiplication by the scalar function $d\lambda/d\mu_2$ carries this latter space isometrically onto $L^1(\Omega_2, \Sigma_2, \mu_2, E)$. Hence $L^1(\mu_1, E) \cong L^1(\mu_2, E)$ and consequently $L^\infty(\mu_1, E^*) \cong L^\infty(\mu_2, E^*)$.

If we assume that $L^1(\mu_1, E)$ and $L^1(\mu_2, E)$ are nearly isometric, then their duals $L^\infty(\mu_1, E^*)$ and $L^\infty(\mu_2, E^*)$ are nearly isometric and the proof follows as above.

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# On symmetric bases in nonseparable Banach spaces

by

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**Abstract.** It is shown that if $E$ and $F$ are nonseparable Banach spaces with symmetric bases and each of these spaces is isomorphic to a subspace of the other space, then the bases are equivalent (and hence the two spaces are isomorphic). In particular, in a nonseparable Banach space with a symmetric basis, any two such bases are equivalent.

The purpose of this paper is to prove the following

THEOREM. *Let $E$ and $F$ be nonseparable Banach spaces with symmetric bases $(u_i)_{i\in I}$ and $(v_j)_{j\in J}$, respectively. If $E \hookrightarrow F$ and $F \hookrightarrow E$ (isomorphic embeddings), then $E$ and $F$ are isomorphic: $E \approx F$. In fact, in this case the bases $(u_i)$ and $(v_j)$ are equivalent, i.e., there exists an isomorphism $T$ from $E$ onto $F$ such that $T(\{u_i: i\in I\}) = \{v_j: j\in J\}$.*

(Thus, for some bijection $\tau: I \to J$, $Tu_i = v_{\tau(i)}$ for all $i \in I$, and every such bijection determines the corresponding isomorphism.)

COROLLARY. *If a nonseparable Banach space $E$ has a symmetric basis, then any two symmetric bases of $E$ are equivalent.*

These results show that there is a sharp distinction between the nonseparable and separable Banach spaces with symmetric bases. Nothing of the above type is valid in the separable case (see [1] and [2]) if we insist on having conclusions that the bases are equivalent. Whether or not the theorem is true in this case if the assertion were merely $E \approx F$, seems to be unknown.

We start with some explanations and a general construction.

A family $(x_\alpha)_{\alpha\in A}$ of elements in a Banach space $X$ is called a *symmetric basis* of $X$ ([4]) if

(a) it is an *unconditional basis* of $X$ ([3]), i.e., for every $x \in X$ there is a unique family of scalars $(t_\alpha)_{\alpha\in A}$ such that $x = \sum_{\alpha\in A} t_\alpha x_\alpha$ (unconditional convergence or summability), and

(b) whenever a series $\sum_{\alpha\in A} t_\alpha x_\alpha$ converges (unconditionally), then so does the series $\sum_{\alpha\in A} t_{\varphi(\alpha)} x_\alpha$; for every bijection $\varphi: A \to A$.

In this definition, condition (b) may be replaced by either of the following two:

(b′) for every sequence $(\alpha_n)$ in $A$, $(x_{\alpha_n})$ is a symmetric basic sequence (in the usual sense [1]);

(b″) for any two sequences $(\alpha_n)$ and $(\alpha'_n)$ in $A$, the basic sequences $(x_{\alpha_n})$ and $(x_{\alpha'_n})$ are equivalent ([1]).

Note. Here, and in what follows, when we speak of a sequence in an index set, we always mean an infinite sequence with pairwise distinct terms.

For example, the family $(e_\alpha)_{\alpha\in A}$ of the unit vectors is a symmetric basis in each of the familiar spaces $c_0(A)$ and $l_p(A)$, $1 \leqslant p < \infty$, and—more generally—in the Orlicz spaces $l_M(A)$ defined by an Orlicz function $M$ satisfying the $\Delta_2$ condition at 0. Each of these spaces (with the indicated basis) is determined by its separable model: $c_0$, $l_p$ or $l_M$ (with its natural unit vector basis $(e_n)$), respectively.

We make this last statement precise by giving a sketch of a general construction:

Let $W$ be a separable Banach space with a symmetric basis $(w_n)$ and a symmetric norm, i.e., for every permutation $\pi$ of $N$,

$$\Big\|\sum_{n=1}^{\infty} t_{\pi(n)} w_n\Big\| = \Big\|\sum_{n=1}^{\infty} t_n w_n\Big\|, \qquad \forall x = \sum_{n=1}^{\infty} t_n w_n \in W.$$

Consider the corresponding sequence space $W(N)$ consisting of all scalar sequences $(t_n)$ for which the series $\sum t_n w_n$ converges, equipped with the norm $\|(t_n)\| = \|\sum t_n w_n\|$. Then, if $A$ is an arbitrary infinite set (possibly uncountable), denote by $W(A)$ the set of all scalar functions $x$ on $A$ such that $|\mathrm{supp}\, x| \leqslant \aleph_0$ and whenever $(\alpha_n)$ is a sequence in $A$ with $\mathrm{supp}\, x \subset \{\alpha_n \colon n\in N\}$, then $(x(\alpha_n))\in W(N)$. Then $W(A)$ is a Banach space under the norm defined by $\|x\| = \|(x(\alpha_n))\|$, where $\mathrm{supp}\, x \subset \{\alpha_n \colon n\in N\}$, and $(e_\alpha)_{\alpha\in A}$ is a symmetric basis of this space. Clearly, for every sequence $(\alpha_n)$ in $A$, the basic sequence $(e_{\alpha_n})$ is isometrically equivalent to the basis $(w_n)$ of the original space $W$.

Conversely, let $X$ be a Banach space with an (infinite) symmetric basis $(x_\alpha)_{\alpha\in A}$. Fix any sequence $(\alpha_n)$ in $A$ and denote $w_n = x_{\alpha_n}$ $(n\in N)$. Finally, equip the subspace $W = \overline{\mathrm{lin}}\{w_n \colon n\in N\}$ with an equivalent symmetric norm. Then $X \approx W(A)$; in fact, the basis $(x_\alpha)$ of $X$ is equivalent to the basis $(e_\alpha)$ of $W(A)$.

Remark. We may say that the Banach space $W(A)$ or, more precisely, the pair $(W(A), (e_\alpha)_{\alpha\in A})$ (or any of its "isomorphs") constructed above, is of symmetric type $(W, (w_n))$. It should be strongly emphasized that $W(A)$ depends not only on the space $W$ but also on the particular choice of a symmetric basis of $W$: From the theorem it follows that nonequivalent symmetric bases in $W$ (with $W$ suitably renormed) yield nonisomorphic spaces $W(A)$ if $A$ is uncountable.



We shall say that a symmetric basis $(x_\alpha)_{\alpha\in A}$ of a Banach space $X$ is an *$l_1$-basis*, or that it is of *$l_1$-type*, if the basis $(x_\alpha)_{\alpha\in A}$ is equivalent to the unit vector basis $(e_\alpha)_{\alpha\in A}$ of $l_1(A)$. This happens precisely when for some (every) sequence $(\alpha_n)$ in $A$, the basic sequence $(x_{\alpha_n})$ is equivalent to the unit vector basis $(e_n)$ of $l_1$.

We now start collecting ingredients needed in the proof of the theorem. The first of these is a result due to Troyanski [4].

LEMMA 1. *Let $X$ be a (nonseparable) Banach space with a symmetric basis $(x_\alpha)_{\alpha\in A}$. If $X$ contains an isomorphic copy of $l_1(\Delta)$ for some uncountable set $\Delta$, then $(x_\alpha)$ is an $l_1$-basis.*

*In particular, symmetric bases of $l_1(A)$ are unique up to equivalence.*

The next result (which is surely well known) is a simple combinatorial fact.

LEMMA 2. *Let $(S_\alpha)_{\alpha\in A}$ be an uncountable family of (at most) countable subsets of a set $S$ such that*

$$|\{\alpha\in A \colon s\in S_\alpha\}| \leqslant \aleph_0, \qquad \forall s\in S.$$

*Then there exists a subset $B$ of $A$ with $|B| = |A|$ such that*

$$(*) \qquad S_\beta \cap S_{\beta'} = \emptyset \quad \text{for all distinct } \beta, \beta'\in B.$$

Proof. Let $B$ be a maximal subset of $A$ satisfying $(*)$ (of course, the Kuratowski–Zorn Lemma is used here), and suppose that $|B| < |A|$. Then for $S' = \bigcup_{\beta\in B} S_\beta$ we have $|S'| \leqslant |B|\cdot\aleph_0 < |A|$. Hence, if

$$B' = \{\alpha\in A \colon S'\cap S_\alpha \neq \emptyset\} = \bigcup_{s\in S'} \{\alpha\in A \colon s\in S_\alpha\},$$

then $|B'| \leqslant |S'|\cdot\aleph_0 < |A|$. In particular, $A\setminus B' \neq \emptyset$, and if we take any $\alpha\in A\setminus B'$, then $B$ is properly contained in $B\cup\{\alpha\}$, and the latter set satisfies $(*)$, contradicting the maximality of $B$. ∎

We shall say that a family $(z_\gamma)_{\gamma\in\Gamma}$ in a Banach space $Z$ is *totally non-$l_1$* if for every sequence $(\gamma_n)$ in $\Gamma$ there exists a scalar sequence $(t_n)$ such that the series $\sum t_n z_{\gamma_n}$ converges unconditionally while $\sum |t_n| = \infty$. Of course, for a family that is a symmetric basis, "to be totally non-$l_1$" means the same as "not to be an $l_1$-basis".

Let $E$ and $F$ be Banach spaces with symmetric bases, as in the theorem. Also, let $(v_j^*)_{j\in J} \subset F^*$ be the dual family, biorthogonal to $(v_j)_{j\in J}$. For $y\in F$, the *support* of $y$ is defined as $s(y) = \{j\in J \colon v_j^*(y) \neq 0\}$; clearly, $|s(y)| \leqslant \aleph_0$.

LEMMA 3. *Let $(y_\alpha)_{\alpha\in A}$ be an uncountable totally non-$l_1$ family in $F$. Then there exists a $B\subset A$ with $|B| = |A|$ such that*

$$s(y_\beta) \cap s(y_{\beta'}) = \emptyset \quad \text{for all distinct } \beta, \beta'\in B.$$

Proof. This will follow from Lemma 2 if we check that the (at most) countable sets $S_\alpha = s(y_\alpha) \subset J = S$ satisfy the condition: $|\{\alpha \in A\colon j \in s(y_\alpha)\}| \leqslant \aleph_0$ for each $j \in J$. (Note that $j \in s(y_\alpha)$ iff $v_j^*(y_\alpha) \neq 0$.) Suppose this is not so for some $j$, i.e., $v_j^*(y_\alpha) \neq 0$ for uncountably many $\alpha \in A$. Then we can find an $r > 0$ and a sequence $(\alpha_n)$ in $A$ such that $|v_j^*(y_{\alpha_n})| \geqslant r$ for all $n \in N$. Now, as $(y_\alpha)$ is totally non-$l_1$, there is a scalar sequence $(t_n)$ for which $\sum |t_n| = \infty$ and $\sum t_n y_{\alpha_n}$ converges unconditionally. But then also the series $\sum t_n v_j^*(y_{\alpha_n})$ converges unconditionally so that $\sum |t_n|\,|v_j^*(y_{\alpha_n})| < \infty$. However, $|t_n|\,|v_j^*(y_{\alpha_n})| \geqslant r|t_n|$ and $\sum |t_n| = \infty$; a contradiction. ■

Proof of the Theorem. Since $E \hookrightarrow F$, the density characters of these spaces satisfy $|I| = \operatorname{dens} E \leqslant \operatorname{dens} F = |J|$. Similarly, $F \hookrightarrow E$ implies $|J| \leqslant |I|$. Thus $|I| = |J|$ and so it remains to show that the bases $(u_i)$ and $(v_j)$ are of the same type. That is, we have to find sequences $(i_n) \subset I$ and $(j_n) \subset J$ such that the basic sequences $(u_{i_n})$ and $(v_{j_n})$ are equivalent. Actually, in view of the symmetry of the assumptions (and the bases), it will suffice to show that $(u_{i_n})$ dominates $(v_{j_n})$, i.e., whenever $\sum t_n u_{i_n}$ converges, then so does $\sum t_n v_{j_n}$.

*Case* 1. If $(u_i)$ is an $l_1$-basis, then from Lemma 1 we conclude that also $(v_j)$ is an $l_1$-basis, and we are done.

*Case* 2. Assume $(u_i)$ is not an $l_1$- basis; in other words, $(u_i)$ is totally non-$l_1$. Let $R\colon E \to F$ be an isomorphic embedding, and consider the family $0 \neq y_i = R(u_i)$, $i \in I$, of elements of $F$. Trivially, $(y_i)_{i \in I}$ is totally non-$l_1$ and by Lemma 3 there is a $K \subset I$ with $|K| = |I|$ such that

$$s(y_k) \cap s(y_{k'}) = \emptyset \quad \text{for all distinct } k, k' \in K.$$

For each $k \in K$ choose a $j(k) \in s(y_k)$. Since $v_{j(k)}^*(y_k) \neq 0$ for all $k \in K$ and $|K| > \aleph_0$, we find an $r > 0$ and a sequence $(i_n) \subset K$ such that for $j_n = j(i_n)$ we have $|v_{j_n}^*(y_{i_n})| \geqslant r$ for all $n \in N$.

Now assume that we have an (unconditionally) convergent series $\sum t_n u_{i_n}$; then also

$$(+)\quad \sum_{n=1}^{\infty} t_n R(u_{i_n}) \equiv \sum_{n=1}^{\infty} t_n y_{i_n} \quad \text{converges unconditionally to some } y \in F.$$

Now $y = \sum_{j \in J} v_j^*(y) \cdot v_j$, and from $(+)$ we have $v_j^*(y) = 0$ for $j \notin \bigcup_n s(y_{i_n})$ and $v_j^*(y) = t_n v_j^*(y_{i_n})$ for $j \in s(y_{i_n})$, $n \in N$. Since the family

$$(t_n v_j^*(y_{i_n}) \cdot v_j\colon\ j \in s(y_{i_n}),\ n \in N)$$

is summable (to $y$), so is its subfamily $(t_n v_{j_n}^*(y_{i_n}) \cdot v_{j_n}\colon n \in N)$. But $|v_{j_n}^*(y_{i_n})| \geqslant r > 0$ for every $n$, so the series $\sum t_n v_{j_n}$ converges (unconditionally). ■

If neither $(u_i)$ nor $(v_j)$ is an $l_1$-basis, we can slightly improve our theorem.

LEMMA 4. *If the basis $(u_i)_{i \in I}$ of $E$ is not of $l_1$-type and if $R\colon E \to F$ is a continuous linear operator with $\dim R(E) = \infty$, then*

$$\operatorname{dens} R(E) = |I_0|, \quad \text{where } I_0 = \{i \in I\colon R(u_i) \neq 0\}.$$



*In particular, if $R$ is injective, then*

$$\operatorname{dens} R(E) = |I| = \operatorname{dens} E.$$

Proof. Denote $m = \operatorname{dens} R(E)$ and $m_0 = |I_0|$. Since $R(E) \subset \overline{\operatorname{lin}}\{Ru_i\colon i \in I_0\}$, we clearly have $\aleph_0 \leqslant m \leqslant m_0$. Suppose $m < m_0$; then by Lemma 3 there exists a $K \subset I_0$ with $|K| = m_0$ such that

$$s(Ru_k) \cap s(Ru_{k'}) = \emptyset \quad \text{for all distinct } k, k' \in K.$$

Hence if $S = \bigcup_{k \in K} s(Ru_k)$, then $m_0 \leqslant |S| \leqslant m_0 \cdot \aleph_0 = m_0$ so that $|S| = m_0$.

On the other hand, since $m = \operatorname{dens} R(E)$, there is a subset $J'$ of $J$ of cardinality $\leqslant m \cdot \aleph_0 = m$ which contains $s(Rx)$ for all $x \in E$. In particular, $S \subset J'$ and hence $m_0 = |S| \leqslant m$; a contradiction. Thus $m = m_0$. ■

COROLLARY. *If $E$ and $F$ are nonseparable Banach spaces with symmetric non-$l_1$-bases $(u_i)_{i \in I}$ and $(v_j)_{j \in J}$, respectively, and if there exist continuous linear injections $E \to F$ and $F \to E$, then the bases $(u_i)$ and $(v_j)$ are equivalent.*

Proof. Use Lemma 4 to see that $|I| = |J|$, then proceed exactly as in the proof of the theorem. ■

Remark. An inspection of the argument used in the proof of the theorem shows that if $E$ and $F$ are as in the above corollary, and if there exist continuous linear operators $E \to F$ and $F \to E$ with nonseparable ranges, then the bases $(u_i)$ and $(v_j)$ are of the same type, i.e., for any sequences $(i_n) \subset I$ and $(j_n) \subset J$, the basic sequences $(u_{i_n})$ and $(v_{j_n})$ are equivalent.

Finally, we would like to inform the reader that it is possible to extend Troyanski's $l_1$-result (Lemma 1) to Banach spaces $X$ with an unconditional basis $(x_\alpha)_{\alpha \in A}$. In this case the assertion is that the basis $(x_\alpha)$ has large $l_1$-subbases. (A $c_0$-version of this is also true, and extends the symmetric $c_0$-result of Troyanski [4].) Details will appear elsewhere.

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