Problem 1. Let A be a complex noncommutative topological algebra. Does it follow that A has generalized topological divisors of zero?

This problem can be answered in the negative in the case of a real algebra. The counterexample is the Banach algebra Q. A positive answer to the following problem would imply a positive solution of Problem 1.

Problem 2. Suppose that A is a complex topological algebra with the property that for arbitrary nets (x_i), (y_i), $i \in I$, of elements of A the condition $\lim_I x_i y_i = 0$ implies $\lim_I y_i x_i = 0$. Does it follow that A is a commutative algebra?

The positive answer to Problem 2 would give a generalization of the following result due to Le Page [2]: If A is a complex Banach algebra and there is a positive constant k such that $\|xy\| \leqslant k\|yx\|$ for all x and y in A then the algebra A is commutative. Using a technique similar to that of [2] one can obtain a positive solution to Problem 2 in the case when A is an m-pseudoconvex algebra.

Problem 3. Suppose that a topological algebra A has generalized topological divisors of zero. Does there exist a commutative subalgebra of A also possessing generalized topological divisors of zero?

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Near isometries of spaces of weak* continuous functions, with an application to Bochner spaces

by

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Abstract. For a Banach dual E and a compact Hausdorff space X we denote by $C(X, E_{\sigma^*})$ the Banach space of continuous functions F from X to E when the latter space is provided with its weak* topology, normed by $\|F\|_\infty = \sup_{x\in X}\|F(x)\|$. Here we show that if X and Y are extremally disconnected compact Hausdorff spaces and E is a uniformly convex Banach space with $C(X, E_{\sigma^*})$ and $C(Y, E_{\sigma^*})$ nearly isometric, then X and Y are homeomorphic. The result has the following immediate consequence for Bochner spaces. If $(\Omega_i, \Sigma_i, \mu_i)$ are σ-finite measure spaces, $i = 1, 2$, and E a uniformly smooth Banach space such that $L^1(\mu_1, E)$ and $L^1(\mu_2, E)$ are nearly isometric or that $L^\infty(\mu_1, E^*)$ and $L^\infty(\mu_2, E^*)$ are nearly isometric, then $L^1(\mu_1, E)$ is isometric to $L^1(\mu_2, E)$ and $L^\infty(\mu_1, E^*)$ is isometric to $L^\infty(\mu_2, E^*)$.

0. Introduction. Throughout this paper the letter E stands for a Banach space, while X and Y denote compact Hausdorff spaces. U denotes the closed unit ball in E and S the surface of U. Interaction between elements of a Banach space and those of its dual will be denoted by $\langle\cdot,\cdot\rangle$. We will write $E_1 \cong E_2$ to indicate that the Banach spaces E_1 and E_2 are isometric.

Given X, assume that E is a Banach dual. Then $C(X, E_{\sigma^*})$ stands for the Banach space of continuous functions F on X to E when the latter space is provided with its weak* topology, normed by $\|F\|_\infty = \sup_{x\in X}\|F(x)\|$.

If (Ω, Σ, μ) is a positive measure space and E is any Banach space then, for $1 \leqslant p \leqslant \infty$, the Bochner spaces $L^p(\Omega, \Sigma, \mu, E)$ will be denoted by $L^p(\mu, E)$ when there is no danger of confusing the underlying measure spaces involved. For the definitions and properties of these spaces we refer to [10].

Following Banach [1, p. 242] we will call the Banach spaces E_1 and E_2 *nearly isometric* if $1 = \inf\{\|T\|\,\|T^{-1}\|\}$, where T runs through all isomorphisms of E_1 onto E_2. It is of course equivalent to suppose that $1 = \inf\{\|T\|\}$, where $\|T^{-1}\| = 1$ and hence T is a norm-increasing isomorphism of E_1 onto E_2. For if T is any continuous isomorphism of one Banach space onto

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another, we obtain an isomorphism $\hat{T}$ having the desired properties by defining $\hat{T}$ to be equal to $\|T^{-1}\|\,T$.

In [6] the isometries of spaces $C(X, E_{\sigma^*})$ were investigated for X extremally disconnected and E uniformly convex. (Spaces $C(X, E_{\sigma^*})$ with X extremally disconnected arise naturally as the biduals of spaces of norm-continuous functions [7], and, more generally, as the duals of spaces of vector measures and of Bochner L^1 spaces [8].) It was shown in [6] that if X and Y are two such compact spaces with $C(X, E_{\sigma^*})$ and $C(Y, E_{\sigma^*})$ isometric, then X and Y are homeomorphic. In Section 1 of this article we show that a modification of the arguments of [6] allows us to replace "isometric" by "nearly isometric". Specifically, we prove the following:

THEOREM 1. *Let X and Y be extremally disconnected compact Hausdorff spaces and E a uniformly convex Banach space. If $C(X, E_{\sigma^*})$ and $C(Y, E_{\sigma^*})$ are nearly isometric then X and Y are homeomorphic.*

In Section 2 we show that our Theorem 1 has an immediate consequence for Bochner spaces. We prove

THEOREM 2. *Let $(\Omega_i, \Sigma_i, \mu_i)$ be σ-finite measure spaces for $i = 1, 2$, and E a uniformly smooth Banach space. Assume that $L^1(\mu_1, E)$ and $L^1(\mu_2, E)$ are nearly isometric, or that $L^\infty(\mu_1, E^*)$ and $L^\infty(\mu_2, E^*)$ are nearly isometric. Then $L^1(\mu_1, E) \cong L^1(\mu_2, E)$ and $L^\infty(\mu_1, E^*) \cong L^\infty(\mu_2, E^*)$.*

This latter result was obtained in [5] by quite different arguments, far more computational in nature, for the special case in which $E = E^*$ is Hilbert space. The initial result of this sort established for E the space of scalars is due to Y. Benyamini [3].

Much of what we do in Section 2 is dependent upon the notions of category measure and hyperstonean space. If X is an extremally disconnected compact Hausdorff space then we will call a nonnegative extended real-valued Borel measure μ on X a *category measure* if

(i) every nonempty clopen set has positive measure,

(ii) every nowhere dense Borel set has measure zero, and

(iii) every nonempty clopen set contains another nonempty clopen set with finite measure.

An extremally disconnected space X that admits a category measure is called *hyperstonean*. This is equivalent to the definition of hyperstonean space obtained via the use of normal measures [2, p. 26]. (In [2], [5] and [8] category measures are referred to as "perfect measures".)

1. Near isometries of weak* continuous functions. Throughout this section X and Y will denote extremally disconnected compact Hausdorff spaces and E a fixed uniformly convex Banach space. We assume that $C(X, E_{\sigma^*})$ and $C(Y, E_{\sigma^*})$ are nearly isometric. Recall that E uniformly convex means that $\delta(\varepsilon) > 0$ when $0 < \varepsilon \leqslant 2$, where

$$\delta(\varepsilon) = \inf_{e_1, e_2 \in U} \{1 - \|(e_1 + e_2)/2\| \colon \|e_1 - e_2\| \geqslant \varepsilon\}.$$

Also recall that E uniformly convex implies that E is reflexive [9, p. 147], and it thus makes sense to consider the weak* topology of E. Here, of course, the weak and weak* topologies coincide, but we state our results for the latter topology since it is precisely the topology on a dual space which arises within the various mathematical contexts considered in [7] and [8]. Also, certain facts about spaces of weak* continuous functions will be needed in Section 2.

The proof of Theorem 1 will be established by means of a sequence of lemmas. The proof of the first lemma is contained in Lemma 1 of [4].

LEMMA 1. *If $\eta > 0$ and $e_1, e_2 \in E$ with $\|e_j\| \geqslant \eta$ for $j = 1, 2$ then there are scalars α_j with $|\alpha_j| \leqslant 1$, $j = 1, 2$, such that*

$$\|\alpha_1 e_1 + \alpha_2 e_2\| \geqslant \eta (1 - \delta(1))^{-1}.$$

From now until the end of this section η will denote a fixed positive number less than one and such that $\eta(1-\delta(1))^{-1} > 1$. T will then denote a fixed isomorphism of $C(X, E_{\sigma^*})$ onto $C(Y, E_{\sigma^*})$ with $\|T^{-1}\| = 1$ satisfying

(1) $\|T\|^2 - \|T\| < \eta$,

(2) $\eta(1-\delta(1))^{-1}/\|T\|^2 > 1$, and

(3) $1 - 1/\|T\| < \delta((1-\eta)/2)$.

For any clopen subset C of X and any $e \in S$ we then define $\varrho_e(C)$ by

$$\varrho_e(C) = \mathrm{cl}(\{y \in Y \colon \|T(e \cdot \chi_C)(y)\| > \eta\}).$$

Since, for $F \in C(Y, E_{\sigma^*})$, $\|F(\cdot)\|$ is lower semicontinuous on Y it follows that $\varrho_e(C)$ is a clopen subset of Y.

LEMMA 2. *If $e_1, e_2 \in S$ then for any clopen subset C of X we have $\varrho_{e_1}(C) = Y - \varrho_{e_2}(X - C)$.*

Proof. We first show that $\varrho_{e_1}(C)$ and $\varrho_{e_2}(X-C)$ are disjoint. Suppose, to the contrary, that $\varrho_{e_1}(C) \cap \varrho_{e_2}(X-C) \neq \emptyset$. Then the fact that if two open subsets of an extremally disconnected space have empty intersection so do their closures would imply the existence of a $y \in Y$ with $\|T(e_1 \cdot \chi_C)(y)\| > \eta$ and $\|T(e_2 \cdot \chi_{X-C})(y)\| > \eta$. By Lemma 1 there exist scalars α_j with $|\alpha_j| \leqslant 1$, $j = 1, 2$, such that

$$\|\alpha_1 T(e_1 \cdot \chi_C)(y) + \alpha_2 T(e_2 \cdot \chi_{X-C})(y)\| > \eta(1-\delta(1))^{-1}.$$

But for all such scalars α_j we have $\|\alpha_1 e_1 \cdot \chi_C + \alpha_2 e_2 \cdot \chi_{X-C}\|_\infty \leqslant 1$, which together with our assumption (2) giving $\|T\| \leqslant \|T\|^2 < \eta(1-\delta(1))^{-1}$, provides a contradiction. Thus $\varrho_{e_1}(C)$ and $\varrho_{e_2}(X-C)$ are indeed disjoint.

If $\varrho_{e_1}(C) \cup \varrho_{e_2}(X-C)$ is not all of Y then its complement, B, is a nonempty clopen subset of Y and on B we have $\|T(e_1 \cdot \chi_C)(y)\| \leqslant \eta$ and

$\|T(e_2\cdot\chi_{X-C})(y)\|\leqslant\eta$. Choose any $e\in S$ and note that $(1-\eta)e\cdot\chi_B$ is an element of $C(Y, E_{\sigma^*})$ with

$$\|T(e_1\cdot\chi_C)/\|T\|\pm(1-\eta)e\cdot\chi_B\|_\infty\leqslant 1$$
$$\text{and}\quad \|T(e_2\cdot\chi_{X-C})/\|T\|\pm(1-\eta)e\cdot\chi_B\|_\infty\leqslant 1$$

so that, since $\|T^{-1}\|=1$,

$$\|e_1\cdot\chi_C/\|T\|\pm T^{-1}((1-\eta)e\cdot\chi_B)\|_\infty\leqslant 1$$
$$\text{and}\quad \|e_2\cdot\chi_{X-C}/\|T\|\pm T^{-1}((1-\eta)e\cdot\chi_B)\|_\infty\leqslant 1.$$

But $\|T^{-1}((1-\eta)e\cdot\chi_B)\|_\infty\geqslant(1-\eta)/\|T\|$ so that there exists an $x\in X$ with

$$\|T^{-1}((1-\eta)e\cdot\chi_B)(x)\|>(1-\eta)/(2\|T\|).$$

Now x belongs to either C or $X-C$, say $x\in C$, so that the segment joining $e_1\cdot\chi_C(x)/\|T\|+T^{-1}((1-\eta)e\cdot\chi_B)(x)$ and $e_1\cdot\chi_C(x)/\|T\|-T^{-1}((1-\eta)e\cdot\chi_B)(x)$ has length greater than $(1-\eta)/\|T\|$. Consequently, one minus the norm of the midpoint of this segment, a quantity which is $1-1/\|T\|$, is greater than $\delta((1-\eta)/\|T\|)$. But since (1) implies that $\|T\|<2$ we have $\delta((1-\eta)/\|T\|)\geqslant\delta((1-\eta)/2)$ which contradicts (3) and completes the proof of the lemma.

LEMMA 3. *Let $e\in S$, $x\in X$, and let $\{C_{x,i}\colon i\in I_x\}$ be the family of clopen neighborhoods of x. Then the family $\{\varrho_e(C_{x,i})\colon i\in I_x\}$ of clopen subsets of Y has the finite intersection property.*

Proof. Suppose, to the contrary, that there exist clopen neighborhoods C_{x,i_k} of x, $k=1,\ldots,n$, such that $\bigcap_{k=1}^{n}\varrho_e(C_{x,i_k})=\emptyset$. In order to simplify the notation, throughout the remainder of this proof we will denote C_{x,i_k} by C_k, $1\leqslant k\leqslant n$. Then

$$Y=Y-\bigcap_{k=1}^{n}\varrho_e(C_k)=\bigcup_{k=1}^{n}[Y-\varrho_e(C_k)]$$

and, by Lemma 2, this latter set is $\bigcup_{k=1}^{n}\varrho_e(X-C_k)$.

Let $C=\bigcap_{k=1}^{n}C_k$ and consider $\varrho_e(C)$. For some k, $1\leqslant k\leqslant n$, we must have that $\varrho_e(C)\cap\varrho_e(X-C_k)$ is nonvoid. Again using the fact that if two open subsets of an extremally disconnected space are disjoint then so are their closures, we conclude that there is a $y\in Y$ with $\|T(e\cdot\chi_C)(y)\|>\eta$ and $\|T(e\cdot\chi_{X-C_k})(y)\|>\eta$. Thus by Lemma 1 there exist scalars α_i with $|\alpha_i|\leqslant 1$, $i=1, 2$, such that

$$\|\alpha_1 T(e\cdot\chi_C)+\alpha_2 T(e\cdot\chi_{X-C_k})\|_\infty\geqslant\|\alpha_1 T(e\cdot\chi_C)(y)+\alpha_2 T(e\cdot\chi_{X-C_k})(y)\|$$
$$>\eta(1-\delta(1))^{-1}.$$

But since C and $X-C_k$ are disjoint, for all choices of such scalars α_i we have $\|\alpha_1 e\cdot\chi_C+\alpha_2 e\cdot\chi_{X-C_k}\|_\infty\leqslant 1$, which, together with the fact that $\|T\|\leqslant\|T\|^2<\eta(1-\delta(1))^{-1}$, again provides a contradiction and completes the proof of the lemma.

Now let e, x, and $\{C_{x,i}\colon i\in I_x\}$ be as in the statement of Lemma 3. By that lemma the set

$$Y_{x,e}=\bigcap_{i\in I_x}\varrho_e(C_{x,i})$$

is a nonvoid subset of Y, and we define the set Y_x by

$$Y_x=\bigcup_{e\in S}Y_{x,e}.$$

LEMMA 4. *If $x_1, x_2\in X$, $x_1\neq x_2$, then $Y_{x_1}\cap Y_{x_2}=\emptyset$.*

Proof. Suppose, to the contrary, that $Y_{x_1}\cap Y_{x_2}\neq\emptyset$; say $y_0\in Y_{x_1}\cap Y_{x_2}$. Let C be a clopen neighborhood of x_1 which does not contain x_2. Then by the definition of Y_{x_1} there is an $e_1\in S$ such that $y_0\in Y_{x_1,e_1}$ and thus $y_0\in\varrho_{e_1}(C)$. Similarly, there is an $e_2\in S$ such that $y_0\in Y_{x_2,e_2}$ and, as $X-C$ is a clopen neighborhood of x_2, we have $x_2\in\varrho_{e_2}(X-C)$. Hence $y_0\in\varrho_{e_1}(C)\cap\varrho_{e_2}(X-C)$ which contradicts Lemma 2 and completes the proof of this lemma.

We now define the subset Y_1 of Y by

$$Y_1=\bigcup_{x\in X}Y_x.$$

It then follows from Lemma 4 that we obtain a well-defined mapping h from Y_1 onto X by setting, for $y\in Y_1$,

$$h(y)=x\quad\text{if}\quad y\in Y_x.$$

Next, consider the isomorphism R from $C(Y, E_{\sigma^*})$ onto $C(X, E_{\sigma^*})$ defined by $R=\|T\|T^{-1}$. Then for any clopen subset B of Y and any $e\in S$ we define the clopen subset $\tau_e(B)$ of X by

$$\tau_e(B)=\mathrm{cl}(\{x\in X\colon \|R(e\cdot\chi_B)(x)\|>\eta\}).$$

Since $\|R\|=\|T\|$, conditions (1), (2) and (3) are satisfied with T replaced by R. Also $\|R^{-1}\|=1$. It thus follows, by interchanging the roles of X and Y and those of T and R, that if $y\in Y$ and $\{B_{y,j}\colon j\in J_y\}$ is the family of clopen neighborhoods of y, the set

$$X_{y,e}=\bigcap_{j\in J_y}\tau_e(B_{y,j})$$

is a nonvoid subset of X. We set

$$X_y=\bigcup_{e\in S}X_{y,e}.$$

Then, by what we have established, $X_{y_1} \cap X_{y_2} = \emptyset$ if $y_1 \neq y_2$ so that if we put

$$X_1 = \bigcup_{y \in Y} X_y,$$

we obtain a well-defined map k of X_1 onto Y by setting, for $x \in X_1$,

$$k(x) = y \quad \text{if} \quad x \in X_y.$$

LEMMA 5. *If $x \in X_y$ and if C is any clopen neighborhood of x then for every clopen neighborhood B of y there exists an $e_B \in S$ such that*

$$\{y' \in Y: \|T(e_B \cdot \chi_C)(y')\| > \eta/\|T\|\} \cap B \neq \emptyset.$$

Proof. Let x, B and C be as given above. Since $x \in X_y$ there is an $e \in S$ with $x \in X_{y,e}$. Thus

$$x \in \tau_e(B) = \mathrm{cl}(\{x' \in X: \|R(e \cdot \chi_B)(x')\| > \eta\}),$$

so there exists an $x_1 \in C$ with

$$\|R(e \cdot \chi_B)(x_1)\| = \|\|T\| T^{-1}(e \cdot \chi_B)(x_1)\| > \eta,$$

i.e. $\|T^{-1}(e \cdot \chi_B)(x_1)\| > \eta/\|T\|$. Let $u = T^{-1}(e \cdot \chi_B)(x_1)$ and let $e_B = u/\|u\|$. Then

$$\|T^{-1}(e \cdot \chi_B) + e_B \cdot \chi_C\|_\infty \geqslant \|T^{-1}(e \cdot \chi_B)(x_1) + e_B \cdot \chi_C(x_1)\| > 1 + \eta/\|T\|,$$

so that (as T is norm-increasing) $\|e \cdot \chi_B + T(e_B \cdot \chi_C)\|_\infty > 1 + \eta/\|T\|$. Since (1) implies that $\|T\| < 1 + \eta/\|T\|$, there must exist a $y_1 \in B$ with $\|T(e_B \cdot \chi_C)(y_1)\| > \eta/\|T\|$.

LEMMA 6. *If $y \in Y_1$ and $h(y) = x$ then $x \in X_1$ and $k(x) = y$.*

Proof. Suppose that $y \in Y_1$, $h(y) = x$, and that either $x \notin X_1$ or $x \in X_1$ but $k(x) \neq y$. Then in either case there would exist an $x' \in X_1$, $x' \neq x$, with $y = k(x')$. Now $h(y) = x$ means $y \in Y_x$ so $y \in Y_{x,e}$ for some $e \in S$. Hence if D is any clopen neighborhood of x we have $y \in \varrho_e(D)$. Choose such a D which does not contain x'. Then as $k(x') = y$ we have $x' \in X_y$, and since $C = X - D$ is a clopen neighborhood of x' and $B = \varrho_e(D)$ a clopen neighborhood of y, by Lemma 5 there is an $e_B \in S$ with

$$\{y' \in Y: \|T(e_B \cdot \chi_C)(y')\| > \eta/\|T\|\} \cap B \neq \emptyset.$$

Choose a point y_1 in this latter intersection and pick $\varphi \in E^*$ with $\|\varphi\| = 1$ such that $\langle T(e_B \cdot \chi_C)(y_1), \varphi \rangle = \|T(e_B \cdot \chi_C)(y_1)\|$. Then

$$W = B \cap \{y' \in Y: \operatorname{Re} \langle T(e_B \cdot \chi_C)(y'), \varphi \rangle > \eta/\|T\|\}$$

is an open neighborhood of y_1 contained in $\varrho_e(D) = \mathrm{cl}(\{y' \in Y: \|T(e \cdot \chi_D)(y')\| > \eta\}$, so that there is a point $y_2 \in W$ with $\|T(e \cdot \chi_D)(y_2)\| > \eta \geqslant \eta/\|T\|$. Hence, by Lemma 1, there exist scalars α_1, α_2 with $|\alpha_j| \leqslant 1$ for $j = 1, 2$ and

$$\|\alpha_1 T(e \cdot \chi_D) + \alpha_2 T(e_B \cdot \chi_C)\|_\infty \geqslant \|\alpha_1 T(e \cdot \chi_D)(y_2) + \alpha_2 T(e_B \cdot \chi_C)(y_2)\|$$
$$\geqslant \eta(1 - \delta(1))^{-1}/\|T\|,$$

a quantity which, by (2), is greater than $\|T\|$. But, as $C = X - D$, for all choices of such scalars α_j we have $\|\alpha_1 e \cdot \chi_D + \alpha_2 e_B \cdot \chi_C\|_\infty \leqslant 1$, and this contradiction completes the proof of the lemma.

The proof of Theorem 1 is then completed by the following:

LEMMA 7. *$Y = Y_1$ and h is a homeomorphism of Y onto X.*

Proof. The previous lemma shows that $X = h(Y_1) \subseteq X_1$. It also shows that $Y = k(X_1) \subseteq Y_1$. (For h maps Y_1 onto X; hence given $x \in X_1 \subseteq X$ there is a $y \in Y_1$ with $h(y) = x$. Then by the previous lemma $k(x) = y \in Y_1$.) Thus h maps Y onto X, h is injective since k is a function, and $k = h^{-1}$.

We must show that h is continuous. Thus suppose A is a closed subset of X. If $y \notin k(A)$ then $y = k(x)$ for some $x \notin A$. Let C_x be a clopen neighborhood of x disjoint from A and let $e \in S$. As we now know that $Y_x = Y_{x,e} = \{y\}$, it follows that $y \in \varrho_e(C_x)$. And since A is contained in the clopen set $X - C_x$, it follows by the same reasoning that $k(A) \subseteq \varrho_e(X - C_x)$ which, by Lemma 2, is equal to $Y - \varrho_e(C_x)$. Hence the open set $\varrho_e(C_x)$ does not meet $k(A)$. Consequently, if we choose such a neighborhood C_x for each $x \notin A$, we have $k(A) = Y - \bigcup_{x \notin A} \varrho_e(C_x)$, a closed set. Thus $h = k^{-1}$ is continuous, and is hence a homeomorphism of Y onto X.

2. An application to Bochner spaces.

Proof of Theorem 2. Assume that $(\Omega_i, \Sigma_i, \mu_i)$, $i = 1, 2$, and E satisfy the hypotheses of Theorem 2. Since E is uniformly smooth, E^* is uniformly convex [9, p. 147]. We may, without loss of generality, suppose that Ω_i is a hyperstonean space, that Σ_i is the σ-field of Borel subsets of Ω_i, and that μ_i is a category measure for $i = 1, 2$. (See, e.g., Section 2.A of [11].)

We first assume that $L^\infty(\mu_1, E^*)$ and $L^\infty(\mu_2, E^*)$ are nearly isometric. It is known that the dual of $L^1(\mu_i, E)$ is $C(\Omega_i, E^*_{\sigma*})$ for $i = 1, 2$ [8, Theorem 1], where the interaction between elements $F \in L^1(\mu_i, E)$ and $G \in C(\Omega_i, E^*_{\sigma*})$ is given by

$$\langle F, G \rangle = \int \langle F(\omega), G(\omega) \rangle \, d\mu_i(\omega),$$

and also known that there exists an isometry of $L^\infty(\mu_i, E^*)$ into $C(\Omega_i, E^*_{\sigma*})$ [11, Proposition 2.4]. But since E^* is reflexive it has the Radon–Nikodým property [10, p. 218], so that (as our measure spaces are σ-finite) $L^\infty(\mu_i, E^*)$ is also the dual of $L^1(\mu_i, E)$ [10, p. 98]. Thus the isometry of Proposition 2.4 of [11] is surjective. It follows that $C(\Omega_1, E^*_{\sigma*})$ and $C(\Omega_2, E^*_{\sigma*})$ are nearly isometric. Hence, by Theorem 1, there exists a homeomorphism k of Ω_1 onto Ω_2.

Next, for Borel sets $B \subseteq \Omega_2$, we define $\lambda(B) = \mu_1[k^{-1}(B)]$. If then A is a Borel subset of Ω_1 we have $\mu_1(A) = \lambda(k(A)) = \int_{k(A)} d\lambda$ so that the map

$$\sum_{j=1}^{n} e_j \chi_{A_j} \to \sum_{j=1}^{n} e_j \chi_{k(A_j)}$$

carries the dense subspace of simple functions in $L^1(\Omega_1, \Sigma_1, \mu_1, E)$ isometrically onto the corresponding subspace of $L^1(\Omega_2, \Sigma_2, \lambda, E)$ and can thus be extended to an isometry of $L^1(\Omega_1, \Sigma_1, \mu_1, E)$ onto $L^1(\Omega_2, \Sigma_2, \lambda, E)$. Then multiplication by the scalar function $d\lambda/d\mu_2$ carries this latter space isometrically onto $L^1(\Omega_2, \Sigma_2, \mu_2, E)$. Hence $L^1(\mu_1, E) \cong L^1(\mu_2, E)$ and consequently $L^\infty(\mu_1, E^*) \cong L^\infty(\mu_2, E^*)$.

If we assume that $L^1(\mu_1, E)$ and $L^1(\mu_2, E)$ are nearly isometric, then their duals $L^\infty(\mu_1, E^*)$ and $L^\infty(\mu_2, E^*)$ are nearly isometric and the proof follows as above.

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On symmetric bases in nonseparable Banach spaces

by

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Abstract. It is shown that if E and F are nonseparable Banach spaces with symmetric bases and each of these spaces is isomorphic to a subspace of the other space, then the bases are equivalent (and hence the two spaces are isomorphic). In particular, in a nonseparable Banach space with a symmetric basis, any two such bases are equivalent.

The purpose of this paper is to prove the following

THEOREM. *Let E and F be nonseparable Banach spaces with symmetric bases $(u_i)_{i\in I}$ and $(v_j)_{j\in J}$, respectively. If $E \hookrightarrow F$ and $F \hookrightarrow E$ (isomorphic embeddings), then E and F are isomorphic: $E \approx F$. In fact, in this case the bases (u_i) and (v_j) are equivalent, i.e., there exists an isomorphism T from E onto F such that $T(\{u_i: i \in I\}) = \{v_j: j \in J\}$.*

(Thus, for some bijection $\tau: I \to J$, $Tu_i = v_{\tau(i)}$ for all $i \in I$, and every such bijection determines the corresponding isomorphism.)

COROLLARY. *If a nonseparable Banach space E has a symmetric basis, then any two symmetric bases of E are equivalent.*

These results show that there is a sharp distinction between the nonseparable and separable Banach spaces with symmetric bases. Nothing of the above type is valid in the separable case (see [1] and [2]) if we insist on having conclusions that the bases are equivalent. Whether or not the theorem is true in this case if the assertion were merely $E \approx F$, seems to be unknown.

We start with some explanations and a general construction.

A family $(x_\alpha)_{\alpha\in A}$ of elements in a Banach space X is called a *symmetric basis* of X ([4]) if

(a) it is an *unconditional basis* of X ([3]), i.e., for every $x \in X$ there is a unique family of scalars $(t_\alpha)_{\alpha\in A}$ such that $x = \sum_{\alpha\in A} t_\alpha x_\alpha$ (unconditional convergence or summability), and

(b) whenever a series $\sum_{\alpha\in A} t_\alpha x_\alpha$ converges (unconditionally), then so does the series $\sum_{\alpha\in A} t_{\varphi(\alpha)} x_\alpha$; for every bijection $\varphi: A \to A$.