STUDIA MATHEMATICA, T. LXXXV. (1987)

GB*-Algebras associated with inductive limits of Hilbert spaces

by

S. J. L. van EIJNDHOVEN and P. KRUSZYŃSKI* (Eindhoven)

Abstract. For a given generating family $\mathscr{R}$ of self-adjoint bounded operators in a Hilbert space $\mathscr{H}$ an inductive limit $\mathscr{S}_{\mathscr{R}} \subset \mathscr{H}$ of Hilbert spaces is constructed. $\mathscr{S}_{\mathscr{R}}$ is the maximal common dense domain for the unbounded operator algebras $\mathscr{R}^c$ and $\mathscr{R}^{cc}$. Both $\mathscr{R}^c$ and $\mathscr{R}^{cc}$ are GB*-algebras. $\mathscr{R}^c$ can be regarded as a strong commutant of $\mathscr{R}$.

Conditions on $\mathscr{R}$ are given such that the inductive limit topology for $\mathscr{S}_{\mathscr{R}}$ is generated by the seminorms $s \mapsto \|Ls\|$, $L \in \mathscr{R}^{cc}$, $s \in \mathscr{S}_{\mathscr{R}}$. A rather general example is included which has been described at length in [EK] and [EGK].

Introduction. Let Φ denote a directed set of bounded nonnegative Borel functions on $\mathbf{R}$, and let A denote a self-adjoint operator in a separable Hilbert space $\mathscr{H}$. With each $\varphi \in \Phi$ we associate the Hilbert space $\varphi(A)\mathscr{H}$ with inner product $(\cdot,\cdot)_\varphi = (\varphi(A)^{-1}\cdot, \varphi(A)^{-1}\cdot)$.

In [EK] we have studied the inductive limit of Hilbert spaces $\mathscr{S}_{\Phi(A)} = \bigcup_{\varphi \in \Phi} \varphi(A)\mathscr{H}$. We have given general conditions on Φ such that the inductive limit topology for $\mathscr{S}_{\Phi(A)}$ can be described by the seminorms $s \mapsto \|f(A)s\|$, $s \in \mathscr{S}_{\Phi(A)}$, where $f \in \Phi^{\ddagger}$. Here $\Phi^{\ddagger}$ denotes a family of Borel functions which is compatible with Φ in a well-defined way.

Further, we have discussed a representation $\mathscr{T}_{\Phi(A)}$ of the strong dual of $\mathscr{S}_{\Phi(A)}$. The topological properties of the spaces in the Gelfand triple $\mathscr{S}_{\Phi(A)} \subset \mathscr{H} \subset \mathscr{T}_{\Phi(A)}$ are completely determined by the set Φ and the operator A. We mention that $\mathscr{S}_{\Phi(A)}$ and $\mathscr{T}_{\Phi(A)}$ are both inductive and projective limits of Hilbert spaces if Φ satisfies the so-called symmetry condition. (Cf. [EK], § 1–2.)

In our paper [EGK] the above-mentioned concepts have been generalized. Thus we developed [EK] in two directions:

– Instead of one self-adjoint operator A we consider an n-tuple of strongly

* On leave from the department of Mathematical Methods in Physics, University of Warsaw, Poland.

AMS Classifications: 46 F 05, 46 F 10, 47 D 40.

commuting self-adjoint operators, and consequently a directed set of Borel functions on $\boldsymbol{R}^n$.

– No boundedness condition is imposed on the elements of Φ. So in general $\mathscr{S}_{\Phi(A)}$, $\mathscr{H}$ and $\mathscr{T}_{\Phi(A)}$ do not establish a Gelfand triple.

We note that the symmetry condition implies that $\mathscr{S}_{\Phi(A)} = \mathscr{T}_{\Phi^\dagger(A)}$ and that $\mathscr{T}_{\Phi(A)} = \mathscr{S}_{\Phi^\dagger(A)}$.

The paper [EGK] contains a further elaboration and refinement of the topological topics of [EK]. The second part of [EK] has been devoted to some algebraic topics. The set of operators $\Phi^\dagger(A) = \{f(A) \mid f \in \Phi^\dagger\}$ admits a GB*-algebra structure. Moreover, under certain conditions on Φ, the algebra $\Phi^\dagger(A)$ is identical with the so-called strong bicommutant of $\Phi(A)$ in $\mathscr{L}(\mathscr{S}_{\Phi(A)})$.

The present paper is almost entirely devoted to a further elaboration of these algebraic features. However, the last sections contain some topological considerations. The starting point is a directed family $\mathscr{R}$ of commuting positive bounded operators on $\mathscr{H}$. On $\mathscr{R}$ we impose very mild conditions. The family $\mathscr{R}$ generates the space $\mathscr{S}_{\mathscr{R}} = \bigcup_{a \in \mathscr{R}} a\mathscr{H}$. In $\mathscr{L}(\mathscr{S}_{\mathscr{R}})$ we consider the strong commutant $\mathscr{R}^c$ and strong bicommutant $\mathscr{R}^{cc}$ of $\mathscr{R}$. Both $\mathscr{R}^c$ and $\mathscr{R}^{cc}$ are GB*-algebras.

Our construction of the commutative GB*-algebra $\mathscr{R}^{cc}$ of unbounded linear operators presents a very natural extension of the usual construction of the von Neumann algebra $\mathscr{W}^*(\mathscr{R})$ of bounded operators generated by $\mathscr{R}$. In this respect we refer to [Pij] and [Ep], where also constructions of unbounded operator commutants can be found.

This paper is organized as follows.

The preliminaries contain the basic theory on GB*-algebras as introduced by G. R. Allan [Al 1–2]. In Section 1 we introduce the concept of generating family of operators and in Section 2 the concept of $\mathscr{R}$-bounded operators. Section 3 is devoted to the construction of the $\mathscr{R}$-commutant $\mathscr{R}^c$ and the $\mathscr{R}$-bicommutant $\mathscr{R}^{cc}$. We prove that $\mathscr{R}^c$ and $\mathscr{R}^{cc}$ are GB*-algebras. In Section 4 we study a functional calculus for the commutative GB*-algebra $\mathscr{R}^{cc}$. We prove a global extension of the Gelfand–Naimark theorem for $\mathscr{R}^{cc}$. Further we discuss relations between $\mathscr{W}^*(\mathscr{R})$ and $\mathscr{R}^{cc}$. Section 5 contains some topological considerations with respect to the inductive limit $\mathscr{S}_{\mathscr{R}}$. At the end of this paper we summarize some results on the space $\mathscr{S}_{\Phi(A)}$.

0. Preliminaries. Here we give a short survey of Allan's theory on GB*-algebras [Al 1–2]. Let $\mathscr{A}$ be a locally convex topological $*$-algebra over the field of complex numbers. This means that the separate multiplication $p \mapsto pq$ (q fixed) and the involution $p \mapsto p^+$ are continuous operations in $\mathscr{A}$. An element $p \in \mathscr{A}$ is said to be *bounded* if there exists $\lambda \in \boldsymbol{C} \setminus \{0\}$ such that the set $\{(\lambda p)^n \mid n \in \boldsymbol{N}\}$ is bounded in $\mathscr{A}$. The set of bounded elements of $\mathscr{A}$ is denoted by $\mathscr{A}_0$.

Let $\mathbf{B}$ denote the family of all bounded, absolutely convex, closed subsets $\mathscr{B}$ of $\mathscr{A}$ with the properties

(0.1) $$\mathscr{B}^2 \subset \mathscr{B},$$

(0.2) $$\mathscr{B}^+ = \mathscr{B}.$$

(0.3) Definition. A locally convex topological $*$-algebra $\mathscr{A}$ with identity $\mathbf{1}$ is called a *GB*-algebra* if the following conditions are satisfied:

(i) $\mathscr{A}$ is sequentially complete.

(ii) $\mathscr{A}$ is symmetric, i.e. for each $p \in \mathscr{A}$ the element $\mathbf{1} + p^+ p$ is invertible in $\mathscr{A}$ with bounded inverse.

(iii) The family $\mathbf{B}$ defined by (0.1) and (0.2) has a maximal element $\mathscr{B}_0$ with respect to set inclusion.

Let $\mathscr{A}$ be a GB*-algebra. Then the $*$-algebra

$$\mathscr{A}(\mathscr{B}_0) = \{\lambda b \mid b \in \mathscr{B}_0,\ \lambda \in \boldsymbol{C}\}$$

is a B*-algebra with respect to the Minkowski norm induced by $\mathscr{B}_0$.

(0.4) Proposition ([Al2], Proposition 2.9). (i) *If $\mathscr{C} \subset \mathscr{A}$ is a closed $*$-subalgebra of $\mathscr{A}$ with $\mathbf{1} \in \mathscr{C}$ then $\mathscr{C}$ is also a GB*-algebra. The maximal element of the family*

$$\mathbf{B}_1 = \{\mathscr{B} \subset \mathscr{C} \mid \mathscr{B} \text{ bounded, absolutely convex, closed, } \mathscr{B}^2 \subset \mathscr{B} \text{ and } \mathscr{B}^+ = \mathscr{B}\}$$

is the set $\mathscr{B}_1 = \mathscr{B}_0 \cap \mathscr{C}$.

(ii) *If $\mathscr{A}$ is commutative, then $\mathscr{A}_0 = \mathscr{A}(\mathscr{B}_0)$.*

Let $\mathscr{A}$ be a commutative GB*-algebra, and let Λ denote the spectrum of the commutative B*-algebra $\mathscr{A}_0$. Then Λ is a compact topological space. The Gelfand transform on $\mathscr{A}_0$ is an isometric $*$-isomorphism from $\mathscr{A}_0$ onto the B*-algebra of continuous complex-valued functions $\mathscr{C}(\Lambda)$ with the usual Banach norm. Allan has proved a "nonbounded" extension of the Gelfand–Naimark theorem.

(0.5) Proposition ([Al2], Proposition 3.1). *Let $\mathscr{A}$ be a commutative GB*-algebra, and let Λ be the spectrum of $\mathscr{A}_0$. Then corresponding to each $\chi \in \Lambda$ there is an extended complex-valued function χ^e: $\mathscr{A} \to \boldsymbol{C}^* (= \boldsymbol{C} \cup \{\infty\})$ such that*

(i) *χ^e is an extension of χ.*

(ii) *For each $p \in \mathscr{A}$ and $\xi \in \boldsymbol{C}$*

$$\chi^e(\xi p) = \xi \chi^e(p)$$

with the convention $0 \cdot \infty = 0$.

(iii) *For each* $p_1, p_2 \in \mathscr{A}$

$$\chi^e(p_1)+\chi^e(p_2) = \chi^e(p_1+p_2)$$

provided that $\chi^e(p_1)$ *and* $\chi^e(p_2)$ *are not both* ∞.

(iv) *For each* $p_1, p_2 \in \mathscr{A}$

$$\chi^e(p_1 p_2) = \chi^e(p_1)\chi^e(p_2)$$

provided that $(\chi^e(p_1), \chi^e(p_2)) \neq (\infty, 0)$ *or* $(0, \infty)$.

(v) *For each* $p \in \mathscr{A}$

$$\chi^e(p^+) = \overline{\chi^e(p)}$$

with the convention $\overline{\infty} = \infty$.

(vi) *The set* $N_p = \{\chi \in \Lambda \mid \chi^e(p) = \infty\}$ *is a nowhere dense closed subset of* $\mathscr{A}$.

(0.6) Definition. A collection $\mathscr{F}$ of C^*-valued continuous functions on a topological space Γ is called a *$*$-algebra of functions* if each $f \in \mathscr{F}$ takes the value ∞ on at most a nowhere dense subset of Γ.

For any $f, g \in \mathscr{F}$ and $\alpha, \beta \in C$, the functions $\alpha f+\beta g$, $f \cdot g$ and $f^* = \bar{f}$ are pointwise well defined on the dense subset of Γ on which f and g are finite. We assume that each of the functions $\alpha f+\beta g$, $f \cdot g$ and f^* has a unique continuous extension to a C^*-valued continuous function which also belongs to $\mathscr{F}$.

(0.7) Theorem ([Al2], Theorem 3.9). *Let $\mathscr{A}$ be a commutative GB*-algebra, and let Λ be the spectrum of $\mathscr{A}_0$. Define the mapping $\hat{}$ on $\mathscr{A}$: $p \mapsto \hat{p}$, $p \in \mathscr{A}$, by $\hat{p}(\chi) = \chi^e(p)$, $\chi \in \Lambda$. Then $\hat{}$ is a $*$-isomorphism of $\mathscr{A}$ onto a $*$-subalgebra $\hat{\mathscr{A}}$ of continuous C^*-valued functions on Λ. The mapping $\hat{}$ extends the usual Gelfand transform of the commutative B*-algebra $\mathscr{A}_0$, i.e. $\hat{\mathscr{A}}_0 = \mathscr{C}(\Lambda) \subset \hat{\mathscr{A}}$.*

1. Generating families of self-adjoint operators. Let $\mathscr{R}$ denote a family of mutually commuting bounded positive operators on a separable Hilbert space $\mathscr{H}$. Then the commutative von Neumann algebra $\mathscr{W}^*(\mathscr{R})$ generated by $\mathscr{R}$ and the identity on $\mathscr{H}$ equals the usual bicommutant $\mathscr{R}'' \subset \mathscr{L}(\mathscr{H})$. In the following sections we describe a way to extend $\mathscr{W}^*(\mathscr{R})$ to a GB*-algebra of unbounded strongly commuting linear operators. It is clear that there is no unique extension. However, our approach seems very natural. We use the family $\mathscr{R}$ to construct an inductive limit of Hilbert spaces $\mathscr{S}_{\mathscr{R}} \subset \mathscr{H}$. First, we define the notion of generating family.

(1.1) Definition. Let $\mathscr{R}$ be a family of bounded self-adjoint operators on $\mathscr{H}$. The family $\mathscr{R}$ is called a *generating family* if it fulfils the following conditions:

(i) $\forall_{a\in\mathscr{R}}$: $0 \leqslant a \leqslant 1$ (positivity and boundedness),
(ii) $\forall_{a,b\in\mathscr{R}}$: $ab = ba$ (commutativity),
(iii) $\forall_{a,b\in\mathscr{R}}\ \exists_{c\in\mathscr{R}}$: $(a \leqslant c) \wedge (b \leqslant c)$ (directedness),
(iv) $\forall_{a\in\mathscr{R}}\ \exists_{b\in\mathscr{R}}$: $a^{1/2} \leqslant b$ (sub-semigroup property).

Remark. Because of Condition (1.1.ii), Condition (1.1.iv) is equivalent to

$$\forall_{a\in\mathscr{R}}\ \exists_{b\in\mathscr{R}}:\quad a \leqslant b^2.$$

Let $a \in \mathscr{R}$. We denote its support by $r(a)$ according to [Sa], Definition (1.10.3). Observe that $a \restriction_{r(a)\mathscr{H}}$ is injective.

(1.2) Lemma. *Let $a \in \mathscr{R}$. Then there exists $b \in \mathscr{R}$ such that $b^{-2} ar(b) \restriction_{r(b)\mathscr{H}}$ is bounded.*

Proof. Take b as indicated in (1.1.iv). Then

$$b^{-1}ab^{-1}r(b)\restriction_{r(b)\mathscr{H}} \leqslant r(b)\restriction_{r(b)\mathscr{H}}. \ \blacksquare$$

For each $a \in \mathscr{R}$ we put $a\mathscr{H} = \{ax \mid x \in \mathscr{H}\}$. If we define in $a\mathscr{H}$ the inner product

$$(ax, ay)_a = (r(a)x, r(a)y)$$

with $(\cdot,\cdot)$ the inner product of $\mathscr{H}$, then $a\mathscr{H}$ becomes a Hilbert space. Observe that the canonical embedding $a\mathscr{H} \hookrightarrow \mathscr{H}$ is continuous. Further, the linear mapping a: $r(a)\mathscr{H} \to a\mathscr{H}$ is a bijective isometry.

(1.3) Definition. By $\mathscr{S}_{\mathscr{R}}$ we denote the inductive limit generated by the Hilbert spaces $a\mathscr{H}$, $a \in \mathscr{R}$, i.e.

$$\mathscr{S}_{\mathscr{R}} = \bigcup_{a\in\mathscr{R}} a\mathscr{H}$$

with the inductive limit topology.

Remark. $\mathscr{S}_{\mathscr{R}}$ is in general a nonstrict inductive limit.

(1.4) Lemma. $\mathscr{S}_{\mathscr{R}} = \bigcup_{b\in\mathscr{R}} b(\mathscr{S}_{\mathscr{R}})$.

Proof. It is clear that $\bigcup_{b\in\mathscr{R}} b(\mathscr{S}_{\mathscr{R}}) \subset \mathscr{S}_{\mathscr{R}}$.

Let $s \in \mathscr{S}_{\mathscr{R}}$. Then $s = ax$ for some $a \in \mathscr{R}$ and $x \in \mathscr{H}$. From Lemma (1.2) it follows that the operator $b^{-1}ab^{-1}r(b)$ is densely defined and bounded on $r(b)\mathscr{H}$. Hence $s = b\tilde{s}$ where $\tilde{s} = b\{b^{-1}ab^{-1}r(b)\}r(b)x \in \mathscr{S}_{\mathscr{R}}$. $\blacksquare$

In this section we do not discuss the topological structure of $\mathscr{S}_{\mathscr{R}}$. Only Section 5 contains some results of this type. We shall show that $\mathscr{S}_{\mathscr{R}}$ is both an inductive limit and a projective limit of Hilbert spaces under certain conditions on $\mathscr{R}$.

2. $\mathscr{R}$-bounded operators. We introduce the notion of $\mathscr{R}$-bounded operator. The generating family $\mathscr{R}$ can be seen as a set of smoothing operators for the $\mathscr{R}$-bounded operators. Here is the definition.

(2.1) Definition. Let L be a densely defined linear operator in $\mathscr{H}$ with $\mathscr{D}(L) \supset \mathscr{S}_{\mathscr{R}}$. Then L is called *$\mathscr{R}$-bounded* if the operator La is bounded for each $a \in \mathscr{R}$.

The vector space of all $\mathscr{R}$-bounded operators is denoted by $\mathscr{R}\mathscr{B}(\mathscr{H})$.

Remark. Each $L \in \mathscr{R}\mathscr{B}(\mathscr{H})$ can be seen as a continuous linear operator from $\mathscr{S}_{\mathscr{R}}$ into $\mathscr{H}$. Also the converse is valid: each continuous linear mapping from $\mathscr{S}_{\mathscr{R}}$ into $\mathscr{H}$ is $\mathscr{R}$-bounded. In the sequel $\mathscr{R}\mathscr{B}(\mathscr{H})$ will be regarded as the set of all continuous linear mappings from $\mathscr{S}_{\mathscr{R}}$ into $\mathscr{H}$.

On the vector space $\mathscr{R}\mathscr{B}(\mathscr{H})$ we define the seminorms p_a, $a \in \mathscr{R}$,

$$p_a(L) = \|La\|, \quad L \in \mathscr{R}\mathscr{B}(\mathscr{H}). \tag{2.2}$$

The family $\{p_a \mid a \in \mathscr{R}\}$ is complete, i.e. $L = 0$ iff $p_a(L) = 0$ for all $a \in \mathscr{R}$.

(2.3) Lemma. *The vector space $\mathscr{R}\mathscr{B}(\mathscr{H})$ endowed with the locally convex topology generated by the seminorms p_a, $a \in \mathscr{R}$, is sequentially complete.*

Proof. Let $(L_n)_{n \in N}$ be a Cauchy sequence in $\mathscr{R}\mathscr{B}(\mathscr{H})$. Let $a \in \mathscr{R}$. Then $(L_n a)_{n \in N}$ is a Cauchy sequence in $\mathscr{L}(\mathscr{H})$. The completeness of $\mathscr{L}(\mathscr{H})$ yields $L_a \in \mathscr{L}(\mathscr{H})$ such that $\|L_n a - L_a\| \to 0$ as $n \to \infty$. Now define the operator L on $\mathscr{S}_{\mathscr{R}}$ by

$$Ls = L_a x, \quad s = ax \in \mathscr{S}_{\mathscr{R}}.$$

Then the definition of L does not depend on the choice of a and x, because for $s = ax = by$ we have

$$L_a x = \lim_{n \to \infty} L_n ax = \lim_{n \to \infty} L_n by = L_b y.$$

Since $La = L_a$ for all $a \in \mathscr{R}$, it follows that $L \in \mathscr{R}\mathscr{B}(\mathscr{H})$. ∎

Next, we introduce the notion of $\mathscr{R}$-commutant.

(2.4) Definition. Let $\mathscr{L} \subset \mathscr{R}\mathscr{B}(\mathscr{H})$. Then we define $\mathscr{L}^c \subset \mathscr{R}\mathscr{B}(\mathscr{H})$ as follows:

$$\mathscr{L}^c = \{L' \in \mathscr{R}\mathscr{B}(\mathscr{H}) \mid \forall_{L \in \mathscr{L}}\ \forall_{a \in \mathscr{R}}:\ L'L \in \mathscr{R}\mathscr{B}(\mathscr{H}),$$
$$LL' \in \mathscr{R}\mathscr{B}(\mathscr{H}) \text{ and } L'La = LL'a\}.$$

The set $\mathscr{L}^c$ is called the *$\mathscr{R}$-commutant* of $\mathscr{L}$.

In the remaining part of this paper we confine ourselves to $\mathscr{R}^c$ and $\mathscr{R}^{cc} = (\mathscr{R}^c)^c$. They are called the *$\mathscr{R}$-commutant* and *$\mathscr{R}$-bicommutant of $\mathscr{R}$*. Clearly, $\mathscr{R}^c$ and $\mathscr{R}^{cc}$ are linear subspaces of $\mathscr{R}\mathscr{B}(\mathscr{H})$. In the next section we show that they admit the structure of a topological algebra.

3. The GB*-algebras $\mathscr{R}^c$ and $\mathscr{R}^{cc}$. Let $L \in \mathscr{R}^c$, and let $s \in \mathscr{S}_{\mathscr{R}}$. By Lemma (1.4), $s = b\tilde{s}$ for some $b \in \mathscr{R}$ and $\tilde{s} \in \mathscr{S}_{\mathscr{R}}$. It follows that

$$Ls = Lb\tilde{s} = bL\tilde{s} \in \mathscr{S}_{\mathscr{R}}.$$

So L maps $\mathscr{S}_{\mathscr{R}}$ into $\mathscr{S}_{\mathscr{R}}$. Therefore, for each $L_1, L_2 \in \mathscr{R}^c$ the composed operator $L_1 L_2$ is well defined on $\mathscr{S}_{\mathscr{R}}$ and maps $\mathscr{S}_{\mathscr{R}}$ into $\mathscr{S}_{\mathscr{R}}$.

Remark. From Section 5 it follows that each $L \in \mathscr{R}^c$ gives rise to a continuous linear mapping from $\mathscr{S}_{\mathscr{R}}$ into $\mathscr{S}_{\mathscr{R}}$.

(3.1) Lemma. *The vector spaces $\mathscr{R}^c$ and $\mathscr{R}^{cc}$ are subalgebras of $\mathscr{R}\mathscr{B}(\mathscr{H})$ which consist of linear mappings from $\mathscr{S}_{\mathscr{R}}$ into $\mathscr{S}_{\mathscr{R}}$. Moreover, $\mathscr{R}^{cc}$ is abelian and $\mathscr{R}^{cc} \subset \mathscr{R}^c$.*

Proof. Let $L_1, L_2 \in \mathscr{R}^c$, and let $a \in \mathscr{R}$. Take $b \in \mathscr{R}$ with $a \leqslant b^2$. Then

$$L_1 L_2 a = (L_1 b)(L_2 b)\big(b^{-1} ab^{-1} r(b)\big).$$

It follows that $L_1 L_2$ is $\mathscr{R}$-bounded. Further, for all $a_1, a_2 \in \mathscr{R}$

$$L_1 L_2 a_1 a_2 = L_1 a_1 L_2 a_2 = a_1 L_1 L_2 a_2.$$

Hence $L_1 L_2 \in \mathscr{R}^c$.

Since $\mathscr{R} \subset \mathscr{R}^c$ it follows that $\mathscr{R}^{cc} \subset \mathscr{R}^c$. $\mathscr{R}^{cc}$ is abelian, because $\mathscr{R}$ consists of mutually commuting operators. ∎

We mention the following useful alternative description:

$$\mathscr{R}^c = \{L \in \mathscr{R}\mathscr{B}(\mathscr{H}) \mid \forall_{a \in \mathscr{R}}\ \forall_{s \in \mathscr{S}_{\mathscr{R}}}:\ Las = aLs\};$$
$$\mathscr{R}^{cc} = \{L' \in \mathscr{R}\mathscr{B}(\mathscr{H}) \mid \forall_{L \in \mathscr{R}^c}\ \forall_{s \in \mathscr{S}_{\mathscr{R}}}:\ L'Ls = LL's\}.$$

We are going to introduce an involution in $\mathscr{R}^c$ by taking the usual Hilbert space adjoint L^* of each $L \in \mathscr{R}^c$.

(3.2) Lemma. *Let $L \in \mathscr{R}^c$. Then $\mathscr{S}_{\mathscr{R}} \subset \mathscr{D}(L^*)$, and $L^* \in \mathscr{R}\mathscr{B}(\mathscr{H})$.*

Proof. Let $s \in \mathscr{S}_{\mathscr{R}}$. Then for all $a \in \mathscr{R}$ and $x \in \mathscr{H}$

$$(Ls, ax) = (Las, x) = \big(s, (La)^* x\big)$$

where $(La)^* \in \mathscr{L}(\mathscr{H})$ because $La \in \mathscr{L}(\mathscr{H})$. Thus we get

$$L^*(ax) = (La)^* x, \quad x \in \mathscr{H}.$$

From this relation the assertions follow. ∎

(3.3) Definition. Let $L \in \mathscr{R}^c$. Then we define L^+ by

$$L^+ := L^* \restriction_{\mathscr{S}_{\mathscr{R}}}.$$

(3.4) Lemma. (i) *Let $K, L \in \mathscr{R}^c$. Then $K^+, L^+ \in \mathscr{R}^c$ and $(KL)^+ = L^+ K^+$, $L^{++} = L$.*

(ii) *Let $L \in \mathscr{R}^{cc}$. Then $L^+ \in \mathscr{R}^{cc}$.*

Proof. (i) Let $a \in \mathscr{R}$ and $s \in \mathscr{S}_{\mathscr{R}}$. Then we have

$$L^+ as = L^* as = (La)^* s = aL^+ s.$$

It is clear that $(KL)^+ = L^+ K^+$, and that $L^{++} = L$.

(ii) Let $L_1 \in \mathscr{R}^c$, and $s \in \mathscr{S}_{\mathscr{R}}$. Since $\mathscr{R}^{cc} \subset \mathscr{R}^c$ and hence $L_1^+ \in \mathscr{R}^c$ we have

$$L_1 L^+ s = (LL_1^+)^+ s = (L_1^+ L)^+ s = L^+ L_1 s. \quad \blacksquare$$

An element L of $\mathscr{R}^c$ is *hermitian* if $L^+ = L$. We have the following nice characterization of the hermitian elements of $\mathscr{R}^c$.

(3.5) LEMMA. *Let $L \in \mathscr{R}^c$. Then L is hermitian iff L is essentially self-adjoint as a linear operator in $\mathscr{H}$ with $\mathscr{D}(L) = \mathscr{S}_{\mathscr{R}}$.*

Proof. $\Leftarrow$ If L is essentially self-adjoint, then L is symmetric on $\mathscr{S}_{\mathscr{R}}$. Hence $L = L^* \upharpoonright_{\mathscr{S}_{\mathscr{R}}} = L^+$.

$\Rightarrow$ Assume $L = L^+$. Let $x \in \mathscr{D}(L^*)$ with $L^* x = \pm ix$. Let $a \in \mathscr{R}$. Then we have

$$(ax, Lax) = (L^* x, a^2 x) = \pm i\|ax\|^2.$$

It follows that $ax = 0$ for all $a \in \mathscr{R}$ and so $x = 0$. $\blacksquare$

On $\mathscr{R}^c$ and $\mathscr{R}^{cc}$ we impose the locally convex topology generated by the seminorms p_a, $a \in \mathscr{R}$, as introduced in (2.2).

(3.6) LEMMA. (i) *Multiplication is jointly continuous in $\mathscr{R}^c$.*

(ii) *The involution $L \mapsto L^+$ is continuous on $\mathscr{R}^c$.*

Proof. (i) Let $a \in \mathscr{R}$. Then there is $b \in \mathscr{R}$ such that $a \leqslant b^2$. So for all $L_1, L_2 \in \mathscr{R}^c$ we have

$$p_a(L_1 L_2) = \|L_1 L_2 a\| \leqslant \|L_1 L_2 b^2\| \leqslant p_b(L_1) p_b(L_2).$$

(ii) For each $a \in \mathscr{R}$ and $L \in \mathscr{R}^c$ we have $\|(La)^*\| = \|La\| = \|L^+ a\|$. $\blacksquare$

(3.7) LEMMA. *The algebra $\mathscr{R}^c$ is a closed subspace of $\mathscr{R}\mathscr{B}(\mathscr{H})$, and the algebra $\mathscr{R}^{cc}$ is closed in $\mathscr{R}^c$.*

Proof. The linear mappings $\Delta_a\colon \mathscr{R}\mathscr{B}(\mathscr{H}) \to \mathscr{R}\mathscr{B}(\mathscr{H})$, $a \in \mathscr{R}$, defined by $\Delta_a(L) = aL - La$, $L \in \mathscr{R}\mathscr{B}(\mathscr{H})$, are continuous. Since $\mathscr{R}^c = \bigcap_{a \in \mathscr{R}} \operatorname{Ker}(\Delta_a)$, $\mathscr{R}^c$ is closed in $\mathscr{R}\mathscr{B}(\mathscr{H})$. Similarly,

$$\mathscr{R}^{cc} = \bigcap_{L \in \mathscr{R}^c} \operatorname{Ker}(D_L)$$

where $D_L\colon \mathscr{R}^c \to \mathscr{R}^c$ denotes the continuous mapping

$$D_L(K) = KL - LK, \qquad K \in \mathscr{R}^c. \quad \blacksquare$$

(3.8) COROLLARY. *$\mathscr{R}^c$ and $\mathscr{R}^{cc}$ are sequentially complete.*

Proof. Cf. Lemma (2.3) and Lemma (3.7). $\blacksquare$

Next, we consider the set $\mathscr{R}_0^c$ of bounded elements of $\mathscr{R}^c$ (cf. Preliminaries). Since $\mathscr{R}^c$ is not commutative, $\mathscr{R}_0^c$ is not even a linear subspace of $\mathscr{R}^c$. However, the bounded normal elements of $\mathscr{R}^c$ admit a useful characterization.

(3.9) LEMMA. *Let $L \in \mathscr{R}^c$ be normal, i.e. $L^+ L = LL^+$. Then L is a bounded element of $\mathscr{R}^c$ iff L is bounded as an operator in $\mathscr{H}$.*

Proof. $\Leftarrow$ Suppose $L \in \mathscr{L}(\mathscr{H})$. Put $\lambda = \|L\|^{-1}$. Then for each $a \in \mathscr{R}$

$$\lambda^n \|L^n a\| \leqslant \|a\|$$

and hence $L \upharpoonright_{\mathscr{S}_{\mathscr{R}}} \in \mathscr{R}_0^c$.

$\Rightarrow$ We have $\|Ls\|^2 \leqslant \|L^+ Ls\|$ for all $s \in \mathscr{S}_{\mathscr{R}}$ with $\|s\| = 1$. Since L is normal, $L^+ L \in \mathscr{R}_0^c$. So we may assume that $L^+ = L$.

Let $x \in \mathscr{H}$ and let $a \in \mathscr{R}$, with $\|ax\| = 1$. Put $s = ax$. Then we compute as follows:

$$\|Ls\|^2 \leqslant \|L^2 s\| \leqslant \ldots \leqslant \|L^{2^n} s\|^{2^{1-n}} \leqslant \|(\lambda L)^{2^n} a\|^{2^{1-n}} (1/\lambda)^2 \|x\|^{2^{1-n}},$$

where we take $\lambda > 0$ such that for all $a \in \mathscr{R}$

$$\sup_{n \in N} (\|(\lambda L)^{2^n} a\|) < \infty.$$

Thus we obtain $\|Ls\| \leqslant 1/\lambda$, for all $s \in \mathscr{S}_{\mathscr{R}}$ with $\|s\| = 1$. $\blacksquare$

Now we come to the main theorem of this section.

(3.10) THEOREM. *Let $\mathscr{R}$ be a generating family of bounded operators on $\mathscr{H}$. Then the $\mathscr{R}$-commutant $\mathscr{R}^c$ and the $\mathscr{R}$-bicommutant $\mathscr{R}^{cc}$ are GB*-algebras.*

Proof. We have already shown that $\mathscr{R}^c$ is a sequentially complete locally convex topological $*$-algebra, and that $\mathscr{R}^{cc}$ is a closed $*$-subalgebra of $\mathscr{R}^c$. So it remains to be proved that $\mathscr{R}^c$ is symmetric and that the family **B** of bounded, absolutely convex, closed, idempotent and symmetric subsets of $\mathscr{R}^c$ has a maximal element with respect to set inclusion.

Symmetry. Let $L \in \mathscr{R}^c$ and put $Q = I + L^+ L$. Since $L^+ L$ is hermitian, it is essentially self-adjoint as an operator in $\mathscr{H}$. So Q^{-1} is well defined and belongs to $L(\mathscr{H})$. By Lemma (3.9) we get $Q^{-1} \upharpoonright_{\mathscr{S}_{\mathscr{R}}} \in \mathscr{R}_0^c$.

Maximal element. The family **B** is defined by

$$\mathbf{B} = \{\mathscr{B} \subset \mathscr{R}^c \mid \mathscr{B} \text{ is bounded, absolutely convex and closed, } \mathscr{B}^2 \subset \mathscr{B} \text{ and } \mathscr{B}^+ = \mathscr{B}\}.$$

Put $\mathscr{B}_0 = \{L \in \mathscr{R}^c \mid \forall_{a \in \mathscr{R}}\colon \|La\| \leqslant 1\}$. We shall prove that $\mathscr{B}_0$ is the maximal element of **B**.

It is clear that $\mathscr{B}_0$ is bounded, absolutely convex and closed. Further, let $L_1, L_2 \in \mathscr{B}_0$ and let $a \in \mathscr{R}$. Then for $b \in \mathscr{R}$ with $a \leqslant b^2$ we get

$$\|L_1 L_2 a\| \leqslant \|L_1 b\| \|L_2 b\| \leqslant 1.$$

Hence $\mathscr{B}_0^2 \subset \mathscr{B}_0$. Moreover for all $L \in \mathscr{B}_0$ and $a \in \mathscr{R}$

$$\|L^+ a\| = \|La\| \leqslant 1$$

and therefore $\mathscr{B}_0^+ = \mathscr{B}_0$.

Now suppose $\mathscr{B}_0$ were not maximal in **B**. It would mean that there exists $\mathscr{B}\in\mathbf{B}$ and $L\in\mathscr{B}\backslash\mathscr{B}_0$, i.e. $\|La\|>1$ for some $a\in\mathscr{R}$. Since also $L^+L\in\mathscr{B}$ we get

$$\|L^+La\|\geqslant\|L^+La^2\|=\|La\|^2>1.$$

The sequence $((L^+L)^{2^n})_{n\in N}$ is not bounded because

$$p_a((L^+L)^{2^n})=\|(L^+L)^{2^n}a\|\geqslant\|L^+La\|^{2^n}\uparrow\infty.$$

However, this sequence is contained in $\mathscr{B}$, which yields a contradiction. Hence $\mathscr{B}_0$ is maximal. ∎

Any von Neumann algebra of bounded linear operators on a separable Hilbert space is a C^*-algebra which is monotonously sequentially closed (cf. [Pe], [Sa]). Here we get a similar result for the commutative GB*-algebra $\mathscr{R}^{cc}$.

(3.11) PROPOSITION. (i) *Let $(L_\alpha)_{\alpha\in I}$ be an increasing net of positive operators in $\mathscr{R}^{cc}$ with the property that the net $(L_\alpha a)_{\alpha\in I}$ is bounded in $\mathscr{L}(\mathscr{H})$ for each $a\in\mathscr{R}$. Then there exists $L\in\mathscr{R}^{cc}$ such that $(L_\alpha a)$ tends strongly to La in $\mathscr{L}(\mathscr{H})$ for each $a\in\mathscr{R}$.*

(ii) *Let the family $\mathscr{R}$ have the additional property that there exists a sequence $(a_n)_{n\in N}$ in $\mathscr{R}$ such that $\bigvee_{n\in N} r(a_n)=\mathbf{1}$. Then for each $L\in\mathscr{R}^{cc}$, $L\geqslant 0$, there exists a sequence $(L_N)_{N\in N}$ in $\mathscr{R}_0^{cc}$ which is monotonously increasing, and for which the sequence $(L_N a)_{N\in N}$ tends strongly to La in $\mathscr{L}(\mathscr{H})$ for each $a\in\mathscr{R}$.*

Proof. (i) Let $L_a\in\mathscr{L}(\mathscr{H})$ be the strong limit of the net $(L_\alpha a)_{\alpha\in I}$. Then $L_a\in\mathscr{R}_0^{cc}$. Define L on $\mathscr{S}_{\mathscr{R}}$ by

$$Ls=L_b\tilde{s}$$

where $s,\tilde{s}\in\mathscr{S}_{\mathscr{R}}$ and $b\in\mathscr{R}$ with $s=b\tilde{s}$. In a standard way it can be shown that L is well defined, and satisfies

$$LL'a=L_aL'=L'L_a=L'La,\qquad a\in\mathscr{R},\ L'\in\mathscr{R}^c.$$

Hence $L\in\mathscr{R}^{cc}$.

(ii) Let $L\in\mathscr{R}^{cc}$, and let $(a_n)_{n\in N}$ be a sequence in $\mathscr{R}$ with $\bigvee_{n\in N} r(a_n)=\mathbf{1}$. Then we have

$$\forall_{N\in N}\ \exists_{b_N\in\mathscr{R}}\ \forall_{n\leqslant N}:\ b_N\geqslant a_n\quad\text{and}\quad b_N\geqslant b_{N-1}.$$

Moreover, $r(b_N)\geqslant\bigvee_{n\leqslant N} r(a_n)$ and

$$b_N^{2^{-m}}\to r(b_N)\quad\text{strongly as } m\to\infty.$$

So the sequence $(b_N^{2^{-N}})_{N\in N}$ tends strongly and monotonously to $\mathbf{1}$ as $N\to\infty$. Put $L_N=Lb_N^{2^{-N}}$, $N\in N$. Then $L_N\in\mathscr{R}_0^{cc}$ and $L_N\geqslant 0$, $N\in N$. Further,

$$L_Na=Lab_N^{2^{-N}}\to La\quad\text{strongly as } N\to\infty.\ ∎$$

4. Functional calculus. Let us consider the bounded part of $\mathscr{R}^{cc}$. By Lemma (3.9), $\mathscr{R}_0^{cc}$ consists of bounded operators. Moreover, it follows that $\mathscr{R}_0^{cc}$ is an abelian C^*-algebra of operators in $\mathscr{H}$, where the norm is the usual operator norm in $\mathscr{L}(\mathscr{H})$. Let Λ be the spectrum of the C^*-algebra $\mathscr{R}_0^{cc}$. Then $\mathscr{R}_0^{cc}$ is isometrically $*$-isomorphic to the C^*-algebra of continuous functions on Λ, i.e. $\mathscr{R}_0^{cc}\cong\mathscr{C}(\Lambda)$. Since $\mathbf{1}\in\mathscr{R}_0^{cc}$ the topological space Λ is compact.

Let the map $\mathscr{R}_0^{cc}\ni a\mapsto\hat{a}\in\mathscr{C}(\Lambda)$ be the Gelfand transform. As we mentioned before (Theorem (0.7)) Allan's theory of GB*-algebras introduces a generalized version of the Gelfand–Naimark theorem. Namely, the extended Gelfand mapping is a homeomorphism into a set of C^*-valued functions. Here we characterize a $*$-algebra of C^*-valued functions such that $\mathscr{R}^{cc}$ is (globally) $*$-isomorphic to this function $*$-algebra.

(4.1) THEOREM. *The GB*-algebra $\mathscr{R}^{cc}$ is $*$-isomorphic to the $*$-algebra of functions on the spectrum Λ of $\mathscr{R}_0^{cc}$ defined by*

$$\mathscr{R}^{\ddagger}(\Lambda):=\{f\in\mathscr{C}^*(\Lambda)\mid\ \forall_{a\in\mathscr{R}}:\ \sup_{\lambda\in\Lambda}|\hat{a}(\lambda)f(\lambda)|\to\infty\},$$

where $\mathscr{C}^*(\Lambda)$ is the $*$-algebra of continuous C^*-valued functions on Λ (cf. Definition (0.6)) *and $a\mapsto\hat{a}$ is the Gelfand transform from $\mathscr{R}_0^{cc}$ onto $\mathscr{C}(\Lambda)$.*

Proof. Let $L\in\mathscr{R}^{cc}$. Then for each $a\in\mathscr{R}$ the operator $L_a:=La$ belongs to $\mathscr{R}_0^{cc}\subset\mathscr{L}(\mathscr{H})$. There exists a number $c_a\geqslant 0$ such that $L_a^*La=(L^*a)La=L^+La^2\leqslant c_a^2\cdot\mathbf{1}_{\mathscr{H}}$. So we have $\|La\|=\sup_{\lambda\in\Lambda}|\hat{L}_a(\lambda)|\leqslant c_a$. Following the result of Allan (cf. Theorem (0.7)) we can represent any element L of the abelian GB*-algebra $\mathscr{R}^{cc}$ by a C^*-valued function $\hat{L}$ on Λ. The mapping $L\mapsto\hat{L}$ is the extended Gelfand transform introduced by Theorem (0.7). It has the property that $(L\cdot a)\hat{}\,(\lambda)=\hat{L}(\lambda)\cdot\hat{a}(\lambda)$, $\lambda\in\Lambda$. Hence $\hat{L}\in\mathscr{R}^{\ddagger}(\Lambda)$.

Let $f\in\mathscr{R}^{\ddagger}(\Lambda)$. Then for all $a\in\mathscr{R}$, f_a: $\lambda\mapsto f(\lambda)\hat{a}(\lambda)$ belongs to $\mathscr{C}(\Lambda)$. For each $a\in\mathscr{R}$, let L_a denote the element of $\mathscr{R}_0^{cc}$ corresponding to f_a. Then for all $a,b\in\mathscr{R}$, $L_{ab}=L_ab=bL_a$. We define L on $\mathscr{S}_{\mathscr{R}}$ by

$$Lw=L_ax$$

where $w=ax\in\mathscr{S}_{\mathscr{R}}$. Then L is well defined because for $w=ax=by$ we have

$$L_ax=b^{-1}L_baa^{-1}by=b^{-1}L_{b^2}y=L_by$$

i.e. $Lw=Lax=Lby$.

Now let $L'\in\mathscr{R}^c$, $s\in\mathscr{S}_{\mathscr{R}}$. By Lemma (1.4) we have $s=b^2x$ for some $b\in\mathscr{R}$, $x\in\mathscr{H}$. Then $LL's=LbL'bx$. By the construction we have $L_b=Lb\in\mathscr{R}_0^{cc}$ and hence $LbL'bx=L'Lb^2x=L'Ls$. Hence $L\in\mathscr{R}^{cc}$. To see that L is unique, use the bijectivity of the usual Gelfand transform on $\mathscr{R}_0^{cc}$. ∎

(4.2) Remark. *The C^*-algebra $\mathscr{R}_0^{cc}$ is the von Neumann algebra generated by the family $\mathscr{R}$ and the identity, i.e. $\mathscr{R}_0^{cc}=\mathscr{R}''$.*

Proof. Let $L\in\mathscr{R}^c$ and $L'\in\mathscr{R}''\subset\mathscr{R}'$. Then for any $s\in\mathscr{S}_{\mathscr{R}}$ we have s

$= ax$ and $L'Ls = L'Lax = LaL'x = LL'ax = LL's$ since $La \in \mathcal{R}_0^c \subset \mathcal{R}'$. Hence $L' \in \mathcal{R}^{cc} \cap \mathcal{L}(\mathcal{H})$ and by Lemma (3.9), $L' \in \mathcal{R}_0^{cc}$. So $\mathcal{R}'' \subset \mathcal{R}_0^{cc}$. On the other hand we have $\mathcal{R}_0^{cc} \subset \mathcal{L}(\mathcal{H})$, because all elements of $\mathcal{R}^{cc}$ are normal. Since $\mathcal{R}' \subset \mathcal{R}^c$, we have for each $L \in \mathcal{R}_0^{cc}, L' \in \mathcal{R}', s \in \mathcal{S}_{\mathcal{R}}$: $LL's = L'Ls$, and, by continuity, $LL' = L'L$ on $\mathcal{H}$. Hence $\mathcal{R}_0^{cc} \subset \mathcal{R}''$.

5. Topological considerations for $\mathcal{S}_{\mathcal{R}}$. In this section we continue the topological investigations of the space $\mathcal{S}_{\mathcal{R}}$ as mentioned in Section 1.

At first we characterize the topology of a general inductive limit $\mathcal{S}_I$ of a family of l.c. topological vector spaces $\{X_\alpha\}_{\alpha \in I}$, recalling the classical result.

(5.1) Lemma. *Let $\mathcal{S}_I$ be an inductive limit of a family $\{X_\alpha\}_{\alpha \in I}$ of l.c. topological vector spaces. Let $\pi_\alpha\colon X_\alpha \to \mathcal{S}_I$ be the canonical embedding. Then a set $O \subset \mathcal{S}_I$ is open in $\mathcal{S}_I$ if for each $\alpha \in I$ the set $\pi_\alpha^{-1}(O)$ is open in X_α.*

Now we introduce a family of seminorms on $\mathcal{S}_{\mathcal{R}}$ that under additional assumptions imposed on $\mathcal{R}$ gives rise to a topology equivalent to the inductive limit topology τ_{ind} in $\mathcal{S}_{\mathcal{R}}$. The family of seminorms is defined by

$$\text{(5.2)} \qquad \mathcal{S}_{\mathcal{R}} \ni s \mapsto \|Ls\|, \quad \text{where } L \in \mathcal{R}^{cc}.$$

It is obvious that these seminorms are continuous with respect to the topology τ_{ind} in $\mathcal{S}_{\mathcal{R}}$ in virtue of the general theory of inductive limit spaces. In particular it follows that the embedding $\mathcal{S}_{\mathcal{R}} \subset \mathcal{H}$ is continuous and hence that the space $\mathcal{S}_{\mathcal{R}}$ is Hausdorff.

Let us denote by τ_c the l.c. topology generated on $\mathcal{S}_{\mathcal{R}}$ by the seminorms (5.2). From the continuity of the seminorms (5.2) it follows that $\tau_{\text{ind}} > \tau_c$.

Let us consider the following condition that may be imposed on a generating family of operators $\mathcal{R}$:

(5.3) Conditions. There exists in $\mathcal{W}^*(\mathcal{R})$ a sequence of mutually orthogonal projections $\{P_n\}_{n \in N}$ such that $\sum_{n=1}^{\infty} P_n = 1$, and

$$\text{(I)} \qquad \forall_{n \in N}\ \exists_{a \in \mathcal{R}}\ \exists_{C_1 > 0}\colon \quad P_n r(a) \leqslant C_1 a.$$

$$\text{(II)} \qquad \forall_{a \in \mathcal{R}}\ \exists_{b \in \mathcal{R}}\ \exists_{C_2 > 0}\ \forall_{n \in N}\colon \quad n^2 \|aP_n\| \leqslant C_2 \inf_{\|y\|=1} \|bP_n y\|.$$

We have the following result:

(5.4) Theorem. *Let $\mathcal{R}$ be a generating family of operators in $\mathcal{H}$ which has the properties (5.3). Then the inductive limit topology τ_{ind} on $\mathcal{S}_{\mathcal{R}}$ is equivalent to the l.c. topology τ_c* (cf. (5.2)).

The proof of this theorem is based on the following crucial lemma:

(5.5) Lemma. *Let O be a convex set in $\mathcal{S}_{\mathcal{R}}$ with the property that $O \cap a\mathcal{H}$ contains an open neighbourhood of 0 in the Hilbert space $a\mathcal{H}$ for each $a \in \mathcal{R}$. Suppose that the conditions (I), (II) of (5.3) are fulfilled. Then there exists $L \in \mathcal{R}^{cc}$ such that $V = \{s \in \mathcal{S}_{\mathcal{R}} \mid \|Ls\| < 1\} \subset O$.*

Proof. For each $n \in N$, put

$$r_n = \sup\{\varrho \in R^+ \mid P_n K(0, \varrho) \subset O\},$$

where $K(0, \varrho) = \{x \in \mathcal{H} \mid \|x\| \leqslant \varrho\}$. Because of Condition (I) and Lemma (5.1) the numbers r_n are well defined and nonzero. Let us define the following unbounded operator in $\mathcal{H}$:

$$L := 2 \sum_{n=1}^{\infty} \frac{n^2}{r_n} P_n.$$

We prove that L is well defined on the dense set $\mathcal{S}_{\mathcal{R}}$ and that it is $\mathcal{R}$-bounded (cf. Definition (2.1)). So let us take $a \in \mathcal{R}$ and choose $b \in \mathcal{R}$ such that (5.3.II) holds. Notice that there exists $\varepsilon > 0$ such that $\{u \in b\mathcal{H} \mid \|u\|_b < \varepsilon\} \subset O \cap b\mathcal{H}$. It is easy to see that for each $n \in N$

$$r_n \geqslant \varepsilon \inf_{\|y\|=1} \|bP_n y\|.$$

Then we have

$$La = 2 \sum_{n=1}^{\infty} \frac{n^2}{r_n} P_n a \leqslant 2 \frac{C_2}{\varepsilon} \sum_{n=1}^{\infty} \frac{1}{\|P_n a\|} P_n a$$

$$\leqslant 2 \frac{C_2}{\varepsilon} \sum_{n=1}^{\infty} P_n = 2 \frac{C_2}{\varepsilon} \cdot \mathbf{1}.$$

Hence $\|La\| < \infty$ for each $a \in \mathcal{R}$ and L is densely defined since $\mathcal{S}_{\mathcal{R}} \subset \mathcal{D}(L)$. Since $P_n \in \mathcal{W}^*(\mathcal{R})$, $L \in \mathcal{R}^{cc}$ (cf. Proposition (3.11)). We will show that

$$V = \{u \in \mathcal{S}_{\mathcal{R}} \mid \|Lu\| < 1\} \subset O.$$

Let $u \in V$, with $u \in a\mathcal{H}$, $a \in \mathcal{R}$. Then $2n^2 P_n u \in O$ since $\|2n^2 P_n u\| < r_n$. The following decomposition holds for each $N \in N$:

$$(*) \qquad u = \sum_{n=1}^{N} \frac{1}{2n^2} 2n^2 P_n u + \Big(\sum_{n=N+1}^{\infty} \frac{1}{2n^2}\Big) u_N$$

where

$$u_N = \Big(\sum_{n=N+1}^{\infty} \frac{1}{2n^2}\Big)^{-1} \sum_{n=N+1}^{\infty} P_n u.$$

The first term in the above convex combination $(*)$ belongs to O for every $N \in N$.

In virtue of (5.3.II) there exists $b \in \mathcal{R}$ such that

$$\forall_{n \in N}\colon \ n^2 \|P_n a\| \leqslant C_2 \inf_{\|y\|=1} \|P_n b y\|.$$

Hence we have for u_N:

$$\|u_N\|_b^2 = \Big(\sum_{n=N+1}^{\infty} \frac{1}{2n^2}\Big)^{-2} \sum_{n=N+1}^{\infty} \|P_n u\|_b^2$$
$$\leqslant 4N^4 C_2 \sum_{n=N+1}^{\infty} \frac{1}{n^4} \|P_n u\|_a^2 \leqslant 4C_2 \sum_{n=N+1}^{\infty} \|P_n u\|_a^2 \to 0$$
$$\text{as } N \to \infty.$$

Since $O \cap b\mathscr{H}$ contains an open neighbourhood of 0 in $b\mathscr{H}$ we have $u_N \in O \cap b\mathscr{H} \subset O$ for sufficiently large $N \in \boldsymbol{N}$. Since O is convex we have $u \in O$, i.e. $V \subset O$. ∎

Now we discuss a characterization of continuous linear maps in $\mathscr{S}_{\mathscr{R}}$ in algebraic terms (automatic continuity):

(5.6) PROPOSITION. *Let* $L\colon \mathscr{S}_{\mathscr{R}} \to \mathscr{S}_{\mathscr{R}}$ *be a linear map which is* $\mathscr{R}$*-bounded. Assume that for each* $b \in \mathscr{R}$ *there exists* $b' \in \mathscr{R}$ *such that* $Lb = b'L$. *Then* L *is a continuous linear mapping from* $\mathscr{S}_{\mathscr{R}}$ *into* $\mathscr{S}_{\mathscr{R}}$.

Proof. Obviously, it is sufficient to show that for any $a \in \mathscr{R}$ the map $La\colon \mathscr{H} \to \mathscr{S}_{\mathscr{R}}$ is continuous. Let $\{x_n\}_{n\in\boldsymbol{N}}$ be a null sequence in $\mathscr{H}$. Then $Lax_n = Lb^2\tilde{x}_n = b'Lb\tilde{x}_n$, where $b, b' \in \mathscr{R}$, $a^{1/2} \leqslant b$, $\tilde{x}_n = b^{-2}ax_n$. By Lemma (1.2), $\tilde{x}_n \to 0$ in $\mathscr{H}$ as $n \to \infty$ so $Lax_n \to 0$ in $b'\mathscr{H}$, hence in $\mathscr{S}_{\mathscr{R}}$, as $n \to \infty$. ∎

In analogy to our notation used in [EK] we define:

(5.7) $$\mathscr{R}^{\ddagger\ddagger} := \{L \in \mathscr{W}^*(\mathscr{R}) \mid \forall_{L' \in \mathscr{R}^{cc}}\colon L'L \in \mathscr{W}^*(\mathscr{R})\}.$$

Consider the following condition:

(III) $$\forall_{R \in \mathscr{R}^{\ddagger\ddagger}}\ \exists_{b \in \mathscr{R}}\ \exists_{c>0}\colon \quad R^*R \leqslant Cb^2.$$

If we impose Conditions (I), (II) and (III) on $\mathscr{R}$, then $\mathscr{S}_{\mathscr{R}}$ is a projective limit of Hilbert spaces. First, we prove the following result which states that bounded subsets of the (nonstrict) inductive limit $\mathscr{S}_{\mathscr{R}}$ admit the same characterization as if $\mathscr{S}_{\mathscr{R}}$ were a strict inductive limit of Hilbert spaces.

(5.8) LEMMA. *Let* $\mathscr{R}$ *be a generating family of operators fulfilling Conditions* (5.3) (I), (II) *and* (III). *Then a set* $\mathscr{B} \subset \mathscr{S}_{\mathscr{R}}$ *is bounded if and only if there exists* $b \in \mathscr{R}$ *such that* $\mathscr{B}$ *is a bounded subset of the Hilbert space* $b\mathscr{H}$.

Proof. Assume that $\mathscr{B} \subset \mathscr{S}_{\mathscr{R}}$ is bounded. In virtue of Theorem (5.4) for each $L \in \mathscr{R}^{cc}$ there exists a constant $K_L > 0$ such that $\sup_{s\in\mathscr{B}} \|Ls\| \leqslant K_L$.

Since $P_n \in \mathscr{R}^{cc}$ for each $n \in \boldsymbol{N}$ we have $q_n := \sup_{s\in\mathscr{B}} \|P_n s\| < \infty$. Put

(5.9) $$R := \sum_{n=1}^{\infty} nq_n P_n.$$

We will show that $R \in \mathscr{R}^{\ddagger\ddagger}$ (cf. (5.7)). Let $L \in \mathscr{R}^{cc}$. Then we prove that $\tilde{L} := \sum_{n=1}^{\infty} n\|P_n L\| P_n$ belongs to $\mathscr{R}^{cc}$. It is enough to show that $\tilde{L}$ is $\mathscr{R}$-bounded (cf. Proposition (3.11)). Let $a \in \mathscr{R}$. Then taking b as indicated in (5.3) we have

$$\|\tilde{L}a\| = \sup_{n\in\boldsymbol{N}} \|\tilde{L}aP_n\| = \sup_{n\in\boldsymbol{N}} n\|P_nL\|\,\|P_na\| \leqslant \sup_{n\in\boldsymbol{N}} \big(\|P_nL\| \inf_{\|y\|=1} \|bP_ny\|\big)$$
$$= \sup_{n\in\boldsymbol{N}, P_nr(b)\neq 0} \Big(\sup_{\xi\in P_n\mathscr{H}, \|\xi\|=1} \frac{\|P_nLr(b)b\xi\|}{\|P_nr(b)b\xi\|} \inf_{\|y\|=1} \|P_nr(b)by\|\Big)$$
$$\leqslant \sup_{n\in\boldsymbol{N}, P_nr(b)\neq 0} \big(\sup_{\|\xi\|=1} \|P_nLb\xi\|\big) \leqslant \|Lb\| < \infty.$$

Now we have the estimation $\|RL\| \leqslant \sup_{n\in\boldsymbol{N}} \sup_{s\in\mathscr{B}} \|\tilde{L}P_ns\| < K_{\tilde{L}}$, hence $R \in \mathscr{R}^{\ddagger\ddagger}$. By Condition (III) we can find $b \in \mathscr{R}$ such that $R^2 \leqslant Cb^2$. Thus $\mathscr{B} \subset R\mathscr{H} \subset b\mathscr{H}$. It follows that for each $s \in \mathscr{B}$

$$\|s\|_b^2 \leqslant C\|R^{-1}r(b)s\|^2 \leqslant C \sum_{n=1, q_n\neq 0}^{\infty} \frac{1}{n^2q_n^2} \|P_ns\|^2 \leqslant C\frac{\pi^2}{6}.$$

In this way we have proved that $\mathscr{B}$ is a bounded subset of $b\mathscr{H}$.

Assume now that $\mathscr{B} \subset \mathscr{S}_{\mathscr{R}}$ is a bounded subset of $b\mathscr{H}$, for some $b \in \mathscr{R}$. Let $L \in \mathscr{R}^{cc}$. Then

$$\sup_{s\in\mathscr{B}} \|Ls\| \leqslant \sup_{s\in\mathscr{B}} \|Lb\|\,\|s\|_b < \infty. \ ∎$$

Thus we arrive at the following result.

(5.10) THEOREM. (i) *Let the family* $\mathscr{R}$ *fulfil Conditions* (5.3) (I), (II). *Then the space* $\mathscr{S}_{\mathscr{R}}$ *is bornological and barrelled.*

(ii) *Let the family* $\mathscr{R}$ *fulfil Conditions* (5.3) (I), (II) *and* (III). *Then* $\mathscr{S}_{\mathscr{R}}$ *is complete and can be represented by* $\bigcap_{L\in\mathscr{R}^{cc}} \mathscr{D}(L)$ *as a projective limit of Hilbert spaces* $\mathscr{D}(L)$, *i.e. the maximal domain of the operator* $L \in \mathscr{R}^{cc}$ *endowed with the graph norm topology.*

Now we can formulate the statement complementary to Proposition (5.6).

(5.11) PROPOSITION. *Let* $\mathscr{R}$ *fulfil Conditions* (5.3) (I), (II) *and* (III). *Then a linear map* $L\colon \mathscr{S}_{\mathscr{R}} \to \mathscr{S}_{\mathscr{R}}$ *is continuous if and only if* L *is* $\mathscr{R}$*-bounded and for each* $b \in \mathscr{R}$ *there exists* $b' \in \mathscr{R}$ *such that* $b'^{-1}Lb$ *is bounded.*

We omit the proof which is a consequence of Lemma (5.8) and Theorem (5.10).

6. An example. In this section we consider the particular case that the family $\mathscr{R}$ is a collection of functions of operators, i.e.

$$\mathscr{R} = \{\varphi(A) \mid \varphi \in \Phi\}.$$

Here Φ denotes a family of Borel functions which will be defined below and $A=(A_1, \ldots, A_n)$ denotes a finite tuple of self-adjoint unbounded mutually strongly commuting operators in the Hilbert space $\mathscr{H}$.

The definition of the family Φ which is given in [EK] is somewhat more general than the one we give here.

Besides the family Φ we introduce the families of functions $\Phi^{\ddagger}$ and $\Phi^{\ddagger\ddagger}$:

(6.1) $$\Phi^{\ddagger}=\{f \mid f \text{ is a Borel function on } \boldsymbol{R}^n,\ \forall_{\varphi\in\Phi}: \sup_{\lambda\in\boldsymbol{R}^n}|f(\lambda)\varphi(\lambda)|<\infty\},$$

(6.2) $$\Phi^{\ddagger\ddagger}=\{\psi \mid \psi \text{ is a Borel function on } \boldsymbol{R}^n,\ \forall_{f\in\Phi^{\ddagger}}: \sup_{\lambda\in\boldsymbol{R}^n}|f(\lambda)\psi(\lambda)|<\infty\}.$$

(6.3) Remark. Observe that $(\Phi^{\ddagger})^{\ddagger}=\Phi^{\ddagger\ddagger}$ and $\Phi\subset\Phi^{\ddagger\ddagger}$.

On Φ we impose the following conditions:

(AI) Φ is directed by the usual order of real functions.

(AII) $\forall_{\varphi\in\Phi}$: $0\leqslant\varphi\leqslant 1$ and the function $\lambda\mapsto\varphi(\lambda)^{-1}\chi_{\underline{\varphi}}(\lambda)$ is bounded on bounded Borel sets, where $\underline{\varphi}=\{\lambda\in\boldsymbol{R}^n \mid \varphi(\lambda)\neq 0\}$.

(AIII) $\forall_{\varphi_1\in\Phi}\ \exists_{\varphi_2\in\Phi}$: $\varphi_1^{1/2}\leqslant\varphi_2$.

(AIV) $\forall_{\varphi_1\in\Phi}\ \forall_{\delta\in\boldsymbol{R}^n}\ \exists_{\varphi_2\in\Phi}\ \exists_{C>0}$: $\varphi_1(\lambda+\delta)\leqslant C\varphi_2(\lambda)\quad\forall_{\lambda\in\boldsymbol{R}^n}$.

(AV) $\forall_{\varphi_1\in\Phi}\ \exists_{\varphi_2\in\Phi}\ \exists_{C>0}$: $(1+|\lambda|^2)\varphi_1(\lambda)<C\varphi_2(\mu)$ if $\lambda, \mu\in Q_m$ where for each $m\in\boldsymbol{Z}^n$

$$Q_m=\{(\xi_1, \ldots, \xi_n)\in\boldsymbol{R}^n \mid m_j-1\leqslant\xi_j<m_j,\ j=1, \ldots, n\}.$$

(AVI) $\forall_{\psi\in\Phi^{\ddagger\ddagger}}\ \exists_{\varphi\in\Phi}\ \exists_{C>0}\ \forall_{\lambda\in\boldsymbol{R}^n}$: $|\psi(\lambda)|<C\varphi(\lambda)$ (symmetry condition).

(AVII) In Φ there exists a countable separating subset $\hat{\Phi}$ which has the property

$$\forall_{\varphi\in\Phi}\ \exists_{\hat{\varphi}\in\hat{\Phi}}\ \exists_{C>0}: \quad \varphi\leqslant C\hat{\varphi}.$$

The real algebra of real-valued bounded Borel functions $\mathscr{B}_r(\boldsymbol{R}^n)$ is the smallest Banach algebra which contains the B*-algebra generated by the family Φ, and which is closed under the operation of taking limits of uniformly bounded monotone sequences of its elements. Then $\mathscr{B}(\boldsymbol{R}^n)=\mathscr{B}_r(\boldsymbol{R}^n)+i\mathscr{B}_r(\boldsymbol{R}^n)$ is a Borel $*$-algebra of bounded complex-valued Borel functions on $\boldsymbol{R}^n$ (cf. [Pe], 4.5).

(6.4) PROPOSITION. *$\mathscr{B}(\boldsymbol{R}^n)$ contains all complex-valued bounded Borel functions on $\boldsymbol{R}^n$.*

Now we consider algebras of operators associated with the family Φ. We start with

$$\mathscr{R}=\Phi(A):=\{\varphi(A) \mid \varphi\in\Phi\}, \quad A=(A_1, \ldots, A_n).$$

Then $\mathscr{R}$ is a generating family of operators in the sense of Definition (1.1). Moreover, $\mathscr{R}$ satisfies Conditions (5.3) (I), (II) and (III). The family of projections $\{P_n\}_{n\in N}$ (cf. (5.3)) can be constructed as follows. Let E denote the joint spectral measure for the commuting system of operators $A_1, \ldots, A_n$. Let Q_m denote the cube as in (AV). Then we put $P_n:=E(Q_n)$.

(6.5) THEOREM. *The GB*-algebra $\mathscr{R}^{cc}$ as defined in Section 3 is equal to the set of operators*

$$\Phi^{\ddagger}(A):=\{f(A) \mid f\in\Phi^{\ddagger}\}, \quad \textit{where } A=(A_1, \ldots, A_n).$$

(6.6) THEOREM. *The inductive limit topology in the space $\mathscr{S}_{\Phi(A)}$ is generated by the family of seminorms*

$$\mathscr{S}_{\Phi(A)}\ni s\mapsto\|f(A)s\|_{\mathscr{H}}, \quad \textit{where } f\in\Phi^{\ddagger}.$$

A number of concrete examples of spaces of type $\mathscr{S}_{\Phi(A)}$ is included in our paper [EGK].

References

[Al1] G. R. Allan, *A spectral theory for locally convex algebras*, Proc. London Math. Soc. (3) 15 (1965), 399–421.

[Al2] —, *On a class of locally convex algebras*, ibid. 17 (1967), 91–114.

[Ar] W. Arveson, *An Invitation to C*-Algebras*, Springer, New York–Heidelberg–Berlin 1976.

[EGK] S. J. L. van Eijndhoven, J. de Graaf and P. Kruszyński, *Dual systems of inductive-projective limits of Hilbert spaces originating from self-adjoint operators*, Nederl. Akad. Wetensch. Proc. (1985), to appear.

[EK] S. J. L. van Eijndhoven and P. Kruszyński, *On Gelfand triples originating from algebras of unbounded operators*, EUT Report 84-WSK-02, Eindhoven University of Technology, 1984.

[Ep] G. Epifanio and C. Trapani, *V*-algebras: A particular class of unbounded operator algebras*, J. Math. Phys. 25 (9) (1984), 2633–2637.

[Pe] G. K. Pedersen, *C*-Algebras and Their Automorphism Groups*, Academic Press, London–New York–San Francisco 1979.

[Pij] H. G. J. Pijls, *Locally convex algebras in spectral theory and eigenfunction expansions*, Thesis, Mathematical Centre, Amsterdam 1975.

[Sa] Sh. Sakai, *C*-Algebras and W*-Algebras*, Springer, Berlin–Heidelberg–New York 1971.

DEPARTMENT OF MATHEMATICS AND COMPUTING SCIENCE
EINDHOVEN UNIVERSITY OF TECHNOLOGY
PO Box 513, 5600 MB Eindhoven, The Netherlands

Received April 29, 1985 (2052)