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The space Weak H^1

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Abstract. This article discusses the properties of some of the basic operators of harmonic analysis in relation to a space of functions whose nontangential maximal function belongs to the space Weak L^1.

Introduction. If we consider weak type inequalities on $L^1(\mathbf{R}^n)$ for some of the most important operators of harmonic analysis, we are led to a question of their sharpness as is illustrated as follows:

For $f \in L^1(\mathbf{R}^1)$ and for H the Hilbert transform of f, it is an extremely well known result that

$$m\{x \in \mathbf{R}^1 \mid |Hf(x)| > \alpha\} \leqslant \frac{C}{\alpha} \int_{\mathbf{R}^1} |f(x)|\, dx \quad \text{for all } \alpha > 0.$$

However, this result is not completely optimal, in the sense that in order for $Hf(x)$ to belong to the class Weak L^1, it is not necessary for f to belong to L^1. For instance, f could be a measure, $d\mu$, and then

$$m\{x \in \mathbf{R}^1 \mid |H\mu(x)| > \alpha\} \leqslant \frac{C}{\alpha} \int_{\mathbf{R}^1} d|\mu|.$$

In order to obtain the best result, we are led to the following definition: Suppose $\varphi(x)$ is a function in $C_c^\infty(\mathbf{R}^n)$ with $\int \varphi \neq 0$ and for $t > 0$, $\varphi_t(x) = t^{-n}\varphi(x/t)$. Define the maximal function $f^*(x) = \sup_{t>0} |f * \varphi_t(x)|$. Then we say that f belongs to "Weak H^1" provided the function f^* belongs to weak $L^1(\mathbf{R}^n)$, i.e., $m\{x \in \mathbf{R}^n \mid f^*(x) > \alpha\} \leqslant C/\alpha$ for all $\alpha > 0$. The smallest C which makes the preceding estimate valid is called the "Weak H^1 norm" (though it does not satisfy the triangle inequality), $\|f\|_{WH^1}$. Of course, since $f^*(x) \leqslant CMf(x)$ for all x, where M is the Hardy–Littlewood maximal function, and since M is of weak type 1–1, we have $\|f\|_{WH^1} \leqslant C\|f\|_{L^1}$, so that L^1 (and the space of complex measures) is continuously embedded as a subspace of Weak H^1. However, the L^1 functions are only a part of our space, and the basic example to think about when considering Weak H^1 is the distribution $1/x$ on $\mathbf{R}^1$. For $f(x) = 1/x$ a trivial computation shows that $f^*(1) < \infty$ whereupon, by dilation invariance, we have $f^*(x) = f^*(1)/|x|$

which is in Weak L^1. So indeed $1/x \in$ Weak H^1. To get a little more insight into things, we should point out that if a distribution f is nonnegative (i.e., when $\eta(x) \geqslant 0$ is a Schwartz function then $\langle f, \eta\rangle \geqslant 0$) and belongs to Weak H^1, then f must be a finite measure. Let us give a sketch of a proof of this fact as follows:

First, as we shall discuss in more detail later, it is of no importance which approximate identity φ_t we use in order to define the space Weak H^1. The class is independent of the choice of φ. If we choose a Gaussian, then if $f \geqslant 0$ is in Weak H^1,

$$(f * \varphi_t) * \varphi_s(x) = f * (\varphi_s * \varphi_t)(x) \leqslant f^*(x),$$

so that, as $t > 0$ varies, the C^∞ positive functions $f * \varphi_t$ are in Weak H^1 with uniformly bounded norms. Since for positive functions $g(x)$, $g^* \geqslant cM(g)$, we see that $M(f * \varphi_t)$ is uniformly in Weak L^1. As is well known (see Stein [8]), by the Calderón–Zygmund decomposition,

$$\frac{1}{m\{M(f*\varphi_t) > \alpha\}} \int_{\{M(f*\varphi_t)>\alpha\}} f * \varphi_t \leqslant C\alpha$$

or $\int_{\{M(f*\varphi_t)>\alpha\}} f * \varphi_t \leqslant C'$; taking the limit as $\alpha \to 0$ we get $\|f * \varphi_t\|_1 \leqslant C'$ for all $t > 0$. A weak* compactness argument now shows that f is a finite measure.

Just as in the case of $H^1(\boldsymbol{R}^n)$ we can characterize Weak H^1 in a number of different ways (see C. Fefferman–E. Stein [3]). Thus, if $S_\psi(f)$ denotes the area integral (with respect to a suitably nontrivial $\psi \in C_c^\infty$ with $\int \psi = 0$) then one can show that $f \in$ Weak H^1 if and only if $S_\psi(f) \in$ Weak L^1. In addition we may take the definition in terms of nontangential or grand maximal functions, and not just radial ones. For a suitably nice function $f(x)$ we also have the singular integral characterization

$$\|f\|_{WH^1} \sim \|f\|_{WL^1} + \sum_{j=1}^{n} \|R_j f\|_{WL^1}$$

where R_j denotes the jth Riesz transform, and where $a \sim b$ means a/b and b/a are bounded above by some constant independent of f.

Our results, after those mentioned above, split into two types. First, as suggested earlier, we wish to show that for many of the operators of classical Fourier analysis, the role of L^1 can just as well be played by Weak H^1. Consider, for example, the Fourier transform. Of course the Fourier transform $\hat{f}(\xi)$ of an L^1 function $f(x)$ on $\boldsymbol{R}^n$ is a bounded function. Unfortunately, even though $(1/x)\hat{}(\xi) = \operatorname{sgn}(\xi)$ is bounded, in general no such claim is valid for arbitrary Weak H^1 functions. Nevertheless, if we are a bit more subtle, we may find a correct estimate. Recall Paley's inequality for Fourier series in one dimension:

$$\Big(\sum_{n\neq 0} |\hat{f}(n)|^p |n|^{p-2}\Big)^{1/p} \leqslant C_p \|f\|_{L^p[0,2\pi]} \quad \text{for } 1 < p \leqslant 2.$$

For $p = 1$, we have the substitute weak type estimate from which the case for $p > 1$ may be derived by interpolation (see Zygmund [10]):

$$(*) \qquad \mu\{n \in \boldsymbol{Z} \mid |\hat{f}(n)|\,|n| > \alpha\} \leqslant \frac{C}{\alpha}\|f\|_{L^1[0,2\pi]},$$

where μ is the measure on $\boldsymbol{Z}$ which assigns mass $1/n^2$ to the point $n \neq 0$. In fact in $(*)$ we may replace $\|f\|_{L^1}$ by $\|f\|_{WH^1}$. This follows from the correct estimate on the Fourier transform of a Weak H^1 function which we now describe. (We switch now to the setting of $\boldsymbol{R}^n$.) We have, for a function in Weak $H^1(\boldsymbol{R}^n)$, if $B(0;r)$ denotes the ball of radius $r > 0$ centered at 0,

$$\frac{1}{m(B(0;r))} \int_{B(0;r)} \exp\big(c\,|\hat{f}(\xi)|/\|f\|_{WH^1}\big)\, d\xi \leqslant C.$$

Thus, although $\hat{f}(\xi)$ may fail to be bounded, its averages over all balls centered at the origin are bounded, and the exponential estimate is sharp, as will be seen below.

If we consider convolution operators $Tf = f * K$ of Calderón–Zygmund type, and ask for weak type inequalities then we consider estimates of the form

$$(**) \qquad m\{x \in \boldsymbol{R}^n \mid |Tf(x)| > \alpha\} \leqslant \frac{C}{\alpha}\|f\|_{WH^1};$$

where classically, of course, $(C/\alpha)\|f\|_{L^1}$ appears on the right-hand side.

In order to obtain such estimates for singular integrals we assume that T is bounded on $L^2(\boldsymbol{R}^n)$, and in addition we require the following Dini condition on K which is a bit more than the usual Hörmander condition.

Suppose, for $0 < \delta < 1$, we set

$$\Gamma(\delta) = \sup_{h\neq 0} \int_{|x| \geqslant \delta^{-1} 2|h|} |K(x+h) - K(x)|\, dx.$$

Then if $\int_0^1 \Gamma(\delta)\frac{d\delta}{\delta} < \infty$, the weak type inequality $(**)$ above is valid.

Both the results above follow from a decomposition theorem which is like the atomic decomposition of $H^1(\boldsymbol{R}^n)$. Here, however, the "atoms" of Weak H^1 are sums of functions supported on cubes and having mean value zero on these cubes, while the cubes themselves have bounded overlap.

In a different direction, we characterize the dual of the closed subspace of Weak H^1 given by the closure in the Weak H^1 norm of L_0^1, the class of L^1 functions with integral zero. We shall denote this subspace by $\overline{L_0^1} = \overline{L_0^1}^{WH^1}$. To motivate this, observe that since $1/x \in$ Weak H^1 and since Weak H^1 is translation invariant, if $\varphi(x)$ belongs to the dual of Weak H^1, then

$$\int_{\boldsymbol{R}^1} \varphi(x) \frac{1}{t-x}\, dx$$

is a bounded function of t. Although we do not characterize the dual of Weak H^1, it is natural from the previous argument to think of our dual space in connection with "real H^∞", i.e., those bounded functions whose Hilbert transform is bounded as well. Trivially (already in the early literature), it is those continuous functions satisfying a Dini condition whose Hilbert transform exists at each point and is bounded. So it is not surprising that we can characterize the functions in the dual of $\overline{L_0^1}$ in terms of a Dini condition. We begin by defining the proper notion of "oscillation of a function over an open set $\Omega \subset \boldsymbol{R}^m$". Now if we set, for $\delta > 0$.

$$\omega(\delta) = \sup_{m(\Omega)=\delta} (\text{oscillation of } \varphi \text{ over } \Omega)$$

then

$$\varphi \in (\overline{L_0^1})^* \quad \text{if and only if} \quad \int_0^\infty \omega(\delta)\frac{d\delta}{\delta} < \infty .$$

It should be noted, finally, that the space Weak H^1 has appeared before in the literature. The most notable example is the work of C. Fefferman, N. Rivière, and Y. Sagher [2], where this space arises through the real method of interpolation as an intermediate space. The results there on interpolation are at the same time a precursor of atomic decompositions, such as ours, and at the same time can be used to obtain some of our results here by interpolating between H^p for $p < 1$, and L^2. More recently, Aleksandrov has also studied some analytic and functional properties of H^p spaces of weak type, including Weak H^1. For his work, we refer the reader to [1].

Throughout this paper, the letters $C, c, C', c', \ldots$ will often denote a constant independent of the main parameters involved and whose value may change from one place to another.

Also, we shall repeatedly use the expression "bounded overlap" when dealing with a collection of subsets (generally cubes) of $\boldsymbol{R}^n$. This will simply mean that there exists a prefixed universal constant $C = C_n \in N$, which depends only on the dimension, so that every point in $\boldsymbol{R}^n$ belongs to no more than C_n elements of that collection. We will sometimes write "bounded C_n-overlap" to specify the role of C_n. As the reader will immediately see, the arguments involved are independent of the exact value of this constant.

1. Definitions, notation, and some preliminaries. Given a bump function φ with $\varphi \in C_c^\infty(\boldsymbol{R}^n)$, and $\int_{\boldsymbol{R}^n} \varphi \neq 0$, we can define, for a distribution f, the following maximal operators:

$$f^+(x) = f_\varphi^+(x) = \sup_{t>0} |f * \varphi_t(x)|;$$

$$f_\alpha^*(x) = f_{\alpha,\varphi}^*(x) = \sup_{|x-y|<\alpha t} |f * \varphi_t(y)|, \qquad \alpha > 0;$$

$$f^{**}(x) = f_\varphi^{**}(x) = \sup_{(y,t)\in \boldsymbol{R}_+^{n+1}} |f * \varphi_t(y)| \left(\frac{t}{|x-y|+t}\right)^N,$$

where $N > n$ and $\varphi_t(x) = t^{-n}\varphi(x/t)$.

We also define the grand maximal function of f as

$$Gf(x) = \sup_{\psi \in \mathscr{A}} \sup_{|x-y|<t} |f * \psi_t(x)|,$$

where

$$\mathscr{A} = \{\psi \in \mathscr{S} \mid \int_{\boldsymbol{R}^n} (1+|x|)^N \big(\sum_{|\alpha|\leqslant N} |\partial^\alpha \psi(x)/\partial x^\alpha|^2\big)\, dx \leqslant 1\}.$$

For $0 < p < \infty$ and $1 \leqslant q \leqslant \infty$ the Hardy space $H(p, q)$ is defined as the collection of all tempered distributions f such that f^+ belongs to the Lorentz class $L(p, q)$. Due to the fundamental work of C. Fefferman–E. M. Stein [3], we know that $H^p = H(p, p)$ can be characterized in terms of the other three maximal operators, f_α^*, f^{**}, and Gf, though we must increase the integer N as p decreases to 0. It is also known that the definition of H^p is independent of our choice of φ.

In this article we shall consider the special case $H(1, \infty)$ which we shall call "Weak H^1". The "norm" will be denoted $\|\cdot\|_{WH^1}$. In fact the same characterizations via maximal functions are valid for Weak H^1: According to the work of C. Fefferman–Rivière–Sagher [2], the spaces $H(p, q)$ occur as intermediate spaces in the real method of interpolation between the H^p spaces. Combining this fact with the equivalence of the different maximal function definitions of the H^p spaces we conclude immediately that

$$\|f^+\|_{L(p,q)} \sim \|f_1^*\|_{L(p,q)} \sim \|f^{**}\|_{L(p,q)} \sim \|Gf\|_{L(p,q)}$$

for all p and q.

It is interesting to notice that $H(p, q)$ also arises as the class of "boundary values" of functions $u(x, t)$, harmonic in the upper half-space $\boldsymbol{R}_+^{n+1}$, satisfying $u^+ \in L(p, q)$ or $u_\alpha^* \in L(p, q)$ or $u^{**} \in L(p, q)$ where u^+, u_α^* and u^{**} are the so-called radial, nontangential, and tangential maximal functions associated to u, respectively. That is,

$$u^+(x) = \sup_{t>0} |u(x, t)|;$$

$$u_\alpha^*(x) = \sup_{|x-y|<\alpha t} |u(y, t)|;$$

$$u^{**}(x) = \sup_{(y,t)\in \boldsymbol{R}_+^{n+1}} |u(y, t)| \left(\frac{t}{|x-y|+t}\right)^N.$$

This comes immediately from the fact that if $f \in H(p, q)$ then it is possible to define its Poisson transform

$$u(x, t) = f * P_t(x),$$

where

$$P(x) = c_n(1+|x|^2)^{-(n+1)/2}.$$

Moreover, by [3], we have the pointwise estimate $u^+(x) \leqslant CG(f)(x)$ (see [3; Th. 11]).

Conversely, if $u(x, t)$ is harmonic and $u^+(x) \in L(p, q)$ then $\lim_{t\to 0} u(x, t) = f$ exists in the sense of distributions and

$$m\{x|\ u_\alpha^*(x) > s\} \leqslant C\alpha m\{x|\ u_1^*(x) > s\}.$$

Summarizing,

$$\|f\|_{H(p,q)} \sim \|u^+\|_{L(p,q)} \sim \|u_1^*\|_{L(p,q)} \sim \|u^{**}\|_{L(p,q)}.$$

As an application of all of this, let us mention a result of Steven Hudson which appears in [4]. This result is related to the most obvious of all Sobolev embedding theorems, namely: If f has one derivative in $L^1(\boldsymbol{R}^1)$, then $f \in L^\infty(\boldsymbol{R}^1)$. What happens when we replace L^∞ by BMO? Hudson's theorem answers this. His result says that if $f \in C_c^\infty(\boldsymbol{R}^1)$ and if

$$F(x) = \int_{-\infty}^{x} f(t)\,dt, \quad \text{then} \quad \|F\|_{\mathrm{BMO}} \leqslant C\|f\|_{WH^1}.$$

The generalization to $\boldsymbol{R}^n$ can be stated as follows:

THEOREM (Hudson). *Let $\Omega(x)$ be a function on $\boldsymbol{R}^n$ which is homogeneous of degree* 0 *and C^∞ away from the origin. Let $f \in C_c^\infty(\boldsymbol{R}^n)$. Then*

$$\|\Omega * f\|_{\mathrm{BMO}(\boldsymbol{R}^n)} \leqslant C\|f\|_{WH^1}.$$

Sketch of the proof. Take $\|f\|_{WH^1} = 1$. Then to show $\|f * \Omega\|_{\mathrm{BMO}} \leqslant C$ it is enough to show that for $a(x)$ an H^1 atom on a cube Q we have

$$\Big|\int_{\boldsymbol{R}^n} (f * \Omega)\,a\Big| \leqslant C.$$

But this is equivalent to

$$\Big|\int_{\boldsymbol{R}^n} f(\Omega * a)\Big| \leqslant C.$$

A trivial computation shows that $\psi(x) = \dfrac{\Omega * a}{|Q|}(x)$ is (if a has sufficiently vanishing moments) a bump function satisfying all the necessary estimates so that

$$\Big|\int_{\boldsymbol{R}^n} f(t)\psi(t)\,dt\Big| \leqslant Gf(x) \quad \text{for all } x \in Q.$$

Then we have

$$G(f)(x) > \frac{1}{|Q|}\Big|\int_{\boldsymbol{R}^n} f(\Omega * a)\Big| \quad \text{on } Q.$$

Since $G(f) \in \text{Weak } L^1$ we have

$$|Q| \leqslant \frac{C|Q|}{|\int f(\Omega * a)|}$$

so that indeed $|\int f(\Omega * a)| \leqslant C$.

Finally, various equivalent definitions can be given in terms of Littlewood–Paley functions. For example, if $f \in \text{Weak } H^1$, then, by interpolation, and by the fact that if $g(x) \in H^p$ then $S(g)(x) \in L^p$ (here

$$S^2(g)(x) = \iint_{\Gamma(x)} |\nabla u|^2(y, t)\,y^{1-n}\,dt\,dy$$

where u is the Poisson integral of g) we see that $S(f) \in \text{Weak } L^1$. Conversely, if $S(f) \in \text{Weak } L^1$ then according to [3],

$$m\{u^*(x) > \beta\} \leqslant Cm\{S(u)(x) > \beta\} + \frac{1}{\beta^2}\int_{S(u) \leqslant \beta} S^2(u)(x)\,dx$$
$$\leqslant \frac{C'}{\beta} + \frac{1}{\beta^2}\int_0^\beta \gamma m\{S(u) > \gamma\}\,d\gamma$$
$$\leqslant C'\left[\frac{1}{\beta} + \frac{1}{\beta^2}\int_0^\beta d\gamma\right] \leqslant \frac{C''}{\beta},$$

and so $f \in \text{Weak } H^1$. (Here u is the Poisson integral of f, so that $S(u)$ means the same as $S(f)$.)

2. The atomic decomposition. In this section we shall decompose a distribution f belonging to Weak H^1 as a sum of functions in $L^\infty \cap L_0^1 = \{g \in L^\infty \cap L^1|\ \int g = 0\}$ each with support in an open set of finite measure whose measure does not exceed the reciprocal of its L^∞ norm. We shall see in the sections which follow that this decomposition has several interesting applications. We also present a converse statement which will be needed in connection with the characterization of the dual of the Weak H^1 closure of L_0^1.

Before starting to discuss the decomposition, let us make some observations and set some notation. To begin with, fix $0 < p_0 < 1$. Then we set $H^{p_0} + L^2 = \{f|\ f$ a distribution which can be written as $f = g + h$ where

$g \in H^{p_0}$ and $h \in L^2\}$. Set $\|f\|_{H^{p_0}+L^2}$ to be $\inf_{f=g+h} (\|g\|_{H^{p_0}}+\|h\|_{L^2})$. Unfortunately, as the reader will clearly observe, the smooth functions are not dense in Weak H^1. This is why we introduce the norm in $H^{p_0}+L^2$.

PROPOSITION. *Given $f \in$ Weak H^1, there exists a sequence of bounded functions $\{f_k\}_{k=-\infty}^{+\infty}$ with the following properties:*

(a) $f - \sum_{|k| \leqslant N} f_k \to 0$ *in the sense of distributions (in fact even in the norm of* $H^{p_0}+L^2$).

(b) *Each f_k may be further decomposed as $f_k = \sum_{i=1}^{\infty} \beta_{ki}$ in L^1, where the β_{ki} satisfy:*

(i) *β_{ki} is supported in a cube Q_{ki} with $\{Q_{ki}\}_i$ having bounded overlap for each k.*

(ii) $\int_{Q_{ki}} \beta_{ki} = 0$.

(iii) $\|\beta_{ki}\|_{L^\infty} \leqslant C2^k$ *and* $\sum_i m(Q_{ki}) \leqslant C_1 2^{-k}$.

Moreover, C_1 is (up to multiplication by an absolute constant) less than the Weak H^1 *norm of f.*

Conversely, if f is a distribution satisfying (a) *and* (b) (i)–(iii), *then $f \in$ Weak H^1 and $\|f\|_{WH^1} \leqslant cC_1$ (where c is some absolute constant).*

Proof. The proof that if $f \in$ Weak H^1 then f can be decomposed as above is a small perturbation of the argument in Latter [6] for H^p functions. For k an integer we set $\Omega_k = \{G(f) > 2^k\}$, and let $\{Q_j^k\}$ be a Whitney decomposition of Ω_k. Following [6], we let φ_j^k be a bump function supported in the double of Q_j^k obtained by translation and dilation of a standard bump function φ. We let

$$m_j^k = \frac{1}{\int \varphi_j^k} \int f \varphi_j^k$$

and write $f = \sum_k f_k$ where

$$f_k = \sum_{i=1}^{\infty} \Big[(f-m_i^k)\varphi_i^k - \sum_{j=1}^{\infty} (f-m_{ij}^{k+1})\varphi_j^k \varphi_j^{k+1}\Big]$$

$$+\sum_{j=1}^{\infty} \Big[\sum_{i=1}^{\infty} (f-m_{ij}^{k+1})\varphi_i^k \varphi_j^{k+1} - (f-m_j^{k+1})\varphi_j^{k+1}\Big]$$

$$= \sum_{i=1}^{\infty} \beta_i^k + \sum_{j=1}^{\infty} \gamma_j^k, \quad \text{where } m_{ij}^{k+1} = \frac{1}{\int \varphi_i^k \varphi_j^{k+1}} \int f \varphi_i^k \varphi_j^{k+1}.$$

(This is line for line taken from Latter [6], p. 96.) As in [6],

$$|\beta_j^k| \leqslant C2^{k+1}, \quad |\gamma_j^k| \leqslant C2^{k+1} \quad \text{and} \quad \int \gamma_j^k = 0 = \int \beta_j^k$$

for all k, j. This proves the decomposition once we observe that $|\Omega_k| = O(2^{-k})$. For the converse, we fix $\alpha > 0$, and choose k_0 so that $2^{k_0} \leqslant \alpha < 2^{k_0+1}$. Write

$$f = \sum_{k=-\infty}^{k_0-1} f_k + \sum_{k=k_0}^{+\infty} f_k = F_1 + F_2.$$

Thus, if φ is a positive smooth bump function supported in the unit ball of $\mathbf{R}^n$ and with $\int \varphi = 1$, we have for the corresponding maximal operator $f_\varphi^+ = f^+$

$$m\{f^+ > 2\alpha\} \leqslant m\{F_2^+ > \alpha\}.$$

Set

$$A_{k_0} = \bigcup_{k=k_0}^{\infty} \bigcup_{i \geqslant 1} 2Q_{ki}$$

where $2Q_{ki}$ denotes the double of Q_{ki}. Observe that $m(A_{k_0}) \leqslant C_1 2^{n+1} 2^{-k_0} \leqslant C_1'/\alpha$ and therefore we need only estimate $I = m\{x \notin A_{k_0} |\ F_2^+(x) > \alpha\}$. Now, an easy computation, using the cancellation of β_{ki}, shows that if $x \notin 2Q_{ki}$ then

$$\beta_{ki}^+(x) \leqslant C2^k \frac{|Q_{ki}|^{(n+1)/n}}{|x-x_{ki}|^{n+1}},$$

where x_{ki} is the center of Q_{ki}.

To finish the proof, we shall use the following simple result in measure theory which was independently founded by Stein–Taibleson–Weiss [9] and by Kalton [5].

LEMMA. *Let g_k be a sequence of measurable functions and let $0 < p < 1$. Assume that $m\{|g_k| > \lambda\} \leqslant C/\lambda^p$ with C independent of k and λ. Then, for every numerical sequence $\{c_k\}$ in l^p we have*

$$m\{x|\ |\sum_k c_k g_k| > \lambda\} \leqslant \frac{2-p}{1-p} \cdot \frac{C}{\lambda^p} \sum_k |c_k|^p.$$

Using this lemma with $g_{k,i} = 1/|x-x_{ki}|^{n+1}$, $p = n/(n+1)$, and $c_{k,i} = 2^k m(Q_{ki})^{(n+1)/n}$ we obtain

$$I \leqslant c_n \frac{1}{\alpha^{n/(n+1)}} \sum_{k \geqslant k_0} \sum_i 2^{k \cdot n/(n+1)} m(Q_{ki})$$

$$\leqslant c_n' \frac{C_1}{\alpha^{n/(n+1)}} (2^{-k_0})^{1/(n+1)} \leqslant c_n'' \frac{C_1}{\alpha}.$$

Hence, $f \in \text{Weak}\,H^1$ and $\|f\|_{WH^1} \leqslant cC_1$.

3. Inequalities for the Fourier transform. As an illustration of how one can apply the atomic decomposition in the previous section, we can obtain the following results on the size of the Fourier transform of Weak H^1 functions:

PROPOSITION 1. *Let $B(0;R)$ denote the ball centered at the origin of radius R. Let $f \in C_c^\infty(\mathbf{R}^n)$. Then we have the following a priori estimate:*

$$(*)\qquad \frac{1}{m(B(0;R))}\int_{B(0;R)} \exp\left(c\,|\hat f(\xi)|/\|f\|_{WH^1}\right) d\xi \leqslant C,$$

for some constants c and C depending only on the dimension n.

By similar considerations, we have also the following:

PROPOSITION 2. *Let $f \in C^\infty(\mathbf{T}^n)$ where $\mathbf{T}^n$ denotes the n-torus. Then*

$$\frac{1}{N^n}\sum_{|m|\leqslant N} \exp\left(c\,|\hat f(m)|/\|f\|_{WH^1}\right) \leqslant C$$

where C and c depend only on n, and where $m = (m_1, m_2, \ldots, m_n)$, $m_k \in \mathbf{Z}$.

The reader may compare these results with Theorems 1.4 and 1.5 in [1, Chapter 4].

Proof of Proposition 1. We may assume $\|f\|_{WH^1} = 1$ without loss of generality. Also, notice that if $f \in \text{Weak}\,H^1$ and $\delta > 0$ then $\delta^{-n} f(x/\delta) \in \text{Weak}\,H^1$ and has the same norm as f, and if $f_\delta(x) = \delta^{-n} f(x/\delta)$, then $\hat f_\delta(\xi) = \hat f(\delta\xi)$. Therefore

$$\frac{1}{m(B(0;R))}\int_{B(0;R)} \exp\left(c\,|\hat f(\xi)|\right) d\xi = \frac{1}{m(B(0;1))}\int_{B(0;1)} \exp\left(c\,|\hat f(R\xi)|\right) d\xi$$
$$= \frac{1}{m(B(0;1))}\int_{B(0;1)} \exp\left(c\,|\hat f_R(\xi)|\right) d\xi.$$

So with no loss of generality, we may assume $R = 1$ in $(*)$.

From the atomic decomposition we write $f = \sum_{k=-\infty}^{+\infty} f_k = \sum_{k=-\infty}^{+\infty}\sum \beta_{ki}$ where the f_k, β_{ki} (and Q_{ki}) have the properties stated above. Write

$$f^{(1)} = \sum_{k\leqslant 0} f_k \quad \text{and} \quad f^{(2)} = \sum_{k>0} f_k.$$

To treat $\int_{|\xi|<1} \exp[|(f^{(1)})\hat{}\,(\xi)|]\,d\xi$ we estimate as follows:

$$(**)\quad \int_{|\xi|<1} \exp[c\,|(f^{(1)})\hat{}\,(\xi)|]\,d\xi \leqslant C\left(1 + c\int_{|\xi|<1} |(f^{(1)})\hat{}\,(\xi)|\,d\xi + \sum_{k\geqslant 2}\frac{c^k}{k!}\|(f^{(1)})\hat{}\,\|_k^k\right).$$

By the Hausdorff–Young inequality $\|(f^{(1)})\hat{}\,\|_k \leqslant \|f^{(1)}\|_{k'}$ for $k > 2$, where $1/k + 1/k' = 1$. Then

$$\|f^{(1)}\|_{k'} \leqslant \sum_{j\leqslant 0}\|f_j\|_{k'} \leqslant \sum_{j\leqslant 0} 2^{j(1-1/k')} = O(k) \quad \text{as } k \to \infty.$$

Substituting this in $(**)$ we get

$$\int_{|\xi|\leqslant 1} \exp[c\,|(f^{(1)})\hat{}\,(\xi)|]\,d\xi \leqslant C\left(\sum_{k=0}^{\infty}\frac{c^k}{k!}k^k\right) < \infty, \quad \text{if } c \text{ is small enough.}$$

To handle $\int_{|\xi|\leqslant 1} \exp(c\,|\hat f(\xi)|)\,d\xi$ we write

$$|(f^{(2)})\hat{}\,(\xi)| \leqslant \sum_{k\geqslant 0}\sum_i \int_{Q_{ki}} |\beta_{ki}(x)|\,|e^{i\xi\cdot x} - e^{i\xi\cdot x_{ki}}|\,dx$$

where x_{ki} is the center of Q_{ki}, and where we have used the cancellation property of β_{ki}. This, in turn, is dominated by

$$|\xi|\sum_{k\geqslant 0}\sum \operatorname{diam}(Q_{ki})\int_{Q_{ki}} |\beta_{ki}| \leqslant \sum_{k\geqslant 0} 2^{-k/n}|\xi|$$

from which obviously

$$\int_{|\xi|\leqslant 1} \exp[c\,|(f^{(1)})\hat{}\,(\xi) + (f^{(2)})\hat{}\,(\xi)|]\,d\xi \leqslant C' \int_{|\xi|\leqslant 1} \exp[c\,|(f^{(1)})\hat{}\,(\xi)|]\,d\xi \leqslant C''.$$

This concludes the proof of Proposition 1 and the proof of Proposition 2 is similar.

We should also remark that the above proof is intimately connected with the method of interpolation. In fact, if one likes, a proof of the proposition can be given directly by interpolating.

Finally, one cannot improve the integrability of $\hat f$ for $f \in \text{Weak}\,H^1$ beyond the exponential class. In fact, for a large integer N

$$f(x) = \sum_{k=0}^{N} 2^{-k} e^{ix}\chi_{\{2^k\cdot 2\pi < |x| < 2^{k+1}\cdot 2\pi\}}(x)$$

belongs to Weak H^1 and has norm bounded independent of N, by the atomic decomposition. On the other hand, a trivial computation shows that for $|1-\xi| < 2^{-N}$, $|\hat f(\xi)| > cN$ so that the exponential estimate $(*)$ is, indeed, sharp.

Next, let us point out that $(*)$ gives an extension of Paley's inequality for functions in Weak H^1. For simplicity, consider the case $n = 1$.

PROPOSITION 3. *Let T be the sequence-valued operator defined by $Tf(m) = \hat f(m)\cdot m$, $m \in \mathbf{Z}$. Then T is a bounded operator from* Weak $H^1(T^1)$ *into* Weak $L^1(d\mu)$ *where μ is the measure on $\mathbf{Z}$ so that $\mu\{n\} = 1/n^2$, $n \neq 0$, that is,*

$$\sum_{\{m\neq 0\,|\,|\hat f(m)|\,|m|>\alpha\}} \frac{1}{m^2} \leqslant \frac{C}{\alpha}\|f\|_{WH^1(T^1)} \quad \textit{for all } \alpha > 0.$$

If we have the larger norm $\|f\|_{L^1}$ replacing $\|f\|_{WH^1}$ then this is a classical inequality (see Zygmund [10]).

Proof. By homogeneity, we may assume again that $\|f\|_{WH^1} = 1$. Now,

$$\sum_{|\hat f(m)|>\alpha/|m|} 1/m^2 \leqslant \sum_{|m|>\alpha} 1/m^2 + \sum_{\alpha/|\hat f(m)|<|m|<\alpha} 1/m^2 \leqslant \frac{1}{\alpha} + \sum_{0<|m|<\alpha} |\hat f(m)|^2/\alpha^2.$$

Using $(*)$, this is

$$\leqslant \frac{1}{\alpha} + \frac{1}{\alpha}(C\|f\|_{WH^1})^2 \leqslant \frac{C'}{\alpha}.$$

4. The dual space. Given a function φ on $\boldsymbol{R}^n$ and an open set $\Omega \subseteq \boldsymbol{R}^n$ of finite measure, we shall define a notion of oscillation $\mathcal{O}(\varphi, \Omega)$ of φ over Ω. This oscillation will then be used to define a "modulus of continuity" of φ by setting, for $\delta > 0$, $\omega(\delta) = \sup_{m(\Omega)=\delta} \mathcal{O}(\varphi, \Omega)$.

More specifically, suppose $\Omega \subseteq \boldsymbol{R}^n$ is an open set of finite measure. We shall now define the oscillation of a function $\varphi(x)$ over Ω, $\mathcal{O}(\varphi, \Omega)$, as follows:

$$\mathcal{O}(\varphi, \Omega) = \sup \frac{1}{m(\Omega)} \sum_k \int_{Q_k} |\varphi(x) - \varphi_{Q_k}|\, dx \qquad \left(\text{here } \varphi_Q = \frac{1}{m(Q)} \int_Q f\right),$$

where the sup is taken over all collections of subcubes of Ω, $\{Q_k\}$, with uniformly bounded C_n-overlap (i.e., $\sum \chi_{Q_k} \leqslant C_n$). As above, we set

$$\omega(\delta) = \omega_\varphi(\delta) = \sup_{m(\Omega)=\delta} \mathcal{O}(\varphi, \Omega).$$

We then have:

THEOREM. *The dual of $\overline{L_0^1}$ can be identified with the class of locally integrable functions φ for which*

$$(\sim) \qquad \|\varphi\|_* = \int_0^\infty \frac{\omega(\delta)}{\delta}\, d\delta < \infty.$$

Remark. As in the case of BMO, the dual of $\overline{L_0^1}$ is a space of classes of functions in which two functions belong to the same class if the difference is constant.

Proof. Assume first that φ satisfies $(\sim)$. Given $f \in C_c^\infty$ with integral 0, consider its atomic decomposition relative to Weak H^1,

$$f = \sum_{k=-\infty}^{+\infty} \Big(\sum_{i\geqslant 1} \beta_{ki}\Big).$$

Then

$$\Big|\int f(x)\varphi(x)\,dx\Big| \leqslant \sum_{k=-\infty}^{+\infty} \sum_i \int_{Q_{ki}} |\beta_{ki}|\,|\varphi - \varphi_{Q_{ki}}|\,dx$$
$$\leqslant C\|f\|_{WH^1} \sum_{k=-\infty}^{+\infty} 2^k \sum_{i\geqslant 1} \int_{Q_{ki}} |\varphi - \varphi_{Q_{ki}}|\,dx$$
$$\leqslant C\|f\|_{WH^1} \sum_{k=-\infty}^{+\infty} \omega(2^{-k}) \leqslant C'\|f\|_{WH^1}\|\varphi\|_*.$$

For the converse, we notice that if L is in the dual of $\overline{L_0^1}$, then, since $L_0^1 = \{f \in L^1 \mid \int f = 0\}$ is by definition continuously embedded in $\overline{L_0^1}$, L must be given on L_0^1 by integration against some L^∞ function $\varphi(x)$. We identify L with φ and we consider, for every integer k, a sequence of cubes $\{Q_{ki}\}_{i\geqslant 1}$ with finite overlapping so that

$$\sum m(Q_{ki}) \leqslant 2^{-k} \quad \text{and} \quad \omega(2^{-k}) \sim 2^k \sum_i \int_{Q_{ki}} |\varphi - \varphi_{Q_{ki}}|\,dx.$$

For every pair of indices (k, i) choose a constant c_{ki} such that

$$m\{x \in Q_{ki} \mid \varphi(x) > c_{ki}\} = m\{x \in Q_{ki} \mid \varphi(x) < c_{ki}\}$$

and set $b_{ki} = 2^k \operatorname{sgn}(\varphi - c_{ki})\chi_{Q_{ki}}$. Then $\int_{Q_{ki}} b_{ki} = 0$ and $\|b_{ki}\|_\infty \leqslant 2^k$. From our results in Section 2, the series $\sum_{k=-N}^{+N} \sum_i b_{ki}$ defines a sequence f_N whose Weak H^1 norms remain bounded as $N \to \infty$. Therefore,

$$|L(f_N)| = |\langle f_N, \varphi\rangle| \leqslant C\|L\|_*.$$

On the other hand

$$\langle f_N, \varphi\rangle = \sum_{k=-N}^{+N} \sum_{i\geqslant 1} \int b_{ki}(x)\varphi(x)\,dx$$
$$= \sum_{k=-N}^{+N} 2^k \sum_{i\geqslant 1} \int_{Q_{ki}} |\varphi - c_{ki}|\,dx \geqslant c \sum_{k=-N}^{+N} \omega(2^{-k}).$$

Thus,

$$\sum_{k=-\infty}^{+\infty} \omega(2^{-k}) \leqslant \frac{C}{c}\|\varphi\|_*$$

and hence

$$\|\varphi\|_* \leqslant C'\|L\|_*.$$

An interesting feature of the dual of $\overline{L_0^1}$ is that, unlike BMO, this class forms an algebra under pointwise multiplication. For, if φ and $\psi \in (\overline{L_0^1})^*$, we have, for every cube Q,

$$\int_Q |\varphi(x)\psi(x) - (\varphi\psi)_Q|\,dx \leqslant \int_Q |\varphi(x)|\,|\psi(x) - \psi_Q|\,dx + \int_Q |\varphi(x) - \varphi_Q|\,|\psi_Q|\,dx + |Q|\,|\psi_Q\varphi_Q - (\psi\varphi)_Q|$$
$$\leqslant 2\Big(\|\varphi\|_\infty \int_Q |\psi(x) - \psi_Q|\,dx + \|\psi\|_\infty \int_Q |\varphi(x) - \varphi_Q|\,dx\Big).$$

Therefore

$$\|\varphi\psi\|_* \leqslant 2(\|\varphi\|_\infty \|\psi\|_* + \|\psi\|_\infty \|\varphi\|_*).$$

We shall end this section by remarking, that if, instead of defining the oscillation ω in terms of open sets, we define it in terms of cubes, i.e.,

$$\omega_*(\delta) = \sup_{m(Q) \leqslant \delta} \frac{1}{m(Q)} \int_Q |\varphi(x) - \varphi_Q|\,dx$$

then the Dini condition $\int (\omega_*(\delta)/\delta)\,d\delta < \infty$ has been studied by Sarason [7].

5. Singular integrals.

THEOREM. *Let $Tf(x) = \int K(x-y) f(y)\,dy$ be a bounded operator on $L^2(\boldsymbol{R}^n)$. Suppose K satisfies the following Dini condition: $\int_0^1 (\Gamma(\delta)/\delta)\,d\delta < \infty$, where*

$$\Gamma(\delta) = \Gamma_K(\delta) = \sup_{h \neq 0} \int_{|x| > \delta^{-1} \cdot 2|h|} |K(x+h) - K(x)|\,dx.$$

Then for $f \in L^1(\boldsymbol{R}^n)$ we have

$$m\{x \mid |Tf(x)| > \alpha\} \leqslant \frac{C}{\alpha} \|f\|_{WH^1} \quad \textit{for all } \alpha > 0.$$

Proof. Let $\sum_{k=-\infty}^{+\infty} f_k = f$ be an atomic decomposition of f. By a simple limiting argument, it suffices to show that

$$m\{x \mid |T(\sum_{|k| \leqslant N} f_k)(x)| > \alpha\} \leqslant \frac{C}{\alpha} \|f\|_{WH^1}.$$

Then by properties of the atomic decomposition,

(i) $\operatorname{supp}(f_k) = \Omega_k = \bigcup_{i \geqslant 1} Q_i^k$.

(ii) $\sum_{i \geqslant 1} \chi_{Q_i^k} \leqslant C$ and $\sum_{i \geqslant 1} m(Q_i^k) < \frac{C}{2^k} \|f\|_{WH^1}$.

(iii) $f_k = \sum_{i \geqslant 1} \beta_i^k$ with $\operatorname{supp} \beta_i^k \subseteq Q_i^k$, $\int \beta_i^k = 0$, and $\|\beta_i^k\|_\infty \leqslant C 2^k$.

In particular, $\|f_k\|_\infty \leqslant C^2 2^k$. Consider $k_0 \in \boldsymbol{Z}$ so that $2^{k_0} \leqslant \alpha < 2^{k_0+1}$ and set

$$f^1 = \sum_{k=-N}^{k_0} f_k \quad \text{and} \quad f^2 = \sum_{k=k_0+1}^{+N} f_k.$$

Then $f^1 \in L^2$, and moreover

$$\|f^1\|_2 \leqslant \sum_{k=-N}^{k_0} c^2 2^k m(\Omega_k)^{1/2} \leqslant C' \|f\|_{WH^1}^{1/2} \sum_{k=-N}^{k_0} 2^{k/2} \leqslant C \|f\|_{WH^1}^{1/2} \alpha^{1/2}.$$

Therefore, since T is bounded on $L^2(\boldsymbol{R}^n)$,

$$m\{x \in \boldsymbol{R}^n \mid |Tf^1(x)| > \alpha\} \leqslant \frac{\|Tf^1\|_2^2}{\alpha^2} \leqslant \|T\|_{L^2}^2 \frac{\|f^1\|_2^2}{\alpha^2} \leqslant C\|T\|_{L^2}^2 \frac{\|f\|_{WH^1}}{\alpha}.$$

Now

$$|Tf^2(x)| = \Big|\sum_{k=k_0+1}^{N} Tf_k(x)\Big| \leqslant C \cdot \sum_{k=k_0+1}^{N} 2^k \sum_{i \geqslant 1} \int_{Q_i^k} |K(x-y) - K(x-x_i^k)|\,dy,$$

where x_i^k is the center of Q_i^k.

Let $\bar{Q}_i^k$ denote the "expansion" of the cube Q_i^k by the factor $(\frac{3}{2})^{(k-k_0)/n}$, and let

$$\bar{\Omega} = \bigcup_{k=k_0+1}^{N} \bigcup_{i \geqslant 1} \bar{Q}_i^k.$$

Then by Fubini's theorem,

$$\int_{^c\bar{\Omega}} |T(f^2)|(x)\,dx \leqslant C \sum_{k=k_0+1}^{N} 2^k \sum_{i \geqslant 1} \int_{Q_i^k} dy \int_{^c\bar{Q}_i^k} |K(x-y) - K(x-x_i^k)|\,dx$$
$$= C \sum_{k=k_0+1}^{N} 2^k \sum_{i \geqslant 1} \int_{Q_i^k} dy \int_{^c(\bar{Q}_i^k - x_i^k)} |K(x' - (y - x_k^i)) - K(x')|\,dx'.$$

Observe that if $x' \notin \bar{Q}_i^k - x_i^k$ and $y \in Q_i^k$ then

$$|x'| > (\tfrac{3}{2})^{(k-k_0)/n} |y - x_i^k|.$$

Hence

$$\int_{^c\bar{\Omega}} |T(f^2)|\,dx \leqslant C \sum_{k=k_0+1}^{N} 2^k \sum_{i \geqslant 1} |Q_i^k|\, \Gamma[(\tfrac{2}{3})^{(k-k_0)/n}]$$
$$\leqslant C^2 \|f\|_{WH^1} \sum_{k=k_0+1}^{N} \Gamma[(\tfrac{2}{3})^{(k-k_0)/n}] \leqslant C \int_0^1 \frac{\Gamma(\delta)}{\delta}\,d\delta \cdot \|f\|_{WH^1}.$$

Finally, one has $m\{x \mid |T(f^2)(x)| > \alpha\} \leqslant m(\bar{\Omega}) + \frac{1}{\alpha} \int_{^c\bar{\Omega}} |T(f^2)|$ and since

$$m(\bar{\Omega}) \leqslant \sum_{k=k_0+1}^{N} \sum_{i \geqslant 1} \left(\tfrac{3}{2}\right)^{k-k_0} |Q_i^k|$$

$$\leqslant \sum_{k \geqslant k_0} 2^{-(k-k_0)} \left(\tfrac{3}{2}\right)^{k-k_0} 2^{-k_0} \|f\|_{WH^1} \leqslant \frac{C}{\alpha} \|f\|_{WH^1},$$

the theorem is proven.

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On regular generators of Z^2-actions in exhaustive partitions

by

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Abstract. It is shown that for every totally ergodic Z^2-action with finite entropy there exists a regular generator in a given exhaustive partition and the set of regular generators is dense in the set of all generators.

1. Introduction. Let $(X, \mathscr{B}, \mu)$ be a Lebesgue probability space, $\mathscr{M}$ the set of all measurable partitions of X and $\mathscr{Z}$ the subset of $\mathscr{M}$ consisting of partitions with finite entropy.

All relations between measurable partitions are to be taken mod 0.

Let ϱ be the metric on $\mathscr{Z}$ defined by the formula

$$\varrho(P, Q) = H(P|Q) + H(Q|P), \qquad P, Q \in \mathscr{Z}.$$

We denote by ε the measurable partition of X into single points and by ν the measurable trivial partition whose only element is X.

Let T be an automorphism of $(X, \mathscr{B}, \mu)$. For $P \in \mathscr{M}$ we define

$$P_T^- = \bigvee_{n=1}^{\infty} T^{-n} P, \qquad P_T = \bigvee_{n=-\infty}^{+\infty} T^n P.$$

If $P_T = \varepsilon$ we say that P is a *generator* of (X, T).

A partition $\zeta \in \mathscr{M}$ is said to be *T-perfect* if

$$T^{-1}\zeta \leqslant \zeta, \quad \zeta_T = \varepsilon, \quad \bigwedge_{n=0}^{\infty} T^{-n}\zeta = \pi(T) \quad \text{and} \quad h(\zeta, T) = h(T)$$

where $\pi(T)$ and $h(T)$ denote the Pinsker partition and the entropy of T respectively.

Rokhlin and Sinai showed in [9] that for every automorphism T there exists a T-perfect partition. If T is aperiodic with $h(T) < \infty$ then for every generator P of (X, T) the partition $\zeta = P \vee P_T^-$ is T-perfect. Rokhlin [7] proved that if $h(T) < \infty$ and ζ is T-perfect then there exists a generator P such that $\zeta = P \vee P_T^-$, i.e. ζ is a past of the process (P, T).

Now, let G be an abelian free group of rank 2 of automorphisms of $(X, \mathscr{B}, \mu)$. We denote by $b(G)$ the set of all ordered pairs of independent generators of G.