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(p, q)-Convexity in quasi-Banach lattices and applications

by

BIENVENIDO CUARTERO and MIGUEL A. TRIANA (Zaragoza)

Dedicated to Prof. Luis Vigil

Abstract. We define (p, q)-convexity in quasi-Banach lattices for $p, q > 0$ and study the values $r, s > 0$ for which (p, q)-convexity implies (r, s)-convexity, showing the difference between this situation and the Banach case.

Finally, we apply our results to a problem of Turpin on the existence of tensor p-norms.

0. Introduction. In the recent development of the theory of Banach lattices the concepts of *p-convexity* and *p-concavity* play a very important role (see Lindenstrauss–Tzafriri [9]). They were first defined by Krivine [8] as "type $\geqslant p$" and "type $\leqslant p$". Maurey [11] introduced the more general notions of "type $\geqslant (p, q)$" and "type $\leqslant (p, q)$" as follows: given a Banach lattice X, we say that X is of "type $\geqslant (p, q)$" or (p, q)-*convex*, $1 \leqslant q \leqslant p < \infty$, if there is some constant C such that for all finite sequences $x_1, \ldots, x_n$ of elements of X we have

$$(*) \qquad \Big\|\Big(\sum_{i=1}^{n} |x_i|^p\Big)^{1/p}\Big\| \leqslant C\Big(\sum_{i=1}^{n} \|x_i\|^q\Big)^{1/q}.$$

((p, q)-*concavity* is dually defined.) Maurey himself showed ([11], especially pp. 11, 12 and 17) that for Banach lattices (p, q)-convexity (resp. (p, q)-concavity) adds nothing to p-convexity (resp. p-concavity), a (p, q)-convex Banach lattice also being r-convex for every $r < q$.

We shall show that the situation is entirely different when one considers quasi-Banach lattices (for the definition, see Kalton [6]). In this case (p, q)-convexity, now defined for $p \geqslant q > 0$, cannot be reduced in general to r-convexity for any $r > 0$.

Observe that on the left side of the inequality $(*)$ the expression $\big(\sum_{i=1}^{n} |x_i|^p\big)^{1/p}$ is used. This element in X is defined by means of a "homogeneous functional calculus", i.e., by proving that for every positive

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integer n and $x_1, \ldots, x_n$ in X there is a unique continuous lattice homomorphism from $\mathscr{H}_n$ into X, where $\mathscr{H}_n$ is the Banach lattice of 1-homogeneous continuous functions $h\colon \boldsymbol{R}^n \to \boldsymbol{R}$ normed by

$$\|h\|_{\mathscr{H}_n} = \sup\{|h(t_1, \ldots, t_n)|;\ \max(|t_1|, \ldots, |t_n|) = 1\}$$

(Krivine [8], Lindenstrauss–Tzafriri [9]), such that it maps the coordinate projections $(t_1, \ldots, t_n) \in \boldsymbol{R}^n \to t_i \in \boldsymbol{R}$ into x_i $(1 \leqslant i \leqslant n)$. We denote this homomorphism by $T_{(x_1, \ldots, x_n)}$ and the image of a function h in it by $h(x_1, \ldots, x_n)$.

This construction also works for p-Banach lattices, $0 < p < 1$ (see Popa [12]) and can be extended even to certain classes of vector lattices without any topology, for example to uniformly complete vector lattices (see Cuartero–Triana [2] for details).

Note. The uniqueness of *each* $T_{(x_1, \ldots, x_n)}$ avoids possible ambiguities and allows in many cases to manage the expressions $h(x_1, \ldots, x_n)$ like the functions $h(t_1, \ldots, t_n)$. For example, if f and g are homogeneous continuous functions on $\boldsymbol{R}^n$ and $\boldsymbol{R}^{n+1}$ respectively, satisfying

$$f(t_1, t_2, \ldots, t_n) = g(t_1, t_1, t_2, \ldots, t_n) \quad \text{for all } (t_1, t_2, \ldots, t_n) \text{ in } \boldsymbol{R}^n,$$

then $f(x_1, x_2, \ldots, x_n) = g(x_1, x_1, x_2, \ldots, x_n)$ for all $x_1, x_2, \ldots, x_n$ in X. (In fact, if $S\colon \mathscr{H}_{n+1} \to \mathscr{H}_n$ is defined by

$$(Sh)(t_1, \ldots, t_n) = h(t_1, t_1, t_2, \ldots, t_n),$$

then $T = T_{(x_1, x_2, \ldots, x_n)} \circ S$ is a lattice homomorphism which maps the coordinate projections into $x_1, x_1, x_2, \ldots, x_n$ and so $T = T_{(x_1, x_1, x_2, \ldots, x_n)}$.) In a similar way, if for each permutation σ of the indices $1, \ldots, n$ we take $S_\sigma\colon \mathscr{H}_n \to \mathscr{H}_n$ such that

$$(S_\sigma h)(t_1, \ldots, t_n) = h(t_{\sigma_1}, \ldots, t_{\sigma_n})$$

we see that $(S_\sigma h)(x_1, \ldots, x_n) = h(x_{\sigma_1}, \ldots, x_{\sigma_n})$, and so on. This result will be used later in § 2.

This paper is an improved version of the first chapter in Triana [13]. Not too surprisingly for us, Kalton [6] has been independently working on p-convexity in quasi-Banach lattices but his results, though interacting with ours, go in a different direction.

In § 1 we define (p, q)-convexity in quasi-Banach lattices L and by means of the s-convexification of L we give several answers to the question: For what values $r, s > 0$ the (p, q)-convexity of L implies its (r, s)-convexity? Taking account of an example of Kalton [6], the more general result (Proposition 1.3) cannot be improved. With supplementary conditions of q-concavity, we can obtain better results (Proposition 1.6).

What about the s-convexifications of Banach lattices? As such s-convexifications are s-convex quasi-Banach lattices, in order to be sure that we have something essentially distinct from Banach lattices we must find quasi-Banach lattices not s-convex for any $s > 0$. We describe here an example; some others of a different kind can be seen in Kalton [6]. Thus we have an important classification of quasi-Banach lattices into two nonvoid groups:

(1) the s-convexifications of Banach lattices, i.e., the s-convex quasi-Banach lattices for some $s > 0$, for which Kalton has given a nice intrinsic characterization, the *L-convexity* (see [6]);

(2) the non-L-convex quasi-Banach lattices, those for which there is no $s > 0$ such that their $(1/s)$-convexification is Banach (equivalently, which are not s-convex for any $s > 0$).

In § 2 we apply our results on (p, q)-convexity to tensor products of p-Banach spaces. Turpin [14], solving a problem which goes back to Waelbroeck, proved that if E is a p-normed space and F is a q-normed space, then a *tensor r-norm* may be given in $E \otimes F$ with $r = pq/(p+q-pq)$ (recall that a tensor r-norm is an r-norm in $E \otimes F$ such that the canonical bilinear map $E \times F \to E \otimes F$ is continuous). We obtain in the general case the same value for r as Turpin does and we can improve it under additional conditions on *one* of the spaces E, F. Moreover, the examples of non-L-convex quasi-Banach lattices suggest that the value $r = pq/(p+q-pq)$ is best possible. After the elaboration of this paper and using very different ideas, Kalton [7] has been able to prove this.

§ **1.** (p, q)-**convexity in quasi-Banach lattices.** Let L be a quasi-Banach lattice, i.e. a complete quasi-normed space $(L, \|\cdot\|)$ where L is a vector lattice and $\|\cdot\|$ is a lattice quasi-norm, i.e., a map $\|\cdot\|\colon L \to \boldsymbol{R}$ such that

$$\begin{aligned} &\|x\| > 0 && \text{if } x \in L \setminus \{0\}, \\ &\|tx\| = |t|\,\|x\| && \text{if } t \in \boldsymbol{R},\ x \in L, \\ &\|x+y\| \leqslant M(\|x\| + \|y\|) && \text{if } x, y \in L, \end{aligned}$$

for some constant M independent of x and y (the best constant M is called the *multiplier* of the quasi-norm) and

$$\|x\| \leqslant \|y\| \quad \text{whenever} \quad |x| \leqslant |y| \text{ in } L.$$

L is said to be (p, q)-*convex* where $0 < q \leqslant p \leqslant \infty$ and $q < \infty$ if there exists a constant $K < \infty$ so that

$$\Big\|\Big(\sum_{i=1}^{n} |x_i|^p\Big)^{1/p}\Big\| \leqslant K\Big(\sum_{i=1}^{n} \|x_i\|^q\Big)^{1/q}$$

for every choice of vectors $\{x_i\}_{i=1}^n$ in L. The smallest possible value of K is called the (p, q)-convexity constant. As usual, for $p = \infty$ we suppose

$$\Big(\sum_{i=1}^{n} |x_i|^p\Big)^{1/p} = \bigvee_{i=1}^{n} |x_i|.$$

When $p = q$ with $1 \leqslant p < \infty$ we find the concept of p-convex Banach lattice (cf. Lindenstrauss–Tzafriri [9]), and if $p = \infty$ the concept of upper q-estimate (cf. Lindenstrauss–Tzafriri [8] and Kalton [6]). Observe that for $p = 1$ we can define $(1, q)$-convexity in each quasi-normed space $(X, \|\cdot\|)$ replacing $\sum_{i=1}^{n} |x_i|$ by $\sum_{i=1}^{n} x_i$. In this case we say that the space X is q-convex, and it is clear that q-convexity is equivalent to q-normability.

Now, we define the *s-convexification* of a quasi-Banach lattice L as in Lindenstrauss–Tzafriri [9], but for all $s > 0$.

We denote, as usual, by $+$, $\cdot$ and $\|\cdot\|$ the algebraic operations and the quasi-norm of L. Let $s \in (0, +\infty)$; for x and y in L and for any scalar α, we define

$$x(+)_s y = (x^{1/s} + y^{1/s})^s, \qquad \alpha(\cdot)_s x = \alpha^s \cdot x$$

where $(x^{1/s} + y^{1/s})^s$ is the element in L corresponding to the function

$$f(t_1, t_2) = \big||t_1|^{1/s}\operatorname{sign} t_1 + |t_2|^{1/s}\operatorname{sign} t_2\big|^s \cdot \operatorname{sign}\big(|t_1|^{1/s}\operatorname{sign} t_1 + |t_2|^{1/s}\operatorname{sign} t_2\big)$$

and α^s is $|\alpha|^s \operatorname{sign}\alpha$ (cf. Popa [12] and Cuartero–Triana [2]).

$(L, (+)_s, (\cdot)_s, \leqslant)$ is a vector lattice denoted by L_s, in which we can define a lattice quasi-norm $\|x\|_s = \|x\|^{1/s}$ (by Hölder's inequality we obtain

$$\|x(+)_s y\|_s \leqslant 2^{|1-1/s|} M^{1/s} \cdot (\|x\|_s + \|y\|_s),$$

where M is the multiplier of the quasi-norm $\|\cdot\|$). $(L_s, \|\cdot\|_s)$ is called the *s-convexification* of L.

1.1. Lemma. *Let $(L, \|\cdot\|)$ be a quasi-Banach lattice. Then for every $0 < \theta < 1$ and $x, y \in L$*

$$\| |x|^\theta |y|^{1-\theta} \| \leqslant M \|x\|^\theta \|y\|^{1-\theta}$$

where M is the multiplier of $\|\cdot\|$.

The proof is similar to that of Proposition 1.d.2 (i) of Lindenstrauss–Tzafriri [9].

1.2. Proposition. *Let $(L, \|\cdot\|)$ be a quasi-Banach lattice. Then $(L_s, \|\cdot\|_s)$ is also quasi-Banach for every $0 < s < \infty$.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in the positive cone L_s^+. We now distinguish two cases:

(a) $0 < s < 1$. Since $\|x_n - x_m\| \leqslant \| |x_n^{1/s} - x_m^{1/s}|^s \|$ $(m, n \in N)$ there is $x \in L^+$ such that the sequence $\{x_n\}$ converges to x in L. We shall prove that $\{x_n\}$ converges to x in L_s; indeed, let M be the multiplier of the quasi-norm $\|\cdot\|$; for every $n \in N$

$$\| |x^{1/s} - x_n^{1/s}|^s \| \leqslant \| |x|^s |x^{(1-s)/s} - x_n^{(1-s)/s}|^s + |x_n|^{1-s} |x - x_n|^s \|$$
$$\leqslant M^2 \big[\|x\|^s \| |x^{(1-s)/s} - x_n^{(1-s)/s}|^{s/(1-s)} \|^{1-s} + \|x_n\|^{1-s} \|x - x_n\|^s\big].$$

When $s \geqslant 1/2$, $|x^{(1-s)/s} - x_n^{(1-s)/s}|^{s/(1-s)} \leqslant |x - x_n|$, consequently $\lim_n x_n = x$ in L_s. If $s \geqslant 1/2^{k+1}$ we repeat the procedure k times.

(b) $1 < s < \infty$. Now

$$\|x_n - x_m\| \leqslant M^2 \big[\|x_n\|^{1/s} \| |x_n^{(s-1)/s} - x_m^{(s-1)/s}|^{s/(s-1)} \|^{(s-1)/s} + \|x_m\|^{(s-1)/s} \| |x_n^{1/s} - x_m^{1/s}|^s \|^{1/s}\big].$$

When $s \leqslant 2$ then $|x_n^{(s-1)/s} - x_m^{(s-1)/s}|^{s/(s-1)} \leqslant |x_n^{1/s} - x_m^{1/s}|^s$ and so $\{x_n\}$ is a Cauchy sequence in L^+. If $s \leqslant 2^{k+1}$ we shall repeat the procedure k times. Hence, there is $x \in L^+$ so that $\lim_n x_n = x$ in L, and also in L_s. In order to complete the proof, we can use Theorem 16.1 of Aliprantis–Burkinshaw [1]. ∎

It is easily verified that if L is (p, q)-convex for $0 < q \leqslant p < \infty$ then L_s is (sp, sq)-convex for every $0 < s < \infty$. In particular, L is (p, p)-convex if and only if $L_{1/p}$ is normable.

The property of being (p, p)-convex for some $p > 0$ or, equivalently, of having a Banach s-convexification for some $s > 0$, has been characterized by Kalton [6] by means of L-convexity: a quasi-Banach lattice has this property if and only if there exists $0 < \varepsilon < 1$ so that if $u \in L^+$ with $\|u\| = 1$ and $0 \leqslant x_i \leqslant u$ $(1 \leqslant i \leqslant n)$ satisfy

$$\frac{1}{n}(x_1 + \ldots + x_n) \geqslant (1 - \varepsilon) u$$

then

$$\max_{1 \leqslant i \leqslant n} \|x_i\| \geqslant \varepsilon.$$

In contrast with the Banach case, there are (q, p)-convex quasi-Banach lattices for some $q \geqslant p > 0$ which are not (r, r)-convex for any $r > 0$, i.e., which are not L-convex. An example of this with $q = \infty$ are the spaces $L^p(\varphi)$ where φ is a suitable pathological submeasure (Kalton [6]). A different example has been supplied to us by G. Pisier (1):

(1) During a collaboration supported by the "Programa general de relaciones científicas hispano-francesas".

Let $(E, \|\cdot\|)$ be a Banach space of Rademacher type $\geqslant 1$ and consider it canonically imbedded in the Banach lattice $\mathscr{C}(K)$ of continuous functions over the unit ball K of the dual E^* with its w^*-topology. Consider the ideal L generated by E in $\mathscr{C}(K)$ endowed with the quasi-norm $|\cdot|$ defined by

$$|\varphi| = \inf\Big\{\Big(\sum_{i=1}^{n} \|x_i\|^p\Big)^{1/p};\ (x_i)_{i=1}^{n} \subset E \text{ and}$$

$$|\varphi| \leqslant \Big(\sum_{i=1}^{n} |x_i|^2\Big)^{1/2}\Big\}.$$

It is easily verified that $(E, |\cdot|)$ is a $(2, p)$-convex quasi-normed lattice.

With a suitable choice of E, (the completion of) L cannot be (r, r)-convex for any $r > 0$. Suppose, to the contrary, that it is (r, r)-convex for some $r > 0$. Consequently there exists a constant K such that

$$\Big|\Big(\sum_{j=1}^{n} |y_j|^r\Big)^{1/r}\Big| \leqslant K\Big(\sum_{j=1}^{n} \|y_j\|^r\Big)^{1/r}$$

for every $(y_j)_{j=1}^{n} \subset E$.

Let $(g_{jk})_{\substack{1\leqslant j\leqslant m\\ 1\leqslant k\leqslant n}}$ be Gaussian random variables with zero mean and variance 1. Then there is a constant C such that

$$\Big\|\sum_{j,k} g_{jk}\, y_{jk}\Big\|_{L^2(E)} = C\Big|\Big(\sum_{j,k} (y_{jk})^2\Big)^{1/2}\Big|$$

(cf. Krivine [8], Lemme 2).

Moreover, by Khintchine's inequality, there are constants C_1 and C_2 such that

$$\Big|\Big(\sum_{j,k} |y_{jk}|^2\Big)^{1/2}\Big| \leqslant C_1\Big|\Big(\mathrm{E}\Big|\sum_{j,k} y_{j,k}\, \varepsilon_j' \varepsilon_k''\Big|^r\Big)^{1/r}\Big|$$

$$\leqslant C_2\Big(\mathrm{E}\Big|\sum_{j,k} y_{jk}\, \varepsilon_j' \varepsilon_k''\Big|^r\Big)^{1/r}$$

where $\{\varepsilon_j'\}_{j=1}^{m}$, $\{\varepsilon_k''\}_{k=1}^{n}$ denote the Rademacher functions. Hence,

$$(*)\qquad \Big\|\sum_{j,k} g_{jk}\, y_{jk}\Big\|_{L^2(E)} \leqslant C\cdot C_2\Big\|\sum_{j,k} y_{jk}\, \varepsilon_j' \varepsilon_k''\Big\|_{L^r(E)}$$

and this is false, for example, when $E = C_p$ with $p \neq 2$, where C_p are the Schatten classes of operators. Indeed, if

$$G_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij}\, e_i \otimes e_j,$$

we know that there exists a constant $\delta > 0$ such that

$$\Big\|\sum_{i,j} g_{ij}\, e_i \otimes e_j\Big\|_{L^2(E)} \geqslant n^{1/2} \int \|G_n\|_{C_p} \geqslant \delta n^{1/2} \|\mathrm{Id}_H\|_{C_p} = \delta n^{1/2} n^{1/p}$$

(cf. Marcus–Pisier [10], Corollary 1.8) where Id_H is the identity operator on H. On the other hand

$$\Big\|\sum_{i,j} \varepsilon_i' \varepsilon_j''\, e_i \otimes e_j\Big\|_{L^r(E)} = n,$$

which is impossible by $(*)$.

Then it is natural to ask for what values of r, s, (p, q)-convexity implies (r, s)-convexity.

1.3. Proposition. *If a quasi-Banach lattice L is (p, q)-convex, then for every $r \leqslant p$, L is also (r, s)-convex, where*

$$\frac{1}{s} - \frac{1}{r} = \frac{1}{q} - \frac{1}{p} \qquad \Big(\frac{1}{s} - \frac{1}{r} = \frac{1}{q} \quad \text{if } p = \infty\Big).$$

In particular,

(a) *L is (r, r)-convex for every $r \leqslant p$ if it is (p, p)-convex.*

(b) *If L is q-normable, $0 < q \leqslant 1$, then it is $(p, pq/(p+q-pq))$-convex for every $p \in (0, 1]$.*

Proof. Let $r < p$, $\{x_i\}_{i=1}^{n} \subset L\setminus\{0\}$. By Hölder's inequality, we have for every $\alpha \in (0, 1)$ if $p < \infty$

$$\Big\|\Big(\sum_{i=1}^{n} |x_i|^r\Big)^{1/r}\Big\| \leqslant \Big(\sum_{i=1}^{n} \|x_i\|^{(r-\alpha)p/(p-r)}\Big)^{(p-r)/(pr)} \Big\|\Big(\sum_{i=1}^{n} \|x_i\|^{(\alpha-r)p/r} |x_i|^p\Big)^{1/p}\Big\|$$

$$\leqslant K\Big(\sum_{i=1}^{n} \|x_i\|^{(r-\alpha)p/(p-r)}\Big)^{(p-r)/(pr)} \Big(\sum_{i=1}^{n} \|x_i\|^{\alpha q/r}\Big)^{1/q}$$

and if $p = \infty$

$$\Big\|\Big(\sum_{i=1}^{n} |x_i|^r\Big)^{1/r}\Big\| \leqslant \Big(\sum_{i=1}^{n} \|x_i\|^{\alpha}\Big)^{1/r} \Big\|\bigvee_{i=1}^{n} \|x_i\|^{-\alpha/r} |x_i|\Big\|$$

$$\leqslant K\Big(\sum_{i=1}^{n} \|x_i\|^{\alpha}\Big)^{1/r} \Big(\sum_{i=1}^{n} \|x_i\|^{q(r-\alpha)/r}\Big)^{1/q}$$

where K is the (p, q)-convexity constant.

Taking $\alpha = \dfrac{r^2 p}{q(p-r)+pr}$ $\Big(\alpha = \dfrac{rq}{q+r}$ if $p = \infty\Big)$, we are done. ∎

This proposition gives the best possible result, as Example 2.4 of Kalton [6] shows (this follows from the fact that we can identify the s-convexification of $L_p(\varphi)$ with $L_{ps}(\varphi)$ in the obvious manner).

We have also

1.4. Proposition. *Let a quasi-Banach lattice L be (p_0, q_0)-convex and (p_1, q_1)-convex $(p_0 < p_1)$. If*

$$\frac{1}{r} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1} \qquad (\theta \in (0, 1))$$

then L is (r, s)-convex for every s with

$$\frac{1}{s} > \frac{\theta}{q_0} + \frac{1-\theta}{q_1}.$$

In particular, if L is L-convex and q-normable $(0 < q \leqslant 1)$, *then L is (s, s)-convex for every s in* $(0, q)$.

Proof. Let $\{x_i\}_{i=1}^n \subset L$ be such that $\|x_i\| \leqslant 1$. By Hölder's inequality

$$\Big\|\Big(\sum_{i=1}^n |x_i|^r\Big)^{1/r}\Big\| \leqslant \Big\|\Big(\sum_{i=1}^n |x_i|^{p_0}\Big)^{\theta/p_0}\Big(\sum_{i=1}^n |x_i|^{p_1}\Big)^{(1-\theta)/p_1}\Big\|$$
$$= M\Big\|\Big(\sum_{i=1}^n |x_i|^{p_0}\Big)^{1/p_0}\Big\|^{\theta}\Big\|\Big(\sum_{i=1}^n |x_i|^{p_1}\Big)^{1/p_1}\Big\|^{1-\theta}$$
$$\leqslant C n^{\theta/q_0 + (1-\theta)/q_1}.$$

Then it follows from Proposition 2.2 of Kalton [3] that $L_{1/r}$ is $(1, s/r)$-convex whenever

$$\frac{1}{s} > \frac{\theta}{q_0} + \frac{1-\theta}{q_1}$$

and so L is (r, s)-convex. ∎

We can obtain better values for s than those in Proposition 1.3 if we suppose that L is (q, q)-concave for some $q \in (0, +\infty)$.

1.5. Definition. Let $p, q \in (0, +\infty)$. We say that L is *(p, q)-concave* if there is a constant K such that

$$\Big(\sum_{i=1}^n \|x_i\|^p\Big)^{1/p} \leqslant K\Big\|\Big(\sum_{i=1}^n |x_i|^q\Big)^{1/q}\Big\|$$

for every choice of vectors $\{x_i\}_{i=1}^n$ in L.

1.6. Proposition. *If a quasi-Banach lattice L is (u, r)-convex* $(u < \infty)$ *and (q, q)-concave for some* $q < \infty$, *then for every* $s > r$, *L is (s, r)-convex.*

In particular, if L is r-normable and (q, q)-concave for some $q < \infty$, *then L is (s, r)-convex for every* $s > r$.

Proof. We may assume that $u = 1$ (otherwise we can use the fact that $L_{1/u}$ is $(1, r/u)$-convex). It is known that a quasi-Banach space E of Rademacher type p is p-convex (cf. Theorem 4.2 of Kalton [4]). Let $p < \min\{1/r, 2\}$; we shall prove that L_p is of Rademacher type rp.

Since L is (q, q)-concave (we may assume without loss of generality that $q > 1/p$) there exists a constant C such that if $\{\varepsilon_i\}_{i=1}^\infty$ is the sequence of the Rademacher functions and $\{x_i\}_{i=1}^n \subset L$ then

$$\Big(\int_0^1 \Big\|\sum_{i=1}^n (+)_p\, \varepsilon_i(t)\, x_i\Big\|_p^{qp} dt\Big)^{1/qp} \leqslant C\Big\|\Big(\int_0^1 \Big|\sum_{i=1}^n (+)_p\, \varepsilon_i(t)\, x_i\Big|^q dt\Big)^{1/q}\Big\|^{1/p}$$
$$= C\Big\|\Big(\int_0^1 \Big|\sum_{i=1}^n \varepsilon_i(t)\, x_i^{1/p}\Big|^{pq} dt\Big)^{1/q}\Big\|^{1/p}$$

by Khintchine's inequality and since L is $(1, r)$-convex, there exist two constants A, B so that this expression is upper bounded by

$$A\Big\|\Big(\sum_{i=1}^n |x_i|^{2/p}\Big)^{p/2}\Big\|^{1/p} \leqslant A\Big\|\sum_{i=1}^n |x_i|\Big\|^{1/p} \leqslant B\Big(\sum_{i=1}^n \|x_i\|^r\Big)^{1/rp} = B\Big(\sum_{i=1}^n \|x_i\|_p^{rp}\Big)^{1/rp}.$$

Hence L_p is of Rademacher type rp, and thus rp-convex. If $1/2^n \leqslant r \leqslant 1/2^{n-1}$ for some $n \in N$, then $L_{2^{n-1}}$ is $2^{n-1}r$-convex and then for every $p < 1/(2^{n-1}r)$ we conclude that $L_{2^{n-1}p}$ is $2^{n-1}pr$-convex and so L is $(1/(2^{n-1}p), r)$-convex. ∎

§ 2. **Applications to tensor products.** It is known (Kalton [5]) that it is possible to find a p-Banach space E such that $E \otimes E$ admits no tensor p-norm. So, the question naturally arises:

Given a p-Banach space E and a q-Banach space F, for what values of $r > 0$ *the tensor product* $E \otimes F$ *admits a tensor r-norm?*

The complete answer to the question is still an open problem.

As we have said in the introduction, Turpin obtains the value $r = pq/(p + q - pq)$. From our results on (p, q)-convexity, we can obtain in the general case the same value as Turpin and we are able to improve it under additional conditions on *one* of the spaces E, F.

To see this, let us consider a quasi-Banach space (E, ϱ) with ϱ a continuous quasi-norm and a quasi-Banach lattice L. Given $\{x_1, \ldots, x_n\} \subset E$, the mapping

$$h_{(x_1,\ldots,x_n)}\colon (t_1, t_2, \ldots, t_n) \in \boldsymbol{R}^n \to h_{(x_1,\ldots,x_n)}(t_1, \ldots, t_n) = \varrho\Big(\sum_{k=1}^n t_k x_k\Big) \in \boldsymbol{R}$$

is a continuous homogeneous function and so we can define (by means of the "homogeneous functional calculus") the corresponding mapping

$$h_{(x_1,\ldots,x_n)}\colon (y_1, \ldots, y_n) \in L \times \ldots \times L \to h_{(x_1,\ldots,x_n)}(y_1, \ldots, y_n) \in L.$$

The following properties are easily verified:

(1) $\quad h_{(x+x', x_2,\ldots,x_n)}(y_1, y_2, \ldots, y_n) = h_{(x, x', x_2,\ldots,x_n)}(y_1, y_1, y_2, \ldots, y_n),$

(2) $\quad h_{(tx_1, x_2,\ldots,x_n)}(y_1, y_2, \ldots, y_n) = h_{(x_1, x_2,\ldots,x_n)}(ty_1, y_2, \ldots, y_n),$

(3) $\quad h_{(x_1, x_2,\ldots,x_n)}(y + y', y_2, \ldots, y_n) = h_{(x_1, x_1, x_2,\ldots,x_n)}(y, y', y_2, \ldots, y_n),$

(4) $\quad h_{(x_1, x_2,\ldots,x_n)}(y_1, y_2, \ldots, y_n) = h_{(x_{\sigma_1}, x_{\sigma_2},\ldots,x_{\sigma_n})}(y_{\sigma_1}, y_{\sigma_2}, \ldots, y_{\sigma_n})$

for every permutation σ of the indices $1, 2, \ldots, n$ (see Note in the introduction).

Thus, the mapping

$$h\colon\ w = \sum_{i=1}^{n} x_i \otimes y_i \in E \otimes L \to h(w) = h_{(x_1,\ldots,x_n)}(y_1, \ldots, y_n) \in L$$

is well defined and satisfies for all $w, w' \in E \otimes L$ and $t \in \boldsymbol{R}$

(i) $h(w) \geqslant 0$.

(ii) $h(w) = 0$ iff $w = 0$.

(iii) $h(w+w') \leqslant M[h(w)+h(w')]$ where M is the multiplier of ϱ.

(iv) $h(tw) = |t|\, h(w)$.

(To see that h is well defined, perhaps the simplest way is to consider a Hamel basis B of the space L and recall that every w in $E \otimes L$ can be uniquely written as $w = \sum_{b \in B} x_b \otimes b$ with $(x_b)_{b \in B} \in E^{(B)}$. Then if

$$w = \sum_{i=1}^{n} x_i \otimes y_i = \sum_{j=1}^{m} x'_j \otimes y'_j,$$

$$y_i = \sum_{k=1}^{N} t_{ik} b_k \quad (1 \leqslant i \leqslant n), \qquad y'_j = \sum_{k=1}^{N} s_{jk} b_k \quad (1 \leqslant j \leqslant m),$$

we have

$$\begin{aligned} h_{(x_1,\ldots,x_n)}&(y_1, \ldots, y_n) \\ &= h_{(x_1,\ldots,x_1,\ldots,x_n,\ldots,x_n)}(t_{11} b_1, \ldots, t_{1N} b_N, \ldots, t_{n1} b_1, \ldots, t_{nN} b_N) \\ &= h_{(t_{11}x_1,\ldots,t_{1N}x_1,\ldots,t_{n1}x_n,\ldots,t_{nN}x_n)}(b_1, \ldots, b_N, \ldots, b_1, \ldots, b_N) \\ &= h_{(\sum_{i=1}^{n} t_{i1}x_i,\ldots,\sum_{i=1}^{n} t_{iN}x_i)}(b_1, \ldots, b_N) \\ &= h_{(\sum_{j=1}^{m} s_{j1}x'_j,\ldots,\sum_{j=1}^{m} s_{jN}x'_j)}(b_1, \ldots, b_N) \\ &= h_{(x'_1,\ldots,x'_m)}(y'_1, \ldots, y'_m). \end{aligned}$$

To prove, for example, that $h(w) = 0$ implies $w = 0$, take $w = \sum_{i=1}^{n} x_i \otimes y_i$ with (x_i) linearly independent. Since

$$h_{(x_1,\ldots,x_n)}(t_1, \ldots, t_n) \neq 0 \quad \text{if} \quad (t_1, \ldots, t_n) \neq (0, \ldots, 0)$$

there exists a constant $C > 0$ such that

$$h_{(x_1,\ldots,x_n)}(t_1, \ldots, t_n) \geqslant C \max_{1 \leqslant i \leqslant n} |t_i| \quad \text{for all } (t_1, \ldots, t_n) \in \boldsymbol{R}^n$$

and so $0 = h(w) \geqslant C(|y_1| \vee \ldots \vee |y_n|) \geqslant 0$. Then $|y_i| = 0$ and $y_i = 0$ $(1 \leqslant i \leqslant n)$ so $w = 0$.)

Moreover, if E is p-convex with constant of p-convexity C, for every finite collection $\{w_1, \ldots, w_n\} \subset E \otimes L$,

$$h\Big(\sum_{i=1}^{n} w_i\Big) \leqslant C\Big(\sum_{i=1}^{n} h^p(w_i)\Big)^{1/p}.$$

Of course, this construction also works when L is merely a Riesz space with a homogeneous functional calculus.

From (i) to (iv), we have (with the same notation):

2.1. Proposition. *Let E be a quasi-Banach space for a continuous quasi-norm ϱ, and $(L, \|\cdot\|_L)$ a quasi-Banach lattice. The mapping*

$$\|\cdot\|\colon\ w \in E \otimes L \to \|w\| = \|h(w)\|_L \in \boldsymbol{R}$$

is a tensor quasi-norm.

Moreover, if E is p-normable and L is (p, q)-convex, then $(E \otimes L, \|\cdot\|)$ is q-normable.

Consequently, using Proposition 1.3 together with Théorème 2.1 of Turpin [14] we can prove, by a different way, the following theorem of Turpin [14]:

2.2. Corollary. *If E and F are p-normed and q-normed (respectively) real vector spaces, then there exists a tensor $pq/(p+q-pq)$-norm in $E \otimes F$.*

Finally, we have

2.3. Proposition. *Let E be a p-normed space continuously imbedded in a p-normable quasi-Banach lattice L and let F be a q-normed space.*

(1) *If L is (p, p)-convex, then $E \otimes F$ is r-normed with $r = \min\{p, q\}$.*

(2) *If L is L-convex, then $E \otimes F$ is r-normed for every $r < \min\{p, q\}$.*

(3) *If L is (s, s)-concave for some $s \in (0, +\infty)$, then $E \otimes F$ is r-normed for every $r < \min\{p, q\}$.*

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DEPARTAMENTO TEORÍA DE FUNCIONES
FACULTAD DE CIENCIAS (MATEMATICAS)
50009 Zaragoza, Spain
and
DEPARTAMENTO MATEMÁTICAS I
ETSII, UNIVERSIDAD DE ZARAGOZA
50009 Zaragoza, Spain

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A direct proof of van der Vaart's theorem

by

J. BOURGAIN (Brussels and Pasadena, Cal.) and H. SATO (Fukuoka)

Abstract. The aim of this paper is to give a direct and simple proof of van der Vaart's theorem [3] determining the absolutely continuous component of a signed measure on $\boldsymbol{R}^d$ from its characteristic functional.

1. Introduction and results. Let

$$d\lambda(t) = d\lambda(t_1, t_2, \ldots, t_d) = (2\pi)^{-d/2}\, dt_1\, dt_2 \ldots dt_d$$

be the modified Lebesgue measure on $\boldsymbol{R}^d$, for a λ-integrable function f on $\boldsymbol{R}^d$ define the Fourier transform by

$$\tilde{f}(\alpha) = \int_{\boldsymbol{R}^d} e^{i(\alpha,t)} f(t)\, d\lambda(t), \qquad \alpha \in \boldsymbol{R}^d,$$

where (α, t) is the inner product of $\boldsymbol{R}^d$, let $\mathscr{K}$ be the collection of all λ-integrable functions $\varkappa$ which satisfy the following conditions:

(1) $\int \varkappa(t)\, d\lambda(t) = 1$.

(2) There exists $a > 1$ such that

$$Q(\varkappa) = \sup_{t \in \boldsymbol{R}^d} (1 + \|t\|^{da})\, |\varkappa(t)| < +\infty,$$

where $\|t\|$ is the Euclidean norm on $\boldsymbol{R}^d$, and define

$$\tilde{\mathscr{K}} = \{\varkappa \in \mathscr{K};\ \tilde{\varkappa} \in L^1(\lambda)\}.$$

Furthermore, for every $\varkappa$ in $\mathscr{K}$ and $T > 0$ define $\varkappa_T(t) = T^d \varkappa(Tt)$. Then evidently we have for every $T > 0$,

$$\int \varkappa_T(t)\, d\lambda(t) = 1 \quad \text{and} \quad \tilde{\varkappa}_T(\alpha) = \tilde{\varkappa}(\alpha/T).$$

Let μ be a signed measure on $\boldsymbol{R}^d$. Then we have the Lebesgue decomposition

$$d\mu(t) = \frac{d\mu}{d\lambda}(t)\, d\lambda(t) + d\mu_s(t),$$

where μ_s is the singular component of μ.

In this paper we shall prove the following theorems.