References

[1] J. Bergh and J. Löfström, *Interpolation Spaces — An Introduction*, Springer, New York 1976.

[2] A. P. Calderón, *Intermediate spaces and interpolation, the complex method*, Studia Math. 24 (1964), 113–190.

[3] A. P. Calderón and A. Torchinsky, *Parabolic maximal functions associated with a distribution, II*, Adv. in Math. 24 (1977), 101–171.

[4] S.-Y. A. Chang and R. Fefferman, *A continuous version of duality of H^1 with* BMO *on the bidisc*, Ann. of Math. 112 (1980), 179–201.

[5] —, —, *The Calderón–Zygmund decomposition on product domains*, Amer. J. Math. 104 (1982), 455–468.

[6] C. Fefferman, N. M. Rivière and Y. Sagher, *Interpolation between H^p spaces: the real method*, Trans. Amer. Math. Soc. 191 (1974), 74–81.

[7] C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. 129 (1972), 137–193.

[8] R. Fefferman, *Bounded mean oscillation on the polydisk*, Ann. of Math. 110 (1979), 395–406.

[9] R. Gundy and E. M. Stein, *H^p theory for the poly-disc*, Proc. Nat. Acad. Sci. U.S.A. 76 (1979), 1026–1029.

[10] S. Janson, *On the interpolation of sublinear operators*, Studia Math. 75 (1982), 51–53.

[11] S. Janson and P. Jones, *Interpolation between H^p spaces: the complex method*, J. Funct. Anal. 48 (1982), 58–80.

[12] P. Jones, *Interpolation between Hardy spaces*, in: Conference on Harmonic Analysis in Honor of A. Zygmund, Vol. 2, 437–451.

[13] K. G. Merryfield, *H^p-spaces in poly-half spaces*, Ph. D. Thesis, University of Chicago, 1980.

[14] T. Wolff, *A note on interpolation spaces*, in: Harmonic Analysis, Proceedings, Minneapolis 1981, Lecture Notes in Math. 908, 199–204.

UNIVERSITY OF CHICAGO
Chicago, Illinois 60637, U.S.A.

Received April 1, 1985 (2044)

STUDIA MATHEMATICA, T. LXXXIV. (1986)

On values of homogeneous polynomials in discrete sets of points

by

P. WOJTASZCZYK (Warszawa)

Abstract. Let $W_N(d)$ denote the space of all homogeneous polynomials on C^d of degree N, restricted to the unit sphere. We show a class of sets Λ of small cardinality such that for every $\varphi\in W_N(d)$ we have $(\int|\varphi|^p)^{1/p}$ comparable to $(|\Lambda|^{-1}\sum_{\lambda\in\Lambda}|\varphi(\lambda)|^p)^{1/p}$. We also show that every subspace $E\subset W_N(d)$ such that $\dim E\geqslant\frac{1}{2}\dim W_N(d)$ contains a polynomial φ such that $\|\varphi\|_\infty\leqslant K(d)\|\varphi\|_2$.

We consider the spaces $W_N(d)$ of all homogeneous polynomials on C^d (the d-dimensional complex space) of degree N. On those spaces we consider the norms inherited from $L_p(S_d)$, i.e. for $\varphi\in W_N(d)$ we put

$$\|\varphi\|_p=\Big(\int_{S_d}|\varphi(\zeta)|^p\,d\sigma(\zeta)\Big)^{1/p}$$

where σ is the normalized rotation-invariant measure on S_d, the unit sphere in C^d. Our main interest in this note is to compare $\|\varphi\|_p$ with its discrete analogue: for a finite subset $\Lambda\subset S_d$ we consider

$$\|\varphi|\Lambda\|_p=\Big(|\Lambda|^{-1}\sum_{\lambda\in\Lambda}|\varphi(\lambda)|^p\Big)^{1/p}$$

(where $|\Lambda|$ denotes the cardinality of Λ). Our main result asserts that it is possible to find relatively small sets Λ such that $\|\varphi|\Lambda\|_p\sim\|\varphi\|_p$ for all $\varphi\in W_N(d)$ (Theorem 1). In the case $p=\infty$ this result was obtained by B. S. Kashin in [5] by a different method. In Theorem 2 we give a simplified version of Kashin's proof which has an additional advantage of giving good constants. As an application of this special case we obtain a strengthening of the main result of [5] and also of Theorem 1 of [9].

The author would like to thank Professor B. S. Kashin for illuminating comments and the referee for extremely valuable criticism which led to the complete revision of the paper.

Preliminaries and notation. The natural scalar product in C^d will be denoted by $\langle\cdot,\cdot\rangle$. We will use the unitarily invariant pseudometric on S_d

defined by

$$\varrho(\zeta_1, \zeta_2) = \sqrt{1-|\langle\zeta_1, \zeta_2\rangle|} \quad \text{for } \zeta_1, \zeta_2 \in S_d.$$

One should note that $\varrho(\zeta_1, \zeta_2) = 0$ if and only if ζ_1 and ζ_2 lie in the same complex line passing through zero. It will also be important that for $\varphi \in W_N(d)$, $|\varphi|$ is constant on the sets of diameter zero in the pseudometric ϱ.

Using (1.4.5) of [7] one easily computes that

$$\sigma(B(r)) = (2r^2 - r^4)^{d-1} \sim r^{2(d-1)} \tag{1}$$

where $B(r) \subset S_d$ is a ball of radius r in the pseudometric ϱ. We will also use the representing formula for $\varphi \in W_N(d)$ (cf. [9], Prop. 1 or [8]):

$$\varphi(z) = D \int_{S_d} \varphi(\zeta) \langle z, \zeta\rangle^N d\sigma(\zeta) \tag{2}$$

where

$$D = \dim W_N(d) = \frac{(N-1+d)!}{(N-1)!\,d!} = \Big[\int_{S_d} |\langle\zeta, \zeta_0\rangle|^{2N} d\sigma(\zeta)\Big]^{-1}. \tag{3}$$

The letter K will be reserved to denote a constant depending on d but on nothing else. It may vary from one occurrence to another.

Results. Let c be a positive number smaller than 1 and let Λ be a maximal $c/\sqrt{N}$ separated subset of S_d, i.e. a maximal set with the property that for all $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1 \neq \lambda_2$ one has $\varrho(\lambda_1, \lambda_2) \geqslant c/\sqrt{N}$.

The next lemma summarizes some obvious properties of such sets.

LEMMA 1. *Let Λ be a maximal $c/\sqrt{N}$ separated set. Then*

(a) *The collection of balls $\{B(\lambda, c/\sqrt{N})\}_{\lambda\in\Lambda}$ covers S_d, so $\sigma(B(c/\sqrt{N})) \geqslant |\Lambda|^{-1}$.*

(b) *The balls $\{B(\lambda, c/2\sqrt{N})\}_{\lambda\in\Lambda}$ are disjoint, so $\sigma(B(c/2\sqrt{N})) \leqslant |\Lambda|^{-1}$.*

(c) $|\Lambda|^{-1} \leqslant \sigma(B(c/\sqrt{N})) \leqslant 4^{d-1}\sigma(B(c/2\sqrt{N})) \leqslant 4^{d-1}|\Lambda|^{-1}$.

Our main result is the following theorem.

THEOREM 1. *There exists a $c_0 = c_0(d) > 0$ such that if $c < c_0$ and $1 \leqslant p \leqslant \infty$ and Λ is a maximal $c/\sqrt{N}$ separated subset of S_d then for all $\varphi \in W_N(d)$*

$$A\,\|\varphi|\Lambda\|_p \leqslant \|\varphi\|_p \leqslant B\,\|\varphi|\Lambda\|_p \tag{4}$$

for some constant A depending on d, p, and c and B depending on c and d.

Proof. Let us start with the left-hand side inequality in (4). It is trivial (with $A = 1$) for $p = \infty$. For $p < \infty$, using Lemma 1 (b) we obtain

$$\int_{S_d} |\varphi(\zeta)|^p d\sigma(\zeta) \geqslant \sum_{\lambda\in\Lambda} \int_{B(\lambda, c/2\sqrt{N})} |\varphi(\zeta)|^p d\sigma(\zeta).$$

Since every ball $B(\lambda, c/2\sqrt{N})$ is invariant with respect to all unitary maps fixing λ, we can average each integral and obtain (cf. [9], Lemma 1.3)

$$\begin{aligned}\int_{B(\lambda, c/2\sqrt{N})} |\varphi(\zeta)|^p d\sigma(\zeta) &\geqslant |\varphi(\lambda)|^p \int_{B(\lambda, c/2\sqrt{N})} |\langle\zeta, \lambda\rangle^N|^p d\sigma(\zeta)\\ &\geqslant |\varphi(\lambda)|^p \sigma(B(\lambda, c/2\sqrt{N}))\Big(1-\frac{c^2}{4N}\Big)^{Np}\\ &\geqslant |\varphi(\lambda)|^p \sigma(B(c/2\sqrt{N}))\,0.75^{c^2 p}.\end{aligned}$$

From Lemma 1 we get the desired conclusion with $A \geqslant 4^{-(d-1)/p}0.75^{c^2}$.

The proof of the right-hand side inequality is more involved. From Lemma 1 (a) we have

$$\|\varphi\|_p \leqslant \Big(\sum_{\lambda\in\Lambda} \int_{B(\lambda, c/\sqrt{N})} |\varphi(\zeta)|^p d\sigma(\zeta)\Big)^{1/p}.$$

Let $\zeta_\lambda \in B(\lambda, c/\sqrt{N})$ be such that

$$|\varphi(\zeta_\lambda)| = \max\{|\varphi(\zeta)|:\ \zeta \in B(\lambda, c/\sqrt{N})\}$$

and $\langle\lambda, \zeta_\lambda\rangle$ is real positive. Obviously we have

$$\begin{aligned}\|\varphi\|_p &\leqslant \Big(\sum_{\lambda\in\Lambda} \sigma(B(c/\sqrt{N}))|\varphi(\zeta_\lambda)|^p\Big)^{1/p}\\ &\leqslant 4^{(d-1)/p}\Big(\sum_{\lambda\in\Lambda} |\Lambda|^{-1}|\varphi(\zeta_\lambda)|^p\Big)^{1/p}\\ &\leqslant 4^{(d-1)/p}\Big(\sum_{\lambda\in\Lambda} |\Lambda|^{-1}|\varphi(\lambda)|^p\Big)^{1/p}\\ &\quad + 4^{(d-1)/p}\Big(\sum_{\lambda\in\Lambda} |\Lambda|^{-1}|\varphi(\lambda)-\varphi(\zeta_\lambda)|^p\Big)^{1/p}.\end{aligned} \tag{5}$$

Now we will concentrate our attention on the last summand in (5). Let us denote by $L_p(\Lambda)$ the space of all functions on the set Λ with the norm $(|\Lambda|^{-1}\sum|a_\lambda|^p)^{1/p}$. For a maximal $c/\sqrt{N}$ separated subset of S_d and ζ_λ as above we define an operator T: $W_N^p(d) \to L_d(\Lambda)$ by the formula $T(\varphi) = \{\varphi(\lambda) - \varphi(\zeta_\lambda)\}_{\lambda\in\Lambda}$. Using (2) we can write

$$T(\varphi) = \{D \int_{S_d} \varphi(\zeta)\Phi_\lambda(\zeta)\,d\sigma(\zeta)\}_{\lambda\in\Lambda} \tag{6}$$

where D is $\dim W_N(d)$ and $\Phi_\lambda(\zeta) = \langle\lambda, \zeta\rangle^N - \langle\zeta_\lambda, \zeta\rangle^N$. Actually (6) defines an operator from $L_p(S_d)$ into $L_p(\Lambda)$.

PROPOSITION 1. *Let T: $L_p(S_d) \to L_d(\Lambda)$ $(1 \leqslant p \leqslant \infty)$ be given by (6). Then $\|T\| \leqslant cK$.*

Clearly (5) and Proposition 1 give the right-hand side estimate in Theorem 1 if c_0 is taken to be the reciprocal of the constant K from Proposition 1 times $4^{-(d-1)/p}$.

The following lemma is crucial for the proof of Proposition 1.

LEMMA 2. *Let λ, $\zeta_\lambda \in S_d$ be such that $\langle\lambda, \zeta_\lambda\rangle$ is real positive and $\varrho(\lambda, \zeta_\lambda) \leqslant c/\sqrt{N}$ with $c \leqslant 1$. Let us put*

$$\gamma_\lambda = (\lambda+\zeta_\lambda)\,\|\lambda+\zeta_\lambda\|^{-1} \in S_d.$$

Then we have

$$(7)\qquad |\langle\lambda, \zeta\rangle^N - \langle\zeta_\lambda, \zeta\rangle^N| \leqslant \begin{cases} c\sqrt{2} & \text{if } \varrho(\zeta, \gamma_\lambda) \leqslant 2/\sqrt{N}, \\ c\varrho(\gamma_\lambda, \zeta)\sqrt{N}\, e^{-2\sqrt{N}\varrho(\gamma_\lambda,\zeta)} & \text{if } \varrho(\zeta, \gamma_\lambda) \geqslant 2/\sqrt{N}. \end{cases}$$

Proof. Using a unitary change of variables we can assume $\lambda = (\alpha, \beta, 0, \ldots, 0)$ and $\zeta_\lambda = (\alpha, -\beta, 0, \ldots, 0)$ with α real positive and $0 \leqslant \beta \leqslant c/\sqrt{2N}$. Obviously in those coordinates $\gamma_\lambda = (1, 0, \ldots, 0)$. Using the binomial expansion and an obvious inequality $\binom{N}{k} \leqslant N\binom{N-1}{k-1}$ we have

$$(8)\qquad \begin{aligned} &|(\alpha\bar{z}_1+\beta\bar{z}_2)^N - (\alpha\bar{z}_1-\beta\bar{z}_2)^N| \\ &\quad = \Big|\sum_{\substack{k=0\\ k\ \text{odd}}}^{N} \binom{N}{k}(\alpha\bar{z}_1)^{N-k}(\beta\bar{z}_2)^k\Big| \\ &\quad \leqslant \beta|z_2| \sum_{\substack{k=0\\ k\ \text{odd}}}^{N} \binom{N}{k}(\alpha|z_1|)^{N-k}(\beta|z_2|)^{k-1} \\ &\quad \leqslant N\beta|z_2| \sum_{k=0}^{N} \binom{N-1}{k}(\alpha|z_1|)^{N-k-1}(\beta|z_2|)^k \\ &\quad \leqslant (c/\sqrt{2})\sqrt{N}\,|z_2|\,(\alpha|z_1|+\beta|z_2|)^{N-1}. \end{aligned}$$

This clearly gives (7) for $N \leqslant 4$. Also for $\varrho(\zeta, \gamma_\lambda) \leqslant 2/\sqrt{N}$, i.e. for $|z_1| \geqslant 1 - 4/N$ one easily checks that (7) holds for arbitrary N. For $\varrho(\zeta, \gamma_\lambda) \geqslant 2/\sqrt{N}$ we put $|z_1| = 1 - a/N$, $a \geqslant 4$, and we have (since $N \geqslant 5$) by (8)

$$\begin{aligned} |(\alpha\bar{z}_1+\beta\bar{z}_2)^N - (\alpha\bar{z}_1-\beta\bar{z}_2)^N| &\leqslant c\sqrt{N/2}\cdot\sqrt{1-|z_1|^2}\,(|z_1|+c|z_2|/\sqrt{2N})^{N-1} \\ &\leqslant c\sqrt{a}\,(1-a/N+c\sqrt{a}/N)^{N-1} \\ &= c\sqrt{a}\,(1-(a-c\sqrt{a})/N)^{N-1} \leqslant c\sqrt{a}\,e^{-2\sqrt{a}} \\ &= c\sqrt{N}\,\varrho(\gamma_\lambda, \zeta)\,e^{-2\sqrt{N}\varrho(\gamma_\lambda,\zeta)}. \end{aligned}$$

Proof of Proposition 1. Since, as is well known, the norm of T as a map from $L_1(S_d)$ into $L_1(\Lambda)$ can be majorized by $D|\Lambda|^{-1}\|\sum_{\lambda\in\Lambda}|\Phi_\lambda|\|_\infty$ and the norm of T as a map from $L_\infty(S_d)$ into $L_\infty(\Lambda)$ equals $\sup\{D\int|\Phi_\lambda|\,d\sigma\colon \lambda \in \Lambda\}$

Proposition 1 will follow by the Riesz–Thorin interpolation theorem (cf. [11]) from the following inequalities:

$$(9)\qquad D|\Lambda|^{-1}\sup_{\zeta\in S_d}\sum_{\lambda\in\Lambda}|\Phi_\lambda(\zeta)| \leqslant Kc,$$

$$(10)\qquad \sup_{\lambda\in\Lambda} D\int|\Phi_\lambda(\zeta)|\,d\sigma(\zeta) \leqslant Kc.$$

Proof of (9). By (1), for a fixed ζ there are at most $4^{d-1}c^{-2(d-1)}k^{2(d-1)}$ points λ in Λ such that $\varrho(\gamma_\lambda, \zeta) \leqslant k/\sqrt{N}$. Using this and (7) we infer that

$$(11)\qquad \begin{aligned} &\sum_{\lambda\in\Lambda}|\Phi_\lambda(\zeta)| \\ &\quad \leqslant c\Big(\sqrt{2}\,4^{d-1}c^{-2(d-1)}4^{2(d-1)} + \sum_{k=1}^{N} ke^{-2k}4^{d-1}c^{-2(d-1)}k^{2(d-1)}\Big) \\ &\quad \leqslant c^{-2d+3}\Big(\sqrt{2}\,4^{3(d-1)} + 4^{d-1}\sum_{k=1}^{\infty} k^{2d-1}e^{-2k}\Big) \\ &\quad \leqslant c^{-2d+3}K. \end{aligned}$$

We see from (3) that $D \leqslant KN^{d-1}$ and we infer from Lemma 1 (a) and (1) that $|\Lambda|^{-1} \leqslant 2^{d-1}c^{2(d-1)}N^{-(d-1)}$. Putting all this together we get (9).

Proof of (10). From (7) and (11) we infer that

$$\int|\Phi_\lambda(\zeta)|\,d\sigma(\zeta) \leqslant |\Lambda|^{-1}c^{-2d+3}K.$$

As previously Lemma 1 (a) and (1) give (10).

Remark. The drawback of Theorem 1 (or rather of our proof of it) is that the constants are bad. Both c_0 and B depend heavily on the dimension d; c_0 gets small and B gets big when d goes to infinity. In the most important case $p = \infty$ we can do better. The following theorem holds.

THEOREM 2. *Let Λ be a maximal $c/\sqrt{N}$ separated subset of S_d with $c < 1/6$. Then for every $\varphi \in W_N(d)$ we have*

$$(12)\qquad \|\varphi|\Lambda\|_\infty \leqslant \|\varphi\|_\infty \leqslant (1-6c)^{-2}\|\varphi|\Lambda\|_\infty.$$

Clearly Theorem 2 follows immediately from the following

LEMMA 3. *Let $\varphi \in W_N(d)$ with $\|\varphi\|_\infty = 1 = \varphi(1, 0, \ldots, 0)$ be given. For every $\zeta \in S_d$ such that $\varrho(\zeta, (1, 0, \ldots, 0)) \leqslant c/\sqrt{N}$ with $c < 1/6$ we have $|\varphi(\zeta)| \geqslant (1-6c)^2$.*

In order to prove Lemma 3 we will use the following elementary consequence of the Möbius invariant Schwarz Lemma (see Lemma 1.2 of [4]).

LEMMA 4. *Let $f(z)$, $z \in C$, $|z| < 1$, be an analytic function such that $|f(z)| \leqslant \sqrt{e}$ for $|z| < 1$ and $f(0) = 1$. Then*

$$|f(z)| \geqslant 1-(\sqrt{e}+1)|z|.$$

Proof of Lemma 3. We can clearly assume that $\zeta = (\alpha, \beta, 0, \ldots, 0)$ with α real positive and $|\beta| \leqslant c\sqrt{2/N}$ and $\alpha \geqslant 1-c^2/N$. Let us define a function of one complex variable by

$$f(z) = \varphi(1, z/\sqrt{N}, 0, \ldots, 0) \quad \text{for } |z| < 1.$$

Since $\|(1, z/\sqrt{N}, 0, \ldots, 0)\| \leqslant \sqrt{1+1/N}$ and $\varphi \in W_N(d)$ we see that $|f(z)| \leqslant \sqrt{e}$ for $|z| < 1$. Moreover, $f(0) = 1$. Since $|\varphi(\zeta)| = \alpha^N |f(\sqrt{N}\beta/\alpha)|$, Lemma 4 gives

$$\begin{aligned}|\varphi(\zeta)| &\geqslant \alpha^N(1-(\sqrt{e}+1)|\sqrt{N}\beta/\alpha|)\\ &\geqslant (1-c^2/N)^N(1-(\sqrt{e}+1)c\sqrt{2}/\alpha) \geqslant e^{-c^2}(1-(\sqrt{e}+1)2c)\\ &\geqslant (1-6c)^2.\end{aligned}$$

Remark. It follows from Lemma 1 that for fixed c and d the cardinality of Λ is proportional to dim $W_N(d)$ independently of N. So in the terminology of [3], $W_N(d)$ is a large subspace of l_∞^n.

Now we will present a strengthening of the main result of [5] and also of Theorem 1 of [9].

THEOREM 3. *There exists a constant $K = K(d)$ such that for every α, $0 < \alpha \leqslant 1$, every $N = 1, 2, \ldots$ and every subspace $E \subset W_N(d)$ with $\dim E \geqslant \alpha \dim W_N(d)$ there exists a polynomial $\varphi \in E$ such that $\|\varphi\|_\infty \leqslant K(d)\alpha^{-1}\|\varphi\|_2$.*

Remark. The very existence of $\varphi \in W_N(d)$ such that $\|\varphi\|_\infty \leqslant K\|\varphi\|_2$ with $K = 2^d/\sqrt{\pi}$ was proved in [9] (see also [8]). Such polynomials turned out to be quite useful in various questions about analytic functions in the ball (see [1], [2], [9], [10]). One can hope that the possibility to find them in some specified subspaces (as in [5] or our Theorem 3) will also be useful. In order to prove Theorem 3 we will use two lemmas.

LEMMA 5. *Let $F \subset L_\infty(\Lambda)$ be a subspace of dimension k. Then there exists $x \in F$, $\|x\| = 1$, such that $|x(\lambda)| = 1$ for at least k indices λ.*

The easy proof of this lemma can be found in [6], p. 214 and in [3], p. 95.

LEMMA 6. *If Λ is a maximal $c/\sqrt{N}$ separated subset of S_d, then*

$$\sum_{\lambda\in\Lambda} |\langle\zeta, \lambda\rangle|^N \leqslant Kc^{-2(d-1)}.$$

The proof is almost identical with the proof of (10) and is omitted.

Proof of Theorem 3. Let us fix $c = (d-1)/6d$ and a maximal $c/\sqrt{N}$ separated subset $\Lambda \subset S_d$. Let $J\colon W_N^\infty(d) \to L_\infty(\Lambda)$ be defined as $J(\varphi) = \{\varphi(\lambda)\}$. From Lemma 5 with $F = J(E)$ we get $x \in J(E)$ such that $\|x\|_\infty = 1$ and $|x(\lambda)| = 1$ for $\lambda \in \Lambda_0 \subset \Lambda$ with $|\Lambda_0| \geqslant \dim E$. We define $\varphi = J^{-1}(x)$ and infer from Theorem 2 that $\|\varphi\|_\infty \leqslant (1-6c)^{-2}$. In order to estimate $\|\varphi\|_2$ we define

$$\psi(\zeta) = \sum_{\lambda\in\Lambda_0} \varphi(\lambda)\langle\zeta, \lambda\rangle^N.$$

Since $|\varphi(\lambda)| = 1$, Lemma 6 gives $\|\psi\|_2 \leqslant Kc^{-2(d-1)}$. Using (2) and (3) we obtain

$$\begin{aligned}Kc^{-2(d-1)}\|\varphi\|_2 &\geqslant \left|\int \varphi(\zeta)\overline{\psi(\zeta)}\,d\sigma(\zeta)\right|\\ &= \sum_{\lambda\in\Lambda_0} \overline{\varphi(\lambda)}\varphi(\lambda)D^{-1} = |\Lambda_0|D^{-1} \geqslant \alpha.\end{aligned}$$

Because of our choice of c we obtain $\|\varphi\|_\infty \leqslant K\alpha^{-1}\|\varphi\|_2$.

Remark. Our Theorem 1 is clearly analogous to the classical Marcinkiewicz theorems from the theory of trigonometric series (see [11], X.7.5 and X.7.28). There is, however, a difference. For the trigonometric polynomials, the number of points required to estimate the L_p-norm ($1 < p < \infty$) equals the dimension of the space and only for $p = 1$ and $p = \infty$ one has to take the number of points which is proportional to the dimension (but the proportionality constant can be an arbitrary number greater than 1). In our case if one takes $|\Lambda| = \dim W_N(2)$ the conclusion of Theorem 1 does not hold. To be more precise: Let us consider numbers $C(p, N)$ such that for some $\Lambda \subset S_2$, $|\Lambda| = \dim W_N(2) = N+1$, we have

$$\Big(\int_{S_2} |\varphi(\zeta)|^p\,d\sigma(\zeta)\Big)^{1/p} \leqslant C(p, N)\Big(|\Lambda|^{-1}\sum_{\lambda\in\Lambda} |\varphi(\lambda)|^p\Big)^{1/p}$$

for all $\varphi \in W_N(2)$.

Then Theorem 2.4 of [9] gives $C(\infty, N) \geqslant \sqrt{N}$ and Proposition 2.2 of [9] gives $C(1, N) \geqslant \sqrt{N}$. By an easy modification of arguments from Section 2 of [9] one can see that $C(p, N) \underset{N\to\infty}{\to} \infty$ at least as fast as some power of N. It is quite likely that $C(p, N) \geqslant \sqrt{N}$ for all p, $1 \leqslant p \leqslant \infty$, and that analogous estimates hold for other d's. Since the computations are likely to be quite involved and we do not see any applications for such a result we decided not to investigate this question in detail.

References

[1] A. B. Aleksandrov, *Inner functions on compact spaces*, Funktsional. Anal. i Prilozhen. 18 (2) (1984), 1–13 (in Russian).

[2] H. Alexander, *On zero sets for the ball algebra*, Proc. Amer. Math. Soc. 86 (1) (1982), 71–74.

[3] T. Figiel and W. B. Johnson, *Large subspaces of l_∞^n and estimates of the Gordon–Lewis constant*, Israel J. Math. 37 (1–2) (1980), 92–112.

[4] J. Garnett, *Bounded Analytic Functions*, Academic Press, 1981.

[5] B. S. Kashin, *On homogeneous polynomials of several variables on the complex sphere*, Mat. Sb. 126 (3) (1985), 420–426 (in Russian).

[6] A. Pietsch, *s-numbers of operators in Banach spaces*, Studia Math. 51 (1974), 201–223.

[7] W. Rudin, *Function Theory in the Unit Ball of C^N*, Springer, 1980.

[8] —, *The Ryll–Wojtaszczyk polynomials*, Ann. Polon. Math. 46 (1985), 291–294.

[9] J. Ryll and P. Wojtaszczyk, *On homogeneous polynomials on complex ball*, Trans. Amer. Math. Soc. 276 (1) (1983), 107–116.

[10] P. Wojtaszczyk, *On functions in the ball algebra*, Proc. Amer. Math. Soc. 85 (2) (1982), 184–186.

[11] A. Zygmund, *Trigonometric Series*, Cambridge University Press, 1979.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES

Received April 16, 1985
Revised version June 20, 1985 (2049)

Added in proof (June 1986). Theorem 2 and its version for more general domains in C^d have been proved by Prof. Lars Hörmander.

STUDIA MATHEMATICA, T. LXXXIV. (1986)

The boundary of Taylor's joint spectrum for two commuting Banach space operators

by

VOLKER WROBEL (Kiel)

Abstract. In this note it is shown that the boundary $\partial\sigma$ of Taylor's joint spectrum for a pair of commuting operators on an arbitrary Banach space is contained in the union of the joint approximate point spectrum $AP\sigma$ and the joint approximate compression spectrum $AC\sigma$, but neither $\partial\sigma \subset AP\sigma$ nor $\partial\sigma \subset AC\sigma$ is true in general. This is in strict contrast to the case of a single operator where $\partial\sigma \subset AP\sigma \cap AC\sigma$.

1. Introduction. In [5] and [6] F.-H. Vasilescu characterized Taylor's joint spectrum [3] for commuting operators on Hilbert spaces by means of the noninvertability of a certain operator acting on a direct sum of copies of the initial space. In this way he succeeded in giving a characterization of Taylor's joint spectrum in terms of classical spectral theory.

Based on Vasilescu's characterization C. Muneo and M. Takaguchi [2] proved that the boundary of Taylor's joint spectrum for a pair of commuting Hilbert space operators is contained in the union of the joint approximate point spectrum and the joint approximate compression spectrum in the sense of A. T. Dash [1]. Since this union is of course contained in Taylor's joint spectrum, the result of Muneo and Takaguchi gives an easy characterization of at least an important part of the spectrum. The method of proof in [2] heavily relies on the Hilbert space setting. It is the purpose of this note to show that the above-mentioned result holds true in the Banach space setting, too. As it seems our proof is completely elementary.

Moreover, we shall show that in general neither $\partial\sigma \subset AP\sigma$ nor $\partial\sigma \subset AC\sigma$, but $AP\sigma \cap AC\sigma$ is nonempty for two commuting operators.

Let X, Y, Z denote complex Banach spaces and let $L(X, Y)$ denote the space of all continuous linear operators from X into Y, writing $L(X)$ for $L(X, X)$ and X' for the dual space $L(X, C)$ instead. Given $S \in L(X, Y)$ we let $S' \in L(Y', X')$ denote the dual operator.

Let $T = (T_1, T_2)$ $(T_i \in L(X),\ i = 1, 2)$ denote a pair of commuting operators. Consider the sequence

$$0 \to X \xrightarrow{\delta_T^0} X \oplus X \xrightarrow{\delta_T^1} X \to 0 \tag{1.1}$$