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Interpolation between Hardy spaces on the bidisc

by

KAI-CHING LIN (Chicago, Ill.)

Abstract. For $0 < p \leqslant \infty$ let $H^p = H^p(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$ be the real variables Hardy spaces on the bi-upper half plane. Let $(\cdot, \cdot)_\theta$ be the Calderón complex interpolation spaces, $(\cdot, \cdot)_{\theta,p}$ the Peetre K real interpolation spaces. We calculate the interpolation spaces between $H^{p_0}(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$ and $H^{p_1}(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$, in both real $(0 \leqslant p_0 < p_1 \leqslant \infty)$ and complex $(1 \leqslant p_0 < p_1 \leqslant \infty)$ methods.

1. Introduction. For $0 < p \leqslant \infty$ let $H^p = H^p(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$ be the real variables Hardy spaces on the bi-upper half plane. Let $(\cdot, \cdot)_\theta$ be the complex interpolation spaces as described in [2], $(\cdot, \cdot)_{\theta,p}$ the Peetre K real interpolation spaces. (We refer the reader to [1] for the general theory of interpolation spaces.) We will calculate the interpolation spaces between $H^{p_0}(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$ and $H^{p_1}(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$, in both real $(0 < p_0 < p_1 \leqslant \infty)$ and complex $(1 \leqslant p_0 < p_1 \leqslant \infty)$ methods.

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Now we describe some basic properties of $H^p(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$. For $1 < p < \infty$, H^p can be identified with $L^p(\mathbf{R}^2)$. For $0 < p \leqslant 1$, H^p can be defined via non-tangential maximal functions as well as square functions. More precisely, let $\varphi \in C_c^\infty(\mathbf{R}^1)$, $\int_{\mathbf{R}^1} \varphi = 1$. For $t = (t_1, t_2)$ with $t_j > 0$ and $y = (y_1, y_2) \in \mathbf{R}^2$, we define

$$\varphi_t(y) = \varphi_{t_1}(y_1)\varphi_{t_2}(y_2) = \frac{1}{t_1 t_2}\varphi\left(\frac{y_1}{t_1}\right)\varphi\left(\frac{y_2}{t_2}\right).$$

If $x = (x_1, x_2)$ in $\mathbf{R}^2$, $\Gamma(x)$ will denote the product cone

$$\Gamma(x) = \Gamma(x_1) \times \Gamma(x_2) = \{(y, t) \in \mathbf{R}_+^2 \times \mathbf{R}_+^2 \colon |y_j - x_j| < t_j, j = 1, 2\}.$$

We say that a tempered distribution f on $\mathbf{R}^2$ is in $H^p(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$ if the nontangential maximal function

$$Nf(x) = \sup_{(y,t) \in \Gamma(x)} |f * \varphi_t(y)|$$

is in $L^p(\mathbf{R}^2)$. To define H^p by square functions, take $\psi \in C_c^\infty(\mathbf{R}^1)$, $\int_{\mathbf{R}^1} \psi = 0$, $\psi \not\equiv 0$, and let $\psi_t(y) = \frac{1}{t_1 t_2}\psi\left(\frac{y_1}{t_1}\right)\psi\left(\frac{y_2}{t_2}\right)$ as before. Then the square function of f is defined by

$$Sf(x) = \left(\int\int_{\Gamma(x)} |f * \psi_t(y)|^2 \, dy \frac{dt}{t_1^2 t_2^2}\right)^{1/2}$$

and we say that $f \in H^p$ if $Sf \in L^p$.

It is well known that these two definitions are equivalent and $\|Nf\|_p$, $\|Sf\|_p$ are comparable up to a constant depending only on the choices of φ and ψ. It is also a fact that both N and S map L^p into itself boundedly for $1 < p < \infty$. See [4], [5], [9] and [13] for further details.

2. The complex method. We will recall here the definition of complex interpolation spaces. For a couple of Banach spaces A^0 and A^1 let $\mathfrak{F} = \mathfrak{F}(A^0, A^1)$ be the family of all functions G from the strip $S = \{z \in C\colon 0 < \operatorname{Re} z < 1\}$ to $A^0 + A^1$ which satisfy the following properties:

(i) $G(z)$ is continuous and bounded on $\bar{S}$.

(ii) $G(z)$ is analytic in S.

(iii) $G(z)$ is a continuous and bounded map from $\{\operatorname{Re} z = j\}$ into A^j, $j = 0, 1$.

We equip $\mathfrak{F}$ with the norm

$$\|G\|_{\mathfrak{F}} = \max\big(\sup_{y \in \mathbf{R}} \|G(iy)\|_{A^0}, \sup_{y \in \mathbf{R}} \|G(1+iy)\|_{A^1}\big).$$

The complex interpolation space is then defined by $(A^0, A^1)_\theta = \{G(\theta)\colon G \in \mathfrak{F}\}$ and the norm is given by

$$\|a\|_{(A^0, A^1)_\theta} = \inf\{\|G\|_{\mathfrak{F}}\colon G(\theta) = a\}.$$

One of the major theorems in interpolation theory states that whenever T is a bounded linear operator from A^j to B^j, $j = 0, 1$, then T is also bounded from $(A^0, A^1)_\theta$ to $(B^0, B^1)_\theta$, for $0 < \theta < 1$.

The following reiteration result is due to Wolff.

THEOREM A (Wolff [14]). *Let A^1, A^2, A^3, A^4 be Banach spaces and assume $A^1 \cap A^4$ is dense in A^2 and A^3. Suppose $(A^2, A^4)_\theta = A^3$, $(A^1, A^3)_\varphi = A^2$, $0 < \theta$, $\varphi < 1$. Then $(A^1, A^4)_\xi = A^2$, $(A^1, A^4)_\psi = A^3$, where*

$$\xi = \frac{\varphi\theta}{1 - \varphi + \varphi\theta}, \quad \psi = \frac{\theta}{1 - \varphi + \varphi\theta}.$$

We are ready to prove the main theorem in this section. Here, we set $H^\infty = L^\infty$.

THEOREM 1. *If $1 \leqslant p_0 < p_1 \leqslant \infty$, then $(H^{p_0}, H^{p_1})_\theta = L^p$, where*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Proof. It suffices to show that $(H^1, L^2)_\theta = L^p$, where $1/p = 1 - \theta/2$. Indeed, if $(H^1, L^2)_\theta = L^p$, Wolff's reiteration Theorem A cited above with $A^1 = H^1$, $A^2 = L^p$ $(p < 2)$, $A^3 = L^2$, $A^4 = L^{p_1}$ $(2 < p_1 \leqslant \infty)$ implies that $(H^1, L^{p_1})_\theta = L^p$ for $p < 2$. Using Wolff's theorem again with $A^1 = H^1$, $A^2 = L^{p_0}$ $(p_0 < 2)$, $A^3 = L^p$ $(p > 2)$, $A^4 = L^{p_1}$ $(p < p_1 \leqslant \infty)$, we have $(H^1, L^{p_1})_\theta = L^p$ for $p > 2$. The other cases are standard L^p interpolation.

The direction $(H^1, L^2)_\theta \subseteq L^p$ is easy. Let Nf be the nontangential maximal function of f as defined in Section 1. Then N maps H^1 into L^1 and L^2 into L^2 continuously. Therefore, by Janson's theorem [10], N maps $(H^1, L^2)_\theta$ into $(L^1, L^2)_\theta = L^p$, since N is sublinear. This means that $\|Nf\|_p \leqslant c\|f\|_{(H^1, L^2)_\theta}$. Because Nf and f have comparable L^p-norms when $1 < p < \infty$, we conclude that $(H^1, L^2)_\theta \subseteq L^p$.

Now we begin to prove the converse direction, i.e. $L^p \subseteq (H^1, L^2)_\theta$. Given a function $f \in L^p$, we will construct an analytic $G(z)$ which maps the strip $\{p/2 \leqslant \operatorname{Re} z \leqslant p\}$ into the space of tempered distributions and which satisfies $\|G(z)\|_{H^1} \leqslant C\|f\|_{L^p}$ when $\operatorname{Re} z = p$, $\|G(z)\|_{L^2} \leqslant C\|f\|_{L^p}$ when $\operatorname{Re} z = p/2$, and $G(1) = f$. This will prove the theorem, because we can then compose G with a linear mapping which takes $\{0 \leqslant \operatorname{Re} z \leqslant 1\}$ onto $\{p/2 \leqslant \operatorname{Re} z \leqslant p\}$ so that the resulting function is in the analytic family $\mathfrak{F}$ with desired bounds. Let Sf be the square function of f with ψ even. Let $O_j = \{Sf > 2^j\}$, $\tilde{O}_j = \{M_s \chi_{O_j} > \frac{1}{100}\}$, where M_s is the strong maximal operator, i.e.

$$M_s g(x) = \sup_{x \in R} |R|^{-1} \int_R |g|,$$

where the sup is taken over all rectangles R containing x with sides parallel to the axes of coordinates. Define

$$\mathscr{R}_j = \{R \text{ dyadic rectangles: } |R \cap O_j| \geqslant \tfrac{1}{2}|R|,\ |R \cap O_{j+1}| < \tfrac{1}{2}|R|\}.$$

For a rectangle $R = I \times J$, R_+ will denote the set

$$\{(y, t) \in \mathbf{R}_+^2 \times \mathbf{R}_+^2\colon y \in R,\ \tfrac{1}{2}|I| < t_1 \leqslant |I|,\ \tfrac{1}{2}|J| < t_2 \leqslant |J|\}.$$

We also set $A_j = \bigcup_{R \in \mathscr{R}_j} R_+$. The following result is contained in the proof of the atomic decomposition for $H^1(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$ functions:

THEOREM B (Chang–R. Fefferman [4], pp. 183–185). *Let $f \in L^p$. With the above notation, we can write*

$$f = \sum_{j=-\infty}^{\infty} f_j, \quad \text{where} \quad f_j(x) = \int\int_{A_j} f * \psi_t(y)\psi_t(x-y)\, dy \frac{dt}{t_1 t_2}$$

with $\|f_j\|_{L^2} \leqslant C 2^j |O_j|^{1/2}$ and $\|f_j\|_{H^1} \leqslant C 2^j |O_j|$.

Following Calderón–Torchinsky [3], we form our analytic function

$$G(z) = \sum_{j=-\infty}^{\infty} f_j 2^{j(z-1)} \|Sf\|_p^{1-z}.$$

It is clear that $G(1) = f$. If $\operatorname{Re} z = p$, then

$$\|G(z)\|_{H^1} \leqslant \sum_{j=-\infty}^{\infty} \|f_j\|_{H^1} 2^{j(p-1)} \|Sf\|_p^{1-p} \leqslant C \sum_{j=-\infty}^{\infty} 2^{jp} |O_j| \, \|Sf\|_p^{1-p}$$
$$\leqslant C \|Sf\|_p^p \|Sf\|_p^{1-p} = C\|Sf\|_p \leqslant C\|f\|_p.$$

If $\operatorname{Re} z = p/2$, we will use duality to prove $\|G(z)\|_2 \leqslant C\|f\|_p$. Take $h \in L^2(\boldsymbol{R}^2)$, $\|h\|_2 \leqslant 1$; we then have

$$\Big|\int_{\boldsymbol{R}^2} \big(\sum_j f_j(x) 2^{j(z-1)}\big) h(x)\,dx\Big|$$
$$= \Big|\int_{\boldsymbol{R}^2} \Big(\sum_j 2^{j(z-1)} \iint_{A_j} f*\psi_t(y)\,\psi_t(x-y)\frac{dy\,dt}{t_1 t_2}\Big) h(x)\,dx\Big|$$
$$= \Big|\sum_j 2^{j(z-1)} \iint_{A_j} f*\psi_t(y)\big(\int_{\boldsymbol{R}^2} \psi_t(x-y)\,h(x)\,dx\big)\frac{dy\,dt}{t_1 t_2}\Big|$$
$$= \Big|\sum_j 2^{j(z-1)} \iint_{A_j} f*\psi_t(y)\cdot h*\psi_t(y)\frac{dy\,dt}{t_1 t_2}\Big|, \quad \text{since } \psi \text{ is even}$$
$$\leqslant \Big(\iint_{\boldsymbol{R}_+^2 \times \boldsymbol{R}_+^2} \big|\sum_j 2^{j(z-1)} \chi_{A_j}(y)\, f*\psi_t(y)\big|^2 \frac{dy\,dt}{t_1 t_2}\Big)^{1/2}$$
$$\times \Big(\iint_{\boldsymbol{R}_+^2 \times \boldsymbol{R}_+^2} |h*\psi_t(y)|^2 \frac{dy\,dt}{t_1 t_2}\Big)^{1/2}$$
$$\leqslant C\Big(\sum_j \iint_{A_j} 2^{j(p-2)} |f*\psi_t(y)|^2 \frac{dy\,dt}{t_1 t_2}\Big)^{1/2} \big(\int_{\boldsymbol{R}^2} |h(x)|^2\,dx\big)^{1/2}$$
$$\leqslant C\Big(\sum_j 2^{j(p-2)} \iint_{A_j} |f*\psi_t(y)|^2 \,|\{x\colon (y,t)\in\Gamma(x),$$
$$x \in \tilde{O}_j - O_{j+1}\}|\frac{dy\,dt}{t_1^2 t_2^2}\Big)^{1/2}$$

(this is because for each (y, t) in A_j, the rectangle centered at y with side lengths t_1 and t_2 has a fixed portion of its area inside $\tilde{O}_j$ and at the same time outside O_{j+1}; see [4], page 186)

$$\leqslant C\big(\sum_j 2^{j(p-2)} \int_{\tilde{O}_j - O_{j+1}} |Sf|^2\big)^{1/2} \leqslant C\big(\sum 2^{j(p-2)} 2^{2j} |O_j|\big)^{1/2}$$
$$\leqslant C\big(\int_{\boldsymbol{R}^2} |Sf|^p\big)^{1/2} = C\|Sf\|_p^{p/2} \leqslant C\|f\|_p^{p/2}.$$

Thus, $\|G(z)\|_{L^2} \leqslant C\|f\|_p$. This finishes the proof of Theorem 1. ■

3. The real method. The result for real interpolation we will prove is

THEOREM 2. *If $0 < p_0 < p_1 \leqslant \infty$, then $(H^{p_0}, H^{p_1})_{\theta,p} = H^p$, where*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Recall that the real interpolation space $(H^{p_0}, H^{p_1})_{\theta,p}$ is defined to be the set of all f in $H^{p_0} + H^{p_1}$ such that

$$\|f\|_{(H^{p_0}, H^{p_1})_{\theta,p}} = \Big(\int_0^\infty [t^{-\theta} K(t,f)]^p \frac{dt}{t}\Big)^{1/p} < \infty,$$

where

$$K(t,f) = K(t,f;\, H^{p_0}, H^{p_1})$$
$$= \inf\{\|f_0\|_{H^{p_0}} + t\|f_1\|_{H^{p_1}}\colon f = f_0 + f_1,\ f_0 \in H^{p_0},\ f_1 \in H^{p_1}\}.$$

We will need the following results.

THEOREM C (Wolff [14]). *Let A^1, A^2, A^3, A^4 be quasi-Banach spaces satisfying $A^1 \cap A^4 \subset A^2 \cap A^3$. Suppose $(A^2, A^4)_{\theta,q} = A^3$ and $(A^1, A^3)_{\varphi,r} = A^2$, $0 < \theta$, $\varphi < 1$, $0 < q$, $r \leqslant \infty$. Then $(A^1, A^4)_{\xi,r} = A^2$ and $(A^1, A^4)_{\psi,q} = A^3$ with*

$$\xi = \frac{\varphi\theta}{1-\varphi+\varphi\theta}, \qquad \psi = \frac{\theta}{1-\varphi+\varphi\theta}.$$

THEOREM D (Chang–R. Fefferman [5]). *Let $0 < p_0 \leqslant 1$, $p_0 < p < 2$ and $\alpha > 0$. If $f \in H^p(\boldsymbol{R}_+^2 \times \boldsymbol{R}_+^2)$, then f can be decomposed as $f = g_\alpha + b_\alpha$, where $g_\alpha \in L^2$, $b_\alpha \in H^{p_0}$ with*

$$\|g_\alpha\|_{L^2}^2 \leqslant C \int_{Sf \leqslant \alpha} |Sf|^2, \qquad \|b_\alpha\|_{H^{p_0}}^{p_0} \leqslant C \int_{Sf > \alpha} |Sf|^{p_0}.$$

THEOREM E (Calderón–Torchinsky [3], p. 156). *Let $g(t) \geqslant 0$ be defined and nonincreasing in $(0, \infty)$, let $u > 0$ and $0 < p \leqslant 1$. Then*

$$(*) \qquad \Big[\int_u^\infty g(t)\,dt\Big]^p \leqslant C \int_{u/2}^\infty g(t)^p\,dt^p.$$

Proof of Theorem 2. With repeated application of Wolff's Theorem C, we only need to show $(H^{p_0}, L^2)_{\theta,p} = H^p$, where $0 < p_0 \leqslant 1$, $1/p = (1-\theta)/p_0 + \theta/2$. Again, the direction $(H^{p_0}, L^2)_{\theta,p} \subseteq H^p$ is easy as in the complex case. For the other direction, fix f in H^p for the moment. For each t, take $\alpha = (Sf)^*(t^{2p_0/(2-p_0)})$, where $(Sf)^*(t)$, the nonincreasing rearrangement of Sf, is defined by $(Sf)^*(t) = \inf\{s\colon |\{|f| > s\}| \leqslant t\}$. We apply the Chang–R. Fefferman decomposition (Theorem D) to f and this α, we obtain

$f = g_\alpha + b_\alpha$ with

$$\|g_\alpha\|_2^2 \leqslant C \int_{Sf\leqslant\alpha} |Sf|^2 \leqslant C\int_0^{\alpha^2} \left|\{|(Sf)^*|^2>\lambda\}\right| d\lambda$$

$$\leqslant Ct^{2p_0/(2-p_0)}[(Sf)^*(t^{2p_0/(2-p_0)})]^2 + C\int_{t^{2p_0/(2-p_0)}}^{\infty} [(Sf)^*(u)]^2\,du$$

$$= I_1^2+I_2^2$$

and

$$\|b_\alpha\|_{H^{p_0}}^{p_0} \leqslant C\int_{Sf>\alpha}|Sf|^{p_0} \leqslant C\int_0^{t^{2p_0/(2-p_0)}} [(Sf)^*(u)]^{p_0}\,du.$$

Now

$$\int_0^\infty t^{-\theta p}(tI_1)^p \frac{dt}{t} = C\int_0^\infty t^{-\theta p+p+pp_0/(2-p_0)}[(Sf)^*(t^{2p_0/(2-p_0)})]^p\frac{dt}{t}$$

$$= C\int_0^\infty |(Sf)^*|^p,$$

by a change of variable $\lambda = t^{2p_0/(2-p_0)}$ and $\frac{1}{p} = \frac{1-\theta}{p_0}+\frac{\theta}{2}$

$$= C\|Sf\|_p^p \leqslant C\|f\|_{H^p}^p,$$

and

$$\int_0^\infty t^{-\theta p}(tI_2)^p\frac{dt}{t} = C\int_0^\infty u^{p((2-p_0)/2p_0)(1-\theta)-1}\Big(\int_u^\infty [(Sf)^*]^2\Big)^{p/2}du$$

$$\leqslant C\int_0^\infty u^{-p/2}\int_{u/2}^\infty |(Sf)^*(t)|^p\,dt^{p/2}\,du, \quad \text{by } (*)$$

$$= C\int_0^\infty |(Sf)^*(t)|^p\Big(\int_0^{2t} u^{-p/2}\,du\Big)dt^{p/2}, \quad \text{by Fubini's theorem}$$

$$= C\int_0^\infty|(Sf)^*|^p \leqslant C\|f\|_{H^p}^p.$$

Moreover,

$$\int_0^\infty t^{-\theta p}\|b_\alpha\|^p_{H^{p_0}}\frac{dt}{t} = \int_0^\infty t^{-\theta p}\Big\{\int_0^{2p_0/(2-p_0)} [(Sf)^*]^{p_0}\Big\}^{p/p_0}\frac{dt}{t}$$

$$= C\int_0^\infty\left\{\frac{1}{u}\int_0^u[(Sf)^*]^{p_0}\right\}^{p/p_0}du,$$

by a change of variable $u = t^{2p_0/(2-p_0)}$

$$\leqslant C\int_0^\infty \{M([(Sf)^*]^p\chi_{(0,\infty)})(u)\}^{p/p_0}\,du,$$

where M is the Hardy–Littlewood maximal operator

$$\leqslant C\int_0^\infty |(Sf)^*|^p \leqslant C\|f\|_{H^p}^p.$$

Combining the three inequalities above, we obtain

$$\|f\|_{(H^{p_0},L^2)_{\theta,p}} \leqslant C\|f\|_{H^p}.$$

Thus, $H^p \subseteq (H^{p_0}, L^2)_{\theta,p}$. The proof is complete. ∎

Remarks. 1. The reason why we can only prove the complex interpolation for the case $1 \leqslant p_0 < p_1 \leqslant \infty$ instead of $0 < p_0 < p_1 \leqslant \infty$ is that Wolff's reiteration theorem is not applicable for quasi-Banach spaces.

2. We refer the reader to [12] for a survey of interpolating Hardy spaces on $\boldsymbol{R}^n$. It is not clear how one can adapt the classical proof to our present setting. For instance, C. Fefferman and Stein in their proof of the result $(L^{p_0}, \mathrm{BMO})_\theta = L^p$ used some properties of the sharp function,

$$f^{\#}(x) = \sup_{\substack{x\in Q\\ Q\text{ cubes}}} |Q|^{-1}\int_Q |f-f_Q|,$$

where f_Q is the average of f over Q. The two key properties they used are:

(a) The map from BMO to L^∞ defined by $f \to f^{\#}$ is a bounded map.

(b) If $f\in L^{p_0}$ for some $p_0 \geqslant 1$, then $f^{\#}$ and f have comparable L^p-norms for $p_0 < p < \infty$.

However, the existence of such an operator on the bidisc is unknown. The most natural analogue of the sharp function is

$$f^{\#}(x) = \sup_{\substack{x\in R\\ R\text{ rectangles}}} |R|^{-1}\int_R |f-f_I-f_J+f_R|,$$

where $R = I\times J$ and f_I, f_J, f_R are the averages of f over I, J and R respectively. But this sharp function fails to satisfy (b) (see [8] for an example). On the other hand, if we let $f^{\#_1}$, $f^{\#_2}$ be the sharp functions along x and y direction, then although the iterated sharp function $f^{\#_1\#_2} = (f^{\#_1})^{\#_2}$ does have comparable L^p norm with f, it does not map BMO into L^∞. It is the idea in Calderón–Torchinsky [3] and Chang–R. Fefferman [4] which makes the above proofs work.

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UNIVERSITY OF CHICAGO
Chicago, Illinois 60637, U.S.A.

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On values of homogeneous polynomials in discrete sets of points

by

P. WOJTASZCZYK (Warszawa)

Abstract. Let $W_N(d)$ denote the space of all homogeneous polynomials on C^d of degree N, restricted to the unit sphere. We show a class of sets Λ of small cardinality such that for every $\varphi \in W_N(d)$ we have $(\int |\varphi|^p)^{1/p}$ comparable to $(|\Lambda|^{-1} \sum_{\lambda\in\Lambda} |\varphi(\lambda)|^p)^{1/p}$. We also show that every subspace $E \subset W_N(d)$ such that $\dim E \geqslant \frac{1}{2}\dim W_N(d)$ contains a polynomial φ such that $\|\varphi\|_\infty \leqslant K(d)\|\varphi\|_2$.

We consider the spaces $W_N(d)$ of all homogeneous polynomials on C^d (the d-dimensional complex space) of degree N. On those spaces we consider the norms inherited from $L_p(S_d)$, i.e. for $\varphi \in W_N(d)$ we put

$$\|\varphi\|_p = \Big(\int_{S_d} |\varphi(\zeta)|^p \, d\sigma(\zeta)\Big)^{1/p}$$

where σ is the normalized rotation-invariant measure on S_d, the unit sphere in C^d. Our main interest in this note is to compare $\|\varphi\|_p$ with its discrete analogue: for a finite subset $\Lambda \subset S_d$ we consider

$$\|\varphi|\Lambda\|_p = \Big(|\Lambda|^{-1} \sum_{\lambda\in\Lambda} |\varphi(\lambda)|^p\Big)^{1/p}$$

(where $|\Lambda|$ denotes the cardinality of Λ). Our main result asserts that it is possible to find relatively small sets Λ such that $\|\varphi|\Lambda\|_p \sim \|\varphi\|_p$ for all $\varphi \in W_N(d)$ (Theorem 1). In the case $p = \infty$ this result was obtained by B. S. Kashin in [5] by a different method. In Theorem 2 we give a simplified version of Kashin's proof which has an additional advantage of giving good constants. As an application of this special case we obtain a strengthening of the main result of [5] and also of Theorem 1 of [9].

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Preliminaries and notation. The natural scalar product in C^d will be denoted by $\langle \cdot, \cdot \rangle$. We will use the unitarily invariant pseudometric on S_d