STUDIA MATHEMATICA, T. LXXXIV. (1986)

The Hölder duality for harmonic functions

by

EWA LIGOCKA (Warszawa)

Abstract. In this paper it is proved that if D is a bounded domain with smooth boundary in $\boldsymbol{R}^n$ then the space of harmonic Hölder functions $\Lambda_\alpha \operatorname{Harm}(D)$ can be represented as the dual space to the space $\mathring{L}^1 \operatorname{Harm}(D, |\varrho|^\alpha)$ which is the closure of $L^2 \operatorname{Harm}(D)$ in $L^1(D, |\varrho|^\alpha)$. The function ϱ is a defining function for D, i.e. $D = \{x \in \boldsymbol{R}^n: \varrho(x) < 0\}$, $\operatorname{grad} \varrho \neq 0$ on ∂D. As a corollary we get the following fact. The Hölder space $\Lambda_\alpha(\partial D)$ can be represented as the dual space to $\mathring{L}^1 \operatorname{Harm}(D, |\varrho|^\alpha)$.

1. Introduction and the statement of results. In [2] S. Bell constructed a family of operators L^s: $C^\infty(\bar{D}) \to C^\infty(\bar{D})$ such that for every $u \in C^\infty(\bar{D})$, $L^s u$ vanishes on ∂D up to order $s-1$ and the function $u - L^s u$ is orthogonal to the space $L^2 \operatorname{Harm}(D)$ of square-integrable harmonic functions on D. Bell uses this construction to establish the duality relation between $\operatorname{Harm}^\infty(\bar{D}) = C^\infty(\bar{D}) \cap \operatorname{Harm}(D)$ and the space

$$\operatorname{Harm}^{-\infty}(D) = \lim\operatorname{ind} \operatorname{Harm}^{-s}(D) \qquad (\operatorname{Harm}^{-s}(D) = W^{-s}(D) \cap \operatorname{Harm}(D)).$$

In [6] it was proved that the operators L^s map continuously the space $\operatorname{Harm}^k(D) = W^k(D) \cap \operatorname{Harm}(D)$ into $\mathring{W}^k(D)$ ($W^k(D)$ denotes the usual Sobolev space, and $\mathring{W}^k(D)$ the closure of $C_0^\infty(D)$ in $W^k(D)$) and that Bell's construction establishes the duality relation between the spaces $\operatorname{Harm}^k(D)$ and $\operatorname{Harm}^{-k}(D)$. This last space was proved to be equal to the space $L^2 \operatorname{Harm}(D, \varrho^{2k})$ of functions harmonic on D and square-integrable with weight ϱ^{2k}, where ϱ is a defining function for the domain D and k is an integer. In Bell's paper and in [6] it is assumed that D is a bounded domain with C^∞-smooth boundary.

The aim of the present note is to extend these ideas to the Hölder spaces of harmonic functions. We shall denote by $\Lambda_\alpha(D)$ the space of functions on D whose kth derivatives satisfy the $\alpha - k$ Hölder condition, $k = [\alpha]$ (the integer part of α), $0 < \alpha - [\alpha] < 1$. Let $\Lambda_\alpha \operatorname{Harm}(D)$ denote the subspace of $\Lambda_\alpha(D)$ consisting of harmonic functions. We shall denote by $L^2 \operatorname{Harm}(D)$ the subspace of $L^2(D)$ consisting of square-integrable harmonic functions, and by P the orthogonal projection from $L^2(D)$ onto $L^2 \operatorname{Harm}(D)$. If D is a bounded domain in $\boldsymbol{R}^n$ then a function $\varrho \in C^\infty(\boldsymbol{R}^n)$ ($C^\lambda(\boldsymbol{R}^n)$) is called *defining* for D iff

$D = \{x \in \mathbf{R}^n: \varrho(x) < 0\}$ and $\operatorname{grad}\varrho \neq 0$ on ∂D. We shall prove the following

Proposition 1. *Let D be a bounded domain with C^∞-smooth boundary. Then Bell's operators L^s map continuously $\Lambda_\alpha \operatorname{Harm}(D)$ into $\Lambda_\alpha(D)$, $\alpha > 0$. If $s = k+1$, $k = [\alpha]$, then for $h \in \Lambda_\alpha \operatorname{Harm}(D)$, $L^s h$ vanishes on ∂D up to order k and $L^s h = |\varrho|^\alpha m$, $m \in L^\infty(D)$, where ϱ is a defining function for D.*

Proposition 2. *Let P denote as above the orthogonal projection from $L^2(D)$ onto $L^2 \operatorname{Harm}(D)$. Let ϱ be a defining function for D. Then the mapping $m \to P(|\varrho|^\alpha m)$ maps continuously $L^\infty(D)$ onto $\Lambda_\alpha \operatorname{Harm}(D)$.* (Note that $|\varrho| = -\varrho$ on $\bar{D}$.)

Propositions 1 and 2 yield the following

Theorem 1. *Let D be a bounded domain with C^∞-smooth boundary. Then $\Lambda_\alpha \operatorname{Harm}(D)$ can be represented as the dual space to the space $\mathring{L}^1 \operatorname{Harm}(D, |\varrho|^\alpha)$ via the pairing $\langle u, v\rangle_s = \langle u, L^s v\rangle$, $s = [\alpha]+1$. The space $\mathring{L}^1 \operatorname{Harm}(D, |\varrho|^\alpha)$ is the closure of $L^2 \operatorname{Harm}(D)$ in the space $L^1(D, |\varrho|^\alpha)$ of functions integrable with weight $|\varrho|^\alpha$, ϱ a defining function for D.*

We do not know whether the space $\mathring{L}^1 \operatorname{Harm}(D, |\varrho|^\alpha)$ is equal to the space of all harmonic functions integrable with weight $|\varrho|^\alpha$.

The next part of this note is devoted to the case where the boundary of D is of the Hölder class $\Lambda_{4+\alpha_0}$.

In this case we cannot take an arbitrary defining function ϱ of D in the construction of Bell's operators and Proposition 1 and 2. We shall consider the function ϱ_0, a biharmonic function on D such that $\varrho_0 = 0$ and $\partial\varrho_0/\partial\eta = 1$ on ∂D. Such a function is of class $C^{4+\alpha_0}$ on $\bar{D}$ (see [1]).

Then Propositions 1 and 2 remain valid if $\alpha < \alpha_0$ and $s = [\alpha]+1$ and we shall get the following

Theorem 2. *Let D be a bounded domain with $\Lambda_{4+\alpha_0}$-smooth boundary. Then for every $\alpha < \alpha_0$, $\Lambda_\alpha \operatorname{Harm}(D)$ can be represented via the pairing*

$$\langle u, v\rangle_s = \langle u, L^s v\rangle, \qquad s = [\alpha]+1,$$

as the dual space to the space $\mathring{L}^1 \operatorname{Harm}(D, |\varrho_0|^\alpha)$.

Theorems 1 and 2 yield the following

Corollary 1. *If Theorem 1 or 2 holds then the Hölder norm of a function f from $\Lambda_\alpha \operatorname{Harm}(D)$ is equivalent to the norm*

$$\|f\| = \sup_{\substack{u \in L^2 \operatorname{Harm}(D) \\ \|u\|_{L^1(D,|\varrho|^\alpha)} \leqslant 1}} |\langle u, f\rangle_s|.$$

The Poisson formula gives an isomorphism between $\Lambda_\alpha(\partial D)$ and $\Lambda_\alpha \operatorname{Harm}(D)$. Thus we get

Corollary 2. *The space $\Lambda_\alpha(\partial D)$ can be represented as the dual space to $\mathring{L}^1 \operatorname{Harm}(D, |\varrho|^\alpha)$.*

Theorems 1 and 2 can also be applied to the study of spaces of

holomorphic and pluriharmonic functions (see [7]). The duality theory was invented by S. Bell primarily for this purpose [3]. The idea of using duality between spaces of harmonic functions comes from the paper of S. Bell and H. Boas [4] (see also G. Komatsu [5]). At the end of this note we shall give some remarks concerning duality with respect to weighted scalar products and indicating some further generalizations of the above results.

2. Proofs.

(a) Proof of Proposition 1. The proof of Proposition 1 is based on the following well-known fact:

If f is a function from $\Lambda_\alpha \operatorname{Harm}(D)$ then

$$|D^\beta f(x)| \leqslant \frac{C_\beta \|f\|_\alpha}{(\operatorname{dist}(x, \partial D))^{|\beta|-[\alpha]-(\alpha-[\alpha])}} = \frac{C_\beta \|f\|_\alpha}{(\operatorname{dist}(x, \partial D))^{|\beta|-\alpha}}$$

for every $x \in D$ and $|\beta| > [\alpha]$.

Since for every defining function ϱ, $c_1 \operatorname{dist}(x, \partial D) \leqslant |\varrho(x)| \leqslant c_2 \operatorname{dist}(x, \partial D)$, it follows that

$$|D^\beta f| \leqslant \frac{C_\beta \|f\|_\alpha}{|\varrho|^{|\beta|-\alpha}}, \qquad |\beta| > [\alpha].$$

Let us now recall the construction of Bell's operators $L^s u$:

$$L^1 u = u - \Delta(\theta_0 \varrho^2), \qquad \theta_0 = \frac{1}{2}\frac{\varphi u}{|\nabla\varrho|^2},$$

$$\theta_t = \frac{\varphi}{(t+2)!}|\nabla\varrho|^{-2}\left(\frac{\partial}{\partial\eta}\right)^t L^t u, \qquad \frac{\partial}{\partial\eta} = \frac{\sum_{i=1}^n \frac{\partial\varrho}{\partial x_i}\cdot\frac{\partial}{\partial x_i}}{|\nabla\varrho|^2},$$

$$L^s u = u - \Delta\Big(\sum_{k=0}^{s-1} \theta_k \varrho^{k+2}\Big).$$

φ denotes here an arbitrarily chosen C^∞-function equal to 1 in a neighborhood of ∂D and equal to zero in a neighborhood of the set $\{\nabla\varrho = 0\}$.

The construction of L^s yields that $L^s u$ consists of terms in which u or its derivatives are multiplied by ϱ to the same power as the order of differentiation in those terms. Thus in order to prove that L^s maps $\Lambda_\alpha \operatorname{Harm}$ into Λ_α it suffices to show that if $p \geqslant |\beta|$ then

$$\varrho^p D^\beta u \in \Lambda_\alpha \quad \text{and} \quad \|\varrho^p D^\beta u\|_\alpha \leqslant C_{p,\beta}\|u\|_\alpha \quad \text{for } u \in \Lambda_\alpha \operatorname{Harm}.$$

The Hardy–Littlewood lemma implies that it suffices to show that if $|\gamma| = [\alpha]+1$ then

$$|D^\gamma \varrho^p D^\beta u| \leqslant \frac{C\|u\|_\alpha}{|\varrho|^{1-\alpha+[\alpha]}}.$$

The derivative on the left can be expressed as the sum of terms of the type

$$\varrho^{p-r} D^{\beta} D^{\delta} u \cdot (\text{smooth function}) \quad \text{where } |\delta| \leqslant |\gamma| - r.$$

We then have

$$|\varrho^{p-r} D^{\beta} D^{\delta} u| \leqslant \frac{C\,\|u\|_{\alpha}\,|\varrho|^{p-r}}{|\varrho|^{|\beta|+|\delta|-\alpha}} \leqslant \frac{C\,\|u\|_{\alpha}\,|\varrho|^{p-r}}{|\varrho|^{|\beta|+[\alpha]+1-\alpha-r}} \leqslant \frac{C\,\|u\|_{\alpha}}{|\varrho|^{1-\alpha+[\alpha]}}$$

since $|\varrho| < 1$ near ∂D.

The above considerations and the construction of L^s imply that if $s = [\alpha]+1$ then $L^s u$ vanishes on ∂D up to order $[\alpha]$. Thus in this case $L^s u = |\varrho|^{\alpha} m$ where $m \in L^{\infty}(D)$ ($|\varrho| = -\varrho$). It can easily be seen that $\|m\|_{\infty} \leqslant c\,\|u\|_{\alpha}$. This ends the proof of Proposition 1.

(b) Proof of Proposition 2. The projection Pf is equal to $f - \Delta G_2 \Delta f$, where G_2 is the operator solving the Dirichlet problem

$$\Delta^2 g = w, \quad g = \frac{\partial g}{\partial n} = 0 \quad \text{on } \partial D.$$

Let $m \in L^{\infty}(D)$ and ϱ be a defining function of D. Let u be the solution of the Dirichlet problem $\Delta u = |\varrho|^{\alpha} m$, $u = 0$ on ∂D. Now, we have

$$P(|\varrho|^{\alpha} m) = |\varrho|^{\alpha} m - \Delta G_2 \Delta (|\varrho|^{\alpha} m) = \Delta (u - G_2 \Delta^2 u) \overset{\text{df}}{=} \Delta v.$$

The function $v = u - G_2 \Delta^2 u$ is the solution of the Dirichlet problem $\Delta^2 v = 0$, $v = u = 0$ on ∂D and $\partial v/\partial n = \partial u/\partial n$ on ∂D. To prove our proposition it suffices to prove that $v \in \Lambda_{2+\alpha}(D)$. It follows from the results of Agmon, Douglis, Nirenberg [1] (especially from Theorem 12.10 and what follows) that $v \in \Lambda_{2+\alpha}(D)$ iff $\partial u/\partial n|_{\partial D} \in \Lambda_{1+\alpha}(\partial D)$. Note that the function u cannot be of class $\Lambda_{2+\alpha}(D)$ if $|\varrho|^{\alpha} m$ does not belong to $\Lambda_{\alpha}(D)$, but, fortunately, the restriction of $\partial u/\partial n$ to the boundary has the needed class of smoothness. This can be proved in the following manner. Let

$$G(x, y) = C\left(\frac{1}{|x-y|^{n-2}} - G_1(x, y)\right)$$

be the Green function of the domain D (C is a constant). We have

$$C^{-1} u(y) = \int_D C^{-1} G(y,x)\,|\varrho(x)|^{\alpha} m(x)\,dV_x = \int_D \frac{|\varrho(x)|^{\alpha} m(x)}{|x-y|^{n-2}}\,dV_x - \int_D G_1(y, x)\,dV_x = u_1(y) - u_2(y).$$

The function u_2 is the harmonic extension to the domain D of the function $u_1|_{\partial D}$. Then in order to prove that $\left.\frac{\partial u}{\partial n}\right|_{\partial D} \in \Lambda_{1+\alpha}(\partial D)$ it suffices to prove that $u_1|_{\partial D} \in \Lambda_{2+\alpha}(\partial D)$ and $\left.\frac{\partial u_1}{\partial n}\right|_{\partial D} \in \Lambda_{1+\alpha}(D)$.

Since

$$u_1(y) = \int_D \frac{|\varrho(x)|^{\alpha} m(x)}{|x-y|^{n-2}}\,dV_x,$$

the boundary values of u_1 are the same as the boundary values of the function

$$w_1(y) = \int_D \frac{|\varrho(x)|^{\alpha} m(x)\,dV_x}{(|x-y|^2 + \varrho(x)\,\varrho(y))^{n/2-1}}$$

and the boundary values of $\partial u_1/\partial n$ are the same as the boundary values of the function

$$w_2(y) = \frac{\partial}{\partial n} w_1(y) + c\int_D \frac{|\varrho(x)|^{\alpha+1}(\partial\varrho/\partial n)(y)\,dV_x}{(|x-y|^2 + \varrho(x)\,\varrho(y))^{n/2}}.$$

The classical gradient estimates for integrals of the above type show that $w_1(y) \in \Lambda_{2+\alpha}(D)$ and $w_2(y) - (\partial/\partial n)\,w_1(y) \in \Lambda_{1+\alpha}(D)$. Thus $u_1|_{\partial D} = w_1|_{\partial D} \in \Lambda_{2+\alpha}(\partial D)$ and $\left.\frac{\partial u_1}{\partial n}\right|_{\partial D} = w_2|_{\partial D} \in \Lambda_{1+\alpha}(\partial D)$ and so $\left.\frac{\partial u}{\partial n}\right|_{\partial D} \in \Lambda_{1+\alpha}(\partial D)$.

Hence for every $m \in L^{\infty}(D)$, $P(|\varrho|^{\alpha} m) \in \Lambda_{\alpha}(D)$ and by the closed graph theorem the operator $P(|\varrho|^{\alpha} m)$ maps continuously $L^{\infty}(D)$ onto $\Lambda_{\alpha}(D)$.

(c) Proof of Theorem 1. Let φ be a functional from the space adjoint to $\mathring{L}^1\,\mathrm{Harm}(D), |\varrho|^{\alpha})$. The functional φ can be extended to a continuous functional $\tilde{\varphi}$ on $L^1(D, |\varrho|^{\alpha})$ and thus there exists a function $m \in L^{\infty}(D)$ such that $\tilde{\varphi}(h) = \int h\bar{m}\,|\varrho|^{\alpha}$. If $h \in L^2\,\mathrm{Harm}(D)$ then

$$\tilde{\varphi}(h) = \int h\bar{m}\,|\varrho|^{\alpha} = \int h\,P\big(\overline{m\,|\varrho|^{\alpha}}\big) = \int h\,L^s P\big(\overline{m\,|\varrho|^{\alpha}}\big) = \langle h, P(m\,|\varrho|^{\alpha})\rangle_s, \qquad s = [\alpha]+1.$$

Since $L^2\,\mathrm{Harm}(D)$ is dense in $\mathring{L}^1\,\mathrm{Harm}(D, |\varrho|^{\alpha})$, the correspondence $\varphi \to P(m\,|\varrho|^{\alpha})$ is independent of the choice of the bounded function m representing φ. Propositions 1 and 2 imply that this correspondence defines a continuous one-to-one mapping from the space $(\mathring{L}^1\,\mathrm{Harm}(D, |\varrho|^{\alpha}))^*$ onto $\Lambda_{\alpha}\,\mathrm{Harm}(D)$. By the open mapping theorem this mapping is an isomorphism. This ends the proof of Theorem 1.

(d) Proof of Theorem 2. We shall begin with the following

LEMMA. *Let u be a biharmonic function on D (i.e. $\Delta^2 u = 0$) such that*

$u \in \Lambda_\alpha(D)$. *Then*

$$|D^\beta u(x)| < \frac{C_\beta \|u\|_\alpha}{\operatorname{dist}(x, \partial D)^{|\beta|-\alpha}} \quad \textit{if } |\beta| > \alpha.$$

By the definition of the spaces Λ_α and the fact that the derivatives of a biharmonic function are biharmonic, it suffices to prove our lemma for $0 < \alpha < 1$.

Let $x_0 \in D$ and $\delta = \operatorname{dist}(x, \partial D)/2$. Without loss of generality we can assume that $x_0 = 0$.

Let $K(0, \delta)$ denote the ball centered at zero with radius δ. Since Δu is harmonic, the mean value theorem implies that

$$\Big(\int_{K(0,\delta)} (|x|^2-\delta^2)^2\Big)|\Delta u(0)| = \Big|\int_{K(0,\delta)} \Delta u(x)(|x|^2-\delta^2)^2\Big|$$
$$= \Big|\int_{K(0,\delta)} u(x)\Delta(|x|^2-\delta^2)^2\Big| = \Big|\int (u(x)-u(0))\Delta(|x|^2-\delta^2)^2\Big|$$
$$\leqslant \delta^\alpha \int |\Delta(|x|^2-\delta^2)^2| \, \|u\|_\alpha.$$

This implies that $|\Delta u(0)| \leqslant c(n)\|u\|_\alpha/\delta^{2-\alpha}$ and therefore there exists a constant c such that

$$|\Delta u(x)| \leqslant \frac{c\|u\|_\alpha}{[\operatorname{dist}(x, \partial D)]^{2-\alpha}}.$$

We can repeat the same procedure for $D^\beta \Delta u$, taking the function $(\delta^2 - |x|^2)^{|\beta|+2}$, and prove that

$$|D^\beta \Delta u(x)| \leqslant \frac{c_\beta \|u\|_\alpha}{\operatorname{dist}(x, \partial D)^{|\beta|+2-\alpha}}.$$

Now $u|_{K(0,\delta)} = h + u_1$, where h is a harmonic function equal to u on $\partial K(0, \delta)$, $\Delta u_1 = \Delta u$ and $u_1 = 0$ on $\partial K(0, \delta)$. Since $\|h\|_{\Lambda_\alpha(K(0,\delta))} \leqslant \|u\|_{\Lambda_\alpha(D)}$, there exists $c(n)$ such that

$$\left|\frac{\partial h}{\partial x_j}(0)\right| \leqslant \frac{c(n)\|u\|_\alpha}{\delta^{1-\alpha}}.$$

We have

$$u_1(x) = \int_{K(0,\delta)} G(x, y)\Delta u(y)\, dV_y$$

where $G(x, y)$ is the Green function of $K(0, \delta)$ and thus

$$\frac{\partial u_1}{\partial x_j}(0) = \int_{K(0,\delta)} \frac{\partial}{\partial x_j} G(0, y)\Delta u(y)\, dV_y$$
$$= c(n) \int_{K(0,\delta)} \left(\frac{y_j}{|y|^n} - \frac{y_j}{\delta^n}\right)\Delta u(y)\, dV_y.$$

Then

$$\left|\frac{\partial u_1}{\partial x_j}(0)\right| \leqslant \frac{c(n)\delta\|u\|_\alpha}{\delta^{2-\alpha}} = \frac{c(n)\|u\|_\alpha}{\delta^{1-\alpha}}.$$

This implies that there exists a constant c_j such that

$$\left|\frac{\partial u}{\partial x_j}(x)\right| \leqslant \frac{c_j\|u\|_\alpha}{(\operatorname{dist}(x, \partial D))^{1-\alpha}}.$$

Now we can apply this procedure to the functions $\partial u/\partial x_j$ and prove in the same way that all second derivatives are bounded by $C\|u\|_\alpha/(\operatorname{dist}(x, \partial D))^{2-\alpha}$ any by induction prove our lemma for derivatives of arbitrary high order.

Now if we use the biharmonic function ϱ_0 in the construction of the operators L^s then Proposition 1 remains valid if $\alpha < \alpha_0$ and can be proved in the same manner as in the case of C^∞-smooth boundary. We must only observe that $\varrho_0^s D^\beta \varrho_0 \in \Lambda_{4+\alpha_0-\beta+s}$, that after each differentiation we get a sum of terms in which u or its derivatives are differentiated and terms in which ϱ or its derivatives are differentiated and that $\Lambda_\alpha(D)$ is a Banach algebra.

Since the estimates from [1] remain valid when ∂D is of class $\Lambda_{4+\alpha_0}$, $\alpha < \alpha_0$, the proof of Proposition 2 is the same as before. Thus we get our Theorem 2 as a consequence of Propositions 1 and 2 in the same way as in the C^∞-smooth case.

3. Remarks.

Remark 1. Propositions 1 and 2 and Theorems 1 and 2 remain valid if we replace the usual scalar product in $L^2(D)$ with a weighted scalar product

$$\langle f, g\rangle_w = \int_D f\bar{g}e^w,$$

where w is a real function from $C^\infty(\bar{D})$. In this case the operators $L^s u$ must be replaced by the operators $L_w^s u = e^{-w}L^s(e^w u)$. It is obvious that Proposition 1 remains valid for $L_w^s u$.

Proposition 2 can be proved in the same way as Theorem 1 in [8]. Let P_w denote the projection from $L^2(D)$ onto $L^2\operatorname{Harm}(D)$, orthogonal with respect to the scalar product $\langle\ ,\ \rangle_w$. We have $P(e^w f) = P(e^w P_w(f))$. In order to prove that P_w maps $|\varrho|^\alpha m$, $m \in L^\infty(D)$, into $\Lambda_\alpha\operatorname{Harm}(D)$ it suffices to show that the operator $Ag = P(e^w g)$ maps isomorphically $\Lambda_\alpha\operatorname{Harm}(D)$ onto $\Lambda_\alpha\operatorname{Harm}(D)$. We can extend A to the whole $\Lambda_\alpha(D)$ by putting

$$Ag = e^w g - \Delta G_2 \Delta e^w Pg = e^w[g - e^{-w}\Delta G_2 \Delta e^w Pg].$$

The operator in square brackets is Fredholm since $\Delta e^w Pg$ is a differential operator of order 1. It is then easy to show that $\ker A = \{0\}$ and A^{-1} and A

map $\Lambda_\alpha \operatorname{Harm}(D)$ onto $\Lambda_\alpha \operatorname{Harm}(D)$. Thus Proposition 2 and Theorems 1 and 2 hold in our case.

In the same manner we can also show that the results of [6] hold for weighted scalar products $\langle\ ,\ \rangle_w$.

We take this opportunity to rectify an error in [8]. At the end of the proof of Theorem 3 of [8] we wrote by mistake that the Sobolev space H^s is dense in $\Lambda_\alpha H$ for large s. This is clearly not true. However, Theorem 3 of [8] remains valid since the operator $P_\varrho(e^h f)$ can be extended to a Fredholm operator on the whole space and thus is invertible as in the proof of Theorem 1 of [8] (or as above).

Remark 2. Let $\mathring{\Lambda}_\alpha(D)$ denote the subspace of $\Lambda_\alpha(D)$ consisting of functions from Λ_α which vanish on ∂D up to order $[\alpha]$. ($\mathring{\Lambda}_\alpha(D)$ is *not* the closure of $C_0^\infty(D)$ in Λ_α.) Let ∂D be C^∞-smooth. Bell's operator L^s, $s = [\alpha]+1$, can be extended to a continuous *projection* from $\Lambda_\alpha(D)$ onto $\mathring{\Lambda}_\alpha(D)$. This fact can be proved in exactly the same manner as its analogue for Sobolev spaces W^s (see [6], Remark 1). First we can prove that there exists a uniquely determined decomposition of $f \in \Lambda_\alpha(D)$,

$$f = h_0 + \varrho h_1 + \ldots + \varrho^s h_s + u, \quad \text{where } s = [\alpha],\ h_k \in \Lambda_{\alpha-k} \operatorname{Harm} \text{ and } u \in \mathring{\Lambda}_\alpha,$$

and define $\tilde{L}^s(f) = \sum_{k=0}^{s} L^s(\varrho^k h_k) + u$.

The details of proof are the same as in the case of Sobolev spaces and therefore can be omitted. Clearly we have $P(f) = P(\tilde{L}^s(f))$.

Remark 3. It is easy to observe that if $L^2 \operatorname{Harm}^m(D)$ is the space of m-polyharmonic square-integrable functions (i.e. such functions $f \in L^2(D)$ that $\Delta^m f = 0$) and P_m is the orthogonal projection from $L^2(D)$ onto $L^2 \operatorname{Harm}^m(D)$ then it is possible to construct for every $u \in C^\infty(\bar{D})$ a function $L_m^s u$ such that $L_m^s u$ vanishes on ∂D up to order $s-1$ and $P_m(L_m^s u) = u$.

We put

$$L_m^1 u = u - \Delta^m(\theta_0 \varrho^{2m}), \qquad \theta_0 = \frac{1}{2}\frac{\varphi u}{|\nabla\varrho|^{2m}},$$

$$\theta_t = \frac{\varphi}{(t+2m)!}|\nabla\varrho|^{-2m}\left(\frac{\partial}{\partial\eta}\right)^t L_m^t u,$$

$$L_m^s u = u - \Delta^m\Big(\sum_{k=0}^{s-1} \theta_k \varrho^{k+2m}\Big).$$

Since the statement of the Lemma in the proof of Theorem 2 remains valid if "biharmonic function" is replaced by "m-polyharmonic function", Proposition 1 holds for the operators L_m^s and Proposition 2 for the projection P_m. Then we get the following analogue of Theorem 1: The space $\Lambda_\alpha \operatorname{Harm}^m(D)$

is the dual of the space $\mathring{L}^1 \operatorname{Harm}^m(D, |\varrho|^\alpha)$, where $\mathring{L}^1 \operatorname{Harm}^m(D, |\varrho|^\alpha)$ is the closure of $L^2 \operatorname{Harm}^m(D)$ in $L^1(D, |\varrho|^\alpha)$, via the pairing $\langle u, v\rangle_s = \langle u, L_m^s v\rangle$, $s = [\alpha]+1$.

Theorem 2 remains valid if ∂D is of class $\Lambda_{4m+\alpha_0}$. The results from [6] on Sobolev spaces also have their analogues for spaces of m-polyharmonic functions. The detailed study of the duality theory for such spaces will be given in a subsequent paper.

References

[1] S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I*, Comm. Pure Appl. Math. 12 (1959), 623–727.

[2] S. Bell, *A duality theorem for harmonic functions*, Michigan Math. J. 29 (1982), 123–128.

[3] —, *A representation theorem in strictly pseudoconvex domains*, Illinois J. Math. 26 (1982), 19–26.

[4] S. Bell and H. Boas, *Regularity of the Bergman projection and duality of holomorphic function spaces*, Math. Ann. 267 (1984), 473–478.

[5] G. Komatsu, *Boundedness of the Bergman projector and Bell's duality theorem*, Tôhoku Math. J. 36 (1984), 453–467.

[6] E. Ligocka, *The Sobolev spaces of harmonic functions*, this volume, 79–87.

[7] —, *On the orthogonal projections onto spaces of pluriharmonic functions and duality*, this volume, 279–295.

[8] —, *The regularity of the weighted Bergman projections*, in: Seminar on Deformations Theory 1982/1984, Lecture Notes in Math. 1165, Springer, 1985, 197–203.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
Śniadeckich 8, 00-950 Warszawa, Poland

Received May 29, 1985 (2061)