satisfy the condition \( D_n \leq M < 1 \) then

\[
\sum_{i=0}^{k} b_i \leq \frac{M}{1-M}.
\]

Proof. We have

\[
D_{n+1} = D_n + (1 - D_n) b_{n+1}, \quad m = 0, 1, \ldots, k-1.
\]

This implies that \( D_n < 1 \) and

\[
D_{n+1} > D_n + (1 - D_n) b_{n+1}.
\]

Hence

\[
D_n > (1 - M) \sum_{i=0}^{k} b_i
\]

which gives (59).

References:


On metric isomorphism of Morse dynamical systems

by

Tadeusz Rojek (Toruń)

Abstract. For each continuous Morse sequence \( x \), the class of all continuous Morse sequences \( y \) such that the dynamical systems induced by \( x \) and \( y \) are metrically isomorphic is described.

Introduction. J. Kwiatkowski in [3] gave sufficient and necessary conditions for two Morse dynamical systems \( \theta(x) \) and \( \theta(y) \) induced by \( x = b^0 \times b^1 \times \cdots \) and \( y = b^0 \times b^1 \times \cdots \) to be metrically isomorphic, assuming that the lengths of the blocks \( b^i \) and \( b^j \) are the same for \( i = 0, 1, 2, \ldots \) and \( x \) and \( y \) are regular sequences. It is also proved in [3] that for a given Morse sequence \( x \) there exist a continuum of Morse sequences \( y \) such that the dynamical systems \( \theta(x) \) and \( \theta(y) \) are metrically isomorphic but the corresponding shift invariant measures on the space \( X = \prod_{i} [0, 1] \) are pairwise orthogonal. For a given regular Morse sequence \( x \) Kwiatkowski defines a class \( \mathcal{M}(x) \) of Morse sequences \( y \) such that the dynamical systems \( \theta(x) \) and \( \theta(y) \) are metrically isomorphic.

However, the procedure of obtaining the class \( \mathcal{M}(x) \) which is described there can be applied to a continuous Morse sequence \( x \) (without the assumption of regularity). In this paper we show that \( \mathcal{M}(x) \) is the class of all continuous Morse sequences \( y \) such that \( \theta(y) \) is metrically isomorphic to \( \theta(x) \).

To prove this, we use the same technique of coding as in [3], but we omit the assumption that the lengths of the blocks \( b^i \) and \( b^j \) are equal and thus codes have different forms. In order to prove the main result, for given Morse sequences \( x = b^0 \times b^1 \times \cdots \) and \( y = b^0 \times b^1 \times \cdots \) such that \( \theta(x) \) is metrically isomorphic to \( \theta(y) \) we construct a Morse sequence \( z = a_0 \times a_1 \times a_2 \times \cdots \) satisfying

\[
|a_0| = |b^0|, \quad |a_0 \times a_1| = |b^1|, \quad |a_1 \times a_2| = |b^2|, \quad \ldots
\]

\[
|a_0 \times a_2| = |b^2|, \quad |a_1 \times a_3| = |b^3|, \quad |a_2 \times a_4| = |b^4|, \quad \ldots
\]
such that $x, z$ and also $y, z$ satisfy conditions (A), (B) of [3] (here $E$ denotes the length of the block $E$). In this construction we use the distance $d(\cdot, \cdot)$ between blocks; however, we calculate it in a different manner than in [3].

In this paper we also announce two additional results. We present a generalization of Kwiatkowski's result for two Morse dynamical systems $\theta(x)$ and $\theta(y)$ induced by $x = b^0 \times b^1 \times \ldots, y = b^0 \times b^1 \times \ldots, | \theta | = | \beta |, t \geq 0$, without assuming their regularity. This generalization is essential as is shown by an example. We also give a necessary and sufficient condition for two Morse dynamical systems to be finitarily isomorphic. It turns out that finitary isomorphism coincides with topological conjugacy in the class of Morse shifts (see [3]). We omit the proofs of these results, because laborious calculations would considerably lengthen the paper.

Since the construction of the sequence $z$ is also laborious, we begin with a special section with a sketch of it. We use the definitions and notation listed in [3].

The author wishes to thank J. Kwiatkowski for helpful conversations on the results of this paper.

§ 1. Outline of the construction. Consider two continuous Morse sequences $x = b^0 \times b^1 \times \ldots$ and $y = b^0 \times b^1 \times \ldots$. We do not assume that $| b | = | | b |$, $t = 0, 1, \ldots$, and we omit the assumption that they are regular. Let $\mathcal{M}(x)$ denote the class defined in the introduction of [3]. The main result of this paper is the following theorem.

**Theorem 1.** The Morse dynamical systems $\theta(x)$ and $\theta(y)$ induced by $x$ and $y$ are metrically isomorphic if $y \in \mathcal{M}(x)$.

To prove the theorem it suffices to show the "if" part because the opposite implication is proved in [3].

Assume that $\theta(x)$ and $\theta(y)$ are metrically isomorphic. We will construct a Morse sequence $z$ such that $x$ and $y$ can be obtained from $z$ by the procedure described in [3, Introduction] (i.e., $x, y \in \mathcal{M}(z)$). Now we give the sketch of the construction of the sequence $z$. To do this we use a coding technique as in [3, § 2]. Let us denote $\lambda_i = | \beta |$, $\lambda_i = | | \beta |$, $i = 0, \ldots, \lambda_i = \lambda_{i+1}$, $\lambda_i = \lambda_{i+1}$, $i \geq 0$. In this paper if $B = b_1 b_2 \ldots b_n$ is a block, then $B[j,k]$, $1 \leq j \leq k \leq n$, denotes the block $b_j b_{j+1} \ldots b_k$, and the symbol $B^{-i}$, $i \in \{0, 1\}$, denotes the block $B'$ (which is equal to $B$ if $i = 0$ and to $\bar{B}$ if $i = 1$).

Let $z : X(y) \to X(x)$ be an isomorphism between $\theta(y)$ and $\theta(x)$. By Keane's results [2] it follows that the eigenvalue group $A$ of $\theta(x)$ consists of all $n$-roots of unity and the eigenvalue group $A_1$ of $\theta(y)$ consists of $n'$-roots of unity. So $A = A_1$. Hence by grouping the blocks $\{ b^0, b^1, \ldots \}$ and $\{ b^0, b^1, \ldots \}$ we may assume that

$n_0 | n_0, n_0 | n_1, n_1 | n_1', n_1 | n_2, \ldots$

So $n_1 = n_1$, $n_{i+1} = n_{i+1}', t \geq 0$. Moreover, we may assume that

$$\sum_{i=0}^{\infty} \frac{1}{n_i} < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} \frac{1}{n_i}' < \infty.$$  

The last conditions yield

$$\lambda_0 = \lambda_0, \quad \lambda_0 = \lambda_0, \quad \lambda_{i+1} = \lambda_{i+1}, \quad t = 0, 1, 2, \ldots$$

If $Q = k^{-1}(P(x))$, where $P(x)$ is the time zero partition of $X(x)$, then we find a sequence $\{ Q \}$ of partitions of $X(y)$ such that $Q \ll \delta$, and $| Q - Q' | \to 0$. Reasoning in the same way as in [3, § 2] we conclude that $Q$ is described by the codes $\{ A_1, A_1 \}$ satisfying

(1) $A_1 = (c_1 \times L), l_1 \vdash \mu_1 n_1, \quad B_1 = A_1, \quad l_{i+1} \equiv l_i (\text{mod } n_i)$

where $c_1 = b^0 \times b^1 \times \ldots \times b^1, L_1$ are blocks of 0 and 1 of lengths $\mu_1 + 1$ and $l_1 \in \{ 0, 1, \ldots, n_1 - 1 \}, t = 0, 1, 2, \ldots$

The codes $\{ A_1, A_1 \}$ satisfy the condition

(2) $\sup_{k \geq 1} \left( \sum_{i=0}^{\infty} \frac{1}{n_i} \right) \rightarrow 0$

where $\lambda_0 = \lambda_0 \times \lambda_0, \lambda_1 \times \lambda_1, \ldots, \lambda_{n_0} \times \lambda_{n_0}$, and $n_0(l) = (1/n_0) (L_1 - l)$. We will prove (3) in Lemma 1.

We are now in a position to present the main steps of the construction of $z$. We consider two cases: $u_k(t) = 0$ for $t \geq 0, k \geq 1$, and $u_0(t)$ arbitrary.

In the sequel we will often use some formulas for calculating the distance $d$ between blocks. First for a given block $E$ with length $n$ and an integer $p > 1$ such that $p | n$ we define blocks $E_0, E_0$ as

$$E_i = \left(\{i - 1\} \frac{n}{p + 1}, i \frac{n}{p}\right),$$

$$E_i = \left[ E[i + p] \right] \frac{n}{p}.$$  

Thus $| E_i | = \frac{| E_i |}{| E_i |} = \frac{n}{p}$. In the sequel we will write $B \times A_{i+p}, B \times A_i$ instead of $B \times (A_{i+p}), B \times (A_i)$. Let $L, L, B, L'$ be blocks with lengths $\mu, \mu', \mu_0, \mu_0, \mu'$ respectively. We
have the following formulas:

\[
d(\beta \times \beta \times b \times L) = \frac{1}{\mu k} \sum_{t \in L} \sum_{j=1}^n d(i, j),
\]

\[
d(\beta \times \beta, b \times L) = \frac{1}{\mu k} \sum_{t \in L} \sum_{j=1}^n d(i, j),
\]

\[
d(\beta \times b \times L) = \frac{1}{\mu k} \sum_{t \in L} \sum_{j=1}^n d(i, j),
\]

\[
d(\beta \times b \times L, b \times L) = \frac{1}{\mu k} \sum_{t \in L} \sum_{j=1}^n d(i, j),
\]

The above formulas are consequences of the definition of the distance \(d\). We explain them using the following illustrations:

![Fig. 1](image1)

![Fig. 2](image2)

The blocks marked by points in Figures 1 and 2 are \(b_\mu = L_i \) and \(b_\mu \times L_i \) respectively. Each of the blocks \(b \times b \) is constructed of \(\mu k\) of them. Hence formula (4) follows. Similarly we can obtain (5) and (6).

**Case 1.** \(u_k(t) = 0\) for all \(t \geq 0\) and \(k \in K\). Throughout this case, by \(L_t, t \geq 0\), we denote the block \(L_t[1, \mu t]\) (i.e. \(L_t\) without the last place).

I. Consider the numbers \(d(L_{t+1} \times \beta \times b \times L_t)\), \(t = 1, 2, \ldots\) Applying formula (4) with \(L' = L_{t-1}, \beta = \beta', b = b', L = L_{t-1}\), we have

\[
d(L_{t-1} \times \beta', b \times L_t) = \frac{1}{\mu t} \sum_{t \in L} \sum_{j=1}^n d(i, j),
\]

where

\[
d_i(i, j) = d((b_\mu \times L_{t-1} [i]), (b_\mu \times L_{t-1} [j]),
\]

\(i = 1, 2, \ldots, \mu_t - 1,
\)

Put

\[
e_i(i, j) = \min \{d_i(i, j), 1 - d_i(i, j)\} \quad \text{and} \quad e_i = \frac{1}{\mu t} \sum_{i=1}^{\mu_t-1} \sum_{j=1}^{\mu_t-1} e_i(i, j).
\]

In Lemma 2 we will show that \(\sum_{i=1}^{\mu_t-1} e_i < \infty\).

II. In the sequel the matrices \(M_i = (e_i(i, j)), i = 1, 2, \ldots, \mu_t, j = 1, 2, \ldots, \mu_t - 1, t > 0\), are considered. Define

\[
F_i = \{ (i, j); e_i(i, j) < \frac{1}{2} \}.
\]

Take the row of \(M_i\) (say the \(i_0\)-th) which contains the largest number of elements of \(F_i\) and denote by \(G_i\) the set of all pairs \((i, j) \in F_i\) such that \((i_0, j) \notin F_i\). Let \(G_i\) be the complement of the set \(G_i\). Then the convergence of the series \(\sum_{i} e_i\) ensures that \(\sum_{i} |G_i|/|\mu_t - 1| < \infty\) (here \(|G_i|\) denotes the number of elements of \(G_i\)). Indeed, the property \(\sum_{i} e_i < \infty\) implies \(\sum_{i} |F_i|/|\mu_t - 1| < \infty\)

and

\[
|G_i| = m_t \{ |1 \leq j \leq \mu_t - 1; e_i(i_0, j) < \frac{1}{2}| \}
\]

\[
= \sum_{i=1}^{\mu_t-1} \{ |1 \leq j \leq \mu_t - 1; e_i(i_0, j) < \frac{1}{2}| \}
\]

\[
\leq \sum_{i=1}^{\mu_t-1} \{ |1 \leq j \leq \mu_t - 1; e_i(i, j) < \frac{1}{2}| \}
\]

III. Now we can define blocks \(\bar{a}_i, t \geq 0\). We set

\[
\bar{Y}_t[i] = \begin{cases} 0 & \text{if } d_0(i_0, j) < \frac{1}{2}, \\ 1 & \text{if } d_0(i_0, j) \geq \frac{1}{2} \end{cases}
\]

and

\[
\bar{a}_0 = \bar{a}_0 \cdots 0 \quad \bar{a}_t[i] = \begin{cases} 0 & \text{if } L_t[i] = Y_{t+1}[i], \\ 1 & \text{if } L_t[i] \neq Y_{t+1}[i] \quad i = 1, 2, \ldots, \mu_t, t \geq 1.
\end{cases}
\]

IV. In order to determine blocks \(a_t, t \geq 0\), we first construct auxiliary blocks \(Y_i\) of lengths \(\mu_t, t \geq 0\). We put

\[
\bar{Y}_t[i] = \begin{cases} 0 & \text{if } d((b_\mu \times L_{t-1} [i]), (b_\mu \times L_{t-1} [j]), \leq \frac{1}{2}, \\ 1 & \text{otherwise}, \\ i = 1, 2, \ldots, \mu_t.
\end{cases}
\]
Using the blocks \( \mathbf{Y} \), we construct blocks \( \gamma' \) (\(|\gamma'| = |\beta'|\)) and \( \varphi' \) (\(|\varphi'| = |\beta'|\)) by putting

\[
(\gamma'_{i \mu_{i-1}} = (\beta'_{i \mu_{i-1}} + \mathbf{Y}[\gamma]), \quad i = 1, 2, \ldots, \mu_i,
(\varphi'_{j \mu_{j-1} - 1} = (\beta'_{j \mu_{j-1} - 1} + \mathbf{Y}[\varphi]), \quad j = 1, 2, \ldots, \mu_{j-1}.
\]

Finally we can define \( a \):

\[
a_i = b^i, \quad a_i[1] = 0 \quad \text{if} \quad d((\varphi')_{i \mu_{i-1} - 1}, L_i) \leq \frac{1}{i},
1 \quad \text{otherwise},
i = 1, \ldots, \alpha_0 - 1, \quad t \geq 1.
\]

We will show (see Lemma 3) that

\[
\sum_{i=1}^{\infty} d(\gamma'_i \bar{a}_{i-1} \times a_i) < \infty, \quad \sum_{i=1}^{\infty} d(\beta'_i \bar{a}_{i-1} \times a_i) < \infty.
\]

If \( a_i[1] = 1 \) (\( a_i[1] = 1 \)) then we put \( a_i[1] = 0 \) (\( a_i[1] = 0 \)) and the conditions (7) remain true. In this way we obtain a binary sequence \( z = a_0 \times \bar{a}_0 \times a_1 \times \bar{a}_1 \times \ldots \) It is easy to verify that the continuity of the sequences \( x \) and \( y \) and the conditions (7) imply that \( z \) is a continuous Morse sequence.

Case 2. \( u_0(k) \) are arbitrary.

V. We recall that \( u_0(k) = (1/n_0) l_1 \). Since \( l_{i+1} = l_i \) (mod \( n_i \)), \( t \geq 0 \), there exist \( q_i, 0 \leq q_i < l_i - 1 \), such that

\[
l_i = q_i + q_i + n_i + q_i + n_i + \ldots + q_i + n_i - 1,
\]

and hence

\[
u_i(k) = q_i + q_i + l_i + \ldots + q_i + l_i + i - 1, \quad i \geq 1, \quad k \geq 1.
\]

We reduce this case to the previous one. To this end we define the blocks \( \delta', \varphi' \), \( t \geq 0 \). The blocks \( \varphi' \) are equal to \( (b' \beta') \) \( [1 + q_i, l_i + \mu_i] \), \( t \geq 0 \), and the blocks \( \delta' \) are determined by the following equalities:

\[
\delta' = b'\beta', \quad (\delta'_{i \mu_{i-1} - 1} = (\beta'_{i \mu_{i-1} - 1} \cup [\mu_i])(\delta'_{i \mu_{i-1} - 1} \cup [\mu_i]), \quad i = 1, 2, \ldots, \mu_{i-1} - 1, \quad i \geq 1,
\]

where \( r_i(i) = 0 \) if \( 1 \leq i \leq (l_i - q_i) / \mu_{i-1} \) and \( r_i(i) = 1 \) if \( i > (l_i - q_i) / \mu_{i-1} \) (here \([a]\) denotes the integer part of \( a \)). The sequences of blocks \( \delta' \), \( \varphi' \), \( t \geq 0 \), satisfy the condition

\[
\sup_{k \geq 1} d(L_k + \delta' + \ldots + \delta' + \varphi' + \ldots + \varphi' + L_k) < \infty,
\]

where \( L_k = L_k \cup [1, \mu_k] \) (see Lemma 4).

VI. The above condition is the same as (3) if we take \( \delta' \) instead of \( \beta' \) and

\( \psi' \) instead of \( b' \) and \( u_0(k) = 0 \). Therefore we can repeat the considerations of Case 1. As a consequence we obtain blocks \( \{K_1, \{\hat{a}_1, \}, |K_1| = c_{k-1}, |a_1 - 1| = \mu_1 - 1, t \geq 1, |a_2| = \lambda_2, \) satisfying

\[
\sum_{i=1}^{\infty} d(\gamma'_i \bar{a}_{i-1} \times K_i) < \infty, \quad \sum_{i=1}^{\infty} d(\beta'_i \bar{a}_{i-1} \times a_i) < \infty.
\]

Further we put \( a_i = (K_i, K_i)[1 + k_i, k_i + n_i] \), \( k_i = (\lambda_i - q_i) / \mu_{i-1} \). As in Case 1 we may assume that \( a_i \) and \( \bar{a}_i \) start with 0. Therefore \( z = a_0 \times \bar{a}_0 \times a_1 \times \bar{a}_1 \times \ldots \) is a Morse sequence and it is not difficult to see that \( z \) is continuous. We will show in Lemma 5 that the pairs of sequences \( x \) and \( y \) satisfy conditions (A) and (B) of [3]. This ends the sketch of the proof of Theorem 1.

Remark 1. It follows from the construction of the sequence \( z \) that \( x \) and \( y \) is continuous. Conditions (A), (B) although they need not be regular. However, Kwiatkowski's Theorem 1 is not valid if the regularity of \( x \) and \( y \) is not assumed. In our paper we formulate (without proof) a necessary and sufficient condition for two Morse dynamical systems \( \theta(x) \) and \( \theta(y) \) induced by \( x = b \times b \times \ldots \) and \( y = b \times b \times \ldots \) and satisfying \( |\beta'| = |\beta'| \), \( t = 0, 1, \ldots \) to be metrically isomorphic (see Theorem 2). In case \( x \) and \( y \) are regular, Kwiatkowski's results are consequences of Theorem 2. Using Theorem 2, we will give an example of two continuous Morse sequences \( x \) and \( y \) such that \( \theta(x) \) is isomorphic to \( \theta(y) \) and such that conditions (A) and (B) of [3] are not satisfied simultaneously.

The fact that the sequence \( z \) satisfies (B) is a consequence of the conditions \( \sum l_\mu < \infty \) and \( \sum l_\alpha < \infty \).

\§ 2. Proof of Theorem 1. Now we give the proofs of the lemmas used in § 1. We assume that \( x = b \times b \times \ldots \) and \( y = b \times b \times \ldots \) are continuous Morse sequences such that \( \theta(y) \neq \theta(x) \) are metrically isomorphic. The lengths \( \lambda_0 \) and \( \lambda_1 \) of \( b' \) and \( \beta' \) satisfy

\[
\lambda_0 = \mu_0 \lambda_0, \quad \lambda_1 = \mu_1 \lambda_1, \quad t \geq 1,
\]

and \( \sum \lambda_\mu < \infty, \sum l_\alpha < \infty \). Codes \( \{A_i, B_i\} \) describing the partition \( Q = b^{-1}(P(x)) \) have the form (1).

**Lemma 1.** The sequence

\[
\{\sup d(l_k[1, \mu_k] \times b^{k+1} \times \ldots \times b^{k+1}, (b^{k+1} \times \ldots \times b^{k+1}) \times L_k\}, \{1 + u_0(k), u_0(k) + s_1(k)\})\}
\]

where \( s_1(k) = \lambda_1 + \ldots + \lambda_k \) and \( u_0(k) = (1/n_0)(l_\mu - l_\mu) \), converges to zero.
Proof. Introduce the following notation:
\[ c_i^j = c_i^j[1, I_j], \quad c_i^j = c_i^j[1 + I_j, n_j], \]
\[ L_i^j = L_i^j[1, \mu_i], \quad L_i^j = L_i^j[2, \mu_i + 1]. \]
\( t = 0, 1, \ldots \)

To prove the lemma divide the block \( A^{(n)} = A_i \times \beta^{i+1} \times \ldots \times \beta^{i+k} \) (for fixed \( t \geq 0 \) and \( k \geq 1 \)) into \( \mu_i \lambda_i^k + 1 \ldots \lambda_{i+k}^k \) consecutive subblocks \( E_i \) of length \( n_i \). Next divide each \( E_i \) into two consecutive blocks \( E_i(t) \), \( r = 1, 2 \), such that \( |E_i(t)| = |c_i^j| = n_i - I_j, |E_i(2)| = |c_i^j| = I_j \). Denote by \( E \) and \( F \) the blocks
\[ E_i(1) E_i(2) \ldots E_i \lambda_i^k+1 \ldots \lambda_{i+k}^k(1) \quad \text{and} \quad E_i(2) E_i(1) \ldots E_i \lambda_{i+k}^k \ldots \lambda_i^k(2) \]
respectively. It is clear that
\[ E = c_i^j \times L_i^j \times \beta^{i+1} \times \ldots \times \beta^{i+k} \quad \text{and} \quad F = c_i^j \times L_i^j \times \beta^{i+1} \times \ldots \times \beta^{i+k}. \]

Let \( E', F \) denote the blocks obtained from \( A_{i+k} \) in the same way as the blocks \( E, F \) from \( A^{(n)} \). We have
\[ E' = c_i^j \times ((\beta^{i+1} \times \ldots \times \beta^{i+k} \times L_{i+k}[1+u_i(k), u_i(k)+s_i(k)]), \]
\[ F' = c_i^j \times ((\beta^{i+1} \times \ldots \times \beta^{i+k} \times L_{i+k}[2+u_i(k), 1+u_i(k)+s_i(k)]). \]

Notice that
\[ d(A_{i+k}, A^{(n)}) = \frac{n_i-1}{n_i} I_i(k) + \frac{1}{n_i} \Pi_i(k), \]
where
\[ \Pi_i(k) = d(L_i^j \times \beta^{i+1} \times \ldots \times \beta^{i+k}, (\beta^{i+1} \times \ldots \times \beta^{i+k} \times L_{i+k}[1+u_i(k), u_i(k)] + s_i(k))), \]
\[ I_i(k) = d(L_i^j \times \beta^{i+1} \times \ldots \times \beta^{i+k}, (\beta^{i+1} \times \ldots \times \beta^{i+k} \times L_{i+k}[2+u_i(k), 1+u_i(k)] + s_i(k))). \]

By (2) it suffices to show that \( \sup \Pi_i(k) - I_i(k) \rightarrow 0 \). This is true by \( \sum_{t} 1/\mu_i < \infty \) and by the equality
\[ d((L_i^j \times \beta^{i+1} \times \ldots \times \beta^{i+k}[2, \mu_i \lambda_i^k+1 \ldots \lambda_{i+k}^k]), \]
\[ (L_i^j \times \beta^{i+1} \times \ldots \times \beta^{i+k}[1, \mu_i \lambda_i^k+1 \ldots \lambda_{i+k}^k-1] \leq 2/\mu_i. \]

In Lemmas 2 and 3 we assume that \( u_i(k) = 0 \) for all \( t \geq 0, k \geq 1 \), and for convenience we write \( L_i \) instead of \( L_i^j \).

**Lemma 2.** The series \( \sum_{i=1}^{\infty} e_i \) is convergent.

---

**Proof.** Let us define
\[ d_i(t) = d(L_i \times \beta^{i+1} \times \ldots \times \beta^{i+k}, (\beta^{i+1} \times \ldots \times \beta^{i+k} \times L_{i+k}[1]),, \]
\[ d_{i}^{(t)} = d(L_i \times \beta^{i+1} \times \ldots \times \beta^{i+k}, (\beta^{i+1} \times \ldots \times \beta^{i+k} \times L_{i+k}[1]),, \]
\[ i = 1, 2, \ldots, \mu_i. \]

First we show the following formula:
\[ (9) \quad d_i(t) = \frac{1}{2} \left\{ \left( \sum_{k=1}^{n_i} (1-2d_i(t_i, t_{i+1})) \right) \right\}, \]
\[ t, k \geq 1. \]

Because of
\[ d_i(k) = \frac{n_i^{k+1}}{i^{k+1}} \sum_{j=1}^{n_i} \frac{d_i^{(t)}}{i^j}, \]

it is sufficient to show that for all \( t, k \geq 1 \) and \( i = 1, 2, \ldots, \mu_i \) the following equalities hold:
\[ (10) \quad 1 - 2d_i^{(t)} = \frac{1}{\mu_i} \sum_{j=1}^{n_i} \left( \prod_{k=1}^{n_i} (1 - 2d_{i+k}^{(t_i, t_{i+1}))} \right) \]
\[ \times \prod_{k=1}^{n_i} (1 - 2d_{i+k}^{(t_i, t_{i+1}))} (1 - 2d_{i+k}^{(t_i, t_{i+1}))} \ldots (1 - 2d_{i+k}^{(t_i, t_{i+1)))}. \]

We prove this by induction on \( k \). Fix \( k \geq 1 \). If \( k = 1 \) then (10) is true by (6). Suppose (10) holds for some \( k \geq 1 \) and each \( 1 \leq j \leq \mu_k. \) Let \( 1 \leq j \leq \mu_k \). \( \mu_k \). Applying (6) with \( L = \beta^{i+1} \times \ldots \times \beta^{i+k}, \beta = \beta^{i+1} \times \beta^{i+2} \times \ldots \times \beta^{i+k}, L' = (\beta^{i+1+k})_{i+k+1} \) we get
\[ 1 - 2d_i^{(t)} = 1 - 2d_i(t) \cdot \beta^{i+1+k} \times \beta^{i+2+k} \times \ldots \times \beta^{i+k} \times L_{i+k+1}[1] \]
\[ = 1 - \frac{2}{\mu_k} \sum_{i=1}^{n_i} d((L_i \times \beta^{i+1} \times \ldots \times \beta^{i+k} \times (\beta^{i+1+k})_{i+k+1}), \]
\[ \beta^{i+1} \times \beta^{i+2+k} \times \ldots \beta^{i+k} \times (\beta^{i+k+1+k}_{i+k+1} \times L_{i+k+1}[1]), \]
\[ = 1 - \frac{2}{\mu_k} \sum_{i=1}^{n_i} \left( \prod_{k=1}^{n_i} \left( 1 - 2d_{i+k}^{(t_i, t_{i+1}))} \times \beta^{i+1+k} \times \beta^{i+2+k} \times \ldots \beta^{i+k} \times L_{i+k+1}[1]\right) \right). \]
Applying the formula
\[(11) \quad d(A \times B, C \times D) = d(B+l, D) + [1 - 2d(B+l, D)]d(A+l, C),\]
(where the blocks \(A, B, C, D\) satisfy \(|A| = |C|, |B| = |D|\) and \(l \in \{0, 1\}\) with
\[A = \beta_{l+1} \times \ldots \times \beta_{l+k} \times (\beta_{l+k+1}), \quad B = (\beta_{l+k+1}) \times \beta_{l+k+1},
\]
\[C = \beta_{l+1} \times \ldots \times \beta_{l+k}, \quad D = (\beta_{l+k+1}) \times \beta_{l+k+1} + L_{l+k+1} [F], \quad I = L_{l+k} [F],\]
we get
\[1 - 2d^{l+1} (j) = \frac{1}{\mu_{k+1}} \sum_{i \neq j} (1 - 2d_{l+i+1} (i, i)) (1 - 2d_{l+i} (i))\]
and by the induction hypothesis it follows that \((10)\) holds.

Now we can show that \(\sum_{i} \xi_i < \infty\). We have \(1 - 2\xi_i (i, j) = |1 - 2d_i (i, j)|\).

Let us define \(\xi_i (k), \quad k \geq 1,\)
\[1 - 2\xi_i = \frac{1}{\mu_{k+1}} \sum_{i \neq j} \frac{1}{a_{l+k} + 2k - 1} \sum_{i = 1}^{\mu_{k+1}} (1 - 2e_i (k, i)) \sum_{i = 1}^{\mu_{k+1}} (1 - 2e_i (k, i)).\]

It is clear by \((9)\) that \(\inf (1 - 2\xi_i (k)) \geq 1\). Let \(t_0\) be an integer such that \(1 - 2\xi_i (k) \geq \frac{1}{2}\) for all \(i \geq t_0\) and \(k \geq 1\). We have
\[1 - 2\xi_i = \frac{1}{\mu_{k+1}} \sum_{i \neq j} (1 - 2e_i (k, i)) \sum_{i = 1}^{\mu_{k+1}} (1 - 2e_i (k, i)).\]

Hence \(\sum_{i} \xi_i < \infty\). Similarly \(\sum_{i} \xi_i < \infty\). This means that the series \(\sum \xi_i\) is convergent.

**Lemma 3.** The sequences of blocks \(\{a_i\}, \{\xi_i\}, r \geq 0\), defined in \(III\) and \(IV\) satisfy the conditions
\[\sum_{i} d(\beta^{l+1}, a_i \times a_i) < \infty, \quad \sum_{i} d(\beta^{l}, a_i \times a_i) < \infty.\]

**Proof.** Notice that for \((i, j) \in G_i\)
\[d((\beta^{l})_{a_i, a_j}, L_i [I], \beta^{l+1} a, \beta^{l+1} a_j, I, [I]) = \frac{1}{2},
\]
\[d((\beta^{l+1})_{a_i, a_j}, L_i [I], \beta^{l+1} a, \beta^{l+1} a_j, I, [I]) = \frac{1}{2}.
\]

The above conditions imply for \((i, j) \in G_i\)
\[d((\beta^{l})_{a_i, a_j}, L_i [I], \beta^{l+1} a, \beta^{l+1} a_j, I, [I]) = \xi_i (i, j), \quad i \geq 1.
\]

Therefore by Lemma 2 and \(4\) we obtain
\[(12) \quad \sum_{i} d(L_{l+1} \times \beta^{l}, \beta^{l} \times L) < \infty.
\]

In view of \((5)\) and \((11)\) we get
\[d(L_{l+1} \times \beta^{l}, \beta^{l} \times L) = \frac{1}{\mu_{k+1}} \sum_{i \neq j} \sum_{i = 1}^{\mu_{k+1}} (1 - 2e_i (k, i)) \sum_{i = 1}^{\mu_{k+1}} (1 - 2e_i (k, i)).\]

Hence in view of \((12)\) we have \(\sum_{i} d(L_{l+1} \times a_i, \phi^r) < \infty\). From \((13)\) we obtain
\[d(\beta^{l+1} a, L_i [I], \beta^{l+1} a_j, I, [I]) = \frac{1}{\mu_{k+1}} \sum_{i \neq j} \sum_{i = 1}^{\mu_{k+1}} (1 - 2e_i (k, i)) \sum_{i = 1}^{\mu_{k+1}} (1 - 2e_i (k, i)).\]

Therefore \(\sum_{i} d(\beta^{l+1} a, L_i [I]) < \infty\). Let us define
\[E_0 = 0 \ldots 0, \quad E_i [I] = \begin{cases} 0 & \text{if } L_i [I] = \bar{Y}_i [I], \\ 1 & \text{otherwise,} \end{cases} \quad i = 1, 2, \ldots, \mu, \quad r \geq 1.
\]

Notice that
\[d(L_{l+1} \times a_i, \phi^r) = d(\beta^{l+1} a, \beta^{l+1} a_j, I, [I]), \quad d(\beta^{l+1} a, \beta^{l+1} a_j, I, [I]) = d(\beta^{l+1} a, \beta^{l+1} a_j, I, [I]).\]

Thus we have \(\sum_{i} d(\beta^{l+1} a, a_j \times a_i) < \infty\), \(\sum_{i} d(\beta^{l+1} a, a_j \times a_i) < \infty\).
To finish the proof it suffices to show that \( \sum_{i} d(\tilde{u}_i, E_i) < \infty \). We may assume that

\[
(14) \quad d(L_{-1}, \tilde{u}_{-1}) \leq \frac{1}{2}
\]

(otherwise we replace \( \tilde{u}, a_i, E_i \) by \( \tilde{u}, a_i, E_i \), respectively and then (11) remains true). Let us define

\[
f_r(k) = d(L_i \times (a_{i+1} \times E_i) \times \ldots \times (a_{i+k} \times E_{i+k}), \\
(\tilde{a}_{i+1} \times \ldots \times (\tilde{a}_{i+k} \times a_{i+k})), \times (a_{i+1} \times \ldots \times (a_{i+k} \times \tilde{a}_{i+k}))
\]

It is easy to see that the triangle inequality and (11) imply \( \sup_{k \geq 1} f_r(k) > 0 \). Hence in view of (10) and (14) we obtain \( \sup_{k \geq 1} f_r(k) > 0 \).

We have \( \sum_{i} d(E_i, \tilde{u}_i) < \infty \).

Now we prove two lemmas which were needed in Case 2.

**Lemma 4.** The sequences of blocks \( \{p_t\}, \{q_t\}, t \geq 0 \), defined in V satisfy the condition

\[
\sup_{k \geq 1} d(L_i \times \hat{p}^{+1} \times \ldots \times \hat{p}^{+k}, \hat{q}^{+1} \times \ldots \times \hat{q}^{+k} \times L_{i+k}) \rightarrow 0.
\]

**Proof.** For convenience introduce the following notation:

\[
E_{\alpha,\beta}(q) = ([E \times (q, \beta)]_{\alpha,\beta}), \quad \text{where } |E| = n, p/n, 1 \leq q \leq n,
\]

\[
\eta_{\alpha,\beta} = \beta^{+1} \times \ldots \times \beta^{+k},
\]

\[
\xi_{\alpha,\beta}(F) = (\hat{p}^{+1} \times \ldots \times \hat{p}^{+k} \times F) \times [1 + u_{i,k}(u_{i,k} + s_k(k))],
\]

where \( |F| = \mu_{i+k} + 1 \).

\[
\xi_{\alpha,\beta}(F; i_1, \ldots, i_k) = (\hat{q}^{+1} \times \ldots \times (\hat{q}^{+k} \times l_{i+k} + 1 + u_{i,k}(u_{i,k} + s_k(k))))
\]

\[
\times (\hat{q}^{+1} \times \ldots \times \hat{q}^{+k} \times l_{i+k} + 1 + u_{i,k}(u_{i,k} + s_k(k)) + 1 + q_{i+k}),
\]

\[
1 \leq i_p \leq \alpha_{i+k-1}, \quad p = 1, \ldots, k, \quad t \geq 0, \quad k \geq 1.
\]

We will prove this lemma in three steps.

1. First we show that if \( L, \beta, b, L' \) are blocks with lengths \( \mu, \mu', \mu_0, \mu_1 \) respectively and \( 0 \leq q \leq \mu_0 - 1,\), then

\[
(15) \quad d(L \times \beta, (b \times L') \times [1 + q \times \mu_0, \mu]) \leq \frac{1}{\alpha_0}
\]

where

\[
r(i) = \begin{cases} 0 & \text{if } 1 \leq i \leq \lceil \mu_0 - q \rceil, \\
1 & \text{if } i > \lceil \mu_0 - q \rceil.
\end{cases}
\]

To this end divide the block \( L \times \beta \) into \( \mu' \) consecutive blocks \( (L \times \beta, \omega, \mu') \). Next divide each block \( L \times \beta, \omega \) into \( \omega \) blocks \( L \times \beta, \omega, \mu \).

Let \( E_k \) denote the block

\[
(L \times \beta, \omega, \mu) \rightarrow \ldots \rightarrow (L \times \beta, \omega, \mu).
\]

It is easy to see that \( E_k = L \times \beta, \omega, \mu \). Let \( E_k' \) be the block obtained from the block \( (b \times L) \times [1 + q \times \mu_0, \mu] \) in the same way as \( E_k \) from \( L \times \beta \). It remains to observe that

\[
E_k = E_k' \times [1 + q \times (L' \times [1 + r(k), \mu])]
\]

for \( k \neq 1 + \lceil \mu_0 - q \rceil \).

2. Next we show that for all \( t, k \geq 1 \)

\[
(16) \quad d_t(k) \leq \frac{1}{\alpha_t} \sum_{i_1 = 1}^{\mu_0} \sum_{i_2 = 1}^{\mu_0} \ldots \sum_{i_k = 1}^{\mu_0} d(\eta_{\alpha,\beta}(i_1, \ldots, i_k), \xi_{\alpha,\beta}(i_1, \ldots, i_k)) \leq \frac{1}{\alpha_t} + \ldots + \frac{1}{\alpha_{i+k-1}},
\]

where

\[
d_t(k) = d(L_i \times \hat{p}^{+1} \times \ldots \times \hat{p}^{+k} \times F \times [1 + u_{i,k}(u_{i,k} + s_k(k)),
\]

where \( L_i \times \hat{p}^{+1} \times \ldots \times \hat{p}^{+k} \times L_{i+k} \times [1 + u_{i,k}(u_{i,k} + s_k(k)),
\]

In order to prove this we show that for each block \( E \) of length \( \mu_{i+k} + 1 \) the number

\[
R_\alpha(E) = d(\eta_{\alpha,\beta}(E), \xi_{\alpha,\beta}(E))
\]

\[
\frac{1}{\alpha_t} \sum_{i_1 = 1}^{\mu_0} \ldots \sum_{i_k = 1}^{\mu_0} d(\eta_{\alpha,\beta}(i_1, \ldots, i_k), \xi_{\alpha,\beta}(E; i_1, \ldots, i_k))
\]

\[
\frac{1}{\alpha_t} \sum_{i_1 = 1}^{\mu_0} \ldots \sum_{i_k = 1}^{\mu_0} d(\eta_{\alpha,\beta}(i_1, \ldots, i_k), \xi_{\alpha,\beta}(E; i_1, \ldots, i_k))
\]

\[
\frac{1}{\alpha_t} \sum_{i_1 = 1}^{\mu_0} \ldots \sum_{i_k = 1}^{\mu_0} d(\eta_{\alpha,\beta}(i_1, \ldots, i_k), \xi_{\alpha,\beta}(E; i_1, \ldots, i_k))
\]
satisfies

$$|R_{<a}(E)| \leq \frac{1}{c_0} + \ldots + \frac{1}{c_{o_{k+1}}}.$$ 

Fix $i > 1$. For $k = 1$, (16) is true in view of (15). Suppose that (16) holds for some $k \geq 1$ and an arbitrary block $E$ of length $\mu_{k+1} + 1$. Let $F$ be a block of length $\mu_{k+1} + 1$. Applying (15) to $L = L' \times \beta^{s+1} \times \ldots \times \beta^{s+k}$, $\beta = \beta^{s+k+1}$, $b = \beta^{s+1} \times \ldots \times \beta^{s+k+1}$, $L = F$, $q = u_k(k+1)$, we get

$$d(\eta_{<a}, \xi_{<a}(F)) = \frac{1}{c_{o_{k+1}}} \sum_{i=1}^{o_{k+1}} d(\eta_{<a} \times (\beta^{s+1} \times \ldots \times \beta^{s+k+1})_{i}, \xi_{<a}(F_{i})),$$

where $F_{i+1} = F[1 + r_{i+1}(i+1), t_{i+1} + \mu_{i+1}]$ and $M \leq 1/c_{o_{k+1}}$. Notice that putting

$$F_{i+1} = (\beta^{s+1} \times \ldots \times \beta^{s+k+1})[1 + u_{k+1}(i+1), t_{i+1} + \mu_{i+1} + 1 + u_{k+1}]$$

we have

$$(\beta^{s+1} \times \ldots \times \beta^{s+k+1})_{i+1} = \xi_{<a}(F_{i+1}).$$

Put $d' = d((\beta^{s+1} \times \ldots \times \beta^{s+k+1})_{i}, \xi_{<a}(F_{i+1}))$. It follows from (17) and the induction hypothesis that

$$d(\eta_{<a} \times (\beta^{s+1} \times \ldots \times \beta^{s+k+1})_{i}, \xi_{<a}(F_{i+1})) = d'(1 + 2d')d(\eta_{<a}, \xi_{<a}(F_{i+1}))$$

$$= d'(1 + 2d')d(\eta_{<a}, \xi_{<a}(F_{i+1}))$$

$$= d'(1 + 2d')d(\eta_{<a}, \xi_{<a}(F_{i+1}))$$

$$= \frac{1}{c_{o_{k+1}}} \sum_{i=1}^{o_{k+1}} d(\eta_{<a}(i_1, \ldots, i_k), \xi_{<a}(F_{i+1})))$$

$$= \frac{1}{c_{o_{k+1}}} \sum_{i=1}^{o_{k+1}} d(\eta_{<a}(i_1, \ldots, i_k), \xi_{<a}(F_{i+1})))$$

$$+ (1 + 2d')R_{<a}(E_{i+1}).$$

Because of $\xi_{<a}(F_{i+1}; i_1, \ldots, i_k) = \xi_{<a}(F; i_1, \ldots, i_k)$ we thus have

$$|R_{<a}(F)| \leq M + \frac{1}{c_{o_{k+1}}} \sum_{i=1}^{o_{k+1}} R_{<a}(E_{i+1}) \leq \frac{1}{c_0} + \ldots + \frac{1}{c_{o_{k+1}}}.$$

3. Now we can show that

$$\sup_{a \in A} d(L', \beta^{s+1} \times \ldots \times \beta^{s+k}) \to 0.$$ 

Put

$$\xi_{<a}(l_1, \ldots, l_k) = (\beta^{s+1})_{l_1} \times \ldots \times (\beta^{s+k})_{l_k} \times L_{i+k},$$

$$\eta_{<a}(l_1, \ldots, l_k) = L' \times (\beta^{s+1})_{l_1} \times \ldots \times (\beta^{s+k})_{l_k} \times L_{i+k}.$$ 

In view of (16) we have

$$d(L', \beta^{s+1} \times \ldots \times \beta^{s+k}) \leq \frac{1}{c_0} \sum_{i=1}^{o_{k+1}} \sum_{i=1}^{o_{k+1}} d(\xi_{<a}(l_1, \ldots, l_k), \eta_{<a}(l_1, \ldots, l_k)) \leq \frac{1}{c_0} + \ldots + \frac{1}{c_{o_{k+1}}}.$$ 

It is not hard to see that

$$(\beta^{s+1})_{l_1} = (\beta^{s+1})_{l_1} + (l_1 + u_{k+1}),$$

$$|d((\beta^{s+1})_{l_1}, (\beta^{s+1})_{l_1} + (l_1 + u_{k+1}))| \leq \frac{1}{\mu_{k+1}},$$

$$|d((\beta^{s+1})_{l_1}, (\beta^{s+1})_{l_1} + (l_1 + u_{k+1}))| \leq \frac{1}{\mu_{k+1}}.$$ 

Therefore, using the formula

$$|d(A_1, \ldots, A_n, B_1, \ldots, B_n) - d(C_1, \ldots, C_n, D_1, \ldots, D_n)| \leq \sum_{i=1}^{n} |d(A_i, B_i) - d(C_i, D_i)|.$$
(where the blocks \( A_i, B_i, C_i, D_i \) satisfy \( |A_i| = |B_i|, |C_i| = |D_i| \)) we obtain
\[
\frac{1}{a_0 \cdots a_{k+1}} \sum_{k=1}^{a_1 + k} \sum_{a_k} \left( d(\delta_i(l_1, \ldots, l_k), \delta_i(l_1, \ldots, l_k)) \right) - \frac{1}{a_0 \cdots a_{k+1}} \sum_{k=1}^{a_1 + k} \sum_{a_k} \left( d(\delta_i(l_1, \ldots, l_k), \delta_i(l_1, \ldots, l_k)) \right) \leq \frac{1}{a_0 \cdots a_{k+1}} \sum_{k=1}^{a_1 + k} \sum_{a_k} \left( \frac{1}{\mu_{i+1}} + \frac{1}{\mu_{i+2}} + \cdots + \frac{1}{\mu_{i+k}} \right).
\]
Thus (18) and (16) imply that
\[
|d_i| - d(L_i \times \delta^{i+1} \times \cdots \times \delta^{i+k}, \psi^{i+1} \times \cdots \times \psi^{i+k} \times L^{i+k}_k) \equiv 2 \left( \frac{1}{a_0} + \cdots + \frac{1}{a_{i+1}} \right) \frac{1}{\mu_{i+1}} + \cdots + \frac{1}{\mu_{i+k}}.
\]
Hence
\[
\sup_{i \geq 1} d(L_i \times \delta^{i+1} \times \cdots \times \delta^{i+k}, \psi^{i+1} \times \cdots \times \psi^{i+k} \times L^{i+k}_k) \not= 0.
\]
Now we prove the last lemma.

**Lemma 5.** Let \( \{a_i\}, \{b_i\}, \{K_i\}, t \geq 0 \), be the blocks defined in VI. There exist sequences of integers \( \{a_i\}, \{b_i\}, 0 < a_i < b_i, 0 < b_i \leq a_i \), such that
\[
\sum_{i} d((k')^{[1]} + q_i, k_i, a_i) < \infty,
\]
\[
\sum_{i} d((\beta')^{[1]} + q_i + b_i, a_i) < \infty,
\]
\[
\sum_{i} \min(1/j_i, n_i/j_i) < \infty,
\]
\[
\sum_{i} \min(1/j_i', n_i/j_i) < \infty,
\]
where
\[
j_0 = q_0, \quad j_i = q_i + q_i n_0 + \cdots + q_i n_{i-1},
\]
\[
q_i = q_i \text{ if } q_i \equiv 1 \text{ (mod } \lambda_i) \text{ otherwise, } i \geq 1,
\]
\[
\lambda_i \text{ is defined similarly (we replace } q_i, \lambda_i, q_i, b_i, a_i, p_i).\]

**Proof.** Let us define \( p_i = \lambda_i' - (a_i - k_i) \text{ (mod } \lambda_i) \), \( t \geq 1 \). (recall that \( k_i = [(a_i - q_i)/\mu_{i-1}] \)). First we show that
\[
d(\delta', K_i \times a_i) = d((\beta')^{[1]} + p_i + k_i + b_i, a_i), \quad t \geq 1.
\]
Suppose first that \( 0 < k_i < a_o - 1 \). Fix \( t \geq 1 \). Put \( a_i' = K_i[1], a_i'' = K_i[1 + k_i + b_i - k_i], \)
\[
+ k_i, a_o - 1 \}. \]
Introduce the following notation:
\[
\beta_i = \beta^{[1]} + (1 - t) a_o - 1, \quad \beta_i = a_o - 1 + k_i, \quad i = 1, \ldots, t,
\]
\[
\beta_i' = \beta_i, \beta_i'' = \beta_i', \quad \beta_i'' = \beta_i''', \quad \beta_i''' = \beta_i''', \quad \beta_i'''' = \beta_i''''.
\]
We have \( |\beta_i| = k_i, |\beta_i''| = a_o - 1 - k_i, i = 1, \ldots, t \). Because of
\[
(\beta_i)^{[1]} + a_o - 1, \quad \beta_i^{[1]} + a_o - 1, \quad \beta_i^{[1]} + a_o - 1, \quad \beta_i^{[1]} + a_o - 1,
\]
we obtain
\[
(\beta_i)^{[1]} + a_o - 1, \quad \beta_i^{[1]} + a_o - 1, \quad \beta_i^{[1]} + a_o - 1, \quad \beta_i^{[1]} + a_o - 1.
\]
On the other hand,
\[
a_i \times a_i = (a_i' + a_i[1]) a_i + [a_i[1]] \times (a_i'' + a_i[1]) a_i + [a_i[1]] a_i + [a_i[1]] a_i.
\]
Therefore we have
\[
d((\beta')^{[1]} + p_i + k_i + b_i, a_i \times a_i)
= k_i \left( \frac{1}{a_o - 1} \right) d((\beta', a_i', a_i') + \left( \frac{1}{a_o - 1} \right) d(\beta_i', a_i'' \times (a_i, a_i)[2, 1 + a_i]).
\]
Notice that
\[
d((\delta'), K_i \times a_i) = \frac{1}{a_o - 1} \sum_{i=1}^{a_i - 1} d((\delta')^{[1]} + K_i[1] + a_i).
\]
\[
= \frac{1}{a_o - 1} \sum_{i=1}^{a_i - 1} d((\delta')^{[1]} + K_i[1] + a_i + K_i[1] + \lambda_i).
\]
\[
= \frac{1}{a_o - 1} \sum_{i=1}^{a_i - 1} d((\delta')^{[1]} + K_i[1] + a_i + K_i[1] + \lambda_i) + \frac{1}{a_o - 1} \sum_{i=1}^{a_i - 1} d((\delta')^{[1]} + K_i[1] + a_i + K_i[1] + \lambda_i) + \frac{1}{a_o - 1} \sum_{i=1}^{a_i - 1} d((\delta')^{[1]} + K_i[1] + a_i + K_i[1] + \lambda_i) + \frac{1}{a_o - 1} \sum_{i=1}^{a_i - 1} d((\delta')^{[1]} + K_i[1] + a_i + K_i[1] + \lambda_i) + \frac{1}{a_o - 1} \sum_{i=1}^{a_i - 1} d((\delta')^{[1]} + K_i[1] + a_i + K_i[1] + \lambda_i).
It suffices to show that
\[
\frac{1}{k_i} \sum_{i=1}^{\lambda_i-1} d((\beta'_{i-1} \sigma_i, \beta_i) + K_i, [1]) = d(\beta', a'_i \times a_i),
\]
\[
\frac{1}{\alpha_i - 1 - k_i} \sum_{i=1}^{\alpha_i - 1 - k_i} d((\beta'_{i-1} \sigma_i, \beta_i) + K_i, [1]) = d(\beta', a'_i \times (a_i, a_i) [2, 1 + \mu_i]).
\]

The first equality is true by the following equalities:
\[
(\beta'_{i-1} \sigma_i, \beta_i) + K_i = \beta [i + \alpha_i] \beta [i + \alpha_i + 1] \ldots \beta [i + (\mu_i - 1) \alpha_i - 1],
\]
\[
(a'_i \times (a_i, a_i) [2, 1 + \mu_i]) \sigma_i = (a_i, a_i) [2, \mu_i + 1] + a'_i, [1]
\]
\[
(\beta'_{i-1} \sigma_i, \beta_i) + K_i = \beta [i + \alpha_i] \beta [i + \alpha_i + 1] \ldots \beta [i + (\mu_i - 1) \alpha_i - 1],
\]

The second is a consequence of the equalities
\[
(\beta'_{i-1} \sigma_i, \beta_i) + K_i = \beta [i + k_i] \beta [i + k_i + \alpha_i - 1] \ldots \beta [i + k_i + (\mu_i - 1) \alpha_i - 1],
\]
\[
(a'_i \times (a_i, a_i) [2, 1 + \mu_i]) \sigma_i = (a_i, a_i) [2, \mu_i + 1] + a'_i, [1]
\]
\[
(\beta'_{i-1} \sigma_i, \beta_i) + K_i = \beta [i + k_i] \beta [i + k_i + \alpha_i - 1] \ldots \beta [i + k_i + (\mu_i - 1) \alpha_i - 1],
\]

If \(k_i = 0\) or \(\alpha_i - 1\), it is not hard to see that (19) also holds. Therefore
\[
\sum_t d((\beta, \beta') [1 + p_t, p_t + \lambda], a_i \times a_i) < \infty.
\]

Put \(\tilde{q}_i = q_i + k_i \mu_i - 1 \mod \lambda_i\). It is easy to see that
\[
\sum_t d((\beta, \beta') [1 + \tilde{q}_i, \tilde{q}_i + \lambda_i], a_i \times a_i) = \sum_t d((\beta, \beta') [1 + q_i, q_i + \lambda_i], a_i \times K_i)
\]

and in view of (8) we get
\[
\sum_t d((\beta, \beta') [1 + \tilde{q}_i, \tilde{q}_i + \lambda_i], a_i \times a_i) < \infty.
\]

It remains to show that \(\sum_{i=0}^{\infty} \min(1 - \frac{\beta_i}{\eta_i}, \frac{j_i}{\eta_i}) < \infty\). Notice that if \(p_i \neq 0\) then \(p_i \geq \lambda_i - \alpha_i - 1\) and if \(q_i \neq 0\) then \(q_i \geq \lambda_i - \mu_i - 1\). Thus it is obvious that if \(p_i \neq 0\) then \(\alpha_i - 1 \leq \mu_i - 1\) and if \(q_i \neq 0\) then \(\lambda_i - \mu_i - 1 \leq q_i \leq \lambda_i - 1\). Since
\[
0 \leq \frac{j_i}{\eta_i}, \frac{p_i}{\lambda_i} < \frac{1}{\lambda_i}, \quad 0 \leq \frac{j_i}{\eta_i}, \frac{q_i}{\lambda_i} < \frac{1}{\lambda_i},
\]
we have
\[
\sum_{i=0}^{\infty} \min\left(1 - \frac{j_i}{\eta_i}, \frac{j_i}{\eta_i}\right) < \infty, \quad \sum_{i=0}^{\infty} \min\left(1 - \frac{p_i}{\eta_i}, \frac{j_i}{\eta_i}\right) < \infty.
\]
This completes the proof of Lemma 5 and finishes the proof of Theorem 1.

\textbf{3. Metric isomorphism in case} \(\lambda_i = \lambda, \ t \geq 0\). Suppose that the lengths \(\lambda_i, \lambda_i'\) of the blocks \(\beta, \beta'\) of continuous Morse sequences \(x = b^a b^b \ldots\) and \(y = b^a b^b \ldots\) are equal for all \(i \geq 0\). From Theorem 1 one can obtain the following theorem, which we give here without proof.

\textbf{Theorem 2.} \(\theta(x)\) is metrically isomorphic to \(\theta(y)\) iff there exist sequences of integers \(\{r_i\}, \{s_i\}, r_i, s_i \in [0, 1], \ t \geq 0\), and a sequence of integers \(\{q_i\}, q_i \in [0, 1, \ldots, \lambda - 1], \ t \geq 0\), such that
\[
\sum_{i=0}^{\infty} \left(1 - \frac{j_i}{\eta_i}, \frac{j_i}{\eta_i}\right) D_{r_i+1} + \frac{j_i}{\eta_i} D_{s_i+1} < \infty,
\]
where
\[
D_{r_i+1} = d((\beta, \beta') [1 + q_i, q_i + \lambda_i], \beta^{r_i+1} + r_i),
\]
\[
D_{s_i+1} = d((\beta, \beta') [1 + q_i, q_i + \lambda_i], \beta^{s_i+1} + s_i),
\]
and \(j_i = q_i + j_i - q_i, j_i - 1 \geq 1\).

Remark 2. Put
\[
q_i = \begin{cases} q_i & \text{if } 2q_i - \lambda_i - 1 \leq \lambda_i - 1, \\ q_i + 1 \mod \lambda_i & \text{otherwise,} \end{cases}
\]
and if \(q_i < \lambda_i - 1\) or \(2q_i - \lambda_i - 1\),
\[
r_i = \begin{cases} r_i & \text{if } 2q_i - \lambda_i - 1 \leq \lambda_i - 1, \\ r_i + s_i + 1 \mod 2 & \text{otherwise,} \end{cases}
\]
\[
s_i = \begin{cases} s_i + 1 \mod 2 & \text{if } 2q_i - \lambda_i - 1 \leq \lambda_i - 1, \\ s_i + s_i + 1 \mod 2 & \text{otherwise.} \end{cases}
\]

If \(q_i = \lambda_i - 1\) and \(2q_i > \lambda_i - 1\) then
\[
r_i + s_i = s_i + 1 \mod 2, \quad s_i = r_i + s_i \mod 2, \quad t \geq 0,
\]
\[
r_i = r_i, \quad s_i = s_i.
\]

Then \(r_i + s_i \mod 2 = r_i + 1 + s_i \mod 2\) and it is not hard to see that (20) is equivalent to the following two conditions:
\[
\sum_{i=0}^{\infty} d((\beta, \beta') [1 + q_i, q_i + \lambda_i], \beta^{r_i+1} + s_i < \infty,
\]
\[
\sum_{i=0}^{\infty} \min\left(1 - \frac{j_i}{\eta_i}, \frac{j_i}{\eta_i}\right) \left(\eta_i, \frac{j_i}{\eta_i}\right) < \infty.
\]
where
\[ \eta_i = \begin{cases} \frac{fr(01, b^i) + fr(10, b^i)}{\lambda_i} & \text{if } r_i = s_i, \\ \frac{fr(00, b^i) + fr(11, b^i)}{\lambda_i} & \text{if } r_i \neq s_i, \end{cases} \]

and \( \eta_i = 0 \) if the number \( b^{i+1} [\lambda_{i+1}] + r_{i+1} + r_i + s_{i+1} + s_i \) is odd and \( \eta_0 = 1 \) otherwise, \( t \geq 0 \).

Condition (A) of [3] is the same as condition (21) and conditions (A), (B) imply conditions (21) and (22). If \( x \) is regular, then obviously condition (22) is the same as (B).

**Example.** We give an example of continuous Morse sequences \( x = b^0 \times b^1 \times \ldots, y = b^0 \times b^1 \times \ldots, |b^i| = |b^i|, t \geq 0, \) such that \( \theta(x) \) and \( \theta(y) \) are metrically isomorphic and condition (B) is not satisfied.

To this end we set
\[ b^0 = 00 \ldots 0 \quad 1^* \ldots 1, \quad b^1 = 0^* 1 \ldots 0 \quad 10 \ldots 1, \quad t \geq 0. \]

Then \( x = b^0 \times b^1 \times \ldots, y = b^0 \times b^1 \times \ldots \) are continuous (but not regular) Morse sequences. Taking \( r_i = s_i = 0, q_i = 1 + (t+1)^2, t \geq 0, \) we see that conditions (21) and (22) are satisfied so \( \theta(x) \) and \( \theta(y) \) are metrically isomorphic. It is not hard to see that if \( q_i \geq \frac{1}{2} (t+1)^2 \) or \( q_i \geq \lambda_i - \frac{1}{2} (t+1)^2 \) then for all \( r_i, s_i \in \{0, 1\} \)

\[ d((b^i)^t) (b^i)^t [1 + q_i, q_i + \lambda_i], b^i) \geq \frac{1}{2} (t+1)^2 \] \[ \frac{3}{2} (t+1)^2 + 1 \geq \frac{1}{2}. \]

Thus if condition (A) holds, then the series in (B) is divergent.

**A note about finitary isomorphism.** We finish this paper by giving (without proof) the necessary and sufficient conditions for Morse dynamical systems to be finitarily isomorphic.

**Theorem 3.** Let \( x = b^0 \times b^1 \times \ldots \) and \( y = b^0 \times b^1 \times \ldots \) be Morse sequences. Then \( \theta(x) \) and \( \theta(y) \) are finitarily isomorphic iff there exist blocks \( A, B, |A| = |B| \), and a Morse sequence \( z \) such that \( x = A \times z \) and \( y = B \times z \). In particular, if \( |b^i| = |b^i| \) for \( t \geq 0 \) then \( \theta(x) \) and \( \theta(y) \) are finitarily isomorphic iff \( b^i = b^j \) for all sufficiently large \( i \).

**Corollary.** Each class of finitary equivalence is countable and coincides with a class of topological conjugacy (see [1]).