satisfy the condition $D_m \leqslant M < 1$ *then*

$$\sum_{i=0}^{k} \bar{b}_i \leqslant \frac{M}{1-M}. \tag{59}$$

Proof. We have

$$D_{m+1} = D_m + (1-D_m)\bar{b}_{m+1}, \quad m = 0, 1, \ldots, k-1.$$

This implies that $D_m < 1$ and

$$D_{m+1} > D_m + (1-M)\bar{b}_{m+1}.$$

Hence

$$D_k > (1-M)\sum_{i=0}^{k} \bar{b}_i$$

which gives (59).

References

[1] G. R. Goodson, *On the spectral multiplicity of a class of finite rank transformations*, Proc. Amer. Math. Soc. 93 (2) (1985), 303–306.

[2] R. M. Gray, D. L. Neuhoff and P. C. Shields, *A generalization of Ornstein's d distance with application to information theory*, Ann. Probab. 3 (2) (1975), 315–328.

[3] A. del Junco, *A transformation with simple spectrum which is not rank one*, Canad. J. Math. 29 (3) (1977), 655–663.

[4] M. Keane, *Generalized Morse sequences*, Z. Wahrsch. Verw. Gebiete 10 (1968), 335–353.

[5] J. Kwiatkowski, *Isomorphism of regular Morse dynamical systems*, Studia Math. 72 (1982), 59–89.

[6] —, *Spectral isomorphism of Morse dynamical systems*, Bull. Acad. Pol. Sci. Sér. Sci. Math. 29 (3-4) (1981), 105–114.

[7] M. Lemańczyk, *The rank of regular Morse dynamical systems*, to appear.

[8] —, *The sequence entropy of Morse shifts and some counterexamples*, Studia Math. 82 (1985), 221–241.

[9] D. S. Ornstein, *An application of ergodic theory to probability theory*, Ann. Probab. 1 (1) (1973), 43–58.

[10] T. Rojek, *On metric isomorphism of Morse dynamical systems*, this volume, 247–267.

INSTYTUT MATEMATYKI UNIWERSYTETU MIKOŁAJA KOPERNIKA
INSTITUTE OF MATHEMATICS, NICHOLAS COPERNICUS UNIVERSITY
Chopina 12/18, 87-100 Toruń, Poland

Received December 17, 1984
Revised version July 30, 1985 (2022)

STUDIA MATHEMATICA, T. LXXXIV. (1986)

On metric isomorphism of Morse dynamical systems

by

TADEUSZ ROJEK (Toruń)

Abstract. For each continuous Morse sequence x, the class of all continuous Morse sequences y such that the dynamical systems induced by x and y are metrically isomorphic is described.

Introduction. J. Kwiatkowski in [3] gave sufficient and necessary conditions for two Morse dynamical systems $\theta(x)$ and $\theta(y)$ induced by $x = b^0 \times b^1 \times \ldots$ and $y = \beta^0 \times \beta^1 \times \ldots$ to be metrically isomorphic, assuming that the lengths of the b^t and β^t are the same for $t = 0, 1, 2, \ldots$ and x and y are regular sequences. It is also proved in [3] that for a given Morse sequence x there exist a continuum of Morse sequences y such that the systems $\theta(x)$ and $\theta(y)$ are metrically isomorphic but the corresponding shift invariant measures on the space $X = \prod_{-\infty}^{+\infty} \{0, 1\}$ are pairwise orthogonal. For a given regular Morse sequence x Kwiatkowski defines a class $\mathcal{M}(x)$ of Morse sequences y such that the dynamical systems $\theta(x)$ and $\theta(y)$ are metrically isomorphic.

However, the procedure of obtaining the class $\mathcal{M}(x)$ which is described there can be applied to a continuous Morse sequence x (without the assumption of regularity). In this paper we show that $\mathcal{M}(x)$ is the class of all continuous Morse sequences y such that $\theta(y)$ is metrically isomorphic to $\theta(x)$.

To prove this, we use the same technique of coding as in [3], but we omit the assumption that the lengths of the blocks b^t and β^t are equal and thus codes have different form. In order to prove the main result, for given Morse sequences $x = b^0 \times b^1 \times \ldots$ and $y = \beta^0 \times \beta^1 \times \ldots$ such that $\theta(x)$ is metrically isomorphic to $\theta(y)$ we construct a Morse sequence $z = a_0 \times \bar{a}_0 \times a_1 \times \bar{a}_1 \times \ldots$ satisfying

$$|a_0| = |b^0|, \quad |\bar{a}_0 \times a_1| = |b^1|, \quad |\bar{a}_1 \times a_2| = |b^2|, \quad \ldots$$
$$|a_0 \times \bar{a}_0| = |\beta^0|, \quad |a_1 \times \bar{a}_1| = |\beta^1|, \quad |a_2 \times \bar{a}_2| = |\beta^2|, \quad \ldots$$

such that x, z and also y, z satisfy conditions (A), (B) of [3] (here $|E|$ denotes the length of the block E). In this construction we use the distance $d(\cdot,\cdot)$ between blocks; however, we calculate it in a different manner than in [3].

In this paper we also announce two additional results. We present a generalization of Kwiatkowski's result for two Morse dynamical systems $\theta(x)$ and $\theta(y)$ induced by $x = b^0 \times b^1 \times \ldots$, $y = \beta^0 \times \beta^1 \times \ldots$, $|b^t| = |\beta^t|$, $t \geqslant 0$, without assuming their regularity. This generalization is essential as is shown by an example. We also give a necessary and sufficient condition for two Morse dynamical systems to be finitarily isomorphic. It turns out that finitary isomorphism coincides with topological conjugacy in the class of Morse shifts (see [1]). We omit the proofs of these results, because laborious calculations would considerably lengthen the paper.

Since the construction of the sequence z is also laborious, we begin with a special section with a sketch of it. We use the definitions and notation listed in [3].

The author wishes to thank J. Kwiatkowski for many helpful conversations on the results of this paper.

§ 1. **Outline of the construction.** Consider two continuous Morse sequences $x = b^0 \times b^1 \times \ldots$ and $y = \beta^0 \times \beta^1 \times \ldots$ We do not assume that $|b^t| = |\beta^t|$, $t = 0, 1, \ldots$, and we omit the assumption that they are regular. Let $\mathcal{M}(x)$ denote the class defined in the introduction of [3]. The main result of this paper is the following theorem.

THEOREM 1. *The Morse dynamical systems $\theta(x)$ and $\theta(y)$ induced by x and y are metrically isomorphic iff $y \in \mathcal{M}(x)$.*

To prove the theorem it suffices to show the "if" part because the opposite implication is proved in [3].

Assume that $\theta(x)$ and $\theta(y)$ are metrically isomorphic. We will construct a Morse sequence z such that x and y can be obtained from z by the procedure described in [3, Introduction] (i.e. $x, y \in \mathcal{M}^*(z)$). Now we give the sketch of the construction of the sequence z. To do this we use a coding technique the same as in [3, § 2]. Let us denote $\lambda_t = |b^t|$, $\lambda'_t = |\beta^t|$, $n_t = \lambda_0 \cdot \ldots \cdot \lambda_t$, $n'_t = \lambda'_0 \cdot \ldots \cdot \lambda'_t$, $t \geqslant 0$. In this paper if $B = b_1 b_2 \ldots b_n$ is a block, then $B[j, k]$, $1 \leqslant j \leqslant k \leqslant n$, denotes the block $b_j b_{j+1} \ldots b_k$, and the symbol $B+i$, $i \in \{0, 1\}$, denotes the block B^i (which is equal to B if $i = 0$ and to $\tilde{B}$ if $i = 1$).

Let $h\colon X(y) \to X(x)$ be an isomorphism between $\theta(y)$ and $\theta(x)$. By Keane's results [2] it follows that the eigenvalue group Λ of $\theta(x)$ consists of all n_t-roots of unity and the eigenvalue group Λ' of $\theta(y)$ consists of n'_t-roots of unity. So $\Lambda = \Lambda'$. Hence by grouping the blocks $\{b^0, b^1, \ldots\}$ and $\{\beta^0, \beta^1, \ldots\}$ we may assume that

$$n_0 | n'_0, \quad n'_0 | n_1, \quad n_1 | n'_1, \quad n'_1 | n_2, \ldots$$

So $n'_t = \mu_t n_t$, $n_{t+1} = \omega_t n'_t$, $t \geqslant 0$. Moreover, we may assume that

$$\sum_{t=0}^{\infty} \frac{1}{\mu_t} < \infty \quad \text{and} \quad \sum_{t=0}^{\infty} \frac{1}{\omega_t} < \infty.$$

The last conditions yield

$$\lambda'_0 = \mu_0 \lambda_0, \quad \lambda_{t+1} = \mu_t \omega_t, \quad \lambda'_{t+1} = \omega_t \mu_{t+1}, \quad t = 0, 1, 2, \ldots$$

If $Q = h^{-1}(P(x))$, where $P(x)$ is the time zero partition of $X(x)$, then we find a sequence $\{Q^t\}$ of partitions of $X(y)$ such that $Q^t \prec \xi_t$ and $|Q - Q^t| \to 0$. Reasoning in the same way as in [3, § 2] we conclude that Q is described by codes $\{A_t, B_t\}$ satisfying

(1) $$A_t = (c_t \times L_t)[1 + l_t, l_t + \mu_t n_t], \quad B_t = \tilde{A}_t, \quad l_{t+1} \equiv l_t \pmod{n_t},$$

where $c_t = b^0 \times b^1 \times \ldots \times b^t$, L_t are blocks of 0 and 1 of lengths $\mu_t + 1$ and $l_t \in \{0, 1, \ldots, n_t - 1\}$, $t = 0, 1, 2, \ldots$

The codes $\{A_t, \tilde{A}_t\}$ satisfy the condition

(2) $$\sup_{k \geqslant 1} d(A_{t+k}, A_t^{(k)}) \underset{t}{\to} 0,$$

where $A_t^{(k)} = A_t \times \beta^{t+1} \times \ldots \times \beta^{t+k}$. This condition and (1) are the main tools in the construction of z. It is convenient to modify (2) to the following condition:

(3) $$\sup_{k \geqslant 1} d\big(L_t[1, \mu_t] \times \beta^{t+1} \times \ldots \times \beta^{t+k},$$
$$(b^{t+1} \times \ldots \times b^{t+k} \times L_{t+k})[1 + u_t(k), u_t(k) + s_t(k)]\big) \underset{t}{\to} 0,$$

where $s_t(k) = \lambda_{t+1} \cdot \ldots \cdot \lambda_{t+k} \cdot \mu_{t+k}$ and $u_t(k) = (1/n_t)(l_{t+k} - l_t)$. We will prove (3) in Lemma 1.

We are now in a position to present the main steps of the construction of z. We consider two cases: $u_t(k) = 0$ for $t \geqslant 0$, $k \geqslant 1$, and $u_t(k)$ arbitrary.

In the sequel we will often use some formulas for calculating the distance d between blocks. First for a given block E with length n and an integer $p > 1$ such that $p | n$ we define blocks $E_{i,p}$ and $E^*_{i,p}$ as

$$E_{i,p} = E[(i-1)n/p + 1, in/p],$$
$$E^*_{i,p} = E[i] E[i+p] \ldots E[i + (n/p - 1)p], \quad i = 1, 2, \ldots, p.$$

Thus $|E_{i,p}| = |E^*_{i,p}| = n/p$. In the sequel we will write $B \times A^*_{i,p}$, $B \times A_{i,p}$ instead of $B \times (A^*_{i,p})$, $B \times (A_{i,p})$.

Let L, β, b, L' be blocks with lengths μ, $\mu'\omega$, $\mu\omega$, μ' respectively. We have the following formulas:

$$(4)\qquad d(L\times\beta,\, b\times L') = \frac{1}{\mu\mu'}\sum_{i=1}^{\mu'}\sum_{j=1}^{\mu} d(\beta_{i,\mu'}+L[j],\, b^{*}_{j,\mu}+L'[i]),$$

$$(5)\qquad d(L\times\beta,\, b\times L') = \frac{1}{\omega}\sum_{i=1}^{\omega} d(L\times\beta^{*}_{i,\omega},\, b_{i,\omega}\times L'),$$

$$(6)\qquad d(L\times\beta_{i,\mu'},\, b+L'[i]) = \frac{1}{\mu}\sum_{j=1}^{\mu} d(\beta_{i,\mu'}+L[j],\, b^{*}_{j,\mu}+L'[i]),$$

$$i = 1, 2, \ldots, \mu'.$$

The above formulas are consequences of the definition of the distance d. We explain them using the following illustrations:

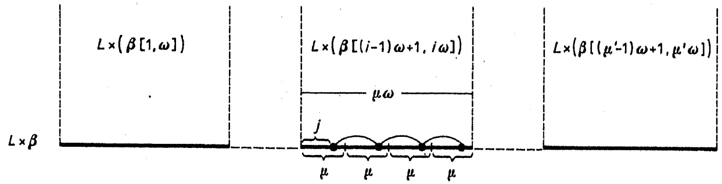

Fig. 1

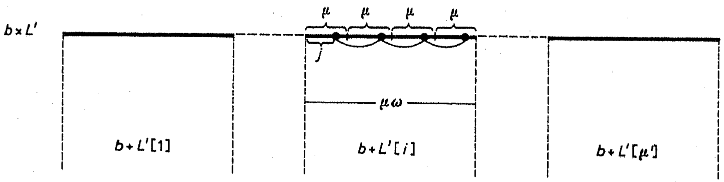

Fig. 2

The blocks marked by points in Figures 1 and 2 are $\beta_{i,\mu'}+L[j]$ and $b^{*}_{j,\mu}+L'[i]$ respectively. Each of the blocks $L\times\beta$ and $b\times L'$ is constructed of $\mu\mu'$ of them. Hence formula (4) follows. Similarly we can obtain (5) and (6).

Case 1. $u_t(k) = 0$ *for all* $t\geqslant 0$, $k\geqslant 1$. Throughout this case, by L_t, $t\geqslant 0$, we denote the block $L_t[1, \mu_t]$ (i.e. L_t without the last place).

I. Consider the numbers $d(L_{t-1}\times\beta^t, b^t\times L_t)$, $t = 1, 2, \ldots$ Applying formula (4) with $L' = L_{t-1}$, $\beta = \beta^t$, $b = b^t$, $L = L_t$, we have

$$d(L_{t-1}\times\beta^t,\, b^t\times L_t) = \frac{1}{\mu_t\mu_{t-1}}\sum_{i=1}^{\mu_t}\sum_{j=1}^{\mu_{t-1}} d_t(i,j),$$

where

$$d_t(i,j) = d\big((\beta^t)_{i,\mu_t}+L_{t-1}[j],\, (b^t)^{*}_{j,\mu_{t-1}}+L_t[i]\big),$$
$$i = 1, 2, \ldots, \mu_t,\; j = 1, 2, \ldots, \mu_{t-1}.$$

Put

$$e_t(i,j) = \min\{d_t(i,j),\, 1-d_t(i,j)\}\quad\text{and}\quad e_t = \frac{1}{\mu_t\mu_{t-1}}\sum_{i=1}^{\mu_t}\sum_{j=1}^{\mu_{t-1}} e_t(i,j).$$

In Lemma 2 we will show that $\sum_{t=1}^{\infty} e_t < \infty$.

II. In the sequel the matrices $M_t = \langle e_t(i,j)\rangle$, $i = 1, 2, \ldots, \mu_t$, $j = 1, 2, \ldots, \mu_{t-1}$, $t > 0$, are considered. Define

$$F_t = \{(i,j);\; e_t(i,j) < \tfrac{1}{5}\}.$$

Take the row of M_t (say the i_0-th) which contains the largest number of elements of F_t and denote by G_t the set of all pairs $(i,j)\in F_t$ such that $(i_0, j)\in F_t$. Let G_t^c be the complement of the set G_t. Then the convergence of the series $\sum_t e_t$ ensures that $\sum_t |G_t^c|/(\mu_t\mu_{t-1}) < \infty$ (here $|G_t^c|$ denotes the number of elements of G_t^c).Indeed, the property $\sum_t e_t < \infty$ implies $\sum_t |F_t^c|/(\mu_t\mu_{t-1}) < \infty$ and

$$\begin{aligned}|G_t^c| &= \mu_t\,|\{1\leqslant j\leqslant \mu_{t-1};\; e_t(i_0,j)\geqslant \tfrac{1}{5}\}|\\ &= \sum_{i=1}^{\mu_t} |\{1\leqslant j\leqslant \mu_{t-1};\; e_t(i_0,j)\geqslant \tfrac{1}{5}\}|\\ &\leqslant \sum_{i=1}^{\mu_t} |\{1\leqslant j\leqslant \mu_{t-1};\; e_t(i,j)\geqslant \tfrac{1}{5}\}| = |F_t^c|.\end{aligned}$$

III. Now we can define blocks $\bar a_t$, $t\geqslant 0$. We set

$$Y_t[j] = \begin{cases}0 & \text{if } d_t(i_0,j) < \tfrac{1}{5},\\ 1 & \text{if } d_t(i_0,j)\geqslant \tfrac{1}{5},\end{cases}\qquad j = 1, 2, \ldots, \mu_{t-1},$$

and

$$\bar a_0 = \underbrace{00\ldots 0}_{\mu_0},\qquad \bar a_t[i] = \begin{cases}0 & \text{if } L_t[i] = Y_{t+1}[i],\\ 1 & \text{if } L_t[i] \neq Y_{t+1}[i],\end{cases}\qquad i = 1, 2, \ldots, \mu_t,\; t\geqslant 1.$$

IV. In order to determine blocks a_t, $t\geqslant 0$, we first construct auxiliary blocks $\bar Y_t$ of lengths μ_t, $t\geqslant 0$. We put

$$\bar Y_t[i] = \begin{cases}0 & \text{if } d\big((\beta^t)_{i,\mu_t}+L_t[i],\, (\beta^t)_{i_0,\mu_t}+L_t[i_0]\big)\leqslant \tfrac{1}{2},\\ 1 & \text{otherwise,}\end{cases}$$
$$i = 1, 2, \ldots, \mu_t.$$

Using the blocks $\bar{Y}_t$ we construct blocks γ^t ($|\gamma^t| = |\beta^t|$) and φ^t ($|\varphi^t| = |b^t|$) by putting

$$(\gamma^t)_{i,\mu_t} = (\beta^t)_{i,\mu_t} + \bar{Y}_t[i], \qquad i = 1, 2, \ldots, \mu_t,$$
$$(\varphi^t)^*_{j,\mu_{t-1}} = (b^t)^*_{j,\mu_{t-1}} + Y_t[j], \quad j = 1, 2, \ldots, \mu_{t-1}.$$

Finally we can define a_t:

$$a_0 = b^0, \quad a_t[i] = \begin{cases} 0 & \text{if } d\big((\varphi^t)_{i,\omega_{t-1}}, L_{t-1}\big) \leqslant \frac{1}{2}, \\ 1 & \text{otherwise,} \end{cases}$$
$$i = 1, \ldots, \omega_{t-1}, t \geqslant 1.$$

We will show (see Lemma 3) that

$$(7) \qquad \sum_{t=1}^{\infty} d(b^t, \bar{a}_{t-1} \times a_t) < \infty, \qquad \sum_{t=1}^{\infty} d(\beta^t, a_t \times \bar{a}_t) < \infty.$$

If $a_t[1] = 1$ ($\bar{a}_t[1] = 1$) then we put $a_t[1] = 0$ ($\bar{a}_t[1] = 0$) and the conditions (7) remain true. In this way we obtain a binary sequence $z = a_0 \times \bar{a}_0 \times a_1 \times \bar{a}_1 \times \ldots$ It is easy to verify that the continuity of the sequences x and y and the conditions (7) imply that z is a continuous Morse sequence.

Case 2. *$u_t(k)$ are arbitrary.*

V. We recall that $u_t(k) = (1/n_t)(l_{t+k} - l_t)$. Since $l_{t+1} \equiv l_t \pmod{n_t}$, $t \geqslant 0$, there exist q_t, $0 \leqslant q_t \leqslant \lambda_t - 1$, such that

$$l_t = q_0 + q_1 n_0 + q_2 n_1 + \ldots + q_t n_{t-1}, \qquad t \geqslant 0,$$

and hence

$$u_t(k) = q_t + q_{t+1}\lambda_t + \ldots + q_{t+k}(\lambda_t \lambda_{t+1} \ldots \lambda_{t+k-1}), \qquad t \geqslant 0, k \geqslant 1.$$

We reduce this case to the previous one. To this end we define the blocks δ^t, ψ^t, $t \geqslant 0$. The blocks ψ^t are equal to $(b^t b^t)$ $[1+q_t, q_t+\lambda_t]$, $t \geqslant 0$, and the blocks δ^t are determined by the following equalities:

$$\delta^0 = \beta^0, \quad (\delta^t)^*_{i,\omega_{t-1}} = \begin{cases} (\beta^t)^*_{i,\omega_{t-1}} & \text{if } r_t(i) = 0, \\ \big((\beta^t)^*_{i,\omega_{t-1}}[\mu_t]\big)\big((\beta^t)^*_{i,\omega_{t-1}}[1, \mu_t - 1]\big) & \text{if } r_t(i) = 1, \end{cases}$$
$$i = 1, 2, \ldots, \omega_{t-1}, t \geqslant 1,$$

where $r_t(i) = 0$ if $1 \leqslant i \leqslant [(\lambda_t - q_t)/\mu_{t-1}]$ and $r_t(i) = 1$ if $i > [(\lambda_t - q_t)/\mu_{t-1}]$ (here $[a]$ denotes the integer part of a). The sequences of blocks $\{\delta^t\}$, $\{\psi^t\}$, $t \geqslant 0$, satisfy the condition

$$\sup_{k \geqslant 1} d(L'_t \times \delta^{t+1} \times \ldots \times \delta^{t+k}, \psi^{t+1} \times \ldots \times \psi^{t+k} \times L'_{t+k}) \xrightarrow{t} 0,$$

where $L'_t = L_t[1, \mu_t]$ (see Lemma 4).

VI. The above condition is the same as (3) if we take δ^t instead of β^t and ψ^t instead of b^t and $u_t(k) = 0$. Therefore we can repeat the considerations of Case 1. As a consequence we obtain blocks $\{K_t\}$, $\{\bar{a}_t\}$, $|K_t| = \omega_{t-1}$, $|\bar{a}_{t-1}| = \mu_{t-1}$, $t \geqslant 1$, $|K_0| = \lambda_0$, satisfying

$$(8) \qquad \sum_{t=1}^{\infty} d(\psi^t, \bar{a}_{t-1} \times K_t) < \infty, \qquad \sum_{t=1}^{\infty} d(\delta^t, K_t \times \bar{a}_t) < \infty.$$

Further we put $a_t = (K_t K_t)[1+k_t, k_t+n_t]$, where $k_t = [(\lambda_t - q_t)/\mu_{t-1}]$. As in Case 1 we may assume that a_t and $\bar{a}_t$ start with 0. Therefore $z = a_0 \times \bar{a}_0 \times a_1 \times \bar{a}_1 \times \ldots$ is a Morse sequence and it is not difficult to see that z is continuous. We will show in Lemma 5 that the pairs of sequences x, z and y, z satisfy conditions (A) and (B) of [3]. This ends the sketch of the proof of Theorem 1.

Remark 1. It follows from the construction of the sequence z that x, z and y, z satisfy conditions (A), (B) although they need not be regular. However, Kwiatkowski's Theorem 1 is not valid if the regularity of x and y is not assumed. In our paper we formulate (without proof) a necessary and sufficient condition for two Morse dynamical systems $\theta(x)$ and $\theta(y)$ induced by $x = b^0 \times b^1 \times \ldots$ and $y = \beta^0 \times \beta^1 \times \ldots$ and satisfying $|b^t| = |\beta^t|$, $t = 0, 1, \ldots$ to be metrically isomorphic (see Theorem 2). In case x and y are regular, Kwiatkowski's results are consequences of Theorem 2. Using Theorem 2, we will give an example of two continuous Morse sequences x and y such that $\theta(x)$ is isomorphic to $\theta(y)$ and such that conditions (A) and (B) of [3] are not satisfied simultaneously.

The fact that the sequence z satisfies (B) is a consequence of the conditions $\sum_t 1/\mu_t < \infty$ and $\sum_t 1/\omega_t < \infty$.

§ 2. Proof of Theorem 1. Now we give the proofs of the lemmas used in § 1. We assume that $x = b^0 \times b^1 \times \ldots$, $y = \beta^0 \times \beta^1 \times \ldots$ are continuous Morse sequences such that $\theta(y)$ and $\theta(x)$ are metrically isomorphic. The lengths λ_t and λ'_t of b^t and β^t satisfy

$$\lambda'_0 = \mu_0 \lambda_0, \qquad \lambda_t = \mu_{t-1}\omega_{t-1}, \qquad \lambda'_t = \mu_t \omega_{t-1}, \qquad t \geqslant 1,$$

and $\sum_t 1/\mu_t < \infty$, $\sum_t 1/\omega_t < \infty$. Codes $\{A_t, B_t\}$ describing the partition $Q = h^{-1}(P(x))$ have the form (1).

Lemma 1. *The sequence*

$$\Big\{\sup_{k \geqslant 1} d\big(L_t[1, \mu_t] \times \beta^{t+1} \times \ldots \times \beta^{t+k}, (b^{t+1} \times \ldots \times b^{t+k} \times L_{t+k})[1 + u_t(k), u_t(k) + s_t(k)]\big)\Big\},$$

where $s_t(k) = \lambda_{t+1} \ldots \lambda_{t+k}\mu_{t+k}$ and $u_t(k) = (1/n_t)(l_{t+k} - l_t)$, converges to zero.

Proof. Introduce the following notation:

$$c_t' = c_t[1, l_t], \qquad c_t'' = c_t[1+l_t, n_t],$$
$$L_t' = L_t[1, \mu_t], \qquad L_t'' = L_t[2, \mu_t+1], \qquad t = 0, 1, \ldots$$

To prove the lemma divide the block $A_t^{(k)} = A_t\times\beta^{t+1}\times\ldots\times\beta^{t+k}$ (for fixed $t \geqslant 0$ and $k \geqslant 1$) into $\mu_t\lambda_{t+1}'\ldots\lambda_{t+k}'$ consecutive subblocks E_i of length n_t. Next divide each E_i into two consecutive blocks $E_i(r)$, $r = 1, 2$, such that $|E_i(1)| = |c_t''| = n_t - l_t$, $|E_i(2)| = |c_t'| = l_t$. Denote by E and F the blocks

$$E_1(1)E_2(1)\ldots E_{\mu_t\lambda_{t+1}'\ldots\lambda_{t+k}'}(1) \quad\text{and}\quad E_1(2)E_2(2)\ldots E_{\mu_t\lambda_{t+1}'\ldots\lambda_{t+k}'}(2)$$

respectively. It is clear that

$$E = c_t''\times L_t'\times\beta^{t+1}\times\ldots\times\beta^{t+k} \quad\text{and}\quad F = c_t'\times L_t''\times\beta^{t+1}\times\ldots\times\beta^{t+k}.$$

Let E', F' denote the blocks obtained from A_{t+k} in the same way as the blocks E, F from $A_t^{(k)}$. We have

$$E' = c_t''\times\big((b^{t+1}\times\ldots\times b^{t+k}\times L_{t+k})[1+u_t(k), u_t(k)+s_t(k)]\big),$$
$$F' = c_t'\times\big((b^{t+1}\times\ldots\times b^{t+k}\times L_{t+k})[2+u_t(k), 1+u_t(k)+s_t(k)]\big).$$

Notice that

$$d(A_{t+k}, A_t^{(k)}) = \frac{n_t-l_t}{n_t}\mathrm{I}_t(k) + \frac{l_t}{n_t}\mathrm{II}_t(k),$$

where

$$\mathrm{I}_t(k) = d\big(L_t'\times\beta^{t+1}\times\ldots\times\beta^{t+k}, (b^{t+1}\times\ldots\times b^{t+k}\times L_{t+k})[1+u_t(k), u_t(k) + s_t(k)]\big),$$
$$\mathrm{II}_t(k) = d\big(L_t''\times\beta^{t+1}\times\ldots\times\beta^{t+k}, (b^{t+1}\times\ldots\times b^{t+k}\times L_{t+k})[2+u_t(k), 1+u_t(k) + s_t(k)]\big).$$

By (2) it suffices to show that $\sup_{k\geqslant 1}|\mathrm{I}_t(k)-\mathrm{II}_t(k)| \underset{t}{\to} 0$. This is true by $\sum_t 1/\mu_t < \infty$ and by the equality

$$d\big((L_t\times\beta^{t+1}\times\ldots\times\beta^{t+k})[2, \mu_t\lambda_{t+1}'\ldots\lambda_{t+k}'],\; (L_t\times\beta^{t+1}\times\ldots\times\beta^{t+k})[1, \mu_t\lambda_{t+1}'\ldots\lambda_{t+k}'-1]\big) \leqslant 2/\mu_t.$$

In Lemmas 2 and 3 we assume that $u_t(k) = 0$ for all $t \geqslant 0$, $k \geqslant 1$, and for convenience we write L_t instead of L_t'.

LEMMA 2. *The series* $\sum_{t=1}^{\infty} e_t$ *is convergent.*

Proof. Let us define

$$d_t(k) = d(L_t\times\beta^{t+1}\times\ldots\times\beta^{t+k}, b^{t+1}\times\ldots\times b^{t+k}\times L_{t+k}),$$
$$d_t^k(i) = d\big(L_t\times\beta^{t+1}\times\ldots\times\beta^{t+k-1}\times(\beta^{t+k})_{i,\mu_{t+k}}, b^{t+1}\times\ldots\times b^{t+k}+L_{t+k}[i]\big),$$
$$i = 1, 2, \ldots, \mu_t.$$

First we show the following formula:

(9)
$$d_t(k) = \frac{1}{2}\Big\{1-\frac{1}{\mu_{t+k}\ldots\mu_t}\sum_{i_1=1}^{\mu_{t+k}}\ldots\sum_{i_{k+1}=1}^{\mu_t}(1-2d_{t+k}(i_1, i_2))\ldots(1-2d_{t+1}(i_k, i_{k+1}))\Big\}$$
$$t, k \geqslant 1.$$

Because of

$$d_t(k) = \frac{1}{\mu_{t+k}}\sum_{i=1}^{\mu_{t+k}} d_t^k(i)$$

it is sufficient to show that for all $t, k \geqslant 1$ and $i = 1, 2, \ldots, \mu_{t+k}$ the following equalities hold:

(10)
$$1-2d_t^k(i) = \frac{1}{\mu_{t+k-1}\ldots\mu_t}\sum_{i_1=1}^{\mu_{t+k-1}}\ldots \sum_{i_k=1}^{\mu_t}(1-2d_{t+k}(i, i_1))(1-2d_{t+k-1}(i_1, i_2))\ldots(1-2d_{t+1}(i_{k-1}, i_k)).$$

We prove this by induction on k. Fix $t \geqslant 1$. If $k = 1$ then (10) is true by (6). Suppose (10) holds for some $k \geqslant 1$ and each $1 \leqslant i \leqslant \mu_{t+k}$. Let $1 \leqslant j \leqslant \mu_{t+k-1}$. Applying (6) with $L = b^{t+1}\times\ldots\times b^{t+k}$, $\beta = b^{t+k+1} + L_{t+k-1}[j]$, $b = L_t\times\beta^{t+1}\times\ldots\times\beta^{t+k}$, $L' = (\beta^{t+k-1})_{j,\mu_{t+k+1}}$ we get

$$1-2d_t^{k+1}(j) = 1-2d\big(L_t\times\beta^{t+1}\times\ldots\times\beta^{t+k}\times(\beta^{t+k+1})_{j,\mu_{t+k+1}}, b^{t+1}\times\ldots\times b^{t+k+1}+L_{t+k+1}[j]\big)$$
$$= 1-\frac{2}{\mu_{t+k}}\sum_{i=1}^{\mu_{t+k}} d\big((L_t\times\beta^{t+1}\times\ldots\times\beta^{t+k})_{i,\mu_{t+k}}\times(\beta^{t+k+1})_{j,\mu_{t+k+1}}, b^{t+1}\times\ldots\times b^{t+k}\times(b^{t+k+1}+L_{t+k+1}[j])^*_{i,\mu_{t+k}}\big)$$
$$= 1-\frac{2}{\mu_{t+k}}\sum_{i=1}^{\mu_{t+k}} d\big(L_t\times\beta^{t+1}\times\ldots\times\beta^{t+k-1}\times(\beta^{t+k})_{i,\mu_{t+k}}\times(\beta^{t+k+1})_{j,\mu_{t+k+1}}, b^{t+1}\times\ldots\times b^{t+k}\times(b^{t+k+1})^*_{i,\mu_{t+k}}+L_{t+k+1}[j]\big).$$

Applying the formula

(11) $$d(A\times B,\ C\times D)=d(B+l,\ D)+[1-2d(B+l,\ D)]\,d(A+l,\ C),$$

(where the blocks A, B, C, D satisfy $|A|=|C|$, $|B|=|D|$ and $l\in\{0,1\}$) with

$$A=L_t\times\beta^{t+1}\times\ldots\times\beta^{t+k-1}\times(\beta^{t+k})_{i,\mu_{t+k}},\qquad B=(\beta^{t+k+1})_{j,\mu_{t+k+1}},$$
$$C=b^{t+1}\times\ldots\times b^{t+k},\qquad D=(b^{t+k+1})^{*}_{i,\mu_{t+k}}+L_{t+k+1}[j],\qquad l=L_{t+k}[j],$$

we get

$$1-2d_t^{k+1}(j)=\frac{1}{\mu_{t+k}}\sum_{i=1}^{\mu_{t+k}}\bigl(1-2d_{t+k+1}(j,i)\bigr)\bigl(1-2d_t^{k}(i)\bigr)$$

and by the induction hypothesis it follows that (10) holds.

Now we can show that $\sum_t e_t<\infty$. We have $1-2e_t(i,j)=|1-2d_t(i,j)|$. Let us define $e_t(k)$, $t\geqslant 1$, $k\geqslant 1$, by

$$1-2e_t(k)=\frac{1}{\mu_{t+k}\cdots\mu_t}\sum_{i_1=1}^{\mu_{t+k}}\cdots\sum_{i_{k+1}=1}^{\mu_t}\bigl(1-2e_{t+k}(i_1,i_2)\bigr)\ldots\bigl(1-2e_{t+1}(i_k,i_{k+1})\bigr).$$

It is clear by (9) that $\inf_{k\geqslant 1}\bigl(1-2e_t(k)\bigr)\underset{t}{\to}1$. Let t_0 be an integer such that $1-2e_t(k)>\frac{1}{2}$ for all $t\geqslant t_0$ and $k\geqslant 1$. We have

$$\begin{aligned}\tfrac{1}{2}<1-2e_{t_0}(2k-1)&\leqslant\Bigl\{\frac{1}{\mu_{t_0+2k-1}\,\mu_{t_0+2k-2}}\sum_{i_1,i_2}\bigl(1-2e_{t_0+2k-1}(i_1,i_2)\bigr)\Bigr\}\\&\quad\cdot\Bigl\{\frac{1}{\mu_{t_0+2k-3}\,\mu_{t_0+2k-4}}\sum_{i_3,i_4}\bigl(1-2e_{t_0+2k-3}(i_3,i_4)\bigr)\Bigr\}\ldots\\&\quad\ldots\Bigl\{\frac{1}{\mu_{t_0+1}\,\mu_{t_0}}\sum_{i_{2k-1},i_{2k}}\bigl(1-2e_{t_0+1}(i_{2k-1},i_{2k})\bigr)\Bigr\}\\&=(1-2e_{t_0+2k-1})(1-2e_{t_0+2k-3})\ldots(1-2e_{t_0+1})\leqslant 1.\end{aligned}$$

Hence $\sum_{r=1}^{\infty}e_{t_0+2r-1}<\infty$. Similarly $\sum_{r=1}^{\infty}e_{t_0+2r}<\infty$. This means that the series $\sum_t e_t$ is convergent.

LEMMA 3. *The sequences of blocks* $\{a_t\}$, $\{\bar a_t\}$, $t\geqslant 0$, *defined in* III *and* IV *satisfy the conditions*

$$\sum_t d(b^{t+1},\ \bar a_t\times a_{t+1})<\infty,\qquad \sum_t d(\beta^t,\ a_t\times\bar a_t)<\infty.$$

Proof. Notice that for $(i,j)\in G_t$

$$d\bigl((\beta^t)_{i_0,\mu_t}+L_t[i_0],\ (b^t)^{*}_{j,\mu_{t-1}}+L_{t-1}[j]+Y_t[j]\bigr)<\tfrac{1}{5},$$
$$d\bigl((\beta^t)_{i_0,\mu_t}+L_t[i_0],\ (\beta^t)_{i,\mu_t}+L_t[i]+\bar Y_t[i]\bigr)\leqslant\tfrac{1}{2}.$$

The above conditions imply for $(i,j)\in G_t$

$$d\bigl((\gamma^t)_{i,\mu_t}+L_t[i],\ (\varphi^t)^{*}_{j,\mu_{t-1}}+L_{t-1}[j]\bigr)=e_t(i,j),\qquad t\geqslant 1.$$

Therefore by Lemma 2 and (4) we obtain

(12) $$\sum_t d(L_{t-1}\times\gamma^t,\ \varphi^t\times L_t)<\infty.$$

In view of (5) and (11) we get

(13) $$\begin{aligned}d(L_{t-1}\times\gamma^t,\ \varphi^t\times L_t)&=\frac{1}{\omega_{t-1}}\sum_{i=1}^{\omega_{t-1}}d\bigl(L_{t-1}\times(\gamma^t)^{*}_{i,\omega_{t-1}},\ (\varphi^t)_{i,\omega_{t-1}}\times L_t\bigr)\\&=\frac{1}{\omega_{t-1}}\sum_i\bigl\{d\bigl(L_{t-1}+a_t[i],\ (\varphi^t)_{i,\omega_{t-1}}\bigr)\\&\quad+\bigl[1-2d\bigl(L_{t-1}+a_t[i],(\varphi^t)_{i,\omega_{t-1}}\bigr)\bigr]\,d\bigl((\gamma^t)^{*}_{i,\omega_{t-1}},\ L_t+a_t[i]\bigr)\bigr\}\\&\geqslant\frac{1}{\omega_{t-1}}\sum_i d\bigl(L_{t-1}+a_t[i],(\varphi^t)_{i,\omega_{t-1}}\bigr)=d(L_{t-1}\times a_t,\ \varphi^t).\end{aligned}$$

Hence in view of (12) we have $\sum_t d(L_{t-1}\times a_t,\ \varphi^t)<\infty$. From (13) we obtain

$$\begin{aligned}d(\gamma^t,\ a_t\times L_t)&=\frac{1}{\omega_{t-1}}\sum_{i=1}^{\omega_{t-1}}d\bigl((\gamma^t)^{*}_{i,\omega_{t-1}},\ L_t+a_t[i]\bigr)\\&\leqslant d(L_{t-1}\times\gamma^t,\ \varphi^t\times L_t)-d(L_{t-1}\times a_t,\ \varphi^t)\\&\quad+2d(L_{t-1}\times a_t,\ \varphi^t)\leqslant 2d(L_{t-1}\times\gamma^t,\ \varphi^t\times L_t).\end{aligned}$$

Therefore $\sum_t d(\gamma^t,\ a_t\times L_t)<\infty$. Let us define

$$E_0=\underbrace{00\ldots0}_{\mu_0},\qquad E_t[i]=\begin{cases}0 & \text{if } L_t[i]=\bar Y_t[i],\\ 1 & \text{otherwise,}\end{cases}\qquad i=1,2,\ldots,\mu_t,\ t\geqslant 1.$$

Notice that

$$d(L_{t-1}\times a_t,\ \varphi^t)=d(b^t,\ \bar a_{t-1}\times a_t),\qquad d(\gamma^t,\ a_t\times L_t)=d(\beta^t,\ a_t\times E_t).$$

Thus we have $\sum_t d(b^{t+1},\ \bar a_t\times a_{t+1})<\infty$, $\sum_t d(\beta^t,\ a_t\times E_t)<\infty$.

To finish the proof it suffices to show that $\sum_t d(\bar{a}_t, E_t) < \infty$. We may assume that

(14) $$d(L_{t-1}, \bar{a}_{t-1}) \leqslant \tfrac{1}{2}$$

(otherwise we replace $\bar{a}_t$, a_t, E_t by $\tilde{\bar{a}}_t$, $\tilde{a}_t$, $\tilde{E}_t$ respectively and then (11) remains true). Let us define

$$f_t(k) = d(L_t \times (a_{t+1} \times E_{t+1}) \times \ldots \times (a_{t+k} \times E_{t+k}),$$
$$(\bar{a}_t \times a_{t+1}) \times \ldots \times (\bar{a}_{t+k-1} \times a_{t+k}) \times L_{t+k}),$$
$$\bar{f}_t(k) = d((a_{t+1} \times E_{t+1}) \times \ldots \times (a_{t+k} \times E_{t+k}),$$
$$(a_{t+1} \times \bar{a}_{t+1}) \times \ldots \times (a_{t+k} \times \bar{a}_{t+k})).$$

It is easy to see that the triangle inequality and (11) imply $\sup_{k \geqslant 1} f_t(k) \underset{t}{\to} 0$. Hence in view of (10) and (14) we obtain $\sup_{k \geqslant 1} \bar{f}_t(k) \underset{t}{\to} 0$. Since

$$1 - 2\bar{f}_t(k) = \big(1 - 2d(E_{t+1}, \bar{a}_{t+1})\big)\big(1 - 2\bar{f}_{t+1}(k)\big) \leqslant \prod_{i=1}^{k-1} \big(1 - 2d(E_{t+i}, \bar{a}_{t+i})\big)$$

we have $\sum_t d(E_t, \bar{a}_t) < \infty$.

Now we prove two lemmas which were needed in Case 2.

Lemma 4. *The sequences of blocks* $\{\delta^t\}$, $\{\psi^t\}$, $t \geqslant 0$, *defined in* V *satisfy the condition*

$$\sup_{k \geqslant 1} d(L_t' \times \delta^{t+1} \times \ldots \times \delta^{t+k}, \psi^{t+1} \times \ldots \times \psi^{t+k} \times L_{t+k}') \underset{t}{\to} 0.$$

Proof. For convenience introduce the following notation:

$$E_{i,p}(q) = \big((EE)[q, q+n]\big)_{i,p}, \quad \text{where } |E| = n,\ p \mid n,\ 1 \leqslant q \leqslant n,$$
$$\eta_{t,k} = L_t' \times \beta^{t+1} \times \ldots \times \beta^{t+k},$$
$$\eta_{t,k}(i_1, \ldots, i_k) = L_t' \times (\beta^{t+1})^*_{i_1,\omega_t} \times \ldots \times (\beta^{t+k})^*_{i_k,\omega_{t+k-1}},$$
$$\xi_{t,k}(F) = (b^{t+1} \times \ldots \times b^{t+k} \times F)[1 + u_t(k), u_t(k) + s_t(k)],$$
$$\text{where } |F| = \mu_{t+k} + 1,$$
$$\xi_{t,k}(F; i_1, \ldots, i_k) = (b^{t+1})_{i_1,\omega_t}(1 + q_{t+1}) \times (b^{t+2})_{i_2,\omega_{t+1}}\big(1 + q_{t+2} + r_{t+1}(i_1)\big)$$
$$\times \ldots \times (b^{t+k})_{i_k,\omega_{t+k-1}}\big(1 + q_{t+k} + r_{t+k-1}(i_{k-1})\big)$$
$$\times \big(F[1 + r_{t+k}(i_k), r_{t+k}(i_k) + \mu_{t+k}]\big),$$
$$\xi_{t,k} = \xi_{t,k}(L_{t+k}), \quad \xi_{t,k}(i_1, \ldots, i_k) = \xi_{t,k}(L_{t+k}; i_1, \ldots, i_k),$$
$$1 \leqslant i_p \leqslant \omega_{t+p-1}, \quad p = 1, \ldots, k, \quad t \geqslant 0,\ k \geqslant 1.$$

We will prove this lemma in three steps.

1. First we show that if L, β, b, L' are blocks with lengths μ, $\mu'\omega$, $\mu\omega$, $\mu'+1$ respectively and $0 \leqslant q \leqslant \mu\omega - 1$, then

(15) $$\Big| d\big(L \times \beta, (b \times L')[1+q, q+\mu\omega\mu']\big) - \frac{1}{\omega}\sum_{i=1}^{\omega} d\big(L \times \beta^*_{i,\omega}, b_{i,\omega}(1+q) \times (L'[1+r(i), r(i)+\mu'])\big)\Big| \leqslant \frac{1}{\omega}$$

where

$$r(i) = \begin{cases} 0 & \text{if } 1 \leqslant i \leqslant [(\mu\omega - q)/\mu], \\ 1 & \text{if } i > [(\mu\omega - q)/\mu]. \end{cases}$$

To this end divide the block $L \times \beta$ into μ' consecutive blocks $(L \times \beta)_{i,\mu'} = L \times \beta_{i,\mu'}$, $i = 1, \ldots, \mu'$. Next divide each block $L \times \beta_{i,\mu'}$ into ω blocks $(L \times \beta_{i,\mu'})_{k,\omega} = L + (\beta_{i,\mu'}[k])$, $k = 1, \ldots, \omega$. Let E_k denote the block

$$(L \times \beta_{1,\mu'})_{k,\omega}(L \times \beta_{2,\mu'})_{k,\omega} \ldots (L \times \beta_{\mu',\mu'})_{k,\omega}.$$

It is easy to see that $E_k = L \times \beta^*_{k,\omega}$. Let E_k' be the block obtained from the block $(b \times L')[1+q, q+\mu\omega\mu']$ in the same way as E_k from $L \times \beta$. It remains to observe that

$$E_k' = b_{k,\omega}(1+q) \times (L'[1+r(k), r(k)+\mu']) \quad \text{for } k \neq 1 + \left[\frac{\mu\omega - q}{\mu}\right].$$

2. Next we show that for all $t, k \geqslant 1$

(16) $$\Big| d_t(k) - \frac{1}{\omega_t \ldots \omega_{t+k-1}} \sum_{i_1=1}^{\omega_t} \ldots \sum_{i_k=1}^{\omega_{t+k-1}} d\big(\eta_{t,k}(i_1, \ldots, i_k), \xi_{t,k}(i_1, \ldots, i_k)\big)\Big| \leqslant \frac{1}{\omega_1} + \ldots + \frac{1}{\omega_{t+k-1}},$$

where

$$d_t(k) = d\big(L_t \times \beta^{t+1} \times \ldots \times \beta^{t+k}, (b^{t+1} \times \ldots \times b^{t+k} \times L_{t+k})[1 + u_t(k), u_t(k) + s_t(k)]\big).$$

In order to prove this we show that for each block E of length $\mu_{t+k}+1$ the number

$$R_{t,k}(E) = d\big(\eta_{t,k}, \xi_{t,k}(E)\big) - \frac{1}{\omega_t \ldots \omega_{t+k-1}} \sum_{i_1=1}^{\omega_t} \ldots \sum_{i_k=1}^{\omega_{t+k-1}} d\big(\eta_{t,k}(i_1, \ldots, i_k), \xi_{t,k}(E; i_1, \ldots, i_k)\big)$$

satisfies

$$|R_{t,k}(E)| \leqslant \frac{1}{\omega_t} + \ldots + \frac{1}{\omega_{t+k-1}}.$$

Fix $t \geqslant 1$. For $k = 1$, (16) is true in view of (15). Suppose that (16) holds for some $k \geqslant 1$ and an arbitrary block E of length $\mu_{t+k}+1$. Let F be a block of length $\mu_{t+k+1}+1$. Applying (15) to $L = L'_t \times \beta^{t+1} \times \ldots \times \beta^{t+k}$, $\beta = \beta^{t+k+1}$, $b = b^{t+1} \times \ldots \times b^{t+k+1}$, $L' = F$, $q = u_t(k+1)$, we get

$$d(\eta_{t,k+1}, \xi_{t,k+1}(F)) = \frac{1}{\omega_{t+k}} \sum_{i_{k+1}=1}^{\omega_{t+k}} d\big(\eta_{t,k} \times (\beta^{t+k+1})^*_{i_{k+1},\omega_{t+k}},$$
$$(b^{t+1} \times \ldots \times b^{t+k+1})_{i_{k+1},\omega_{t+k}}(1+u_t(k+1)) \times F_{i_{k+1}}\big) + M,$$

where $F_{i_{k+1}} = F[1+r_{t+k+1}(i_{k+1}), r_{t+k+1}(i_{k+1})+\mu_{t+k+1}]$ and $|M| \leqslant 1/\omega_{t+k}$. Notice that putting

$$E_{i_{k+1}} = (b^{t+k+1}b^{t+k+1})[1+q_{t+k+1}+(i_{k+1}-1)\mu_{t+k},\ 1+q_{t+k+1}+i_{k+1}\mu_{t+k}]$$

we have

(17) $$(b^{t+1} \times \ldots \times b^{t+k+1})_{i_{k+1},\omega_{t+k}}(1+u_t(k+1)) = \xi_{t,k}(E_{i_{k+1}}).$$

Put $d' = d((\beta^{t+k+1})^*_{i_{k+1},\omega_{t+k}}, F_{i_{k+1}})$. It follows from (17) and the induction hypothesis that

$$d\big(\eta_{t,k} \times (\beta^{t+k+1})^*_{i_{k+1},\omega_{t+k}}, (b^{t+1} \times \ldots \times b^{t+k+1})_{i_{k+1},\omega_{t+k}}(1+u_t(k+1)) \times F_{i_{k+1}}\big)$$
$$= d\big(\eta_{t,k} \times (\beta^{t+k+1})^*_{i_{k+1},\omega_{t+k}}, \xi_{t,k}(E_{i_{k+1}}) \times F_{i_{k+1}}\big)$$
$$= d' + (1-2d')\, d(\eta_{t,k}, \xi_{t,k}(E_{i_{k+1}}))$$
$$= d' + (1-2d')\Big\{\frac{1}{\omega_t \ldots \omega_{t+k-1}} \sum_{i_1=1}^{\omega_t} \ldots \sum_{i_k=1}^{\omega_{t+k-1}} d(\eta_{t,k}(i_1, \ldots, i_k),$$
$$\xi_{t,k}(E_{i_{k+1}}; i_1, \ldots, i_k)) + R_{t,k}(E_{i_{k+1}})\Big\}$$
$$= \frac{1}{\omega_t \ldots \omega_{t+k-1}} \sum_{i_1} \ldots \sum_{i_k} [d' + (1-2d')\, d(\eta_{t,k}(i_1, \ldots, i_k),$$
$$\xi_{t,k}(E_{i_{k+1}}; i_1, \ldots, i_k))] + (1-2d')\, R_{t,k}(E_{i_{k+1}})$$
$$= \frac{1}{\omega_t \ldots \omega_{t+k-1}} \sum_{i_1} \ldots \sum_{i_k} d(\eta_{t,k+1}(i_1, \ldots, i_{k+1}),$$
$$\xi_{t,k}(E_{i_{k+1}}; i_1, \ldots, i_k) \times F_{i_{k+1}}) + (1-2d')\, R_{t,k}(E_{i_{k+1}}).$$

Because of $\xi_{t,k}(E_{i_{k+1}}; i_1, \ldots, i_k) \times F_{i_{k+1}} = \xi_{t,k+1}(F; i_1, \ldots, i_{k+1})$ we thus have

$$|R_{t,k+1}(F)| \leqslant M + \frac{1}{\omega_{t+k}} \sum_{i_{k+1}=1}^{\omega_{t+k}} R_{t,k}(E_{i_{k+1}}) \leqslant \frac{1}{\omega_t} + \ldots + \frac{1}{\omega_{t+k}}.$$

3. Now we can show that

$$\sup_{k \geqslant 1} d(L'_t \times \delta^{t+1} \times \ldots \times \delta^{t+k}, \psi^{t+1} \times \ldots \times \psi^{t+k} \times L'_{t+k}) \underset{t}{\to} 0.$$

Put

$$\bar{\xi}_{t,k}(i_1, \ldots, i_k) = (\psi^{t+1})_{i_1,\omega_t} \times \ldots \times (\psi^{t+k})_{i_k,\omega_{t+k-1}} \times L'_{t+k},$$
$$\bar{\eta}_{t,k}(i_1, \ldots, i_k) = L'_t \times (\delta^{t+1})^*_{i_1,\omega_t} \times \ldots \times (\delta^{t+k})^*_{i_k,\omega_{t+k-1}},$$
$$i_s = 1, \ldots, \omega_{t+s-1},\ s = 1, \ldots, k.$$

In view of (16) we have

(18) $$\Big|d(L'_t \times \delta^{t+1} \times \ldots \times \delta^{t+k}, \psi^{t+1} \times \ldots \times \psi^{t+k} \times L'_{t+k})$$
$$- \frac{1}{\omega_t \ldots \omega_{t+k-1}} \sum_{i_1=1}^{\omega_t} \ldots \sum_{i_k=1}^{\omega_{t+k-1}} d(\bar{\eta}_{t,k}(i_1, \ldots, i_k), \bar{\xi}_{t,k}(i_1, \ldots, i_k))\Big|$$
$$\leqslant \frac{1}{\omega_t} + \ldots + \frac{1}{\omega_{t+k-1}}.$$

It is not hard to see that

$$(\psi^{t+1})_{i_1,\omega_t} = (b^{t+1})_{i_1,\omega_t}(1+q_{t+1}),$$
$$\big|d((\psi^{t+s})_{i_s,\omega_{t+s-1}}, (\delta^{t+s-1})^*_{i_{s-1},\omega_{t+s-2}})$$
$$- d((b^{t+s})_{i_s,\omega_{t+s-1}}(1+r_{t+s-1}(i_{s-1})+q_{t+s}), (\beta^{t+s-1})^*_{i_{s-1},\omega_{t+s-2}})\big|$$
$$\leqslant \frac{1}{\mu_{t+s-1}}, \quad s = 2, \ldots, k,$$
$$\big|d(L'_{t+k}, (\delta^{t+k})^*_{i_k,\omega_{t+k-1}})$$
$$- d(L_{t+k}[1+r_{t+k}(i_k), r_{t+k}(i_k)+\mu_{t+k}], (\beta^{t+k})^*_{i_k,\omega_{t+k-1}})\big| \leqslant \frac{1}{\mu_{t+k}}.$$

Therefore, using the formula

$$|d(A_1 \times \ldots \times A_n, B_1 \times \ldots \times B_n) - d(C_1 \times \ldots \times C_n, D_1 \times \ldots \times D_n)|$$
$$\leqslant \sum_{i=1}^{n} |d(A_i, B_i) - d(C_i, D_i)|$$

(where the blocks A_i, B_i, C_i, D_i satisfy $|A_i| = |B_i|$, $|C_i| = |D_i|$) we obtain

$$\Big|\frac{1}{\omega_t\ldots\omega_{t+k-1}}\sum_{i_1=1}^{\omega_t}\ldots\sum_{i_k=1}^{\omega_{t+k-1}} d\big(\bar{\eta}_{t,k}(i_1,\ldots,i_k),\bar{\xi}_{t,k}(i_1,\ldots,i_k)\big)$$
$$-\frac{1}{\omega_t\ldots\omega_{t+k-1}}\sum_{i_1}\ldots\sum_{i_k} d\big(\xi_{t,k}(i_1,\ldots,i_k),\eta_{t,k}(i_1,\ldots,i_k)\big)\Big|$$
$$\leqslant \frac{1}{\omega_t\ldots\omega_{t+k-1}}\sum_{i_1}\ldots\sum_{i_k}\Big(\frac{1}{\mu_{t+1}}+\ldots+\frac{1}{\mu_{t+k}}\Big) = \frac{1}{\mu_{t+1}}+\ldots+\frac{1}{\mu_{t+k}}.$$

Thus (18) and (16) imply that

$$|d_t(k) - d(L'_t\times\delta^{t+1}\times\ldots\times\delta^{t+k},\psi^{t+1}\times\ldots\times\psi^{t+k}\times L'_{t+k})|$$
$$\leqslant 2\Big(\frac{1}{\omega_t}+\ldots+\frac{1}{\omega_{t+k-1}}\Big)+\frac{1}{\mu_{t+1}}+\ldots+\frac{1}{\mu_{t+k}}.$$

Hence

$$\sup_{k\geqslant 1} d(L'_t\times\delta^{t+1}\times\ldots\times\delta^{t+k},\psi^{t+1}\times\ldots\times\psi^{t+k}\times L'_{t+k}) \underset{t}{\to} 0.$$

Now we prove the last lemma.

LEMMA 5. *Let $\{a_t\}$, $\{\bar{a}_t\}$, $\{K_t\}$, $t\geqslant 0$, be the blocks defined in VI. There exist sequences of integers $\{\bar{q}_t\}$, $\{p_t\}$, $0\leqslant \bar{q}_t<\lambda_t$, $0\leqslant p_t<\lambda'_t$, $t\geqslant 0$, such that*

$$\sum_t d\big((b^t b^t)[1+\bar{q}_t,\bar{q}_t+\lambda_t],\bar{a}_{t-1}\times a_t\big)<\infty,$$

$$\sum_t d\big((\beta^t\beta^t)[1+p_t,p_t+\lambda'_t],a_t\times\bar{a}_t\big)<\infty,$$

$$\sum_t \min(1-j_t/n_t,j_t/n_t)<\infty,$$

$$\sum_t \min(1-j'_t/n'_t,j'_t/n'_t)<\infty,$$

where

$$j_0=\bar{q}_0,\quad j_t=\bar{q}'_0+\bar{q}'_1 n_0+\ldots+\bar{q}'_t n_{t-1},$$
$$\bar{q}'_t=\bar{q}_t\ \text{ if } \bar{q}_{t-1}\leqslant\lambda_{t-1}-\bar{q}_{t-1}-1,\quad \bar{q}'_t=\bar{q}_t-1 \pmod{\lambda_t}\ \text{ otherwise},\quad t\geqslant 1,$$

and j'_t is defined similarly (we replace $\bar{q}_t$, λ_t, $\bar{q}'_t$ by p_t, λ'_t, p'_t).

Proof. Let us define $p_t=\lambda'_t-(\omega_{t-1}-k_t) \pmod{\lambda'_t}$, $t\geqslant 1$. (recall that $k_t=[(\lambda_t-q_t)/\mu_{t-1}]$). First we show that

(19) $$d(\delta^t,K_t\times\bar{a}_t)=d\big((\beta^t\beta^t)[1+p_t,p_t+\lambda'_t],a_t\times\bar{a}_t\big),\qquad t\geqslant 1.$$

Suppose first that $0<k_t<\omega_{t-1}$. Fix $t\geqslant 1$. Put $a'_t=K_t[1,k_t]$, $a''_t=K_t[1$ $+k_t,\omega_{t-1}]$. Introduce the following notation:

$$\beta^{\mathrm{I}}_i=\beta^t[1+(i-1)\omega_{t-1},(i-1)\omega_{t-1}+k_t],$$
$$\beta^{\mathrm{II}}_i=\beta^t[(i-1)\omega_{t-1}+k_t+1,i\omega_{t-1}],\qquad i=1,\ldots,\mu_t,$$
$$\beta^{\mathrm{I}}=\beta^{\mathrm{I}}_1\beta^{\mathrm{I}}_2\ldots\beta^{\mathrm{I}}_{\mu_t},\qquad \beta^{\mathrm{II}}=\beta^{\mathrm{II}}_1\beta^{\mathrm{II}}_2\ldots\beta^{\mathrm{II}}_{\mu_t}.$$

We have $|\beta^{\mathrm{I}}_i|=k_t$, $|\beta^{\mathrm{II}}_i|=\omega_{t-1}-k_t$, $i=1,\ldots,\mu_t$, $\beta^t=\beta^{\mathrm{I}}_1\beta^{\mathrm{II}}_1\ldots\beta^{\mathrm{I}}_{\mu_t}\beta^{\mathrm{II}}_{\mu_t}$. Because of

$$(\beta^t\beta^t)[1+p_t,p_t+\lambda'_t]=\beta^t[1+p_t,\lambda'_t]\beta^t[1,p_t],$$
$$\beta^t[1,p_t]\beta^t[1+p_t,\lambda'_t]=\beta^t=\beta^{\mathrm{I}}_1\beta^{\mathrm{II}}_1\ldots\beta^{\mathrm{I}}_{\mu_t}\beta^{\mathrm{II}}_{\mu_t},$$
$$|\beta^t[1+p_t,\lambda'_t]|=\lambda'_t-p_t=|\beta^{\mathrm{II}}_{\mu_t}|,$$

we obtain

$$(\beta^t\beta^t)[1+p_t,p_t+\lambda'_t]=\beta^{\mathrm{II}}_{\mu_t}\beta^{\mathrm{I}}_1\beta^{\mathrm{II}}_1\ldots\beta^{\mathrm{I}}_{\mu_t}.$$

On the other hand,

$$a_t\times\bar{a}_t=(a''_t+\bar{a}_t[1])(a'_t+\bar{a}_t[1])\times\ldots\times(a''_t+\bar{a}_t[\mu_t])(a'_t+\bar{a}_t[\mu_t]).$$

Therefore we have

$$d\big((\beta^t\beta^t)[1+p_t,p_t+\lambda'_t],a_t\times\bar{a}_t\big)$$
$$=\frac{k_t}{\omega_{t-1}}d(\beta^{\mathrm{I}},a'_t\times\bar{a}_t)+\Big(1-\frac{k_t}{\omega_{t-1}}\Big)d\big(\beta^{\mathrm{II}},a''_t\times((\bar{a}_t\bar{a}_t)[2,1+\mu_t])\big).$$

Notice that

$$d(\delta^t,K_t\times\bar{a}_t)=\frac{1}{\omega_{t-1}}\sum_{i=1}^{\omega_{t-1}} d\big((\delta^t)^*_{i,\omega_{t-1}},K_t[i]+\bar{a}_t\big)$$
$$=\frac{1}{\omega_{t-1}}\sum_i d\big((\beta^t)^*_{i,\omega_{t-1}},(\bar{a}_t\bar{a}_t)[1+r_t(i),r_t(i)+\mu_t]+K_t[i]\big)$$
$$=\frac{1}{\omega_{t-1}}\sum_{i=1}^{k_t} d\big((\beta^t)^*_{i,\omega_{t-1}},(\bar{a}_t\bar{a}_t)[1+r_t(i),r_t(i)+\mu_t]+K_t[i]\big)$$
$$+\frac{1}{\omega_{t-1}}\sum_{i=1}^{\omega_{t-1}-k_t} d\big((\beta^t)^*_{i+k_t,\omega_{t-1}},(\bar{a}_t\bar{a}_t)[1+r_t(i+k_t),r_t(i+k_t)+\mu_t]+K_t[i]\big)$$
$$=\frac{k_t}{\omega_{t-1}}\cdot\frac{1}{k_t}\sum_{i=1}^{k_t} d\big((\beta^t)^*_{i,\omega_{t-1}},\bar{a}_t+K_t[i]\big)$$
$$+\Big(1-\frac{k_t}{\omega_{t-1}}\Big)\frac{1}{\omega_{t-1}-k_t}\sum_{i=1}^{\omega_{t-1}-k_t} d\big((\beta^t)^*_{i+k_t,\omega_{t-1}},(\bar{a}_t[2,\mu_t]\bar{a}_t[1])+K_t[i]\big).$$

It suffices to show that

$$\frac{1}{k_t}\sum_{i=1}^{k_t} d\big((\beta^t)^*_{i,\omega_{t-1}},\ \bar a_t+K_t[i]\big) = d(\beta^{\mathrm{I}},\ a'_t\times\bar a_t),$$

$$\frac{1}{\omega_{t-1}-k_t}\sum_{i=1}^{\omega_{t-1}-k_t} d\big((\beta^t)^*_{i+k_t,\omega_{t-1}},\ (\bar a_t[2,\mu_t]\,\bar a_t[1])+K_t[i]\big) = d\big(\beta^{\mathrm{II}},\ a''_t\times((\bar a_t\bar a_t)[2,1+\mu_t])\big).$$

The first equality is true by the following equalities:

$$(\beta^{\mathrm{I}})^*_{i,k_t} = \beta^{\mathrm{I}}_1[i]\,\beta^{\mathrm{I}}_2[i]\dots\beta^{\mathrm{I}}_{\mu_t}[i] = \beta^t[i]\,\beta^t[i+\omega_t]\dots\beta^t[i+(\mu_t-1)\,\omega_{t-1}],$$

$$(a'_t\times\bar a_t)^*_{i,k_t} = \bar a_t+a'_t[i] = \bar a_t+K_t[i],$$

$$(\beta^t)^*_{i,\omega_{t-1}} = \beta^t[i]\,\beta^t[i+\omega_{t-1}]\dots\beta^t[i+(\mu_t-1)\,\omega_{t-1}],$$

$$i=1,\dots,k_t.$$

The second is a consequence of the equalities

$$(\beta^{\mathrm{II}})^*_{i,\omega_{t-1}-k_t} = \beta^{\mathrm{II}}_1[i]\,\beta^{\mathrm{II}}_2[i]\dots\beta^{\mathrm{II}}_{\mu_t}[i]$$

$$= \beta^t[i+k_t]\,\beta^t[i+k_t+\omega_{t-1}]\dots\beta^t[i+k_t+(\mu_t-1)\,\omega_{t-1}],$$

$$(a''_t\times(\bar a_t\bar a_t)[2,\mu_t+1])^*_{i,\omega_{t-1}-k_t} = (\bar a_t\bar a_t)[2,\mu_t+1]+a''_t[i]$$

$$= (\bar a_t[2,\mu_t]\,\bar a_t[1])+a_t[i+k_t],$$

$$(\beta^t)^*_{i+k_t,\omega_{t-1}} = \beta^t[i+k_t]\,\beta^t[i+k_t+\omega_{t-1}]\dots\beta^t[i+k_t+(\mu_t-1)\,\omega_{t-1}],$$

$$i=1,\dots,\omega_{t-1}-k_t.$$

If $k_t=0$ or ω_{t-1} it is not hard to see that (19) also holds. Therefore

$$\sum_t d\big((\beta^t\beta^t)[1+p_t,\ p_t+\lambda'_t],\ a_t\times\bar a_t\big)<\infty.$$

Put $\bar q_t = q_t+k_t\mu_{t-1} \pmod{\lambda_t}$. It is easy to see that

$$d\big((b^tb^t)[1+\bar q_t,\ \bar q_t+\lambda_t],\ \bar a_{t-1}\times a_t\big) = d\big((b^tb^t)[1+q_t,\ q_t+\lambda_t],\ \bar a_{t-1}\times K_t\big)$$

and in view of (8) we get

$$\sum_t d\big((b^tb^t)[1+\bar q_t,\ \bar q_t+\lambda_t],\ \bar a_{t-1}\times a_t\big)<\infty.$$

It remains to show that $\sum_t\min(1-j_t/n_t,\ j_t/n_t)<\infty$, $\sum_t\min(1-j'_t/n_t,\ j'_t/n_t)<\infty$. Notice that if $p_t\neq 0$ then $p_t\geqslant\lambda'_t-\omega_{t-1}$ and if $\bar q_t\neq 0$ then $\bar q_t\geqslant\lambda_t-\mu_{t-1}$. Thus it is obvious that if $p_t\neq 0$ then $\lambda'_t-\omega_{t-1}-1\leqslant p'_t\leqslant\lambda'_t-1$ and if $\bar q_t\neq 0$ then $\lambda_t-\mu_{t-1}-1\leqslant\bar q'_t\leqslant\lambda_t-1$. Since

$$0\leqslant\frac{j'_t}{n'_t}-\frac{p'_t}{\lambda'_t}<\frac{1}{\lambda'_t},\qquad 0\leqslant\frac{j_t}{n_t}-\frac{\bar q'_t}{n_t}<\frac{1}{\lambda_t}$$

we have

$$\sum_t\min\Big(1-\frac{j_t}{n_t},\ \frac{j_t}{n_t}\Big)<\infty,\qquad \sum_t\min\Big(1-\frac{j'_t}{n'_t},\ \frac{j'_t}{n'_t}\Big)<\infty.$$

This completes the proof of Lemma 5 and finishes the proof of Theorem 1.

§ 3. Metric isomorphism in case $\lambda_t=\lambda'_t$, $t\geqslant 0$. Suppose that the lengths λ_t, λ'_t of the blocks b^t, β^t of continuous Morse sequences $x=b^0\times b^1\times\dots$ and $y=\beta^0\times\beta^1\times\dots$ are equal for all $t\geqslant 0$. From Theorem 1 one can obtain the following theorem, which we give here without proof.

THEOREM 2. *$\theta(x)$ is metrically isomorphic to $\theta(y)$ iff there exist sequences of integers $\{r_t\}$, $\{s_t\}$, $r_t, s_t\in\{0,1\}$, $t\geqslant 0$, and a sequence of integers $\{q_t\}$, $q_t\in\{0,1,\dots,\lambda_t-1\}$, $t\geqslant 0$, such that*

$$\sum_{t=0}^{\infty}\Big[\Big(1-\frac{j_t}{n_t}\Big)D_{t+1}+\frac{j_t}{n_t}\bar D_{t+1}\Big]<\infty, \tag{20}$$

where

$$D_{t+1} = d\big((b^{t+1})^{r_{t+1}}(b^{t+1})^{s_{t+1}}[1+q_{t+1},\ q_{t+1}+\lambda_{t+1}],\ \beta^{t+1}+r_t\big),$$

$$\bar D_{t+1} = d\big((b^{t+1})^{r_{t+1}}(b^{t+1})^{s_{t+1}}[2+q_{t+1},\ 1+q_{t+1}+\lambda_{t+1}],\ \beta^{t+1}+s_t\big),$$

and $j_0=q_0$, $j_t=j_{t-1}+q_t n_{t-1}$, $t\geqslant 1$.

Remark 2. Put

$$q'_t=\begin{cases} q_t & \text{if } 2q_{t-1}\leqslant\lambda_{t-1}-1,\\ q_t+1 \pmod{\lambda_t} & \text{otherwise,}\end{cases}$$

and if $q_t<\lambda_t-1$ or $2q_{t-1}\leqslant\lambda_{t-1}-1$,

$$r'_{t+1}=\begin{cases} r_{t+1}+r_t \pmod 2 & \text{if } 2q_{t-1}\leqslant\lambda_{t-1}-1,\\ r_{t+1}+s_t \pmod 2 & \text{otherwise,}\end{cases}$$

$$s'_{t+1}=\begin{cases} s_{t+1}+r_t \pmod 2 & \text{if } 2q_{t-1}\leqslant\lambda_{t-1}-1,\\ s_{t+1}+s_t \pmod 2 & \text{otherwise;}\end{cases}$$

if $q_t=\lambda_t-1$ and $2q_{t-1}>\lambda_{t-1}-1$ then

$$r'_{t+1}=s_{t+1}+s_t \pmod 2,\qquad s'_{t+1}=r_{t+1}+s_t \pmod 2,\qquad t\geqslant 0,$$

$$r'_0=r_0,\qquad s'_0=s_0.$$

Then $r'_{t+1}+s'_{t+1} \pmod 2 = r_{t+1}+s_{t+1} \pmod 2$ and it is not hard to see that (20) is equivalent to the following two conditions:

$$\sum_{t=0}^{\infty} d\big((b^{t+1})^{r'_{t+1}}(b^{t+1})^{s'_{t+1}}[1+q'_{t+1},\ q'_{t+1}+\lambda_{t+1}],\ \beta^{t+1}\big)<\infty, \tag{21}$$

$$\sum_{t=0}^{\infty}\min\Big(1-\frac{j_t}{n_t},\ \frac{j_t}{n_t}\Big)\Big(\eta_t+\frac{v_t}{\lambda_t}\Big)<\infty, \tag{22}$$

where

$$\eta_t = \begin{cases} \dfrac{\mathrm{fr}(01, b^t)+\mathrm{fr}(10, b^t)}{\lambda_t} & \text{if } r_t = s_t, \\ \dfrac{\mathrm{fr}(00, b^t)+\mathrm{fr}(11, b^t)}{\lambda_t} & \text{if } r_t \neq s_t, \end{cases}$$

and $v_t = 0$ if the number $b^{t+1}[\lambda_{t+1}]+r'_{t+1}+r'_t+s'_{t+1}+s'_t$ is odd and $v_t = 1$ otherwise, $t \geqslant 0$.

Condition (A) of [3] is the same as condition (21) and conditions (A), (B) imply conditions (21) and (22). If x is regular, then obviously condition (22) is the same as (B).

EXAMPLE. We give an example of continuous Morse sequences $x = b^0 \times b^1 \times \ldots$, $y = \beta^0 \times \beta^1 \times \ldots$, $|b^t| = |\beta^t|$, $t \geqslant 0$, such that $\theta(x)$ and $\theta(y)$ are metrically isomorphic and condition (B) is not satisfied.

To this end we set

$$\beta^t = \underbrace{00\ldots0}_{(t+1)^2}\underbrace{11\ldots1}_{2(t+1)^2+1}, \quad b^t = 0\underbrace{11\ldots1}_{(t+1)^2}\underbrace{00\ldots0}_{(t+1)^2}\underbrace{11\ldots1}_{(t+1)^2}, \quad t \geqslant 0.$$

Then $x = b^0 \times b^1 \times \ldots$, $y = \beta^0 \times \beta^1 \times \ldots$ are continuous (but not regular) Morse sequences. Taking $r'_t = s'_t = 0$, $q'_t = 1+(t+1)^2$, $t \geqslant 0$, we see that conditions (21) and (22) are satisfied so $\theta(x)$ and $\theta(y)$ are metrically isomorphic. It is not hard to see that if $q_t \leqslant \frac{1}{2}(t+1)^2$ or $q_t \geqslant \lambda_t - \frac{1}{2}(t+1)^2$ then for all $r_t, s_t \in \{0, 1\}$

$$d\left((b^t)^{r_t}(b^t)^{s_t}[1+q_t, q_t+\lambda_t], \beta^t\right) \geqslant \frac{\frac{1}{2}(t+1)^2}{3(t+1)^2+1} \geqslant \frac{1}{8}.$$

Thus if condition (A) holds, then the series in (B) is divergent.

A note about finitary isomorphism. We finish this paper by giving (without proof) the necessary and sufficient conditions for Morse dynamical systems to be finitarily isomorphic.

THEOREM 3. *Let $x = b^0 \times b^1 \times \ldots$ and $y = \beta^0 \times \beta^1 \times \ldots$ be Morse sequences. Then $\theta(x)$ and $\theta(y)$ are finitarily isomorphic iff there exist blocks A, B, $|A| = |B|$, and a Morse sequence z such that $x = A \times z$ and $y = B \times z$. In particular, if $|b^t| = |\beta^t|$ for $t \geqslant 0$ then $\theta(x)$ and $\theta(y)$ are finitarily isomorphic iff $b^t = \beta^t$ for all sufficiently large t.*

COROLLARY. *Each class of finitary equivalence is countable and coincides with a class of topological conjugacy* (see [1]).

References

[1] W. Bułatek, *Topological conjugacy of Morse flows over finite abelian groups*, to appear.
[2] M. Keane, *Generalized Morse sequences*, Z. Wahrsch. Verw. Gebiete 10 (1968), 335–353.
[3] J. Kwiatkowski, *Isomorphism of regular Morse dynamical systems induced by arbitrary blocks*, this volume, 219–246.

INSTYTUT MATEMATYKI UNIWERSYTETU MIKOŁAJA KOPERNIKA
INSTITUTE OF MATHEMATICS, NICHOLAS COPERNICUS UNIVERSITY
Chopina 12/18, 87-100 Toruń, Poland

Received October 29, 1984 (2007)
Revised version July 30, 1985