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Some remarks on Suslin sections

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Abstract. We consider the action of a group of homeomorphisms on a topological space, and establish conditions for the existence of a Suslin section of the corresponding equivalence relation. We focus on first category situations; we exhibit for instance a natural class of Banach spaces – containing e.g. $c_0(N)$ and $\mathscr{K}(H)$ – which act in a somehow "ergodic" way in their bidual E^{**} equipped with its w^*-topology. The proofs are short and simple; a number of examples, belonging to various domains of analysis, are given.

I. Introduction. It has been known for a very long time that some very simple equivalence relations may not admit measurable sections. Actually, the first examples of nonmeasurable sets have been constructed in this manner ([13]). This fact has been connected more recently with the "size" of the quotient space, in the articles [12], [7] and [5]. The present note is an attempt to develop this natural idea. A general result (Theorem 3), concerning the action of a group G of homeomorphisms on a Polish space, is shown; let us notice that no topological assumption on G is needed in order to obtain this result – in contrast with [12], [7], and [5]. We consider also "first category" situations (Theorem 7) and this requires the use of some new techniques.

A general theory can be completed in the setting of equivalence relations with Suslin graph. For the sake of shortness, we will restrict ourselves to the action of a group of homeomorphisms on topological spaces; indeed, most of the interesting examples occur in this setting. For the same reason, we did not systematically try to give the more general results, but only the useful ones for natural applications. A number of examples, belonging to various areas of analysis, are given.

Notation. A topological space X is *Suslin* – resp. *standard* – if it is the continuous image – resp. continuous injective image – of a Polish space. We denote by $\mathscr{B}(X)$ the σ-field of the Borel sets of X. If G is a group of homeomorphisms acting on a topological space X, we denote by τ_G the σ-field of G-invariant Borel sets; the quotient space will be denoted by X/G, and will be equipped with the quotient Borel structure. X/G is *countably separated* if there exists a countable subfamily $(B_n)_{n\geq 1}$ of τ_G which separates X/G. A *section* Σ of X/G is a subset of X which meets every equivalence class in exactly one point. A probability ν on a measurable space (Ω, τ) is an *atom* if $\nu(X) \in \{0, 1\}$ for every $X \in \tau$. This atom is *nontrivial* if $\nu(\Omega) = 1$ and $\nu(A) = 0$ for every atom A of τ. The characteristic function of S is denoted $\mathbf{1}_S$.

A subset M of a topological space X is *meager*—resp. *co-meager*—if M—resp. $X \setminus M$—is contained in a countable union of closed sets with empty interior. M is *Baire-measurable* if there exists an open set O such that $M \Delta O$ is meager. The class of Baire-measurable sets is a σ-field which contains $\mathscr{B}(X)$ and is stable under the Suslin operation (see [3]).

$\mathscr{L}(H)$—resp. $\mathscr{K}(H)$—is the space of bounded—resp. compact—operators on a separable Hilbert space H. The weak-star topology on a dual Banach space is denoted by w^*. All the linear operators we consider are strongly continuous.

Our main reference is [3]; see also the very comprehensive and useful surveys [15] and [16].

II. The general results. Let us begin with the easy

PROPOSITION 1. *Let P be a Polish topological group, and H a Suslin subgroup of P. Then the following are equivalent:*

(1) *H is closed.*

(2) *P/H is countably separated.*

(3) *P/H admits a Suslin section Σ.*

(4) *P/H admits a Borel section Σ.*

Proof. (1) ⇒ (4) is a consequence of ([3], Th. 4.3) (see [3], p. 84) or of the theorem of Kuratowski–Ryll-Nardzewski.

(4) ⇒ (3) is obvious.

(3) ⇒ (2). Let $(O_n)_{n\geq 1}$ be a basis of the topology of P, and $B_n = O_n \cap \Sigma$; we let $\tilde{B}_n = B_n + H$. Since H is Suslin, $\tilde{B}_n$ is Suslin; moreover, $P \setminus \tilde{B}_n = (\Sigma \setminus O_n) + H$ and thus $P \setminus \tilde{B}_n$ is Suslin. Now the separation theorem (see [3], Th. 2.2) shows that $\tilde{B}_n \in \mathscr{B}(P)$, and it is clear that $(\tilde{B}_n)_{n\geq 1}$ separates P/H.

(2) ⇒ (1). Let us assume that $\bar{H} \neq H$. The set $\bar{H}/H$ is a subset of P/H and is, therefore, countably separated. The next lemma will be useful.

LEMMA 2. *Let X be a Baire topological space, and G a group of homeomorphisms of X such that for every pair (O, O') of nonempty open sets in X, there exists $g \in G$ such that $g(O) \cap O' \neq \emptyset$. Then every G-invariant Baire-measurable subset B of X is meager or co-meager.*

Proof. If B and $X \setminus B$ are both nonmeager, there exist O_1 and O_2, two nonempty open subsets of X such that $O_1 \setminus B$ and $O_2 \setminus (X \setminus B)$ are meager. By assumption, there exists $g \in G$ such that $g(O_1) \cap O_2 \neq \emptyset$. The set $(g(O_1) \cap O_2) \setminus B$ is contained in $g(O_1) \setminus B = g(O_1 \setminus B)$ and, therefore, it is meager. The set $g(O_1) \cap O_2 \cap B$ is contained in $O_2 \setminus (X \setminus B)$ and thus it is meager too. But $g(O_1) \cap O_2$ is a nonempty open set and thus is nonmeager; this is a contradiction. ∎

By Lemma 2, the H-saturated Baire-measurable subsets of $\bar{H}$ are meager or co-meager in $\bar{H}$. We define an atom ν on τ_H by

$$\nu(B) = 0 \Leftrightarrow B \cap \bar{H} \text{ is meager in } \bar{H},$$

$$\nu(B) = 1 \Leftrightarrow B \cap \bar{H} \text{ is co-meager in } \bar{H}.$$

Since P/H is countably separated, the atom ν has to be trivial and, therefore, there exists $x_0 \in \bar{H}$ such that $x_0 + H$ is co-meager in $\bar{H}$; this implies that H is co-meager in $\bar{H}$ and Pettis's lemma (see [3], Th. 5.1) concludes the proof. ∎

Remarks. (1) The implication (4) ⇒ (1) is claimed in ([15], p. 884), where it is considered as a consequence of ([1], Th. 2) and ([12], Th. 7.2). It is not clear to me that these results could give (4) ⇒ (1) since [1] and [12] deal with continuous actions of locally compact groups; and we are clearly not allowed to assume that H is of second category in itself in order to prove (4) ⇒ (1).

(2) Let us notice that the assumption "H Suslin"—which does not figure in ([15], p. 884)—is necessary, as shown by the example: $P = \mathbf{R}$, H an hyperplane of $\mathbf{R}$ considered as a $\mathbf{Q}$-vector space.

(3) Proposition 1 is an improvement of ([3], Th. 5.5).

Our next result is essentially an improvement of ([5], Th. 2.6):

THEOREM 3. *Let P be a Polish space. Let G be a group of homeomorphisms of P such that $G(x)$ is F_σ for every $x \in P$. Then the following are equivalent:*

(1) *$G(x)$ is locally closed for every $x \in P$.*

(2) *P/G is countably separated.*

(3) *τ_G has no nontrivial atoms.*

If G is Suslin for the topology of simple convergence on P, then (1·2·3) *are implied by:*

(4) *There exists a Suslin section Σ of P/G.*

If $G(B) \in \mathscr{B}(P)$ for every $B \in \mathscr{B}(P)$, then (4) *is a consequence of* (1·2·3).

Let us notice that no topological assumption on G is needed in order to obtain the equivalence (1·2·3).

Proof. (1) ⇒ (2). Let $(O_n)_{n\geq 1}$ be a basis of the topology of P. The sets $B_n = G(O_n)$ are open and thus $B_n \in \tau_G$. Let x and y be such that $G(x) \cap G(y) = \emptyset$.

If $G(x) \cap \overline{G(y)} = \emptyset$, there exists O_{n_0} such that $x \in O_{n_0}$ and $O_{n_0} \cap \overline{G(y)} = \emptyset$ and $B_{n_0} = G(O_{n_0})$ separates $G(x)$ and $G(y)$. If $G(x) \subseteq \overline{G(y)}$, there exists O_{n_1} containing y such that $O_{n_1} \cap G(x) = \emptyset$ since $G(y)$ is open in $\overline{G(y)}$; and then $B_{n_1} = G(O_{n_1})$ separates $G(x)$ and $G(y)$.

Therefore, the countable family $(B_n)_{n\geq 1}$ of τ_G separates P/G.

(2) ⇒ (3) is clear.

$(3)\Rightarrow(1)$. Let x be a point of P, and $P'=\overline{G(x)}$. It is clear that $X=P'$ satisfies the assumptions of Lemma 2, and thus the G-invariant Baire-measurable subsets of P' are meager or co-meager. Since τ_G has no nontrivial atoms, there exists x_0 such that $G(x_0)$ is co-meager in P'; but $G(x_0)$ is F_σ and thus $\overset{\circ}{\overline{G(x_0)}}\neq\emptyset$; this implies $G(x_0)$ is open in $\overline{G(x)}$. But the only class which can be open in $\overline{G(x)}$ is $G(x)$ itself, and thus $G(x_0)=G(x)$ is open in $\overline{G(x)}$.

$(4)\Rightarrow(2)$. Let Σ be a Suslin section of P/G, and assume that G is Suslin for the topology of simple convergence. The application φ: $(g, x)\to g(x)$ from $G\times P$ into P is separately continuous on a product of two Suslin spaces, and thus $\varphi(S)$ is Suslin in P for every Suslin subset S of $G\times P$. We let

$$X_n = G(O_n\cap\Sigma)=\varphi(G\times(O_n\cap\Sigma)).$$

By the above remark, X_n is Suslin in P. Moreover, the set $P\setminus X_n=G(\Sigma\setminus O_n)$ is also Suslin and then the separation theorem (see [3], Th. 2.2) shows that X_n is actually Borel in P. It is clear that the countable subset $(X_n)_{n\geq 1}$ of τ_G separates P/G.

$(2)\Rightarrow(4)$. We suppose now that $G(B)$ is Borel for every Borel set B. Let $(X_n)_{n\geq 1}$ be a separating subset of τ_G. We define the distance d_1 by

$$d_1(x, y)=d(x, y)+\sum_{n=1}^{+\infty}2^{-n}|\mathbf{1}_{X_n}(x)-\mathbf{1}_{X_n}(y)|$$

where d defines the topology of P. It is easy to see that (P, d_1) is still separable; since the Borel structures of d and d_1 coincide, the space (P, d_1) is Suslin (see [3]) and thus there exists a distance d_2 on P, finer than d_1, such that (P, d_2) is Polish. We consider now

$$\psi\colon (P, d_2)\to\{0, 1\}^{\mathbf{N}},\qquad x\to(\mathbf{1}_{X_n}(x))_{n\geq 1}$$

ψ is continuous from (P, d_2) onto the Suslin set $S=\psi(P)$. Since $G(B)\in\mathscr{B}(P)$ for every $B\in\mathscr{B}(P)$, the application $y\to\psi^{-1}(y)$ is measurable from S into the set $\hat{P}$ of closed subsets of (P, d_2) equipped with the Effros–Borel structure (see [3], p. 82); indeed the Borel structures of d_2 and d coincide (see [3], Th. 2.4) and thus every d_2-open set is d-Borel. By ([3], Th. 4.2) there exists a measurable choice function σ: $\hat{P}\to P$; we define Σ_1: $P\to P$ by $\Sigma_1(x)=\sigma\circ\psi^{-1}(\psi(x))$. It is clear that Σ_1 is Borel and, therefore, $\Sigma_1(P)=\Sigma$ is Suslin in P; and by construction, the set Σ is a section of P/G. ∎

A natural example of the above situation is the continuous action of a K_σ-group on a Polish space. Let us point out that if G is countable, then $(1\cdot 2\cdot 3\cdot 4)$ are equivalent.

If P is now assumed to be compact, we can prove

COROLLARY 4. *Let K be a metrizable compact space, and G a Suslin group of homeomorphisms of K such that $G(x)$ is F_σ for every x. If K/G admits a Suslin section Σ, then for every x in K, the set $\overline{G(x)}$ contains a closed equivalence class $G(x_0)$.*

Proof. We define $\mathscr{K}$ by

$$\mathscr{K}=\{F \text{ closed in } \overline{G(x)}\mid g(F)\subseteq F\ \forall g\in G\}$$

Since $\overline{G(x)}\in\mathscr{K}$, this is a nonempty family. The compactness of $\overline{G(x)}$ shows that $\mathscr{K}$ is inductive for the inclusion. By Zorn's lemma, there exists K_0 minimal in $\mathscr{K}$. Let x_0 be a point of K_0; the set $\overline{G(x_0)}\setminus G(x_0)$ is closed by Theorem 3, G-stable and strictly contained in K_0; since K_0 is minimal in $\mathscr{K}$, this implies $G(x_0)=\overline{G(x_0)}$. ∎

Let us point out an interesting special case:

COROLLARY 5. *Let K be a metrizable compact space, and G a Suslin group of homeomorphisms such that $G(x)$ is countable infinite for every x in K. Then K/G does not admit a Suslin section Σ.*

Proof. If there exists a Suslin section Σ, then by Corollary 4 there exists a closed class $G(x_0)$. Since G acts transitively on $G(x_0)$, $G(x_0)$ is a homogeneous compact space; and such a compact space cannot be countable infinite. ∎

The next situation we will consider arises from Banach space theory. If E is an infinite-dimensional Banach space, then E is a *meager* subgroup of the *meager* group (E^{**}, w^*). We will show that a natural class of Banach spaces act by translation in a somehow "ergodic" way on their bidual E^{**}. Moreover, in this case, a nontrivial linear analogue will be available.

LEMMA 6. *Let X be a norm-closed and w^*-analytic proper subspace of $l^\infty(\mathbf{N})$ containing $c_0(\mathbf{N})$. Then:*

(1) *The measurable space $(l^\infty(\mathbf{N}), w^*)/X$ is not countably separated.*

(2) *There does not exist a linear injection from l^∞/X into $l^\infty(\mathbf{N})$.*

Proof. (1) The unit ball of $l^\infty(\mathbf{N})$ equipped with the w^*-topology is homeomorphic to the Hilbert cube $[-1, 1]^{\mathbf{N}}$. By the "0-1" topological law ([3], Th. 5.6), for every X-saturated w^*-Borel set B of $l^\infty(\mathbf{N})$, the set $B\cap[-1, 1]^{\mathbf{N}}$ is meager or co-meager. If $(l^\infty, w^*)/X$ is countably separated, every atom of τ_X must be trivial; therefore there exists $x_0\in[-1, 1]^{\mathbf{N}}$ such that $(x_0+X)\cap[-1, 1]^{\mathbf{N}}$ is co-meager in $[-1, 1]^{\mathbf{N}}$. By the Hahn–Banach theorem, there exists $\varphi\in X^\perp\setminus\{0\}$; such a φ will be constant on the co-meager set $(x_0+X)\cap[-1, 1]^{\mathbf{N}}$ and thus will be Baire-measurable; by [14], this implies $\varphi\in l^1(\mathbf{N})$, and this is a contradiction with $c_0(\mathbf{N})\subseteq X$.

(2) Let us assume that there exists a linear injection from l^∞/X into $l^\infty(\mathbf{N})$. This means that there exists a countable set $(\varphi_n)_{n\geq 1}$ in $l^{\infty *}$ such that $X=\bigcap_{n=1}^{\infty}\operatorname{Ker}\varphi_n$. Since X is a proper subspace of $l^\infty(\mathbf{N})$, at least one of the (φ_n), let us say φ_1, must be nonzero. We consider the finitely additive

measures μ and μ_1 defined on N by

$$\mu(A) = \sum_{n=1}^{\infty} 2^{-n}|\varphi_n|(1_A); \qquad \mu_1(A) = \varphi_1(1_A).$$

Since $X \supseteq c_0(N)$, μ and μ_1 are zero on finite sets. We let

$$A = \{B \subseteq N|\ \mu(B) = 0\}, \qquad A_1 = \{B \subseteq N|\ \mu_1(B) = 0\}.$$

By [8], the subsets A and A_1 of $\mathscr{P}(N)$ cannot be Baire-measurable in $\mathscr{P}(N)$. We have $A \subseteq X \cap \mathscr{P}(N) \subseteq A_1$ and thus $X \cap \mathscr{P}(N)$ can be neither meager nor co-meager in $\mathscr{P}(N)$; but $X \cap \mathscr{P}(N)$ is Suslin in $\mathscr{P}(N)$ and does not depend upon a finite number of coordinates. Thus, by the "0-1" topological law ([3], Th. 5.6) or by Lemma 2, $X \cap \mathscr{P}(N)$ must be meager or co-meager; this is a contradiction. ∎

Let us recall that a function f defined on a compact space K is said to be *strictly of the first Baire class* if there exists a sequence $(x_n)_{n\geq 1}$ in $\mathscr{C}(K)$ with $f = \sum_{n=1}^{+\infty} x_n$ pointwise and $\sum_{n=1}^{\infty} |x_n(x)| < \infty$ for every $x \in K$.

With this terminology, one has:

THEOREM 7. *Let E be a Banach space with separable dual such that every $f \in E^{**}$, considered as a function on (E^*, w^*), is strictly of the first Baire class. Let X be a w^*-Suslin and norm-closed proper subspace of E^{**} containing E. Then*

(1) *$(E^{**}, w^*)/X$ does not admit a w^*-Suslin section Σ.*

(2) *X is not complemented in E^{**}.*

Proof. (1) We consider $f \in E^{**} \setminus X$. Since f is strictly of the first Baire class, there exists a subspace Y of E isomorphic to $c_0(N)$ such that $f \in Y^{\perp\perp}$ ([11], vol. I, p. 98). By Lemma 6 (1), the set $(Y^{\perp\perp}, w^*)/Y^{\perp\perp} \cap X$ is not countably separated; but $Y^{\perp\perp}/Y^{\perp\perp} \cap X$ is a subset of $(E^{**}, w^*)/X$ and thus $(E^{**}, w^*)/X$ is not countably separated.

If there exists a w^*-Suslin section Σ, then for every w^*-Borel set B in E^{**} the set $(B \cap \Sigma) + X$ is w^*-Suslin since X is w^*-Suslin; we have

$$E^{**} \setminus \big((B \cap \Sigma) + X\big) = (\Sigma \setminus B) + X$$

and thus the separation theorem ([3], Th. 2.2) shows that $(B \cap \Sigma) + X$ is actually w^*-Borel for every w^*-Borel set B. Now, if $(B_n)_{n\geq 1}$ is a basis of the Borel structure of the standard space (E^{**}, w^*), the family $(B_n \cap \Sigma) + X$ separates E^{**}/X and this is a contradiction.

(2) Let $(g_n)_{n\geq 1}$ be a countable norm-dense subset of E^*. If there exists a linear right-inverse σ: $E^{**}/X \to E^{**}$ of the quotient map Q: $E^{**} \to E^{**}/X$, then the operator

$$\psi\colon E^{**}/X \to l^\infty(N), \qquad x \to \big(g_n(\sigma(x))\big)_{n\geq 1}$$

is a linear injection from E^{**}/X into $l^\infty(N)$. With the notation of (1), there exists a linear—canonical—injection from $Y^{\perp\perp}/Y^{\perp\perp} \cap X$ into E^{**}/X; and by Lemma 6 (2) there is no linear injection from $Y^{\perp\perp}/Y^{\perp\perp} \cap X$ into $l^\infty(N)$; this is a contradiction. ∎

Remark. The first example of a Banach space E to which Theorem 7 applies is $E = c_0(N)$. In this case, Theorem 7 (2) is essentially an improvement of ([3], Th. 5.8). Theorem 7 applies also to the noncommutative analogue $E = \mathscr{K}(H)$. Actually, Theorem 7 applies exactly to the Banach spaces with separable dual which have the property (u) of Pełczyński ([11], vol. II, p. 31). This class contains e.g. every separable σ-complete Banach lattice which does not contain $l^1(N)$. Moreover, it is easily shown that this class is stable under subspaces, quotient maps and c_0-direct sums (see [11], vol. II, p. 31).

III. Examples and remarks.

1. If E is a metrizable complete separable t.v.s., and X a subspace of E of countable infinite dimension, then Proposition 1 shows that E/X does not admit a Suslin section.

For instance, if $E = \mathscr{H}(C)$ is the space of analytic functions on C, equipped with the topology of compact convergence, and $X = C[z]$, then $\dot{g}$ in E/X describes somehow the "behavior at infinity" of g. Therefore, there does not exist a Suslin subset Σ of $\mathscr{H}(C)$ such that every $g \in \mathscr{H}(C)$ has the same "behavior at infinity" as exactly one $h \in \Sigma$.

2. In the deep article [5], E.G. Effros considers the Polish space $\mathscr{U}^i_\infty$ of irreducible representations of a given C^*-algebra A on an infinite-dimensional Hilbert space H (see [4]). The group $U(H)$ of unitary operators of H is a Polish group which acts continuously on $\mathscr{U}^i_\infty$ by $\psi_U(L) = U^{-1}LU$. It is shown in [5] that the equivalence classes are F_σ and that the conditions $(1\cdot 2\cdot 3)$ of Theorem 3 are satisfied if and only if A is a C^*-algebra of type I.

3. The Polish group $U(H)$ acts continuously on the unit ball B of $\mathscr{L}(H)$, equipped with the weak operator topology w_0, by $U(T) = U \circ T$. In this case, the polar decomposition $T = US$ provides us with a w_0-closed section of $B/U(H)$. The equivalence class $U(H)$ of Id is neither F_σ nor locally closed. This example shows that the hypothesis "$G(x)$ is F_σ for every x" cannot be removed from Theorem 3.

4. If G is a locally compact separable and noncompact topological group which acts continuously on K compact, and if $G_x = \{g \in G|\ g(x) = x\}$ is compact for every x in K, then it is easy to deduce from Corollary 4 that K/G does not admit a Suslin section Σ.

This shows that if the conditions of ([2], Th. 2 or Th. 3) are satisfied, then—in the notation of [2]—the group G must be compact if $\mathscr{H}$ is compact.

5. Under the assumptions of Corollary 4, K/G may admit a Suslin

section even if there exist "very few" closed classes. An example is given by $K = [0, 1]$ and G the group of homeomorphisms generated by $f(x) = x^2$.

6. Let K be a metrizable compact space, and T a homeomorphism of K. We let $G = \{T^n |\ n \in Z\}$. By Corollary 5, if K/G has a Suslin section Σ, then T must admit "periodic points" x such that $T^n(x) = x$ for some $n \geqslant 1$.

7. Let E be a nonreflexive separable dual space with a separable dual, e.g. the space J constructed in [10]. Then $E^{**} = E \oplus X$ where X is a w^*-closed subspace of E^{**}. This shows that some assumption on E is needed in order to obtain Theorem 7. Moreover, this provides us with natural examples of K_σ groups G with a proper dense $K_{\sigma\delta}$ subgroup H such that G/H has a closed section.

8. It is not clear whether or not "Σ Suslin" may be replaced by "Σ co-Suslin" in Theorem 3 (4). If one assumes Martin's axiom and the negation of the continuum hypothesis, then every PCA-set is Baire-measurable (see [6]), and the proof of Theorem 3 shows that, under these axioms, we may replace "Σ Suslin" by "Σ PCA" in (4). On the contrary, if we assume Gödel's axiom of constructibility (G) ([9]), a PCA section always exists, and thus we may not replace "Σ Suslin" by "Σ PCA" in (4).

Let us finally point out that in many situations, it is easy to show without set-theoretic axioms that $(1 \cdot 2 \cdot 3)$ of Theorem 3 are equivalent to:

(4′) There exists a section Σ which belongs to the σ-field generated by the Suslin sets.

This is for instance the case when G is a countable group acting on a Polish space P.

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