[7] N. J. Kalton, *Plurisubharmonic functions on quasi-Banach spaces*, Studia Math. 84 (3) (1986), to appear.

[8] J. L. Krivine, *Théorèmes de factorisation dans les espaces réticulés*, Sém. Maurey–Schwartz 1973–74, Exposés 22–23, École Polytechnique, Paris.

[9] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II. Function Spaces*, Springer, 1979.

[10] M. R. Marcus and G. Pisier, *Random Fourier Series with Applications to Harmonic Analysis*, Ann. of Math. Stud., Princeton University Press, 1981.

[11] B. Maurey, *Théorèmes de factorisation pour les opérateurs linéaires à valeurs dans les espaces $L^p$*, Astérisque 11 (1974), Soc. Math. France.

[12] N. Popa, *Uniqueness of the symmetric structure in $L_p(\mu)$ for $0 < p < 1$*, Rev. Roumaine Math. Pures Appl. 27 (10) (1982), 1061–1083.

[13] M. A. Triana, *Espacios F-normados de funciones y sucesiones vectoriales*, Doctoral Thesis, Zaragoza 1984.

[14] P. Turpin, *Représentation fonctionnelle des espaces vectoriels topologiques*, Studia Math. 73 (1982), 1–10.



DEPARTAMENTO TEORÍA DE FUNCIONES
FACULTAD DE CIENCIAS (MATEMATICAS)
50009 Zaragoza, Spain
and
DEPARTAMENTO MATEMÁTICAS I
ETSII, UNIVERSIDAD DE ZARAGOZA
50009 Zaragoza, Spain



*Received December 24, 1984*
*Revised version August 22, 1985* (2024)


STUDIA MATHEMATICA, T. LXXXIV. (1986)

# A direct proof of van der Vaart's theorem

by

J. BOURGAIN (Brussels and Pasadena, Cal.) and H. SATO (Fukuoka)

**Abstract.** The aim of this paper is to give a direct and simple proof of van der Vaart's theorem [3] determining the absolutely continuous component of a signed measure on $\boldsymbol{R}^d$ from its characteristic functional.

**1. Introduction and results.** Let

$$d\lambda(t) = d\lambda(t_1, t_2, \ldots, t_d) = (2\pi)^{-d/2}\, dt_1\, dt_2 \ldots dt_d$$

be the modified Lebesgue measure on $\boldsymbol{R}^d$, for a $\lambda$-integrable function $f$ on $\boldsymbol{R}^d$ define the Fourier transform by

$$\tilde{f}(\alpha) = \int_{\boldsymbol{R}^d} e^{i(\alpha,t)} f(t)\, d\lambda(t), \qquad \alpha \in \boldsymbol{R}^d,$$

where $(\alpha, t)$ is the inner product of $\boldsymbol{R}^d$, let $\mathscr{K}$ be the collection of all $\lambda$-integrable functions $\varkappa$ which satisfy the following conditions:

(1) $\int \varkappa(t)\, d\lambda(t) = 1$.

(2) There exists $a > 1$ such that

$$Q(\varkappa) = \sup_{t\in\boldsymbol{R}^d} (1 + \|t\|^{da})\, |\varkappa(t)| < +\infty,$$

where $\|t\|$ is the Euclidean norm on $\boldsymbol{R}^d$, and define

$$\tilde{\mathscr{K}} = \{\varkappa \in \mathscr{K};\ \tilde{\varkappa} \in L^1(\lambda)\}.$$

Furthermore, for every $\varkappa$ in $\mathscr{K}$ and $T > 0$ define $\varkappa_T(t) = T^d \varkappa(Tt)$. Then evidently we have for every $T > 0$,

$$\int \varkappa_T(t)\, d\lambda(t) = 1 \quad \text{and} \quad \tilde{\varkappa}_T(\alpha) = \tilde{\varkappa}(\alpha/T).$$

Let $\mu$ be a signed measure on $\boldsymbol{R}^d$. Then we have the Lebesgue decomposition

$$d\mu(t) = \frac{d\mu}{d\lambda}(t)\, d\lambda(t) + d\mu_s(t),$$

where $\mu_s$ is the singular component of $\mu$.

In this paper we shall prove the following theorems.

THEOREM 1. *Let $\mu$ be a signed measure on $\boldsymbol{R}^d$. Then we have for every $\varkappa$ in $\mathscr{K}$*

$$\frac{d\mu}{d\lambda}(t) = \lim_{T\to+\infty} \int_{\boldsymbol{R}^d} \varkappa_T(t-s)\, d\mu(s), \qquad a.e.\ (d\lambda),$$

*and the exceptional null set does not depend on the choice of $\varkappa$.*

As an application of Theorem 1, we have the following theorem.

THEOREM 2. *Let $\mu$ be a signed measure on $\boldsymbol{R}^d$ and $\tilde{\mu}(\alpha)$ the characteristic functional of $\mu$ defined by*

$$\tilde{\mu}(\alpha) = \int_{\boldsymbol{R}^d} e^{i(\alpha,t)}\, d\mu(t), \qquad \alpha \in \boldsymbol{R}^d.$$

*Then we have for every $\varkappa$ in $\mathscr{K}$*

$$\frac{d\mu}{d\lambda}(t) = \lim_{T\to+\infty} \int_{\boldsymbol{R}^d} e^{-i(\alpha,t)}\, \tilde{\mu}(\alpha)\, \tilde{\varkappa}(\alpha/T)\, d\lambda(\alpha), \qquad a.e.\ (d\lambda),$$

*and the exceptional null set does not depend on the choice of $\varkappa$.*

COROLLARY. *For every $\lambda$-integrable function $f$ and every $\varkappa$ in $\mathscr{K}$ we have*

$$f(t) = \lim_{T\to+\infty} \int_{\boldsymbol{R}^d} e^{-i(\alpha,t)}\, \tilde{f}(\alpha)\, \tilde{\varkappa}(\alpha/T)\, d\lambda(\alpha), \qquad a.e.\ (d\lambda),$$

*and the exceptional null set does not depend on the choice of $\varkappa$.*

The above theorems were first proved by van der Vaart ([3], Lemma 2.6 and Theorem 1) for a slightly more general class of kernel functions. J. Bourgain gave a direct proof for the kernel function $\varkappa(t) = \exp(-\frac{1}{2}\|t\|^2)$ (see H. Sato [2], Lemma 3). Combining the idea of J. Bourgain and the arguments of W. Rudin [1], we shall give the direct proofs in simplified formulations. The authors consider that the class $\mathscr{K}$ of kernel functions is sufficient for the practical use.

**2. Proofs of theorems.** For any signed measure $\mu$ on $\boldsymbol{R}^d$ and any $\varkappa$ in $\mathscr{K}$ define

$$(D_\varkappa \mu)(t) = \lim_{T\to+\infty} \int_{\boldsymbol{R}^d} \varkappa_T(t-s)\, d\mu(s)$$

if the right side converges. Then, since we have $D_\varkappa = c^+ D_{(1/c^+)\varkappa^+} - c^- D_{(1/c^-)\varkappa^-}$ where

$$\varkappa = \varkappa^+ - \varkappa^-, \quad c^+ = \int \varkappa^+ d\lambda, \quad c^- = \int \varkappa^- d\lambda \quad \text{and} \quad c^+ - c^- = 1,$$

without loss of generality we may assume that $\varkappa(t) \geqslant 0$, a.e. $(d\lambda)$. Therefore in the remaining part of the paper we always assume that every kernel function $\varkappa$ in $\mathscr{K}$ is nonnegative.



Fix a positive number $a > 1$ and define

$$\mathscr{K}_a = \{\varkappa \in \mathscr{K}\colon\ Q(\varkappa) = \sup_t (1+\|t\|^{da})\,\varkappa(t) < +\infty\},$$

$$\varphi(t) = \frac{1}{1+\|t\|^{da}},$$

and $\varphi_T(t) = T^d \varphi(Tt)$, $t \in \boldsymbol{R}^d$, $T > 0$. Then for every finite measure $\mu$ on $\boldsymbol{R}^d$ the function

$$(D\mu)(t) = \limsup_{T\to+\infty} \int_{\boldsymbol{R}^d} \varphi_T(t-s)\, d\mu(s)$$

is Borel measurable since for every $T_0 > 0$

$$\sup_{T>T_0} \int_{\boldsymbol{R}^d} \varphi_T(t-s)\, d\mu(s)$$

is lower semi-continuous.

LEMMA 1. *Let $\mu$ be a finite measure on $\boldsymbol{R}^d$ and $A$ a Borel subset such that $\mu(A) = 0$. Then there exists a Borel subset $\Omega_a \subset A$ such that $\lambda(A \setminus \Omega_a) = 0$ and*

$$(D_\varkappa \mu)(t) = 0, \qquad t \in \Omega_a,$$

*for every $\varkappa$ in $\mathscr{K}_a$.*

Proof. For every $\varkappa$ in $\mathscr{K}_a$ and every $t$ in $\boldsymbol{R}^d$ we have

$$0 \leqslant \liminf_T \int \varkappa_T(t-s)\, d\mu(s) \leqslant \limsup_T \int \varkappa_T(t-s)\, d\mu(s)$$
$$\leqslant Q(\varkappa) \limsup_T \int \varphi_T(t-s)\, d\mu(s) = Q(\varkappa)\, \bar{D}\mu(t).$$

Therefore, if we show that $\bar{D}\mu(t) = 0$ for $\lambda$-almost all $t$ in $A$, then we have

$$D_\varkappa \mu(t) = 0, \qquad t \in \Omega_a = \{t \in A;\ \bar{D}\mu(t) = 0\},$$

for every $\varkappa$ in $\mathscr{K}_a$.

Without loss of generality we may assume that $\mu$ is a probability measure and define $\mathscr{A} = \{t \in A;\ \bar{D}\mu(t) > 0\}$ and for every positive number $\gamma$, $\mathscr{A}_\gamma = \{t \in A;\ \bar{D}\mu(t) > \gamma\}$. Then in order to show that $\lambda(\mathscr{A}) = 0$ it is sufficient to show that $\lambda(\mathscr{A}_\gamma) = 0$ for every $\gamma > 0$.

For every $t$ in $\mathscr{A}_\gamma$ there exists a sequence of positive numbers $T_k = T_k(t) \nearrow +\infty$ such that

$$\int \varphi_{T_k}(t-s)\, d\mu(s) > \gamma$$

for every $k$ in $\boldsymbol{N} = \{0, 1, 2, 3, \ldots\}$.

Assume that for $t$ in $\mathscr{A}_\gamma$ and $T > 0$ we have

$$\int \varphi_T(t-s)\, d\mu(s) > \gamma,$$

and define

$$B_k = \{s \in \mathbf{R}^d;\ e^{-k} \geqslant \varphi(T(t-s)) > e^{-(k+1)}\}, \qquad k \in \mathbf{N}.$$

Then we have

$$\gamma/T^d \leqslant \int \varphi(T(t-s))\,d\mu(s) \leqslant \sum_{k=0}^{+\infty} e^{-k}\mu(B_k).$$

Let $l = l(T)$ be the maximal natural number which does not exceed $\log[2e^2/(e-1)\gamma] + d(\log T)$. Then we have

$$\sum_{k \geqslant l} e^{-k} = \frac{e^{-l}}{1-e^{-1}} \leqslant \frac{\gamma}{2T^d}$$

so that

$$\gamma/T^d \leqslant \sum_{k=0}^{\infty} e^{-k}\mu(B_k) \leqslant \sum_{k<l} e^{-k}\mu(B_k) + \sum_{k \geqslant l} e^{-k}$$

$$\leqslant \sum_{k<l} e^{-k}\mu(B_k) + \gamma/(2T^d)$$

and consequently

$$\sum_{k<l} e^{-k}\mu(B_k) \geqslant \gamma/(2T^d).$$

Define $b = (a-1)/(2a) > 0$, $L = (1-e^{-b})^{-1}$ and $M = \gamma/(2L)$. Then there exists $k(T) < l = l(T)$ such that

$$\mu(B_{k(T)}) \geqslant \frac{M}{T^d} e^{(1-b)k(T)}.$$

For assume the contrary. Then we have

$$\sum_{k<l} e^{-k}\mu(B_k) < \sum_{k<l} \frac{M}{T^d} e^{-bk} \leqslant \frac{ML}{T^d} = \frac{\gamma}{2T^d},$$

which is a contradiction.

For every $k$ in $N$ we have

$$C_k(t) = \{s \in \mathbf{R}^d;\ \varphi(T(t-s)) > e^{-(k+1)}\}$$

$$= \{s \in \mathbf{R}^d;\ (1+T^{da}\|t-s\|^{da})^{-1} > e^{-(k+1)}\}$$

$$\subset \left\{s \in \mathbf{R}^d;\ \|t-s\| < \frac{1}{T}\exp\left[\frac{k+1}{da}\right]\right\} = S_k(t),$$

$$\lambda(S_k(t)) = \frac{\lambda(V)}{T^d}\exp\left[\frac{k+1}{a}\right],$$



where $V$ is the unit ball of $\mathbf{R}^d$, and

$$\mu(S_{k(T)}(t)) \geqslant \mu(C_{k(T)}(t)) \geqslant \mu(B_{(k(T)}(t)) \geqslant \frac{M}{T^d} e^{(1-b)k(T)}.$$

Since we have

$$\delta = \inf_k \frac{M\exp[(1-b)k]}{\lambda(V)\exp[(k+1)/a]} = \inf_k \frac{M}{\lambda(V)}\exp\left[\frac{a-1}{2a}k - \frac{1}{a}\right] > 0,$$

therefore

$$\mu(S_{k(T)}(t)) \geqslant \mu(B_{k(T)}(t)) \geqslant \delta\lambda(S_{k(T)}(t)).$$

On the other hand we have

$$\text{radius } S_{k(T)}(t) = \frac{1}{T}\exp\left[\frac{k(T)+1}{da}\right] \leqslant \frac{1}{T}\exp\left[\frac{l(T)+1}{da}\right]$$

$$\leqslant \frac{2e^2}{\gamma(e-1)}\left(\frac{1}{T}\right)^{\frac{a-1}{a}} \to 0 \quad \text{as } T \to +\infty.$$

Thus we have proved that the collection of open subsets

$$\Sigma = \{S_{k(T)}(t);\ t \in \mathscr{A}_\gamma,\ T > 0,\ \int \varphi_T(t-s)\,d\mu(s) > \gamma\}$$

is a *substantial family* of W. Rudin [1], Definition 8.2, and that every $S$ in $\Sigma$ satisfies the inequality

$$\mu(S) \geqslant \delta\lambda(S).$$

Let $K$ be any compact subset of $\mathscr{A}_\gamma$ and for every natural number $p$ define

$$\Sigma_p = \{S \in \Sigma;\ \text{radius}\, S < 1/p\}.$$

Then, since $\Sigma_p$ is an open covering of $K$, there are $S^1, S^2, S^3, \ldots, S^n$ in $\Sigma_p$ such that $\bigcup_{i=1}^n S^i \supset K$. By Theorem 8.5 of W. Rudin [1] we can extract $S^{i_1}, S^{i_2}, \ldots, S^{i_m}$, a disjoint subcollection of $S^1, S^2, \ldots, S^n$ such that

$$\lambda\left(\bigcup_{i=1}^n S^i\right) \leqslant 3^d \lambda\left(\bigcup_{j=1}^m S^{i_j}\right).$$

Let $K_p$ be the $1/p$-neighborhood of $K$. Then we have

$$\lambda(K) \leqslant \lambda\left(\bigcup_{i=1}^n S^i\right) \leqslant 3^d\lambda\left(\bigcup_{j=1}^m S^{i_j}\right) = 3^d\sum_{i=1}^m \lambda(S^{i_j})$$

$$\leqslant \frac{1}{\delta}3^d\sum_{i=1}^m \mu(S^{i_j}) = \frac{3^d}{\delta}\mu\left(\bigcup_{j=1}^m S^{i_j}\right) \leqslant \frac{3^d}{\delta}\mu(K_p),$$

so that

$$\mu(K) \leqslant \frac{3^d}{\delta} \lim_{p\to+\infty} \mu(K_p) = \frac{3^d}{\delta}\mu(K) < \frac{3^d}{\delta}\mu(A) = 0.$$

Since the compact subset $K$ of $\mathscr{A}_\gamma$ is arbitrary, the regularity of $\lambda$ shows that $\lambda(\mathscr{A}_\gamma) = 0$.

It is obvious that $\Omega_a = A \setminus \bigcup_n \mathscr{A}_{1/n}$ is the desired set. ∎

The above lemma implies immediately the following lemma.

LEMMA 2. *Let $\mu$ be a finite measure on $\boldsymbol{R}^d$ and $A$ a Borel subset such that $\mu(A) = 0$. Then there exists a Borel subset $\Omega \subset A$ such that $\lambda(A\setminus\Omega) = 0$ and for every $\varkappa$ in $\mathscr{K}$*

$$D_\varkappa \mu(t) = 0, \quad t \in \Omega.$$

Proof. Define $\Omega = \bigcap_n \Omega_{(1+1/n)}$ where $\Omega_{(1+1/n)}$ is defined by Lemma 1. Then, since $\mathscr{K} = \bigcup_n \mathscr{K}_{(1+1/n)}$, we have the desired result. ∎

Proof of Theorem 1. Let $\mu$ be a signed measure on $\boldsymbol{R}^d$ with the Lebesgue decomposition

$$d\mu(t) = \frac{d\mu}{d\lambda}(t)\,d\lambda(t) + d\mu_s(t).$$

Then, since $\mu_s$ is singular to $\lambda$, there exists a Borel subset $A$ such that

$$\lambda(A^c) = \mu_s^+(A) = \mu_s^-(A) = 0$$

where $\mu_s^+$ and $\mu_s^-$ are the positive and negative variation of $\mu_s$, respectively. Therefore by Lemma 2 there exists a Borel subset $\mathscr{N}_s$ of $A$ such that

$$D_\varkappa \mu_s(t) = 0, \quad t \in A\setminus\mathscr{N}_s$$

for every $\varkappa$ in $\mathscr{K}$. On the other hand, define for every rational number $r$

$$F_r = \left\{t \in \boldsymbol{R}^d;\ \frac{d\mu}{d\lambda}(t) \geqslant r\right\},$$

for every natural number $m$

$$G_m = \{t \in \boldsymbol{R}^d;\ \|t\| < m\},$$

and for every Borel subset $E$

$$\nu_r^m(E) = \int_{E\cap F_r\cap G_m} \left(\frac{d\mu}{d\lambda}(t) - r\right) d\lambda(t).$$



Then $\nu_r^m$ is a finite measure on $\boldsymbol{R}^d$ and by the same arguments as in the proof of Theorem 8.6 of W. Rudin [1] there exists a Borel subset $\mathscr{N}_m \subset G_m$ such that $\lambda(\mathscr{N}_m) = 0$ and

$$D_\varkappa \mu(t) = \frac{d\mu}{d\lambda}(t), \quad t \in G_m\setminus\mathscr{N}_m$$

for every $\varkappa$ in $\mathscr{K}$.

Finally define $\mathscr{N} = \bigcup_m \mathscr{N}_m$. Then we have $\lambda(\mathscr{N}) = 0$ and

$$D_\varkappa \mu(t) = \frac{d\mu}{d\lambda}(t), \quad t \in \boldsymbol{R}^d\setminus\mathscr{N},$$

for every $\varkappa$ in $\mathscr{K}$, which proves Theorem 1. ∎

Proof of Theorem 2. For every signed measure $\mu$ on $\boldsymbol{R}^d$, every $\varkappa$ in $\tilde{\mathscr{K}}$ and every $T > 0$ we have

$$\int_{\boldsymbol{R}^d} e^{-i(\alpha,t)}\,\tilde{\mu}(\alpha)\,\tilde{\varkappa}(\alpha/T)\,d\lambda(\alpha) = \int_{\boldsymbol{R}^d} \varkappa_T(t-s)\,d\mu(s)$$

and Theorem 1 proves Theorem 2. ∎

## References


[1] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York 1966.

[2] H. Sato, *Characteristic functional of a probability measure absolutely continuous with respect to a Gaussian Radon measure*, J. Funct. Anal. 61 (1985), 222–245.

[3] H. R. van der Vaart, *Determining the absolutely continuous component of a probability distribution from its Fourier–Stieltjes transform*, Ark. Mat. 7 (1967), 331–342.



DEPARTMENT OF MATHEMATICS,
VRIJE UNIVERSITEIT
Pleinlaan, 2-F7, 1050 Brussels, Belgium

DEPARTMENT OF MATHEMATICS,
CALIFORNIA INSTITUTE OF TECHNOLOGY
Pasadena, California 91125, U.S.A.

and

DEPARTMENT OF MATHEMATICS,
KYUSHU UNIVERSITY
Hakozaki, Fukuoka, 812 Japan


*Received February 27, 1985* (2035)