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Quasi-radial Fourier multipliers

by

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Abstract. We give sufficient conditions on functions $m\colon [0, \infty) \to C$ and $\varrho\colon R^n \to [0, \infty)$ so that their composition $m \circ \varrho(x) = m(\varrho(x))$, called a *quasi-radial function*, will be a Fourier multiplier on $L^p = L^p(R^n)$. The condition on m is stated, as in the radial case, in terms of localized Bessel potential spaces $WBV_{q,\gamma}$ and is for $\gamma \in N$ of the type

(0.1) $$\|m\|_\infty + \sup_{r>0} \left(\int_r^{2r} |t^\gamma m^{(\gamma)}(t)|^q \frac{dt}{t} \right)^{1/q} < \infty, \quad 1 \leqslant q < \infty.$$

The conditions on ϱ generalize the basic properties of the Euclidean distance $|x|$ on R^n and read:

$\varrho(x) > 0$ if $x \in R_0^n = R^n \setminus \{0\}$ (positive definiteness),

$\varrho(A_t x) = t\varrho(x)$ if $t > 0$ and $x \in R^n$ (homogeneity),

$\varrho \in C(R^n)$ (continuity).

Here the dilation matrix A_t is defined by $A_t = t^P$ where P is a real $n \times n$-matrix and its eigenvalues have positive real parts. Functions ϱ with the above properties are called A_t-*homogeneous distance functions.*

1. Quasi-radial Fourier multipliers do not seem to have been systematically treated yet. There are some single results in the literature concerning only special instances of quasi-radial Fourier multiplier criteria. One may encounter them e.g. in the papers of Ashurov [1], Löfström [18], Peetre [20], Sjöstrand [25] and Peral and Torchinsky [21]. Among all these criteria Theorem 1.4 of the last authors has to be emphasized, for it directed our attention to the problem dealt with in our Theorem 1 which is a generalization and an improvement of their result.

The paper is organized as follows. In Section 2 the principal notation and $WBV_{q,\gamma}$-spaces are introduced and some properties of homogeneous distance functions are stated. The exposition of the main results and their discussion in the following remarks constitute the content of Section 3. In Section 4 the essential results needed for proving our main results are listed together with some auxiliary lemmas. In Section 5 Theorem 1 is proved. Section 6 contains the main technical contribution of this paper, the proof of Theorem 2. In Section 7 auxiliary lemmas are proved.

2. The Fourier transformation $\mathscr{F}$ is defined on $\mathscr{S} = \mathscr{S}(\boldsymbol{R}^n)$, the space of all rapidly decreasing C^∞-functions on $\boldsymbol{R}^n$, by

$$\mathscr{F}f(\xi) = f\hat{\ }(\xi) = \int f(x) e^{-ix\xi}\,dx$$

where $x\cdot\xi = x\xi = x_1\xi_1 + \ldots + x_n\xi_n$ and the region of integration is usually omitted if it is the whole $\boldsymbol{R}^n$. $\mathscr{S}' = \mathscr{S}'(\boldsymbol{R}^n)$ denotes the space of all tempered distributions, $M_p = M_p(\boldsymbol{R}^n) \subset \mathscr{S}'$ the space of all Fourier multipliers on L^p endowed with the norm

$$\|m\|_{M_p} = \inf\{C\colon \|\mathscr{F}^{-1}[m\hat{\varphi}]\|_p \leqslant C\|\varphi\|_p,\ \varphi\in\mathscr{S}\};$$

here $\mathscr{F}^{-1}$ denotes the inverse Fourier transformation. The letters c or C denote constants which may vary from line to line. If $E\subset\boldsymbol{R}^n$ is (Lebesgue) measurable then $|E| = \int\chi_E(x)\,dx$ where χ_E is the characteristic function of E. The differential operator is denoted by $D^\sigma = (\partial/\partial x)^\sigma = \partial^{|\sigma|}/\partial x_1^{\sigma_1}\ldots\partial x_n^{\sigma_n}$ where $\sigma = (\sigma_1, \ldots, \sigma_n)\in\boldsymbol{N}^n$, $\boldsymbol{N} = \{0, 1, \ldots\}$, $|\sigma| = \sigma_1 + \ldots + \sigma_n$ and for $x\in\boldsymbol{R}^n$, $x^\sigma = x_1^{\sigma_1}\ldots x_n^{\sigma_n}$. If $t\in\boldsymbol{R}$ then $[t]$ is the integer part of t, $t_+ = \max\{0, t\}$. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of P, $\operatorname{Re}\lambda_j > 0$, then we set

$$\alpha_{\mathrm{m}} = \min \operatorname{Re}\lambda_j,\qquad \alpha_{\mathrm{M}} = \max\operatorname{Re}\lambda_j.$$

In [28] it is shown that for $x\neq 0$ the mapping

$$\text{(2.1)}\quad \begin{cases}(0,\infty)\to(0,\infty)\\ t \to BA_t x\cdot A_t x\end{cases}\quad \text{is onto and strongly monotone increasing,}$$

where $A_t = t^P$, $B = \int_0^\infty e^{-tP^*}e^{-tP}\,dt$ and P^* is the transpose of P with respect to the usual scalar product on $\boldsymbol{R}^n$. From (2.1) it follows that the equations $BA_t x\cdot A_t x = 1$ and $tr(x) = 1$, $x\neq 0$, $t>0$, uniquely determine an A_t-homogeneous distance function $r\in C^\infty(\boldsymbol{R}_0^n)$. Otherwise the letter ϱ will always be used to denote an arbitrary A_t-homogeneous distance function. We need the following properties of ϱ:

$$\text{(2.2)}\qquad \varrho(x+y)\leqslant c(\varrho(x)+\varrho(y))\quad \text{for some } c \text{ and all } x, y\in\boldsymbol{R}^n;$$

for every $0<\varepsilon<\alpha_{\mathrm{m}}$ there are positive constants c_1 and c_2 such that for all x

$$\text{(2.3)}\quad c_1\min\{\varrho(x)^{\alpha_{\mathrm{m}}-\varepsilon}, \varrho(x)^{\alpha_{\mathrm{M}}+\varepsilon}\}\leqslant|x|\leqslant c_2\max\{\varrho(x)^{\alpha_{\mathrm{m}}-\varepsilon}, \varrho(x)^{\alpha_{\mathrm{M}}+\varepsilon}\};$$

if $\varrho\in C^k(\boldsymbol{R}_0^n)$ then for each $|\sigma|\leqslant k$ and $0<\varepsilon<\alpha_{\mathrm{m}}$ there is a constant c such that for all $x\neq 0$

$$\text{(2.4)}\qquad |D^\sigma\varrho(x)|\leqslant c\max\{\varrho(x)^{1-(\alpha_{\mathrm{m}}-\varepsilon)|\sigma|}, \varrho(x)^{1-(\alpha_{\mathrm{M}}+\varepsilon)|\sigma|}\}.$$

Next we need the notion of the ϱ-polar coordinates defined if $\varrho\in C^1(\boldsymbol{R}_0^n)$. Differentiating both sides of the equality $\varrho(A_t x) = t\varrho(x)$ with respect to t and setting $t = 1/\varrho(x)$ gives

$$\text{(2.5)}\qquad Px\cdot\operatorname{grad}\varrho(x) = 1\quad \text{for all } x\in\Sigma = \{y\colon \varrho(y) = 1\}.$$

Thus, Σ is a C^1-surface (i.e. $(n-1)$-dimensional manifold) in $\boldsymbol{R}^n$. It can be easily verified that the mapping

$$\text{(2.6)}\qquad \begin{cases}\boldsymbol{R}_0^n\to(0,\infty)\times\Sigma, & \Sigma = \{y\colon \varrho(y) = 1\},\\ x\to(t, x'), & t = \varrho(x),\ x' = A_{1/t}x\end{cases}$$

is a diffeomorphism and we use it to define the ϱ-polar coordinates. Furthermore, it is not hard to check that the Lebesgue measure on $\boldsymbol{R}^n$ admits the representation

$$\text{(2.7)}\qquad dx = t^{\nu-1}\,dt\,d\omega(x')$$

where $\nu = \operatorname{trace}(P)$ and if $d\sigma(x')$ is the surface Lebesgue measure on Σ, $N(x')$ the unit outer normal vector of Σ at x', then

$$\text{(2.8)}\qquad d\omega(x') = |\operatorname{grad}\varrho(x')|^{-1}\,d\sigma(x') = Px'\cdot N(x')\,d\sigma(x').$$

All these results are already stated and proved in the case of a particular homogeneous distance function e.g. in [28; pp. 1256–57], [23; p. 261] or [3; pp. 6–7]. The adaptation of the proofs given there to the case of a general ϱ is a routine matter and therefore omitted.

For the precise formulation of our main results we need the notion of the fractional derivative. Let $0<\delta<1$ and $m\in L^1_{\mathrm{loc}}(0,\infty)$, set

$$I_\omega^\delta(m)(t) = \begin{cases}\dfrac{1}{\Gamma(\delta)}\displaystyle\int_0^\omega (s-t)_+^{\delta-1}m(s)\,ds, & 0<t<\omega,\\ 0, & \text{otherwise}.\end{cases}$$

The *fractional derivative* is defined by

$$m^{(\delta)}(t) = -\lim_{\omega\to\infty}\frac{d}{dt}I_\omega^{1-\delta}(m)(t)$$

and of order $\gamma\geqslant 1$ by

$$m^{(\gamma)}(t) = \left(\frac{d}{dt}\right)^k m^{(\delta)}(t),\qquad k = [\gamma],\ \delta = \gamma-k.$$

Following Gasper and Trebels [12], we define for $\gamma>0$, $1\leqslant q\leqslant\infty$,

$$\begin{aligned} WBV_{q,\gamma} = \{m\in L^\infty\cap C(0,\infty)\colon\ & I_\omega^{1-\delta}(m)\in AC_{\mathrm{loc}} \text{ if } \delta = \gamma-[\gamma]>0,\\ & m^{(\delta)}, \ldots, m^{(\gamma-1)}\in AC_{\mathrm{loc}} \text{ if } \gamma\geqslant 1\\ & \text{and } \|m\|_{q,\gamma}<\infty\}\end{aligned}$$

where for $1\leqslant q<\infty$,

$$\|m\|_{q,\gamma} = \|m\|_\infty + \sup_{k\in\mathbf{Z}} \Big(\int_{2^k}^{2^{k+1}} |t^\gamma m^{(\gamma)}(t)|^q \frac{dt}{t} \Big)^{1/q},$$
$$\|m\|_{\infty,\gamma} = \|m\|_\infty + \|t^\gamma m^{(\gamma)}(t)\|_\infty$$

and AC_{loc} is the class of all locally absolutely continuous functions in $(0, \infty)$.

3. Our main results can now be stated as follows.

THEOREM 1. *Let $m \in WBV_{1,\gamma}$ and $\varrho \in C^\infty(\mathbf{R}_0^n)$ be an A_t-homogeneous distance function satisfying*

(3.1) $$\Big|\int_\Sigma e^{ixz}\, d\omega(z)\Big| \leqslant c(1+|x|)^{-\mu}$$

for some positive constants c and μ. Assume further

$$\gamma > \frac{\nu}{\alpha_m}\left|\frac{1}{p}-\frac{1}{2}\right|+\frac{1}{2} \quad \text{if } 1 < p < \frac{2(\mu+1)}{\mu+2} \text{ or } \frac{2(\mu+1)}{\mu} < p < \infty.$$

Then $m\circ\varrho \in M_p(\mathbf{R}^n)$ and $\|m\circ\varrho\|_{M_p} \leqslant c\|m\|_{1,\gamma}$ with c independent of m.

If $\varrho(x) = |x|$ we recover the result in [7].

COROLLARY 1. *Let $m \in WBV_{1,\gamma}$ and*

$$\gamma > n\left|\frac{1}{p}-\frac{1}{2}\right|+\frac{1}{2} \quad \text{if } 1 < p < \frac{2(n+1)}{n+3} \text{ or } \frac{2(n+1)}{n-1} < p < \infty.$$

Then $m(|\cdot|) \in M_p(\mathbf{R}^n)$ and $\|m(|\cdot|)\|_{M_p} \leqslant c\|m\|_{1,\gamma}$ with c independent of m.

A variant of Theorem 1 reads

COROLLARY 2. *We have*

$$\|m\circ\varrho\|_{M_p} \leqslant c\,\{\|m\|_\infty + \sup_{r>0} \int_r^{2r} t^{\gamma-1}|dm^{(\gamma-1)}(t)|\}$$

with γ, p and ϱ as in Theorem 1.

For the proof see e.g. [27; p. 109] and [5; p. 25].

Set $\mathscr{L}_k = \{\varphi \in C[0,\infty) \cap C^k(0,\infty)$: φ is nonnegative and strictly monotone increasing, $\varphi(0) = 0$ and $\varphi(\infty) = \infty$, $t^j|\varphi^{(j)}(t)| \leqslant ct\varphi'(t)$ if $j = 2, \ldots, k$ and $t > 0\}$ (cf. [30; p. 28] or [31; p. 15]). Possible choices for φ are:

$$t^{\varkappa_1}(t_0+t)^{\varkappa_2}, \quad t^{\varkappa_1}\log^{\varkappa_2}(1+t_0+t^{\varkappa_3}), \quad \log^{\varkappa_1}(1+t^{\varkappa_2})$$

where $t_0 \geqslant 0$ and $\varkappa_j > 0$. It is shown in Sec. 7 that

(3.2) $$\|m\circ\varphi\|_{q,\gamma} \leqslant c\|m\|_{q,\gamma}, \quad 1 < q < \infty,\ \gamma \geqslant 1.$$

Combining this with the embedding properties of WBV-spaces one obtains

COROLLARY 3. *Let $m \in WBV_{1,\gamma}$ and $\varphi \in \mathscr{L}_k$ $(\gamma \leqslant k \in \mathbf{N})$ and otherwise the hypotheses of Theorem 1 be valid; then $m\circ\varphi\circ\varrho \in M_p(\mathbf{R}^n)$ and $\|m\circ\varphi\circ\varrho\|_{M_p} \leqslant c\|m\|_{1,\gamma}$ with c independent of m.*

In view of applications simple sufficient criteria for (3.1) are desirable.

LEMMA A. *Let $\varrho \in C^\infty(\mathbf{R}_0^n)$ be a homogeneous distance function. If there is an integer k such that for all $x \in \Sigma = \{z\colon \varrho(z) = 1\}$, k of the principal curvatures do not vanish at x, then (3.1) holds for $\mu = k/2$.*

This criterion is a particular case of a more general result in [17]. An example of ϱ falling under the scope of this criterion is

$$\varrho(x) = (x_1^{\alpha_1} + \ldots + x_n^{\alpha_n})^\beta$$

where $\beta > 0$ and the α_j's are even and l of them equal 2. It can be elementarily verified that for this ϱ, $k = l-1$.

LEMMA B. *Let $\varrho \in C^\infty(\mathbf{R}_0^n)$ be a homogeneous distance function. Suppose there is an integer $N \geqslant 2$ such that for each $x \in \Sigma = \{y\colon \varrho(y) = 1\}$ there are some integers $1 \leqslant j < k \leqslant n$ and $2 \leqslant l \leqslant N$ such that*

$$\left(\left(a_j(x)\frac{\partial}{\partial x_j} + a_k(x)\frac{\partial}{\partial x_k}\right)^l \varrho\right)(x) \neq 0, \qquad a_j(x) = \frac{\partial\varrho}{\partial x_k},\ a_k(x) = -\frac{\partial\varrho}{\partial x_j}.$$

Then (3.1) is valid for $\mu = 1/N$.

This condition means that the order of contact of T_x, the tangent of Σ at x which is parallel to the x_k, x_j-plane, is at most $l-1$. With this in mind it is not hard to derive Lemma B from the van der Corput Lemma (cf. [32]; I, p. 197). Estimates of this kind are known and occur e.g. in [24]. An example of ϱ falling under the scope of Lemma B is given when ϱ equals a positive definite (i.e. elliptic) polynomial of degree k on $\mathbf{R}^n$ satisfying $\varrho(t^{\alpha_1}x_1, \ldots, t^{\alpha_n}x_n) = t\varrho(x)$ for some $\alpha_j > 0$ and all x. An easy calculation shows that one can take $N = k$. However, if $\varrho(x) = (x_1^{\alpha_1} + \ldots + x_n^{\alpha_n})^\beta$, $\beta > 0$ and the α_j's are even, one may choose $N = \min(\{\alpha_1, \ldots, \alpha_n\} \setminus \{\min_j \alpha_j\})$. This result seems to be improvable for that particular ϱ. We conjecture that $\mu = 1/\alpha_2 + \ldots + 1/\alpha_n$ is best possible if we assume $\alpha_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_n$.

The proof of Theorem 1 follows Stein's approach [26] via Littlewood–Paley functions; the most important one in our context is a quasi-radial generalization of the g_λ-function of Bonami and Clerc [2], which reads as follows:

$$g_\lambda(f)(x) = \Big(\int_0^\infty \int_{s/2}^\infty |(\Theta_t * W_{\lambda,s} * f)(x)|^2\, dt/t^2\, v(s)\, ds\Big)^{1/2}, \quad f \in \mathscr{S},$$

where Θ_t and $W_{\lambda,s}$ are defined by

$$\hat{\Theta_t}(\xi) = \theta(\varrho(\xi)/t), \quad \hat{W_{\lambda,s}}(\xi) = w_\lambda(\varrho(\xi)/s), \quad w_\lambda(t) = -t(1-t)_+^\lambda,$$
$$\theta \in C^\infty(\mathbf{R}) \text{ with } \operatorname{supp}(\theta) \subset [1/2, 1] \text{ and } 0 \leqslant \theta(t) \leqslant 1,\ \theta(t) \not\equiv 0,$$

$v\colon [0, \infty) \to [0, \infty)$ is measurable and there exists a constant $b \geqslant 1$ such that

$$t \leqslant V(t) = \int_0^t v(s)\,ds \leqslant bt, \qquad t > 0. \tag{3.3}$$

THEOREM 2. *Let ϱ satisfy the same assumptions as in Theorem 1 and λ, p be such that*

$$\lambda > \frac{\nu}{\alpha_m}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{2} \quad \text{and} \quad 1 < p < \frac{2(\mu+1)}{\mu+2} = p_\mu. \tag{3.4}$$

Then there exists a constant C such that

$$\|g_\lambda(f)\|_p \leqslant C\|f\|_p, \quad f \in \mathcal{S}.$$

Compared with the corresponding result of Bonami and Clerc [2] the L^p-estimate in our Theorem 2 is better since our method of proof is more subtle than the one in [2] based only on interpolation. Indeed, our method is a quasi-radial modification of Fefferman's [10] techniques and consists of appropriate restriction type theorems for the Fourier transform, a theorem on fractional integration (cf. [4]), an anisotropic decomposition theorem (cf. [10], [23]) etc. Finally, let us mention that the approach via Littlewood–Paley functions also gives Hörmander criteria for quasi-radial Fourier multipliers based on condition (0.1) as is shown in [8].

Remarks. (i) Theorem 1.4 of Peral and Torchinsky [21] may be stated as follows in our terminology:

LEMMA F. *Let $m \in WBV_{q,k}$, $1 \leqslant q \leqslant \infty$, $0 < p < \infty$, $k \in N$ with $k > \nu|1/p - 1/2| + 1/q$. Then $m \circ \varrho$ is a Fourier multiplier on $H^p(\mathbf{R}^n)$ $(0 < p < \infty)$ and its multiplier norm does not exceed $c\|m\|_{q,k}$. Here ϱ is defined by $t\varrho(x) = 1$ and $|A_t x| = 1$, $x \neq 0$, $t > 0$ and the matrix P with $A_t = t^P$ satisfies $Px \cdot x \geqslant |x|^2$.* (Note that this implies $\alpha_m \geqslant 1$.)

It is interesting to note that in our Theorem 1 quite general dilation matrices are admitted, the class of admissible distance functions ϱ homogeneous with respect to a diagonal dilation matrix A_t is quite rich. Moreover, in view of the interpolation and embedding properties of $WBV_{q,\gamma}$ (cf. [5], [12]) the differentiation order γ is considerably lower than that in Lemma F. Indeed, interpolating between our Theorem 1 and the trivial L^2-result yields that for $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, $\gamma > (\nu/\alpha_m)|1/p - 1/2| + 1/(2q)$, $m \in WBV_{q,\gamma}$ and ϱ as in Theorem 1 we have $m \circ \varrho \in M_p(\mathbf{R}^n)$.

(ii) In the isotropic case, i.e. $A_t = \operatorname{diag}(t^\alpha, \ldots, t^\alpha)$ $(\alpha > 0)$, the condition on γ in Theorem 1: $\gamma > n|1/p - 1/2| + 1/2$ provided $1 < p < 2(\mu+1)/(\mu+2)$ or $2(\mu+1)/\mu < p < \infty$, is best possible in the sense that

$$(1-\varrho(\cdot))_+^{\lambda-1} \notin M_p(\mathbf{R}^n) \quad \text{for } \lambda \leqslant n\left|\frac{1}{p}-\frac{1}{2}\right|+\frac{1}{2}.$$

The latter statement follows from the fact that

$$\mathcal{F}^{-1}\left[(1-\varrho(\cdot))_+^\lambda f\hat{\ }\right] \notin L^p(\mathbf{R}^n) \quad \text{for } \lambda \leqslant n\left|\frac{1}{p}-\frac{1}{2}\right|-\frac{1}{2}$$

if one chooses $f\hat{\ } \in C^\infty(\mathbf{R}^n)$ with a small support in the neighbourhood of a point $x_0 \in \Sigma_\varrho$ where the Gaussian curvature does not vanish. The proof is essentially the same as in Randol [22] and therefore omitted.

(iii) In the general case the condition posed on γ in Theorem 1 is not satisfactory, which is a consequence of some L^1-methods used in the proof of Theorem 2. The following criterion induces to conjecture that $\gamma > n|1/p - 1/2| + 1/2$ is sufficient.

LEMMA C. *If ϱ satisfies the same condition as in Theorem 1, $1 \leqslant p \leqslant 2(\mu+1)/(\mu+2)$ and $\gamma > n(1/p - 1/2) + 1/2$ then there is a constant c such that for appropriate m we have*

$$\|m \circ \varrho\|_{M_p} \leqslant c\left\{\|m\|_\infty + \int_0^\infty t^{\gamma-1}|dm^{(\gamma-1)}(t)|\right\} < \infty.$$

(iv) A simple consequence of [8] and the embedding properties is

LEMMA D. *Suppose $\varrho \in C^N(\mathbf{R}_0^n)$ $(N = [n/2]+1)$ is a homogeneous distance function, $1 < p < \infty$ and $\gamma > (n-1)|1/p - 1/2| + 1$. Then there exists a constant c such that for any $m \in WBV_{1,\gamma}$ we have $m \circ \varrho \in M_p(\mathbf{R}^n)$ and $\|m \circ \varrho\|_{M_p} \leqslant c\|m\|_{1,\gamma}$.*

Here more distance functions ϱ are admitted, (3.1) is not required but in the isotropic case (see Remark (ii)), the differentiation order γ is increased. We conjecture that the hypotheses of Theorem 1 can be weakened to $\varrho \in C^N(\mathbf{R}_0^n)$ and $\gamma > \max\{n|1/p - 1/2| + 1/2;\ 1\}$.

(v) An application of Theorem 1 gives in particular an improvement of the following result of Ashurov [1].

LEMMA E. *Let ϱ be a strictly convex polynomial (i.e. the Gaussian curvature of $\Sigma = \{z\colon \varrho(z) = 1\}$ never vanishes) with $\varrho(tx) = t^k\varrho(x)$ for some $k \in N$ and all $t > 0$, $x \in \mathbf{R}^n$. Then for any $1 < p < 4n/(3n+1)$ and $\lambda > n(1/p - 1/2) - 1/2$ we have $(1-\varrho(x))_+^\lambda \in M_p(\mathbf{R}^n)$.*

4. In the proof of Theorem 2 we make essentially use of the following theorems.

RESTRICTION THEOREM. *Let ϱ satisfy the condition of Theorem 1 and $1 \leqslant p \leqslant 2(\mu+1)/(\mu+2)$. Then for every $f \in L^p$, $f\hat{\ }$ can be restricted to Σ and there is a constant c independent of f such that*

$$\left(\int_\Sigma |f\hat{\ }(\xi')|^2\,d\omega(\xi')\right)^{1/2} \leqslant c\|f\|_p.$$

This theorem is proved in a more general setting in [15]. For the next theorem we set $N = [n/2+1]$.

FRACTIONAL INTEGRATION THEOREM. *Let* $\varrho \in C^N(\mathbf{R}_0^n)$ *be an* A_t*-homogeneous distance function; define the operator* I_α, $0 < \alpha < v$, *by* $\mathscr{F} I_\alpha f = \varrho^{-\alpha} \mathscr{F} f$ *where* $f \in \mathscr{S}$. *If* $1 < p < \infty$ *and* $1/q = 1/p - \alpha/v$ *then* I_α *extends to all of* L^p *and there is a constant* C *such that for all* $f \in L^p$

$$\|I_\alpha f\|_q \leqslant C \|f\|_p.$$

This theorem is a consequence of [4; Theorem 4.1] and [9; Lemma 4a].

The following Decomposition Theorem is a special instance of a general result of Rivière [23; Theorem (2.1)] (see also [19; Theorem 1]).

DECOMPOSITION THEOREM. *Let* $1 \leqslant p < \infty$, $f \in L^p$ *and* $\gamma > 0$ *be given. Then we can write* $f = f_1 + f_2$ *where*

(i) $\|f_j\|_p \leqslant C\|f\|_p$, $j = 1, 2$.

(ii) $\|f_2\|_\infty \leqslant C\gamma$.

(iii) $f_1 = \sum_{j=1}^{\infty} b_j$ *where*

(a) $\operatorname{supp}(b_j) \subset I_j = \{x\colon r(x-z_j) < 2^{k_j}\}$, $k_j \in \mathbf{Z}$; *here* r *is defined by* $BA_t x \cdot A_t x = 1$, $tr(x) = 1$ (see (2.1));

(b) $\|b_j\|_p^p \leqslant C\gamma^p |I_j|$, $j \in \mathbf{N}$;

(c) $\int b_j(x)\,dx = 0$, $j \in \mathbf{N}$.

(iv) $\sum_{j=1}^{\infty} |I_j| \leqslant C\gamma^{-p} \|f\|_p^p$.

Let $I_j^* = \{x\colon r(x-z_j) < a2^{k_j+1}\}$, a *defined by* $r(x+y) \leqslant a(r(x)+r(y))$. *Then*

(v) *each* $x \in \mathbf{R}^n$ *can belong to at most* N *of the* I_j^**'s.*

The constants C *and* N *are independent of* f *and* γ.

An easy consequence of Madych [19; Theorem 7 and its Corollary] is

LEMMA 1. *Let* $\eta \in C^{n+1}[0, \infty)$ *have compact support in* $[0, \infty)$, $\eta \not\equiv 0$, *and*

(4.1) $$\lim_{r \to 0} r^{k-(n+1)\delta} \eta^{(k)}(r) = 0 \quad \text{for some } \delta > \alpha_M \text{ and all } k = 0, \ldots, n+1.$$

Set $K_t\hat{\ }(\xi) = \eta(\varrho(\xi)/t)$ *where* $\varrho \in C^{n+1}(\mathbf{R}_0^n)$ *is an* A_t*-homogeneous distance function. Then for any* $1 < p < \infty$ *there are positive constants* c_1, c_2 *such that for every* $f \in L^p \cap L^2$ *we have*

$$c_2 \|f\|_p \leqslant \left\| \left(\int_0^\infty |K_t * f|^2 \frac{dt}{t} \right)^{1/2} \right\|_p \leqslant c_1 \|f\|_p.$$

For the next lemma define

$$h_\gamma(f)(x) = \left(\int_0^\infty \left| \int_{2t}^\infty s^{\gamma-1} m^{(\gamma)}(s) (\Lambda_{\gamma-1,s,t} * f)(x)\, ds \right|^2 \frac{dt}{t} \right)^{1/2}, \quad f \in \mathscr{S},$$

$$\Lambda_{\gamma,s,t}\hat{\ }(\xi) = \varrho(\xi)/t \cdot \theta(\varrho(\xi)/t)(1 - \varrho(\xi)/s)_+^\gamma.$$

LEMMA 2. *Let* $\varrho \in C^\infty(\mathbf{R}_0^n)$ *be an* A_t*-homogeneous distance function and* $m \in WBV_{1,\gamma}$ *be compactly supported away from the origin. Then for any* $1 < p < \infty$ *and* $\gamma > 1$ *we have*

$$\|h_\gamma(f)\|_p \leqslant C \|m\|_{1,\gamma} \|f\|_p$$

where C *is independent of* m *and* $f \in \mathscr{S}$.

LEMMA 3. (a) *The expression* $\|m\|_\infty + \sup_{r>0} (r^{-1} \int_0^r |t^\gamma m^{(\gamma)}(t)|^q\, dt)^{1/q}$ *is an equivalent norm on* $WBV_{q,\gamma}$, $1 \leqslant q < \infty$.

(b) *If* $m \in WBV_{q,\gamma}$ *has compact support in* $(0, \infty)$, *then*

$$m(r) = \frac{1}{\Gamma(\gamma)} \int_0^\infty (s-r)_+^{\gamma-1} m^{(\gamma)}(s)\, ds \quad \text{for almost every } r > 0.$$

(c) *Let* $G \in C^\infty(\mathbf{R})$ *be monotone increasing with* $G(t) = 0$ *if* $t \leqslant 1$ *and* $G(t) = 1$ *if* $t \geqslant 2$. *Set* $m_\varepsilon(t) = m(t)(G(t/\varepsilon) - G(t\varepsilon))$, $\varepsilon > 0$, $m \in WBV_{q,\gamma}$. *Then* $m_\varepsilon \in WBV_{q,\gamma}$ *and* $\sup_{\varepsilon>0} \|m_\varepsilon\|_{q,\gamma} \leqslant c\|m\|_{q,\gamma}$ *with* c *independent of* m.

The proof of Lemma 3 (a) is trivial. Lemma 3 (b) and (c) is proved in [12; pp. 250, 255–258].

5. Proof of Theorem 1. The space of all $\mathscr{S}$-functions whose Fourier transforms are compactly supported away from the origin is dense in L^p, $1 < p < \infty$. If f is such a function, then there is an $\varepsilon > 0$ such that

$$F\hat{\ }(\xi) = m(\varrho(\xi)) f\hat{\ }(\xi) = m_\varepsilon(\varrho(\xi)) f\hat{\ }(\xi)$$

where m_ε is defined in Lemma 3 (c). Hence, we may assume in the following that m is compactly supported away from the origin.

Let $K_t\hat{\ }(\xi) = \dfrac{\varrho(\xi)}{t} \theta\left(\dfrac{\varrho(\xi)}{t}\right)$ with θ as in Theorem 2. We shall show the existence of positive constants c_1 and c_2 such that uniformly for f and m,

$$\|F\|_p \leqslant c_1 \left\| \left(\int_0^\infty |(K_t * F)(x)|^2 \frac{dt}{t} \right)^{1/2} \right\|_p \leqslant c_2 \|m\|_{1,\gamma} \|f\|_p.$$

The first inequality holds on account of Lemma 1, the second one is the aim of the further proof. Introducing the ϱ-polar coordinates we can write

$$(K_t * F)(x) = c\int \frac{\varrho(\xi)}{t}\theta\left(\frac{\varrho(\xi)}{t}\right)m(\varrho(\xi))f\hat{\ }(\xi)\exp(i\xi\cdot x)\,d\xi$$

$$= c\int_0^\infty \frac{r}{t}\theta\left(\frac{r}{t}\right)m(r)\varphi(r, x)\,dr,$$

where $\varphi(r, x) = r^{\nu-1}\int_\Sigma f\hat{\ }(A_r\xi')\exp(iA_r\xi'\cdot x)\,d\omega(\xi')$.

By Lemma 3 (b) and the Fubini theorem,

$$(K_t * F)(x) = c\int_0^\infty \frac{r}{t}\theta\left(\frac{r}{t}\right)\varphi(r, x)\int_0^\infty (s-r)_+^{\gamma-1}m^{(\gamma)}(s)\,ds\,dr$$

$$= -\frac{c}{t}\int_0^{2t} s^\gamma m^{(\gamma)}(s)\int_0^\infty \theta\left(\frac{r}{t}\right)w_{\gamma-1}\left(\frac{r}{s}\right)\varphi(r, x)\,dr\,ds$$

$$+c\int_{2t}^\infty s^{\gamma-1}m^{(\gamma)}(s)\int_0^\infty \frac{r}{t}\theta\left(\frac{r}{t}\right)\left(1-\frac{r}{s}\right)_+^{\gamma-1}\varphi(r, x)\,dr\,ds$$

$$= T_1+T_2.$$

By Minkowski's inequality,

$$\left\|\left(\int_0^\infty |K_t * F|^2\frac{dt}{t}\right)^{1/2}\right\|_p \leqslant \sum_{j=1}^2 \left\|\left(\int_0^\infty |T_j|^2\frac{dt}{t}\right)^{1/2}\right\|_p. \tag{5.1}$$

Introducing the ϱ-polar coordinates we can write, using the notation of Theorem 2,

$$\int_0^\infty \theta\left(\frac{r}{t}\right)w_{\gamma-1}\left(\frac{r}{s}\right)\varphi(r, x)\,dr = (\Theta_t * W_{\gamma-1,s} * f)(x)$$

and the one of Lemma 2,

$$\int_0^\infty \frac{r}{t}\theta\left(\frac{r}{t}\right)\left(1-\frac{r}{s}\right)_+^{\gamma-1}\varphi(r, x)\,dr = (\Lambda_{\gamma-1,t,s} * f)(x).$$

Thus, it follows that

$$T_1 = -\frac{c}{t}\int_0^{2t} s^\gamma m^{(\gamma)}(s)(\Theta_t * W_{\gamma-1,s} * f)(x)\,ds,$$

$$T_2 = c\int_{2t}^\infty s^{\gamma-1}m^{(\gamma)}(s)(\Lambda_{\gamma-1,t,s} * f)(x)\,ds.$$

Since $1 < p < \infty$ and $\gamma > 1$ we deduce from Lemma 2

$$\left\|\left(\int_0^\infty |T_2|^2\frac{dt}{t}\right)^{1/2}\right\|_p = c\|h_\gamma(f)\|_p \leqslant c\|m\|_{1,\gamma}\|f\|_p.$$

In view of (5.1) it remains to show that

$$\left\|\left(\int_0^\infty |T_1|^2\frac{dt}{t}\right)^{1/2}\right\|_p \leqslant c\|m\|_{1,\gamma}\|f\|_p. \tag{5.2}$$

To this end set $v(s) = (|s^\gamma m^{(\gamma)}(s)|+\|m\|_{1,\gamma})/\|m\|_{1,\gamma}$ and observe that by Lemma 3 (a), $r \leqslant V(r) = \int_0^r v(s)\,ds \leqslant br$ with b independent of m and r. Hence,

$$|T_1| \leqslant \frac{bc\|m\|_{1,\gamma}}{t}\int_0^{2t}|(\Theta_t * W_{\gamma-1,s} * f)(x)|\,v(s)\,ds$$

and we obtain by Hölder's inequality

$$|T_1|^2 \leqslant \frac{c\|m\|_{1,\gamma}^2}{t^2}\int_0^{2t} v(s)\,ds\int_0^{2t}|(\Theta_t * W_{\gamma-1,s} * f)(x)|^2 v(s)\,ds$$

$$\leqslant \frac{c\|m\|_{1,\gamma}^2}{t}\int_0^{2t}|(\Theta_t * W_{\gamma-1,s} * f)(x)|^2 v(s)\,ds.$$

Changing the integration order gives

$$\left(\int_0^\infty |T_1|^2\frac{dt}{t}\right)^{1/2} \leqslant c\|m\|_{1,\gamma}\left(\int_0^\infty\int_{s/2}^\infty |(\Theta_t * W_{\gamma-1,s} * f)(x)|^2\frac{dt}{t^2}v(s)\,ds\right)^{1/2}$$

$$= c\|m\|_{1,\gamma}\,g_{\gamma-1}(f)(x)$$

where $g_{\gamma-1}(f)$ is the same as in Theorem 2. On account of this theorem we obtain (5.2) if $\gamma > \frac{\nu}{\alpha_m}\left(\frac{1}{p}-\frac{1}{2}\right)+\frac{1}{2}$, $1 < p < p_\mu$.

The duality, $M_p = M_{p'}$ if $1/p+1/p' = 1$, and the Riesz–Thorin interpolation theorem yield the whole assertion of Theorem 1.

6. Proof of Theorem 2. By the Marcinkiewicz interpolation theorem it is enough to prove

$$|\{g_\lambda(f) > \gamma > 0\}| \leqslant C\gamma^{-p}\|f\|_p^p, \quad C \text{ is independent of } \gamma \text{ and } f \in \mathscr{S}. \tag{6.1}$$

Write $f = f_1+f_2$ with f_1 and f_2 as in the Decomposition Theorem where we let r be A_t^*-homogeneous. The subadditivity of g_λ gives

$$|\{g_\lambda(f) > \gamma\}| \leqslant |\{g_\lambda(f_1) > \gamma/2\}|+|\{g_\lambda(f_2) > \gamma/2\}|.$$

Hence, the inequalities

$$|\{g_\lambda(f_i) > \gamma\}| \leqslant C\gamma^{-p}\|f\|_p^p, \quad i = 1, 2, \tag{6.2}$$

will imply (6.1). By the Chebyshev inequality,

$$|\{g_\lambda(f_2) > \gamma\}| \leqslant \gamma^{-2}\|g_\lambda(f_2)\|_2^2 \leqslant C\gamma^{-p}\|f\|_p^p,$$

provided we can show that

(6.3) $$\|g_\lambda(f_2)\|_2^2 \leqslant C\gamma^{2-p}\|f\|_p^p.$$

Changing the order of integration and applying Plancherel's theorem we obtain

$$\|g_\lambda(f_2)\|_2^2 = \int_0^\infty \int_{s/2}^\infty \|\Theta_t * W_{\lambda,s} * f_2\|_2^2 \frac{dt}{t^2} v(s)\,ds$$
$$\leqslant 2\|\theta\|_\infty^2 \int |f_2\hat{}(\xi)|^2 \left\{\int_0^\infty \left|w_\lambda\left(\frac{\varrho(\xi)}{s}\right)\right|^2 \frac{v(s)\,ds}{s}\right\} d\xi.$$

Using (3.3) we derive

$$\{\ldots\} = \int_0^\infty \left(\frac{\varrho(\xi)}{s}\right)^2 \left(1-\frac{\varrho(\xi)}{s}\right)_+^{2\lambda} \frac{v(s)\,ds}{s} \leqslant C_\gamma < \infty.$$

Thus, Plancherel's theorem and the Decomposition Theorem imply

$$\|g_\lambda(f_2)\|_2^2 \leqslant C\|f_2\hat{}\|_2^2 = C\int |f_2(x)|^2\,dx \leqslant C\gamma^{2-p}\|f\|_p^p,$$

which proves (6.3). The proof of (6.2) for $i = 1$ is based on a decomposition of f_1 and $\Theta_t * W_{\lambda,s}$. To this end take an even $\varphi \in C^\infty(\boldsymbol{R})$ with $\operatorname{supp}(\varphi) \subset \subset [-\frac{1}{2}, \frac{1}{2}]$ and $\varphi(t) = 1$ if $|t| \leqslant \frac{1}{4}$ and set

$$\varphi_k(t) = \varphi(2^{\varkappa(1-\delta)k} t), \qquad \Phi_{k,s}\hat{}(\xi) = \varphi_k\left(1-\frac{\varrho(\xi)}{s}\right)$$

where $0 < \delta < 1$ and $\varkappa > 0$ will be determined later. Observe that $\Phi_{k,s} \in \mathscr{S}$ and

(6.4) $$\operatorname{supp}(\Phi_{k,s}\hat{}) \subset \{\xi\colon |1-\varrho(\xi)/s| \leqslant \tfrac{1}{2}2^{-\varkappa(1-\delta)k}\}.$$

The decomposition of $\Theta_t * W_{\lambda,s}$ reads:

(6.5) $$(\Theta_t * W_{\lambda,s})\hat{}(\xi) = (\Theta_t * U_{\lambda,k,s} * \Phi_{k,s})\hat{}(\xi) + \psi_{\lambda,k,s,t}\hat{}(\xi)$$

where $\psi_{\lambda,k,s,t}\hat{}(\xi) = \theta\left(\frac{\varrho(\xi)}{t}\right) w_\lambda\left(\frac{\varrho(\xi)}{s}\right)\{1-\Phi_{k,s}\hat{}(\xi)\}$ and

$$U_{\lambda,k,s}\hat{}(\xi) = u_{\lambda,k}\left(\frac{\varrho(\xi)}{s}\right), \qquad u_{\lambda,k}(s) = \begin{cases} w_\lambda(s) & \text{if } |s-1| \leqslant \frac{1}{2}2^{-\varkappa(1-\delta)k}, \\ 0 & \text{otherwise.}\end{cases}$$

By the Decomposition Theorem, $f_1 = \sum_{j\in N} b_j$. We collect the b_j's according to the size of their supports $I_1, I_2, \ldots$, which are r-balls with radius $\operatorname{rad}(I_j) = 2^k$ for some $k = k(j) \in \boldsymbol{Z}$ depending on the ball. Set

$$Q_k = \{j \in N\colon \operatorname{rad}(I_j) = 2^k\}, \qquad \beta_k = \sum_{j\in Q_k} b_j;$$

clearly $f_1 = \sum_{k\in\boldsymbol{Z}} \beta_k$ and we obtain with the aid of (6.5)

$$\Theta_t * W_{\lambda,s} * f_1 = \sum_{k\in\boldsymbol{Z}} \Theta_t * U_{\lambda,k_s,s} * \Phi_{k_s,s} * \beta_k + \sum_{k\in\boldsymbol{Z}} \psi_{\lambda,k_s,s,t} * \beta_k,$$

where we have used the abbreviations:

$$k_s = (k-[\lg 1/s])_+, \quad \lg \equiv \log_2, \quad k \in \boldsymbol{Z},\ s > 0.$$

Note that there are $c_1, c_2 > 0$ such that

$$c_1 s2^k \leqslant 2^{k_s} \leqslant c_2 s2^k \quad \text{if } k \geqslant [\lg 1/s],\ k \in \boldsymbol{Z},\ s > 0.$$

Using this decomposition and Minkowski's integral inequality we obtain

$$g_\lambda(f_1)(x) \leqslant \left(\int_0^\infty \int_{s/2}^\infty \left|\left(\Theta_t * \left(\sum_{k\in\boldsymbol{Z}} U_{\lambda,k_s,s} * \Phi_{k_s,s} * \beta_k\right)\right)(x)\right|^2 \frac{dt}{t^2} v(s)\,ds\right)^{1/2}$$
$$+ \left(\int_0^\infty \int_{s/2}^\infty \left|\sum_{k\in\boldsymbol{Z}} (\psi_{\lambda,k_s,s,t} * \beta_k)(x)\right|^2 \frac{dt}{t^2} v(s)\,ds\right)^{1/2}$$
$$= g_{\lambda,1}(x) + g_{\lambda,2}(x).$$

The further proof is divided into three parts. In Part I we are concerned with the estimate of $g_{\lambda,1}$ and in Part II with that of $g_{\lambda,2}$. It turns out that Part I is essential and in particular requires the assumption (3.4) on p and λ. Part II seems to be only technical and needs the assumptions $1 < p < \infty$, $\lambda \geqslant 0$, $\varrho \in C^\infty(\boldsymbol{R}_0^n)$.

I. The assertion reads:

(6.6) $$\|g_{\lambda,1}\|_2^2 \leqslant C\gamma^{2-p}\|f\|_p^p.$$

This gives by the Chebyshev inequality

$$|\{x\colon g_{\lambda,1}(x) > \gamma\}| \leqslant C\gamma^{-p}\|f\|_p^p,$$

which is the desired estimate. Now change the order of integration and apply Plancherel's theorem to obtain

$$\|g_{\lambda,1}\|_2^2 = C\int_0^\infty \int_{s/2}^\infty \int \left|\theta\left(\frac{\varrho(\xi)}{t}\right)\right|^2 \left|\left\{\sum_{k\in\boldsymbol{Z}} U_{\lambda,k_s,s} * \Phi_{k_s,s} * \beta_k\right\}\hat{}(\xi)\right|^2 d\xi \frac{dt}{t^2} v(s)\,ds.$$

Since $\|\theta\|_\infty \leqslant 1$ it follows by performing the t-integration and using Plancherel's theorem again that $\|g_{\lambda,1}\|_2^2 \leqslant C\|g^*\|_2^2$ where

$$g^*(x) = \left(\int_0^\infty \left|\sum_{k\in\boldsymbol{Z}} (U_{\lambda,k_s,s} * \Phi_{k_s,s} * \beta_k)(x)\right|^2 \frac{v(s)\,ds}{s}\right)^{1/2}.$$

Assertion (6.6) is now a consequence of $\|g^*\|_2^2 \leqslant C\gamma^{2-p}\|f\|_p^p$.

An application of Minkowski's inequality yields

$$g^*(x) \leqslant \Big(\int_0^\infty \Big|\sum_{k>[\lg 1/s]} \big(U_{\lambda,k_s,s} * \big(\sum_{j\in Q_k} (\Phi_{k_s,s} * b_j)\chi_{I_j^*}\big)\big)(x)\Big|^2 \frac{v(s)\,ds}{s}\Big)^{1/2}$$

$$+\Big(\int_0^\infty \Big|\sum_{k>[\lg 1/s]} \big(U_{\lambda,k_s,s} * \big(\sum_{j\in Q_k} (\Phi_{k_s,s} * b_j)\chi_{\mathbf{R}^n\setminus I_j^*}\big)\big)(x)\Big|^2 \frac{v(s)\,ds}{s}\Big)^{1/2}$$

$$+\Big(\int_0^\infty \Big|\big(U_{\lambda,0,s} * \big(\sum_{k\leqslant[\lg 1/s]}\sum_{j\in Q_k} (\Phi_{0,s} * b_j)\chi_{I_j^*}\big)\big)(x)\Big|^2 \frac{v(s)\,ds}{s}\Big)^{1/2}$$

$$+\Big(\int_0^\infty \Big|\big(U_{\lambda,0,s} * \big(\sum_{k\leqslant[\lg 1/s]}\sum_{j\in Q_k} (\Phi_{0,s} * b_j)\chi_{\mathbf{R}^n\setminus I_j^*}\big)\big)(x)\Big|^2 \frac{v(s)\,ds}{s}\Big)^{1/2}$$

$$= \sum_{i=1}^{4} g_i^*(x).$$

Hence, it is sufficient to show that $\|g_i^*\|_2^2 \leqslant C\gamma^{2-p}\|f\|_p^p$, $i = 1, \ldots, 4$.

g_1^* is estimated with the aid of the Restriction Theorem and requires the assumptions $\lambda > (\nu/\varkappa)(1/p-1/2)-1/2$, $1 \leqslant p < p_\mu$, $\varkappa > 0$.

g_2^* is estimated by L^1-arguments. Here we essentially utilize the geometrical facts stated in the Decomposition Theorem and have to choose $\varkappa = \alpha_*$ where α_* is any fixed positive number with $\alpha_* < \alpha_m$.

g_3^* is estimated by the Fractional Integration Theorem for which $1 < p < \infty$ is necessary.

g_4^* is estimated by arguments analogous to the ones used for g_2^*, which imply the restriction $1 < p < \infty$. In what follows we estimate the g_i^*'s separately.

I(a). *Estimate of* g_1^*. Hölder's inequality gives for any $\varepsilon > 0$

$$\Big|\sum_{k>[\lg 1/s]} (U_{\lambda,k_s,s} * h)(x)\Big|^2 \leqslant C_1 \sum_{k>[\lg 1/s]} 2^{2\varepsilon k_s}|(U_{\lambda,k_s,s} * h)(x)|^2, \qquad h \in L^2.$$

By the definition of $U_{\lambda,k_s,s}$ and Plancherel's theorem

$$\|U_{\lambda,k_s,s} * h\|_2 = C\Big\|u_{\lambda,k_s}\Big(\frac{\varrho(\cdot)}{s}\Big)h\hat{}\Big\|_2 \leqslant C\|u_{\lambda,k_s}\|_\infty \|h\hat{}\|_2 \leqslant C2^{-\varkappa\lambda(1-\delta)k_s}\|h\|_2.$$

Hence,

$$\Big\|\sum_{k>[\lg 1/s]} U_{\lambda,k_s,s} * h\Big\|_2^2 \leqslant C \sum_{k>[\lg 1/s]} 2^{2\varepsilon k_s - 2\varkappa\lambda(1-\delta)k_s}\|h\|_2^2. \tag{6.7}$$

Set

$$h = \sum_{j\in Q_k} (\Phi_{k_s,s} * b_j)\chi_{I_j^*},$$

use the Decomposition Theorem and Hölder's inequality to obtain

$$|h(x)|^2 \leqslant N \sum_{j\in Q_k} |((\Phi_{k_s,s} * b_j)\chi_{I_j^*})(x)|^2.$$

Invoking the definition of g_1^* and applying the above inequalities we obtain

$$\|g_1^*\|_2^2 = \int_0^\infty \Big\|\sum_{k>[\lg 1/s]} U_{\lambda,k_s,s} * h\Big\|_2^2 \frac{v(s)\,ds}{s} \tag{6.8}$$

$$\leqslant C \sum_{k\in \mathbf{Z}}\sum_{j\in Q_k} \int_{2^{-k}}^\infty 2^{2\varepsilon k_s - 2\varkappa\lambda(1-\delta)k_s}\|\Phi_{k_s,s} * b_j\|_2^2 \frac{v(s)\,ds}{s}.$$

By Plancherel's theorem we obtain introducing the ϱ-polar coordinates

$$\|\Phi_{k_s,s} * b_j\|_2^2 = C\int_0^\infty |\varphi_{k_s}(1-r/s)|^2 \Big\{\int_\Sigma |b_j\hat{}(A_r\xi')|^2\,d\omega(\xi')\Big\} r^{\nu-1}\,dr.$$

Since $[r^{-\nu} b_j(A_{1/r}^* \cdot)]\hat{}(\xi) = b_j\hat{}(A_r\xi)$, we derive by the Restriction Theorem

$$\{\ldots\} = \int_\Sigma |b_j\hat{}(A_r\xi')|^2\,d\omega(\xi') \leqslant C\|r^{-\nu} b_j(A_{1/r}^*\cdot)\|_p^2, \qquad 1 \leqslant p < p_\mu,$$

and by the Decomposition Theorem,

$$\{\ldots\} \leqslant Cr^{-2\nu+2\nu/p}\|b_j\|_p^2 \leqslant C\gamma^2 r^{-2\nu+2\nu/p}2^{2k\nu/p}.$$

Collecting the estimates gives

$$\|\Phi_{k_s,s} * b_j\|_2^2 \leqslant C\gamma^2 2^{2k\nu/p}\int_0^\infty |\varphi_{k_s}(1-r/s)|^2 r^{-\nu-1+2\nu/p}\,dr$$

$$\leqslant C\gamma^2 2^{2k\nu/p - \varkappa(1-\delta)k_s} s^{-\nu+2\nu/p}$$

where we have used (6.4) for the last inequality. Putting this estimate into (6.8) gives

$$\|g_1^*\|_2^2 \leqslant C\gamma^2 \sum_{k\in\mathbf{Z}}\sum_{j\in Q_k} 2^{k\nu}\int_{2^{-k}}^\infty (s2^k)^{2\varepsilon-2\varkappa\lambda(1-\delta)-\varkappa(1-\delta)+2\nu/p-\nu}\frac{v(s)\,ds}{s}. \tag{6.9}$$

By (3.3), the integral on the right hand side of (6.9) exists if $\lambda > (\nu/\varkappa)\times$ $\times(1/p-1/2)-1/2$ and $\varepsilon, \delta > 0$ are sufficiently small. This integral is then bounded uniformly for $k \in \mathbf{Z}$ and using the Decomposition Theorem we arrive at

$$\|g_1^*\|_2^2 \leqslant C\gamma^2 \sum_{k\in\mathbf{Z}}\sum_{j\in Q_k} |I_j| \leqslant C\gamma^{2-p}\|f\|_p^p.$$

I(b). *Estimate of* g_2^*. Changing the order of integration and setting

$$h = \sum_{j\in Q_k} (\Phi_{k_s,s} * b_j)\chi_{\mathbf{R}^n\setminus I_j^*}$$

gives

$$\|g_2^*\|_2^2 = \int_0^\infty \Big\| \sum_{k>[\lg 1/s]} U_{\lambda,k_s,s} * h\Big\|_2^2 \frac{v(s)\,ds}{s}.$$

If $\varepsilon = \varkappa\lambda(1-\delta)$ then by (6.7)

$$\|g_2^*\|_2^2 \leqslant C \int_0^\infty \sum_{k>[\lg 1/s]} \|h\|_2^2 \frac{v(s)\,ds}{s}.$$

In Part III it is shown that

(6.10) $$\|h\|_\infty = \Big\| \sum_{j\in Q_k} (\Phi_{k_s,s} * b_j)\, \chi_{\boldsymbol{R}^n\setminus I_j^*}\Big\|_\infty \leqslant C\gamma,$$

where C is independent of k, s and γ. Interchanging the sums and integrals we obtain

$$\|g_2^*\|_2^2 \leqslant C\gamma \sum_{k\in \boldsymbol{Z}} \sum_{j\in Q_k} \int |b_j(y)| \Big\{ \int_{\boldsymbol{R}^n\setminus I_j^*} \int_{2^{-k}}^\infty |\Phi_{k_s,s}(x-y)| \frac{v(s)\,ds}{s}\, dx\Big\}\, dy.$$

Suppose that

(6.11) $$\{\ldots\} \leqslant C, \quad \text{uniformly for } k\in \boldsymbol{Z}.$$

Then by $\int |b_j(y)|\,dy \leqslant C\gamma |I_j|$, which is a consequence of the Decomposition Theorem and Hölder's inequality, we derive

$$\|g_2^*\|_2^2 \leqslant C\gamma^2 \sum_{k\in\boldsymbol{Z}} \sum_{j\in Q_k} |I_j| \leqslant C\gamma^{2-p} \|f\|_p^p,$$

which is the desired inequality. To prove (6.11) we need

(6.12) $$|\Phi_{k,s}(x)| \leqslant C_m s^\nu \big(1+s2^{-\varkappa(1-\delta)k/\alpha_*} r(x)\big)^{-m\alpha_*}, \quad m\in\boldsymbol{N}, \quad 0<\alpha_*<\alpha_m,$$

where C_m is independent of k, s, x. The above estimate is proved in Part III. Furthermore we need

$$\boldsymbol{R}^n\setminus I_j^* \subset \{x\colon r(x-y)\geqslant 2^k\} \quad \text{for each } j\in Q_k \text{ and } y\in I_j.$$

This and (6.12) imply (6.11) as is shown below by taking $\varkappa=\alpha_*$ and $m\geqslant \nu/(\delta\alpha_*)$:

$$\{\ldots\} \leqslant C_m \int_{r(x-y)\geqslant 2^k} r(x-y)^{-m\alpha_*}\,dx \int_{2^{-k}}^\infty 2^{\varkappa(1-\delta)mk} s^{\nu-m\alpha_*} \frac{v(s)\,ds}{s}$$
$$\leqslant C_m 2^{-km\alpha_*+\nu k} 2^{\varkappa(1-\delta)mk} \int_{2^{-k}}^\infty s^{\varkappa(1-\delta)m+\nu-m\alpha_*} \frac{v(s)\,ds}{s} \leqslant C < \infty$$

uniformly for $k\in\boldsymbol{Z}$. We have to take $\varkappa=\alpha_*$ since we have to admit every $\delta>0$. This completes the proof of I(b).

I(c). *Estimate of* g_3^*. Let $h = \sum_{k\leqslant[\lg 1/s]} \sum_{j\in Q_k} (\Phi_{0,s} * b_j)\chi_{I_j^*}$; then, by the Decomposition Theorem and Hölder's inequality,

$$|h(x)|^2 \leqslant N \sum_{k\leqslant[\lg 1/s]} \sum_{j\in Q_k} \big|\big((\Phi_{0,s} * b_j)\chi_{I_j^*}\big)(x)\big|^2.$$

Interchanging the sums and integrals and using Plancherel's theorem yields

$$\|g_3^*\|_2^2 \leqslant C \int_0^\infty \|h\|_2^2 \frac{v(s)\,ds}{s}$$
$$\leqslant C \sum_{k\in\boldsymbol{Z}} \sum_{j\in Q_k} \int |\hat{b_j}(\xi)|^2 \Big\{\int_0^{2^{-k+1}} |\Phi_{0,s}(\xi)|^2 \frac{v(s)\,ds}{s}\Big\}\, d\xi.$$

Since $\varphi(1-\varrho(\xi)/s)=0$ unless $\frac{2}{3}\varrho(\xi)\leqslant s\leqslant 2\varrho(\xi)$, we deduce that

$$\int_0^{2^{-k+1}} \Big|\varphi\Big(1-\frac{\varrho(\xi)}{s}\Big)\Big|^2 \frac{v(s)\,ds}{s} \leqslant \begin{cases} 0 & \text{if } 2^k\varrho(\xi)\geqslant 3,\\ C\int_{2/3\cdot\varrho(\xi)}^{2\varrho(\xi)} dV(s)/V(s) \leqslant C & \text{otherwise}\end{cases}$$
$$\leqslant C\big(2^k\varrho(\xi)\big)^{-2\mu},$$

where $\mu = \nu(1/p-1/2)$ and C is independent of k and ξ. The Fractional Integration Theorem and Plancherel's theorem imply

$$\int |\varrho(\xi)^{-\mu}\hat{b_j}(\xi)|^2\,d\xi = C\|I_\mu(b_j)\|_2^2 \leqslant C\|b_j\|_p^2, \quad p>1.$$

Collecting the estimates and applying the Decomposition Theorem gives

$$\|g_3^*\|_2^2 \leqslant C\gamma^2 \sum_{k\in\boldsymbol{Z}} \sum_{j\in Q_k} 2^{-2\mu k}|I_j|^{2/p} \leqslant C\gamma^2 \sum_{j\in\boldsymbol{N}} |I_j| \leqslant C\gamma^{2-p}\|f\|_p^p.$$

I(d). *Estimate of* g_4^*. Plancherel's theorem and an interchange of sums and integrals imply

(6.13) $$\|g_4^*\|_2^2 \leqslant C\gamma \sum_{k\in\boldsymbol{Z}} \sum_{j\in Q_k} \int_0^{2^{-k+1}} \|(\Phi_{0,s} * b_j)\chi_{\boldsymbol{R}^n\setminus I_j^*}\|_1 \frac{v(s)\,ds}{s},$$

provided that

$$\Big\| \sum_{k\leqslant[\lg 1/s]} \sum_{j\in Q_k} (\Phi_{0,s} * b_j)\chi_{\boldsymbol{R}^n\setminus I_j^*}\Big\|_\infty \leqslant C\gamma,$$

which is proved in Part III. By the Decomposition Theorem,

$$(\Phi_{0,s} * b_j)(x) = \int \big(\Phi_{0,s}(x-y) - \Phi_{0,s}(x-z_j)\big)\, b_j(y)\,dy.$$

Hence,

(6.14) $$\|(\Phi_{0,s} * b_j)\chi_{\boldsymbol{R}^n\setminus I_j^*}\|_1 \leqslant \int |b_j(y)| \int_{\boldsymbol{R}^n\setminus I_j^*} |\Phi_{0,s}(x-y) - \Phi_{0,s}(x-z_j)|\,dx\,dy.$$

If we show that

$$(6.15)\qquad \Big\{\int_{\mathbf{R}^n\setminus I_j^*}\int_0^\infty |\Phi_{0,s}(x-y)-\Phi_{0,s}(x-z_j)|\frac{v(s)\,ds}{s}\,dx\Big\}\leqslant C,\qquad y\in I_j,$$

uniformly for y and j, then we can conclude from (6.13) and (6.14) that

$$\|g_4^*\|_2^2 \leqslant C\gamma\sum_{k\in\mathbf{Z}}\sum_{j\in Q_k}\int |b_j(y)|\,dy \leqslant C\gamma^{2-p}\|f\|_p^p,$$

which is the desired inequality. We prove (6.15). By $\Phi_{0,s}(x)=s^\nu\Phi_{0,1}(A_s^*x)$ and the mean value theorem,

$$|\Phi_{0,s}(x-y)-\Phi_{0,s}(x-z_j)|\leqslant s^\nu|A_s^*(z_j-y)|\,\big|(\operatorname{grad}\Phi_{0,1})\big(A_s^*(x-z_j)-qA_s^*(y-z_j)\big)\big|,$$

where $0<q<1$. Observe that for $x\notin I_j^*$ and $y\in I_j$

$$r\big(A_s^*(x-z_j)+qA_s^*(z_j-y)\big)\geqslant\frac{1}{a}r\big(A_s^*(x-z_j)\big)-r\big(qA_s^*(y-z_j)\big)\geqslant\frac{s}{2a}r(x-z_j);$$

$\Phi_{0,1}\in\mathscr{S}$ gives $|(\operatorname{grad}\Phi_{0,1})(x)|\leqslant C(1+r(x))^{-2\nu}$. Hence,

$$|\Phi_{0,s}(x-y)-\Phi_{0,s}(x-z_j)|\leqslant Cs^\nu|A_s^*(z_j-y)|\,(1+sr(x-z_j))^{-2\nu}.$$

Since $\mathbf{R}^n\setminus I_j^*\subset\{\xi\colon r(\xi-z_j)\geqslant 2ar(y-z_j)\}$ for any fixed $y\in I_j$, we infer that the expression $\{\ldots\}$ in (6.15) satisfies

(6.16)

$$\{\ldots\}\leqslant C\int_{r(x-z_j)\geqslant 2ar(y-z_j)}\Big[\int_0^\infty s^\nu|A_s^*(z_j-y)|\,(1+sr(x-z_j))^{-2\nu}\frac{v(s)\,ds}{s}\Big]dx.$$

By (2.1) and (3.3),

$$|A_s^*(z_j-y)|^2\leqslant CBA_{V(s)}^*(z_j-y)\cdot A_{V(s)}^*(z_j-y)\leqslant\tilde{C}\,|A_{V(s)}^*(z_j-y)|^2.$$

This gives that the expression $[\ldots]$ in (6.16) satisfies

$$(6.17)\qquad [\ldots]\leqslant C\int_0^\infty (V(s))^\nu|A_{V(s)}^*(z_j-y)|\,(1+V(s)r(x-z_j))^{-2\nu}\frac{dV(s)}{V(s)}$$
$$=Cr(x-z_j)^{-\nu}\int_0^\infty t^\nu|A_{t/r(x-z_j)}^*(z_j-y)|\,(1+t)^{-2\nu}\frac{dt}{t}.$$

Since $A_{1/r(z_j-y)}(z_j-y)\in\Sigma_r=\{\xi\colon r(\xi)=1\}$ and Σ_r is compact it follows that

$$|A_{t/r(x-z_j)}^*(z_j-y)|\leqslant|A_t^*|\,|A_{r(z_j-y)/r(x-z_j)}^*|\,|A_{1/r(z_j-y)}^*(z_j-y)|$$
$$\leqslant C|A_t^*|\,|A_{r(z_j-y)/r(x-z_j)}^*|.$$

Using this and $|t^{P^*}|=|A_t^*|\leqslant Ct^{\alpha_*}$, $0<t<c$, we obtain from (6.17)

$$[\ldots]\leqslant C|A_{r(z_j-y)/r(x-z_j)}^*|\,r(x-z_j)^{-\nu},$$

and from this and (6.16),

$$\{\ldots\}\leqslant C\int_{r(x-z_j)/r(z_j-y)\geqslant 2a}|A_{r(z_j-y)/r(x-z_j)}^*|\,r(x-z_j)^{-\nu}\,dx$$
$$=C\int_{r(u)\geqslant 2a}|A_{1/r(u)}^*|\,r(u)^{-\nu}\,du\leqslant C.$$

This completes the proof of I(d) and also of Part I.

II. The assertion reads:

$$|\{x\colon g_{\lambda,2}(x)>\gamma\}|\leqslant C\gamma^{-p}\|f\|_p^p,\qquad C\text{ independent of }\gamma\text{ and }f.$$

Note that by the Decomposition Theorem we have for $\Omega^*=\bigcup_{j\in\mathbf{N}}I_j^*$,

$$|\Omega^*|\leqslant\sum_{j\in\mathbf{N}}|I_j^*|=(2a)^\nu\sum_{j\in\mathbf{N}}|I_j|\leqslant C\gamma^{-p}\|f\|_p^p,$$

so that the above assertion can be reduced to

$$|\{x\in\mathbf{R}^n\setminus\Omega^*\colon g_{\lambda,2}(x)>\gamma\}|\leqslant C\gamma^{-p}\|f\|_p^p.$$

But by Chebyshev's inequality, this is a consequence of

$$\|g_{\lambda,2}\chi_{\mathbf{R}^n\setminus\Omega^*}\|_2^2\leqslant C\gamma^{2-p}\|f\|_p^p,\qquad C\text{ independent of }\gamma\text{ and }f.$$

By the Minkowski inequality

$$g_{\lambda,2}(x)\leqslant\Big(\int_0^\infty v(s)\int_{s/2}^\infty\Big|\sum_{k>[\lg 1/s]}\sum_{j\in Q_k}(\psi_{\lambda,k_s,s,t}*b_j)(x)\Big|^2\frac{dt}{t^2}\,ds\Big)^{1/2}$$
$$+\Big(\int_0^\infty v(s)\int_{s/2}^\infty\Big|\sum_{k\leqslant[\lg 1/s]}\sum_{j\in Q_k}(\psi_{\lambda,k_s,s,t}*b_j)(x)\Big|^2\frac{dt}{t^2}\,ds\Big)^{1/2}$$
$$=g_3(x)+g_4(x).$$

Hence, we only need to prove that

$$\|g_i\chi_{\mathbf{R}^n\setminus\Omega^*}\|_2^2\leqslant C\gamma^{2-p}\|f\|_p^p,\qquad i=3,4.$$

But this follows for g_3 by the techniques of I(b) and for g_4 by the methods of I(d).

This completes Part II.

III(a). *Proof of* $|\Phi_{k,s}(x)|\leqslant C_m(1+s2^{-\varkappa(1-\delta)k/\alpha_*}r(x))^{-m\alpha_*}$, $m\in\mathbf{N}$. By $\Phi_{k,s}(x)=s^\nu\Phi_{k,1}(A_s^*x)$, (6.4) and (2.3) the above assertion reduces to

$$|\Phi_{k,1}(x)|\leqslant C_m2^{\varkappa(1-\delta)m}|x|^{-m},\qquad m=0,1,\ldots,|x|\geqslant 1,$$

which for itself follows from (6.4) and

$$\left(\frac{\partial}{\partial\xi}\right)^{\sigma}\hat{\Phi_{k,1}}(\xi) = O(2^{\varkappa(1-\delta)k|\sigma|}),\quad \text{uniformly for } \xi.$$

The last estimate is an immediate consequence of the definition of $\hat{\Phi_{k,1}}$.

III(b). *Proof of* $\|\sum_{j\in Q_k}(\Phi_{k_s,s}*b_j)\chi_{\mathbf{R}^n\setminus I_j^*}\|_\infty \leqslant C\gamma$ *and*

$$\|\sum_{k\leqslant[\lg 1/s]}\sum_{j\in Q_k}(\Phi_{0,s}*b_j)\chi_{\mathbf{R}^n\setminus I_j^*}\|_\infty \leqslant C\gamma.$$

Setting

$$P_{k,s}(x) = C_m s^{\nu}(1+s2^{-(1-\delta)k}r(x))^{-m\alpha_*} \geqslant |\Phi_{k,s}(x)|$$

and observing that

$$\frac{1}{2a}r(x-z_j) \leqslant r(x-y) \leqslant 2ar(x-z_j),\quad x\notin I_j^*,\ y\in I_j,$$

gives

$$\sup_{y\in I_j} P_{k_s,s}(x-y) \leqslant c\inf_{y\in I_j} P_{k_s,s}(x-y) \leqslant c|I_j|^{-1}\int_{I_j} P_{k_s,s}(x-y)\,dy,\quad x\notin I_j^*.$$

For a fixed $x\in\mathbf{R}^n$ put $Q_{k,x} = \{j\in Q_k\colon x\notin I_j^*\}$; it follows that

$$\begin{aligned}\Sigma_{k,x} &= \Big|\sum_{j\in Q_k}((\Phi_{k_s,s}*b_j)\chi_{\mathbf{R}^n\setminus I_j^*})(x)\Big| \leqslant \sum_{j\in Q_{k,x}}|(\Phi_{k_s,s}*b_j)|\\ &\leqslant \sum_{j\in Q_{k,x}}\sup_{y\in I_j}P_{k_s,s}(x-y)\int|b_j(y)|\,dy.\end{aligned}$$

By the Decomposition Theorem we have, setting $U_{k,x} = \bigcup_{j\in Q_{k,x}} I_j$,

$$\begin{aligned}\Sigma_{k,x} &\leqslant C\gamma\sum_{j\in Q_{k,x}}\int_{I_j}P_{k_s,s}(x-y)\,dy \leqslant C\gamma\int_{U_{k,x}}P_{k_s,s}(x-y)\sum_{j\in Q_{k,x}}\chi_{I_j}(y)\,dy\\ &\leqslant CN\gamma\int_{r(x-y)\geqslant 2^k}P_{k_s,s}(x-y)\,dy \leqslant C_m' 2^{-(m\alpha_*\delta-\nu)k_s}.\end{aligned}$$

This proves the first inequality. The proof of the second one is quite analogous and therefore omitted.

7. Proof of Lemma 2. Setting

$$\Phi(r,x) = r^{\nu-1}\int_\Sigma \hat{f}(A_r\xi')\exp(iA_r\xi'\cdot x)\,d\omega(\xi'),\quad f\in\mathcal{S},$$

we can write using the ϱ-polar coordinates

$$\begin{aligned}H_\gamma(t,x) &= \int_{2t}^{\infty}s^{\gamma-1}m^{(\gamma)}(s)(A_{\gamma-1,t,s}*f)(x)\,ds\\ &= \int_{2t}^{\infty}m^{(\gamma)}(s)\int_0^\infty\frac{r}{t}\theta\left(\frac{r}{t}\right)(s-r)^{\gamma-1}\Phi(r,x)\,dr\,ds.\end{aligned}$$

It is useful to introduce the following notation:

$$b(r,s,t) = \left(\frac{r}{t}\right)^{1-l}\theta\left(\frac{r}{t}\right)(s-r)_+^{\gamma-1},\quad l\geqslant 1 \text{ arbitrary},$$

$$\psi(r,t,x) = \left(\frac{r}{t}\right)^{l}r^{\nu-1}\int_\Sigma \hat{f}(A_r\xi')\exp(iA_r\xi'\cdot x)\,d\omega(\xi').$$

Observe that $\operatorname{supp}(b(\cdot,s,t))\subset[t/2,t]$, and $r\to b(r,s,t)$ is infinitely differentiable if $s\geqslant 2t$. By our notation

$$H_\gamma(t,x) = \int_{2t}^{\infty}m^{(\gamma)}(s)\int_{t/2}^{t}b(r,s,t)\psi(r,t,x)\,dr\,ds. \tag{7.1}$$

Integrating N times by parts we obtain

$$\begin{aligned}&\int_{t/2}^{t}b(r,s,t)\psi(r,t,x)\,dr\\ &\quad= \frac{(-1)^N}{l^N N!}\int_{t/2}^{t}\frac{\partial}{\partial r}\left(\frac{\partial}{r^{l-1}\partial r}\right)^N b(r,s,t)\int_0^r(r^l-u^l)^N\psi(u,t,x)\,du\,dr.\end{aligned}$$

Setting $\hat{K_t}(\xi) = (\varrho(\xi)/t)^l(1-(\varrho(\xi)/t)^l)_+^N$ we can write

$$\int_0^r(r^l-u^l)^N\psi(u,t,x)\,du = \frac{r^{l(N+1)}}{t^l}(K_r*f)(x).$$

Putting this into (7.1) gives

$$H_\gamma(t,x) = \frac{c}{t}\int_{t/2}^{t}(K_r*f)(x)\int_{2t}^{\infty}m^{(\gamma)}(s)\frac{r^{l(N+1)}}{t^{l-1}}\frac{\partial}{\partial r}\left(\frac{\partial}{r^{l-1}\partial r}\right)^N b(r,s,t)\,ds\,dr.$$

Assume for a moment that

$$\mathrm{I} = \left|\int_{2t}^{\infty}m^{(\gamma)}(s)\frac{r^{l(N+1)}}{t^{l-1}}\frac{\partial}{\partial r}\left(\frac{\partial}{r^{l-1}\partial r}\right)^N b(r,s,t)\,ds\right| \leqslant c\|m\|_{1,\gamma},\quad \gamma>1, \tag{7.2}$$

c independent of r, t and m. Applying this and Hölder's inequality we derive

$$|H_\gamma(t,x)| \leqslant c\,\|m\|_{1,\gamma}\left(\frac{1}{t}\int_0^t |(K_r * f)(x)|^2\,dr\right)^{1/2}$$

and by the definition of h_γ,

$$h_\gamma(f)(x) \leqslant c\,\|m\|_{1,\gamma}\left(\int_0^\infty |(K_r * f)(x)|^2\,\frac{dr}{r}\right)^{1/2}.$$

This proves Lemma 2 since $\eta(t) = t^l(1-t^l)_+^N$ satisfies the hypotheses of Lemma 1 if l and N are large enough. We shall prove (7.2). It is easily verified that

$$(7.3)\qquad \frac{r^{l(N+1)}}{t^{l-1}}\,\frac{\partial}{\partial r}\left(\frac{\cdot\,\partial}{r^{l-1}\,\partial r}\right)^N b(r,s,t) = \left(\frac{r}{t}\right)^{l-1}\sum_{\mu=1}^{N+1} c_\mu r^\mu \left(\frac{\partial}{\partial r}\right)^\mu b(r,s,t).$$

Setting $\bar\theta(t) = t^{1-l}\theta(t)$ and observing that $b(r,s,t) = \bar\theta(r/t)(s-r)^{\gamma-1}$ we can write by the Leibniz rule

$$r^\mu\left(\frac{\partial}{\partial r}\right)^\mu b(r,s,t) = \sum_{\lambda=0}^{\mu} d_\lambda r^\mu \left(\frac{r}{t}\right)^{\mu-\lambda} \bar\theta^{(\mu-\lambda)}\left(\frac{r}{t}\right)(s-r)^{\gamma-1-\lambda}.$$

Putting this into (7.3) and the result into (7.2) we obtain

$$\mathrm{I} = \left|\sum_{\mu=1}^{N+1}\sum_{\lambda=0}^{\mu} c_\mu \left(\frac{r}{t}\right)^{\mu+l-1} \bar\theta^{(\mu-\lambda)}\left(\frac{r}{t}\right) M_\lambda(r,t)\right|$$

where we set $M_\lambda(r,t) = d_\lambda t^\lambda \int_{2t}^\infty (s-r)^{\gamma-1-\lambda} m^{(\gamma)}(s)\,ds$. If we show that

$$(7.4)\qquad |M_\lambda(r,t)| \leqslant c\,\|m\|_{1,\gamma}, \qquad \gamma > 1,\ 0 \leqslant \lambda \leqslant \mu,$$

then (7.2) will follow since $\bar\theta \in C^\infty(\boldsymbol{R})$ has compact support. To prove (7.4) we consider two cases.

(a) *The case* $\lambda = 0$. By Lemma 3(b)

$$\left|\int_{2t}^\infty (s-r)^{\gamma-1} m^{(\gamma)}(s)\,ds\right| \leqslant c\,|m(r)| + \int_r^{2t} (s-r)^{\gamma-1}|m^{(\gamma)}(s)|\,ds.$$

There must be $r/t \in \operatorname{supp}(\bar\theta) \subset [\frac{1}{2}, 1]$ unless $\mathrm{I} = 0$. Thus, since $\gamma > 1$,

$$\int_r^{2t} (s-r)^{\gamma-1}|m^{(\gamma)}(s)|\,ds \leqslant C\,\|m\|_{1,\gamma},$$

which yields (7.4) for $\lambda = 0$.

(b) *The case* $\lambda > 0$. Choosing $\tau \in \boldsymbol{Z}$ with $2^\tau \leqslant 2t < 2^{\tau+1}$ we obtain

$$|M_\lambda(r,t)| \leqslant ct^\lambda\left|\int_{2t}^{2^{\tau+1}} (s-r)^{\gamma-1-\lambda} m^{(\gamma)}(s)\,ds\right|$$
$$+\,c2^{\tau\lambda}\sum_{k=\tau+1}^{\infty}\left|\int_{2^k}^{2^{k+1}} (s-r)^{\gamma-1-\lambda} m^{(\gamma)}(s)\,ds\right| = M_{\lambda,1} + M_{\lambda,2}.$$

As in case (a) we have $|M_{\lambda,1}| \leqslant c\,\|m\|_{1,\gamma}$. Furthermore,

$$M_{\lambda,2} \leqslant c2^{\tau\lambda}\sum_{k=\tau+1}^{\infty}\int_{2^k}^{2^{k+1}} s^{\gamma-1-\lambda}|m^{(\gamma)}(s)|\,ds$$
$$\leqslant c2^{\tau\lambda}\sup_{k\in\boldsymbol{Z}}\int_{2^k}^{2^{k+1}} |s^\gamma m^{(\gamma)}(s)|\frac{ds}{s}\sum_{k=\tau+1}^{\infty} 2^{-k\lambda} \leqslant c\,\|m\|_{1,\gamma}.$$

The proof of Lemma 2 is now complete.

Proof of (3.2). The case $\gamma \in \boldsymbol{N}$ is essentially proved in [31; p. 15]. By the identification of the WBV-spaces with the localized Bessel potential spaces of Connett and Schwartz [5] in [12] one has

$$(7.5)\qquad \|m \circ \varphi\|_{S(q,\gamma)} \leqslant C\,\|m\|_{S(q,\gamma)}, \qquad \gamma \in \boldsymbol{N},\ 1 < q < \infty.$$

We consider the linear operator

$$T\colon S(q,\gamma_0) \to S(q,\gamma_0), \qquad Tm = m \circ \gamma,$$

which, by (7.5), is continuous on $S(q,\gamma)$, $\gamma \in \boldsymbol{N}$. Then complex interpolation (see [5]) gives (7.5) for fractional $\gamma \geqslant 1$.

References

[1] R. Ashurov, *Conditions for Riesz means of spectral resolutions to be a basis in $L^p(\boldsymbol{R}^n)$*, Math. Notes 29 (1981), 342–348.

[2] A. Bonami and J. L. Clerc, *Sommes de Cesàro et multiplicateurs des développements en harmoniques sphériques*, Trans. Amer. Math. Soc. 183 (1973), 223–263.

[3] A. Calderón and A. Torchinsky, *Parabolic maximal functions associated with a distribution, I*, Adv. in Math. 16 (1975), 1–64.

[4] —, —, *Parabolic maximal functions associated with a distribution, II*, ibid. 24 (1977), 101–171.

[5] W. Connett and A. Schwartz, *The theory of ultraspherical multipliers*, Mem. Amer. Math. Soc. 9 (1977), no. 183.

[6] A. Carbery, G. Gasper and W. Trebels, *Radial Fourier multipliers of $L^p(\boldsymbol{R}^2)$*, Proc. Nat. Acad. Sci. U.S.A. 81 (1984), 3254–3255.

[7] H. Dappa, *A Marcinkiewicz criterion for L^p-multipliers*, Pacific J. Math. 111 (1984), 9–21.

[8] H. Dappa and H. Luers, *A Hörmander type criterion for quasi-radial Fourier multipliers*, Proc. Amer. Math. Soc. 95 (1985), 419–424.

[9] H. Dappa and W. Trebels, *On L^1-criteria for quasi-radial Fourier multipliers with applications to some anisotropic function spaces*, Analysis Math. 9 (1983), 275–289.

[10] C. Fefferman, *Inequalities for strongly singular convolution operators*, Acta Math. 124 (1970), 9–36.
[11] —, *A note on spherical summation multipliers*, Israel J. Math. 15 (1973), 44–52.
[12] G. Gasper and W. Trebels, *A characterization of localized Bessel potential spaces and applications to Jacobi and Hankel multipliers*, Studia Math. 65 (1979), 243–278.
[13] —, —, *Multiplier criteria of Hörmander type for Fourier series and applications to Jacobi series and Hankel transforms*, Math. Ann. 242 (1979), 225–240.
[14] —, —, *On sharp necessary conditions for radial Fourier multipliers*, in: Quantitative Approximation, Academic Press, New York 1980, 133–142.
[15] A. Greenleaf, *Principal curvature and harmonic analysis*, Indiana Univ. Math. J. 30 (1981), 519–537.
[16] S. Igari and S. Kuratsubo, *A sufficient condition for L^p-multipliers*, Pacific J. Math. 38 (1971), 85–88.
[17] W. Littman, *Fourier transforms of surface-carried measures and differentiability of surface averages*, Bull. Amer. Math. Soc. 69 (1963), 766–770.
[18] J. Löfström, *Some theorems on interpolation spaces with applications to approximation in L_p*, Math. Ann. 172 (1967), 176–196.
[19] W. Madych, *On Littlewood–Paley functions*, Studia Math. 50 (1974), 43–63.
[20] J. Peetre, *Applications de la théorie des espaces d'interpolation dans l'Analyse Harmonique*, Ricerche Mat. 15 (1966), 3–36.
[21] J. Peral and A. Torchinsky, *Multpliers in $H^p(R^n)$, $0 < p < \infty$*, Ark. Mat. 17 (1979), 224–235.
[22] B. Randol, *The asymptotic behavior of a Fourier transform and the localization property for eigenfunction expansions for some partial differential operators*, Trans. Amer. Math. Soc. 168 (1972), 265–271.
[23] N. Rivière, *Singular integrals and multiplier operators*, Ark. Mat. 9 (1971), 243–278.
[24] P. Sjölin, *Multipliers and restrictions of Fourier transforms in the plane*, unpublished manuscript, 1972.
[25] S. Sjöstrand, *On the Riesz means of the solution of the Schrödinger equation*, Ann. Scuola Norm. Sup. Pisa 24 (1970), 331–348.
[26] E. Stein, *Classes H^p, multiplicateurs et fonctions de Littlewood–Paley*, C. R. Acad. Sci. Paris 263 (1966), 716–719; also 264 (1967), 107–108.
[27] —, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, N. J. 1970.
[28] E. Stein and S. Wainger, *Problems in harmonic analysis related to curvature*, Bull. Amer. Math. Soc. 84 (1978), 1236–1295.
[29] P. Tomas, *A restriction theorem for the Fourier transform*, ibid. 81 (1975), 477–478.
[30] W. Trebels, *Multipliers for (C, α)-Bounded Fourier Expansions in Banach Spaces and Approximation Theory*, Lecture Notes in Math. 329, Springer, Berlin 1973.
[31] —, *On Fourier M_p^q multiplier criteria of Marcinkiewicz type*, Studia Math. 58 (1976), 7–19.
[32] A. Zygmund, *Trigonometric Series*, Cambridge University Press, 1959.

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On bounded biorthogonal systems in some function spaces

by

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Abstract. In this paper biorthogonal systems in the space of continuous functions $C(K)$ (K an infinite metric compact) and in the space B_p, $1 < p < \infty$, of almost periodic Besicovitch functions are considered. It is shown that there is a separable subspace $F \subset C(K)^*$ for which there is no biorthogonal system x_n, f_n, $x_n \in C(K)$, $f_n \in C(K)^*$ with $\|x_n\| = \|f_n\| = 1$ and $[f_n]_1^\infty \supset F$. It is proved that under the continuum hypothesis there is a decomposition of the real line $R = \bigcup_n R_n$, $n \in N$, for which the system $e^{i\lambda t} \in B_p$, $\lambda \in R_n$, is equivalent to the standard basis of the Hilbert space $l_2(R_n)$ for arbitrary n.

Introduction. Let X be a Banach space, X^* its dual and I some set of indices. A system x_i, f_i, $i \in I$, $x_i \in X$, $f_i \in X^*$, is called *biorthogonal* if $f_i(x_j) = 0$ for $i \neq j$ and 1 for $i = j$. A biorthogonal system is called *fundamental* if the closed linear span $[x_i\colon i \in I]$ is equal to X, and *total* if for any element $x \in X$, $x \neq 0$, there is an index i such that $f_i(x) \neq 0$. A fundamental and total biorthogonal system is said to be a *Markushevich basis* (an *M-basis*). A biorthogonal system is *bounded by a number c* if $\sup_i \|x_i\| \|f_i\| \leqslant c$. It is known (cf. [10]) that for any separable Banach space X, any separable subspace $F \subset X^*$ and any $\varepsilon > 0$ there exists an M-basis x_n, f_n bounded by $1+\varepsilon$ with $[f_n]_1^\infty \supset F$. Although the question whether every separable Banach space has an M-basis bounded by 1 is still open, we show that in the result of [10] quoted above $\varepsilon > 0$ is essential in some sense. Let us formulate the exact statement. Let K be a metric compact and let $C(K)$ be the space of real continuous functions on K. Its dual is the space $M(K)$ of Borel measures on the set K with bounded variation. Let δ_t, $t \in K$, be the atomic measure defined by $\delta_t\{t\} = 1$, $\delta_t\{K \setminus t\} = 0$.

THEOREM 1. *Let $(t_n)_1^\infty$ be a dense set in a nice metric compact K. The space $C(K)$ fails to have a biorthogonal system x_n, f_n bounded by 1 for which $[f_n]_1^\infty \supset (\delta_{t_n})_1^\infty$.*

This answers in the negative a question from [16, problem 8.2b)], where it is written that the question was raised by A. Pełczyński. Not every Banach space has an M-basis [16, p. 691], but if it has an M-basis then it has a