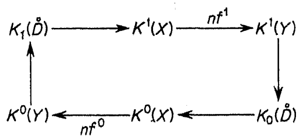
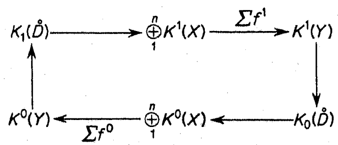
vertex of the cone. Then $K_*(D) \cong K_*(\mathring{D})$ and we get the following exact sequence:



Finally, if we drop the condition on the blocks, we get

EXAMPLE 5. Let X and Y be compact spaces and $f: Y \to X$ a continuous function. Let C_f be the mapping cone of f and D the C^*-algebra of maps from C_f into M_{nk} whose values on the canonical image of X in C_f are block diagonal matrices with blocks of size $k \times k$. Let $\mathring{D} := \ker \mathrm{ev}$, where ev is the evaluation at the vertex $y_0 \in C_f$. Then $K_1(D) \cong K_1(\mathring{D})$ and $K_0(D) \cong \boldsymbol{Z} \oplus K_0(\mathring{D})$. Moreover, we have the following exact sequence:



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τ-smooth linear functionals on vector lattices of real-valued functions

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Abstract. A vector lattice E of real-valued functions is said to be a strong Daniell lattice if every positive linear functional Φ: $E \to \boldsymbol{R}$ is τ-smooth (i.e. $\lim_\alpha \Phi(f_\alpha) = 0$ for every net (f_α) in E with $f_\alpha \downarrow 0$). Under some additional assumptions which, in general, cannot be omitted, several characterizations of strong Daniell lattices are given. These results are then applied to the vector lattices $\mathscr{C}(\mathscr{L}, \mathscr{B})$ and $\mathscr{C}^b(\mathscr{L}, \mathscr{B})$ of $\mathscr{L}$-continuous (and bounded) functions with $\mathscr{B}$-bounded support, where $\mathscr{L}$ denotes a lattice of sets and $\mathscr{B}$ is an $\mathscr{L}$-bounding system.

1. Introduction. This paper is a continuation of [4]. However, whereas in [4] we are concerned with the characterization of *Daniell lattices* (i.e. vector lattices E of real-valued functions having the property that every positive linear functional on E is σ-smooth), we consider in this paper only such vector lattices on which every positive linear functional is τ-smooth. Under some additional assumptions which, in general, cannot be omitted, we give several characterizations of these so-called *strong Daniell lattices*. As application of these general characterization theorems, we can prove, among others, the following results:

(1) *For a completely regular space X the following statements are equivalent*:

(a) *X is realcompact.*

(b) *The space of all continuous functions on X is a strong Daniell lattice.*

(c) *The space of all Baire-measurable functions on X is a strong Daniell lattice.*

(2) *If $(X, \mathscr{A})$ is a measurable space, then X is $\mathscr{A}$-complete* ([1]) *iff the space of all $\mathscr{A}$-measurable functions on X is a strong Daniell lattice. In particular, a topological space X is Borel-complete* ([11]) *iff the space of all Borel-measurable functions on X is a strong Daniell lattice.*

Some special cases of our results can be found in [10] and [17]. However, the methods of proof are different. Our proceeding seems to be more direct; in contrast to [10] and [17], we do not make use of any compactification.

Throughout this paper X will denote an arbitrary nonvoid set and $E \subset \boldsymbol{R}^X$ a vector lattice (with respect to pointwise operations). 1_Q denotes the indicator function of a subset Q of X. For $f \in E$ we put $\|f\|$

$:= \sup\{|f(x)|: x \in X\}$. We write $\Gamma(E)$ for the subfamily of $\boldsymbol{R}^E$ consisting of all positive linear functionals. For $x \in X$ we denote by I_x the evaluation functional on E pertaining to the point $x \in X$. For a net $(f_\alpha) \subset E$ we write $f_\alpha \downarrow 0$ if (f_α) is decreasing (i.e. $\alpha \geqslant \beta$ implies $f_\alpha \leqslant f_\beta$) and converges pointwise to zero.

E is said to be a *strong Daniell lattice* if every $\Phi \in \Gamma(E)$ is τ-smooth (i.e. $\lim_\alpha \Phi(f_\alpha) = 0$ for every net $(f_\alpha) \subset E$ with $f_\alpha \downarrow 0$).

A family $\mathscr{H}$ of subsets of X is said

(i) to have the *finite* [*countable*] *intersection property* if every finite [countable] subfamily of $\mathscr{H}$ has nonvoid intersection;

(ii) to be a *compact* [*semicompact*] *class* if every [countable] subfamily of $\mathscr{H}$ having the finite intersection property has nonvoid intersection.

Let $\mathscr{H}$ be a nonvoid family of subsets of X, closed under finite unions and finite intersections. A nonempty subfamily $\mathscr{D}$ of $\mathscr{H}$ is called an *$\mathscr{H}$-filter* [*$\mathscr{H}$-ultrafilter*] if $\mathscr{D}$ satisfies the following conditions (α)–(γ) [(α)–(δ)]:

(α) $\emptyset \notin \mathscr{D}$;

(β) $\mathscr{D}$ is closed under finite intersections;

(γ) $D_1 \in \mathscr{D}$, $D_2 \in \mathscr{H}$ and $D_1 \subset D_2$ imply $D_2 \in \mathscr{D}$;

(δ) $H \in \mathscr{H}$ and $H \cap D \neq \emptyset$ for all $D \in \mathscr{D}$ imply $H \in \mathscr{D}$.

If each $\mathscr{H}$-ultrafilter with the countable intersection property has nonvoid intersection, we say, following [1], that X is *$\mathscr{H}$-complete*.

Let $\mathscr{H}(E) := \{\{f \geqslant 1\}: f \in E\}$. Then $\mathscr{H}(E)$ is closed under finite unions and finite intersections. In addition, we have $\emptyset \in \mathscr{H}(E)$.

If X is a topological space, then $\mathscr{C}(X)$ [$\mathscr{C}^b(X)$] denotes the collection of all [bounded] continuous real-valued functions on X. A *zero-set* in X is a set of the form $\{f = 0\}$ with $f \in \mathscr{C}(X)$. We write $\mathscr{Z}(X)$, $\mathscr{F}(X)$, $\mathscr{G}(X)$ for the family of all zero-, closed, open sets in X, respectively. $\mathscr{B}_0(X)$ [$\mathscr{B}(X)$] denotes the Baire [Borel] σ-algebra in X.

2. Some characterizations of strong Daniell lattices. The following four properties (P1)–(P4) will play an important role in the sequel:

(P1) For every $f \in E$ there is some $g \in E$ such that $\{f \neq 0\} \subset \{g \geqslant 1\}$.

(P2) Every $f \in E$ is bounded.

(P3) $\min(1, f) \in E$ for every $f \in E$ (Stone's condition).

(P4) For every $x \in X$ there is some $f \in E$ with $f(x) \neq 0$.

The following basic result can be proved in the same way as Theorem 1 in [4].

2.1. THEOREM. *Consider the following four statements*:

(S1) *E is a strong Daniell lattice.*

(S2) *The family $\mathscr{H}(E)$ is a compact class.*

(S3) *Every net $(f_\alpha) \subset E$ with $f_\alpha \downarrow 0$ converges uniformly on X to zero.*

(S4) *For every $\varepsilon > 0$, for every net $(f_\alpha)_{\alpha \in A} \subset E$ with $f_\alpha \downarrow 0$, and for every $g \in E_+$ satisfying $\{f_{\alpha_0} \neq 0\} \subset \{g \geqslant 1\}$ for some $\alpha_0 \in A$, there exist a sequence $(\alpha_n) \subset A$ and a function $h \in E$ with $h \geqslant \sum_{n \in N} (f_{\alpha_n} - \min(\varepsilon g, f_{\alpha_n}))$.*

Then the following implications are valid:

(a) (S2) $\Rightarrow$ (S3) $\Rightarrow$ (S4).

(b) *If E satisfies* (P1), *then* (S4) $\Rightarrow$ (S1).

(c) *If E satisfies* (P2), *then* (S1) $\Rightarrow$ (S3).

(d) *If E satisfies* (P3), *then* (S3) $\Rightarrow$ (S2).

2.2. COROLLARY. *If E satisfies* (P1), (P2) [*and* (P3)], *then the statements* (S1), (S3), (S4) [*and* (S2)] *are equivalent.*

2.3. Remarks. (a) By means of the examples 1–3 in [4] one can see that, in general, the assumptions (P1)–(P3) cannot be omitted from the three last parts of 2.1.

(b) It is a consequence of Zorn's lemma that every subfamily of $\mathscr{H}(E)$ with the finite intersection property is contained in an $\mathscr{H}(E)$-ultrafilter. Thus statement (S2) is equivalent to

(S2′) *Every $\mathscr{H}(E)$-ultrafilter has nonvoid intersection.*

2.4. EXAMPLE. Let $\mathscr{R}$ be a ring of subsets of X. If E denotes the class of all step functions for $\mathscr{R}$ (cf. [14], I.7.8), then E is a vector lattice satisfying the conditions (P1)–(P3). In addition, we have $\mathscr{H}(E) = \mathscr{R}$. Thus 2.2 implies that E is a strong Daniell lattice iff $\mathscr{R}$ is a compact class. In the same way, it follows from [4], Corollary 1, that E is a Daniell lattice iff $\mathscr{R}$ is semicompact. If X is uncountable, then the algebra $\mathscr{A} = \{A \subset X: A \text{ or } X - A \text{ is finite}\}$ is a semicompact, non-compact class. Consequently, the space of all step functions for $\mathscr{A}$ is in this case a Daniell lattice which fails to be a strong Daniell lattice.

In the following we denote by $\mathscr{T}_E$ the topology on X generated by E (i.e. $\mathscr{T}_E$ is the coarsest topology on X making all functions $f \in E$ continuous).

A functional $\Phi \in \Gamma(E)$ is said to be *tight* if for every $H \in \mathscr{H}(E)$ and for every net $(f_\alpha) \subset E$ satisfying $\{f_\alpha \neq 0\} \subset H$ for all α and $\sup_\alpha \|f_\alpha\| < \infty$, we have $\lim_\alpha \Phi(f_\alpha) = 0$ provided that (f_α) converges uniformly on all $\mathscr{T}_E$-compact subsets of H to zero.

We can now prove the following new result.

2.5. THEOREM. *Assume that E satisfies the conditions* (P1)–(P4). *Then the following assertions are equivalent*:

(S1) *E is a strong Daniell lattice.*

(S5) *Every $\Phi \in \Gamma(E)$ is tight.*

(S6) *Every set $H \in \mathscr{H}(E)$ is $\mathscr{T}_E$-compact.*

(S7) *Every function $f \in E$ has $\mathscr{T}_E$-compact support.*

Proof. (S5)$\Rightarrow$(S1). It suffices to show that every tight $\Phi \in \Gamma(E)$ is τ-smooth. Let $\Phi \in \Gamma(E)$ be tight and let $(f_\alpha)_{\alpha \in A}$ be a net in E with $f_\alpha \downarrow 0$. Fix some $\alpha_0 \in A$ and put $A_0 := \{\alpha \in A:\ \alpha \geqslant \alpha_0\}$. Then $\sup_{\alpha \in A_0} \|f_\alpha\| = \|f_{\alpha_0}\| < \infty$ and, by (P1), there is a set $H_0 \in \mathscr{H}(E)$ with $\{f_{\alpha_0} \neq 0\} \subset H_0$. By Dini's theorem, the net $(f_\alpha)_{\alpha \in A_0}$ converges uniformly on all $\mathscr{T}_E$-compact subsets of H_0 to zero. This implies $\lim_{\alpha \in A_0} \Phi(f_\alpha) = 0$ and hence $\lim_{\alpha \in A} \Phi(f_\alpha) = 0$.

(S1)$\Rightarrow$(S6). Let $H_0 \in \mathscr{H}(E)$ be given. Furthermore, let $(F_i)_{i \in I}$ be a family of $\mathscr{T}_E$-closed subsets of X such that $\bigcap_{i \in I} (H_0 \cap F_i) = \emptyset$. By [2], 3.2, there exists, for every $i \in I$, a subfamily $\mathscr{F}_i$ of E with $H_0 \cap F_i = \bigcap_{f \in \mathscr{F}_i} \{f \geqslant 1\}$. Putting

$$\tilde{\mathscr{F}} := \bigcup_{i \in I} \mathscr{F}_i \quad \text{and} \quad \mathscr{F} := \{\min(f_1, \ldots, f_n):\ f_i \in \tilde{\mathscr{F}} \text{ for } i = 1, \ldots, n;\ n \in \mathbf{N}\},$$

we have $\bigcap_{f \in \mathscr{F}} \{f \geqslant 1\} = \emptyset$. Since $\mathscr{H}(E)$ is a compact class by 2.2, there exist finitely many functions $f_1, \ldots, f_n \in \tilde{\mathscr{F}}$ such that $\bigcap_{j=1}^{n} \{f_j \geqslant 1\} = \emptyset$. Choosing $i_j \in I$ with $f_j \in \mathscr{F}_{i_j}$ for $j = 1, \ldots, n$, we obtain $\bigcap_{j=1}^{n} (H_0 \cap F_{i_j}) = \emptyset$.

(S6)$\Rightarrow$(S7). Let $f \in E$ be given. By (P1), there is an $\mathscr{H}(E)$-set $H \supset \{f \neq 0\}$. As H is $\mathscr{T}_E$-compact by (S6), so is the support of f.

(S7)$\Rightarrow$(S5). Let $\Phi \in \Gamma(E)$, $H \in \mathscr{H}(E)$ and $\varepsilon > 0$ be given. Furthermore, let $(f_\alpha)_{\alpha \in A}$ be a net in E such that $\sup_\alpha \|f_\alpha\| < \infty$ and $\{f_\alpha \neq 0\} \subset H$ for all α. Assume that (f_α) converges uniformly on all $\mathscr{T}_E$-compact subsets of H to zero. As H is $\mathscr{T}_E$-compact by (S7), (f_α) converges uniformly on X to zero. Let $f \in E_+$ be such that $H = \{f \geqslant 1\}$. Putting $\varepsilon' := \varepsilon/\max(1, \Phi(f))$, there exists an index α_0 such that $|f_\alpha| \leqslant \varepsilon' f$ for all $\alpha \geqslant \alpha_0$. This implies $|\Phi(f_\alpha)| \leqslant \Phi(|f_\alpha|) \leqslant \varepsilon' \Phi(f) \leqslant \varepsilon$ for all $\alpha \geqslant \alpha_0$. Hence Φ is tight. ∎

If X is a topological space and E is a sublattice of $\mathscr{C}(X)$ such that the topology $\mathscr{T}_E$ is coarser than the given one, then, in general, Theorem 2.5 does not remain true if in the statements (S5), (S6) and (S7) the topology $\mathscr{T}_E$ is replaced by the given one:

2.6. Examples. (a) Let X be the real line (with the Euclidean topology) and let E be the lattice consisting of the constant functions. Then E is a strong Daniell lattice by 2.1. However, $X \in \mathscr{H}(E)$ is not compact.

(b) Let $X := [0, 1]$ be equipped with the discrete topology and let E be the space of all real-valued functions on X continuous w.r.t. the Euclidean topology. By 2.1, E is a strong Daniell lattice. Let $\mathscr{F}$ be the family of all finite subsets of X. Put $g_F := 1_{X-F}$ for $F \in \mathscr{F}$ and $\Phi(f) := \int f\, d\lambda$, $f \in E$, where λ denotes Lebesgue measure on X. We have $\int g_F\, d\lambda = 1$ and $g_F = \sup\{f \in E:\ 0 \leqslant f \leqslant g_F\}$ for all $F \in \mathscr{F}$. As Φ is τ-smooth, there exists for every $F \in \mathscr{F}$ a function $f_F \in E$ such that $0 \leqslant f_F \leqslant g_F$ and $\Phi(f_F) > 0.5$. Then $(f_F)_{F \in \mathscr{F}}$ is a net in E with $\sup\{\|f_F\|:\ F \in \mathscr{F}\} \leqslant 1$ converging pointwise (i.e. uniformly on all compact subsets of the discrete space X) to zero. On the other hand, the net $(\Phi(f_F))_{F \in \mathscr{F}}$ does not converge to zero.

3. The vector lattice $\mathscr{C}(\mathscr{L}, \mathscr{B})$. In the following let $\mathscr{L}$ be a family of subsets of X containing $\emptyset$ and X and closed under finite unions and countable intersections. Let $\mathscr{C}(\mathscr{L}) := \{f \in \mathbf{R}^X:\ f^{-1}(F) \in \mathscr{L}$ for all closed $F \subset \mathbf{R}\}$ denote the family of so-called *$\mathscr{L}$-continuous functions*. $\mathscr{C}(\mathscr{L})$ is a vector lattice and an algebra containing the constants ([6]). $\mathscr{C}^b(\mathscr{L})$ denotes the space of all bounded functions in $\mathscr{C}(\mathscr{L})$.

Furthermore, let $\mathscr{B}$ be an $\mathscr{L}$-bounding system ([10], [17]), i.e. $\mathscr{B}$ is a nonvoid family of subsets of X satisfying the following two conditions:

(i) $\mathscr{B} \uparrow X$ (i.e. $\bigcup \mathscr{B} = X$ and for $B_1, B_2 \in \mathscr{B}$ there is a set $B_3 \in \mathscr{B}$ such that $B_1 \cup B_2 \subset B_3$).

(ii) For every set $B_1 \in \mathscr{B}$ there exist a function $f \in \mathscr{C}(\mathscr{L})$ and a set $B_2 \in \mathscr{B}$ such that $f|B_1 = 1$ and $\{f \neq 0\} \subset B_2$.

Put $\mathscr{C}(\mathscr{L}, \mathscr{B}) := \{f \in \mathscr{C}(\mathscr{L}):\ \{f \neq 0\} \subset B$ for some $B \in \mathscr{B}\}$. Then $\mathscr{C}(\mathscr{L}, \mathscr{B})$ is a vector lattice satisfying conditions (P1), (P3) and (P4). Moreover, $\mathscr{C}(\mathscr{L}, \mathscr{B})$ is an ideal in $\mathscr{C}(\mathscr{L})$.

Notice that $X \in \mathscr{B}$ implies $\mathscr{C}(\mathscr{L}, \mathscr{B}) = \mathscr{C}(\mathscr{L})$.

It has been shown in [4] that $\mathscr{C}(\mathscr{L}, \mathscr{B})$ is a Daniell lattice. However, in general, $\mathscr{C}(\mathscr{L}, \mathscr{B})$ fails to be a strong Daniell lattice (see [18], Remark 4, p. 178). In the sequel we shall give several conditions necessary and sufficient for $\mathscr{C}(\mathscr{L}, \mathscr{B})$ to be a strong Daniell lattice. For this purpose we need

3.1. Lemma. *Let $\mathscr{A}$ be the σ-algebra generated by $\mathscr{C}(\mathscr{L}, \mathscr{B})$, and let μ be a measure on $\mathscr{A}$ such that $\mathscr{C}(\mathscr{L}, \mathscr{B}) \subset \mathscr{L}_1(X, \mathscr{A}, \mu)$. Then, for any $f \in \mathscr{C}(\mathscr{L}, \mathscr{B})$, there is a constant $r \in (0, \infty)$ such that $\mu(\{|f| > r\}) = 0$.*

Proof. Assume that there is a function $f \in \mathscr{C}(\mathscr{L}, \mathscr{B})$ with $\mu(\{|f| > r\}) > 0$ for all $r > 0$. Then one can inductively define two sequences (a_n), (b_n) of real numbers such that $n < a_n < b_n < a_{n+1}$ and $\mu(A_n) > 0$ holds for all $n \in \mathbf{N}$, where $A_n := \{a_n \leqslant |f| \leqslant b_n\}$. Choose a continuous function g: $[0, \infty) \to \mathbf{R}$ such that $g(0) = 0$ and $g(t) = (\mu(A_n))^{-1}$ for $t \in [a_n, b_n]$ and all $n \in \mathbf{N}$. Then $h := g \circ |f| \in \mathscr{C}(\mathscr{L}, \mathscr{B})$ and $\int h\, d\mu \geqslant \sum_{n \in \mathbf{N}} \int_{A_n} h\, d\mu = \infty$ which contradicts the assumption $h \in \mathscr{L}_1(X, \mathscr{A}, \mu)$. ∎

Now consider a maximal ideal $\mathscr{J}$ in $\mathscr{C}(\mathscr{L}, \mathscr{B})$. Then the residue-class ring $F := \mathscr{C}(\mathscr{L}, \mathscr{B})/\mathscr{J}$ is a field (see [10], section 9). If the mapping T: $\mathbf{R} \to F$ defined by $T(r) := [rf^*]$, where $[f^*]$ denotes the unit element of F, is surjective, then $\mathscr{I}$ is said to be *real*. On the other hand, $\mathscr{I}$ is said to be *fixed* if $\mathscr{I} = \{f\in \mathscr{C}(\mathscr{L}, \mathscr{B})\colon f(x_0) = 0\}$ for some $x_0 \in X$.

We can prove the main theorem of this section which is a common generalization of several well-known results.

3.2. THEOREM. *For $E := \mathscr{C}(\mathscr{L}, \mathscr{B})$ the following four statements are equivalent:*

(S1) *E is a strong Daniell lattice.*

(S8) *Every nonzero multiplicative linear functional on E is an evaluation.*

(S9) *Every real maximal ideal in E is fixed.*

(S10) *X is $\mathscr{H}(E)$-complete.*

Proof. (S1)$\Rightarrow$(S8). We consider X as the topological space equipped with the topology $\mathscr{T}_E$, and we denote by $\mathrm{int}(Q)$ $[\mathrm{cl}(Q)]$ the interior [closure] of a subset Q of X. Let $\Phi \neq 0$ be a multiplicative linear functional on E. If $f\in E_+$, then $f = (f^{1/2})^2$ with $f^{1/2}\in E$ and so $\Phi(f) = (\Phi(f^{1/2}))^2 \geqslant 0$. Thus Φ is positive and hence τ-smooth by (S1). By [8], 72E, there is a Borel measure μ on X with the following properties:

(3.1) $\Phi(f) = \int f\,d\mu$ for all $f\in E$;

(3.2) $\mu(A) = \sup\{\mu(F)\colon F\subset A,\ F\in\mathscr{F}(X)\}$ for all Borel sets A;

(3.3) $\mu(\bigcup_\alpha G_\alpha) = \sup_\alpha \mu(G_\alpha)$ for every increasing net $(G_\alpha)\subset \mathscr{G}(X)$.

There is a function $f_0\in E$ with $\Phi(f_0)\neq 0$. Choosing $B_0\in\mathscr{B}$ and $f\in E$ such that $\{f_0\neq 0\}\subset B_0$, $0\leqslant f\leqslant 1$ and $f|B_0 = 1$, we have $f_0 = f\cdot f_0$, hence $0\neq \Phi(f_0) = \Phi(f)\cdot\Phi(f_0)$ and so

$$1 = \Phi(f) = \int f\,d\mu \leqslant \mu(X). \tag{3.4}$$

For a given $B\in\mathscr{B}$ choose a $\mathscr{B}$-set B_1 with $B\cup B_0\subset B_1$ and a function $f\in E$ with $0\leqslant f\leqslant 1$ and $f|B_1 = 1$. Then (3.4) implies $1 = \int f\,d\mu \geqslant \int f 1_{\mathrm{cl}(B_1)}\,d\mu = \mu(\mathrm{cl}(B_1)) \geqslant \mu(\mathrm{cl}(B))$. Thus we have

$$\mu(\mathrm{cl}(B)) \leqslant 1 \quad \text{for all } B\in\mathscr{B}. \tag{3.5}$$

For any $x\in X$, there is a $\mathscr{B}$-set B_1 with $x\in B_1$. Choose $f\in E$ and $B_2\in\mathscr{B}$ such that $f|B_1 = 1$ and $\{f\neq 0\}\subset B_2$. Then $x\in\{f\neq 0\}\subset \mathrm{int}(B_2)$. Thus we have $\{\mathrm{int}(B)\colon B\in\mathscr{B}\}\uparrow X$ which, together with (3.3), (3.4) and (3.5), implies $1\leqslant \mu(X) = \sup\{\mu(\mathrm{int}(B))\colon B\in\mathscr{B}\} \leqslant \sup\{\mu(\mathrm{cl}(B))\colon B\in\mathscr{B}\}\leqslant 1$. Hence μ is a probability measure.

Let $G\in\mathscr{G}(X)$ with $\mu(G) > 0$. Then $1_G = \sup\{f\in E\colon 0\leqslant f\leqslant 1_G\}$ by [2], 3.4. Hence, for any $t\in(0, 1)$, we have $G = \bigcup\{\{f>t\}\colon f\in E,\ 0\leqslant f\leqslant 1_G\}$ and so, by (3.1) and (3.3),

$$t\mu(G) = \sup\{t\mu(\{f>t\})\colon f\in E,\ 0\leqslant f\leqslant 1_G\} \leqslant \sup\{\Phi(f)\colon f\in E,\ 0\leqslant f\leqslant 1_G\}.$$

For $t\uparrow 1$ we thus obtain $0<\mu(G) = \sup\{\Phi(f)\colon f\in E,\ 0\leqslant f\leqslant 1_G\}$. In

particular, there is a function $f\in E$ with $0\leqslant f\leqslant 1_G$ and $\Phi(f) > 0$. Then $f^{1/n}\in E$, $f^{1/n}\leqslant 1_G$ and $(\Phi(f))^{1/n} = \Phi(f^{1/n}) \leqslant \mu(G) \leqslant 1$ for all $n\in N$. As $(\Phi(f))^{1/n}\to 1$, we obtain $\mu(G) = 1$.

Thus we have shown $\mu(G)\in\{0, 1\}$ for all $G\in\mathscr{G}(X)$ which, together with (3.2), implies that μ is $\{0, 1\}$-valued. Let $F_0 := X - \bigcup\{G\in\mathscr{G}(X)\colon \mu(G) = 0\}$. By (3.3), we have $\mu(F_0) = 1$, in particular $F_0\neq\emptyset$. Fix some $x_0\in F_0$. Then it follows from (3.2) that $\mu(A) = 0$ for any Borel set A with $x_0\notin A$. Thus μ is the Dirac measure pertaining to the point x_0; hence $\Phi = I_{x_0}$ by (3.1).

(S8)$\Rightarrow$(S9). Let $\mathscr{I}$ be a real maximal ideal in E. For any $f\in E$ put $\Phi(f) := r$ if $[f] = [rf^*]$, where $[f^*]$ denotes the unit element of the residue-class field generated by $\mathscr{I}$. Φ is a nonzero multiplicative linear functional on E. By (S8), we have $\Phi = I_{x_0}$ for some $x_0\in X$ which implies $\mathscr{I} = \{f\in E\colon \Phi(f) = 0\} = \{f\in E\colon f(x_0) = 0\}$. Hence $\mathscr{I}$ is fixed.

(S9)$\Rightarrow$(S8). Let $\Phi\neq 0$ be a multiplicative linear functional on E. It can be shown by routine arguments (cf. [9]) that $\mathscr{I} := \{f\in E\colon \Phi(f) = 0\}$ is a real maximal ideal in E. By (S9), we then have $\mathscr{I} = \{f\in E\colon f(x_0) = 0\}$ for some $x_0\in X$ which implies $\Phi = I_{x_0}$.

(S8)$\Rightarrow$(S10). Let $\mathscr{U}$ be an $\mathscr{H}(E)$-ultrafilter with the countable intersection property. For any $H\in\mathscr{H}(E)$ put $\lambda(H) = 1$ or 0 according as $H\in\mathscr{U}$ or not. Then λ can be extended to a $\{0, 1\}$-valued measure μ on the σ-algebra $\mathscr{A}$ generated by E (cf. the proof of Theorem 2.1 in [1]), and we have $E\subset\mathscr{L}_1(X, \mathscr{A}, \mu)$ (cf. [13], (12.60)). In addition, $\Phi(f) := \int f\,d\mu$, $f\in E$, is a nonzero multiplicative linear functional on E (cf. [13], (20.52) (c)). Thus (S8) implies $\Phi = I_{x_0}$ for some $x_0\in X$. Now let $U\in\mathscr{U}$ be given. Then $U = \{f\geqslant 1\}$ for some $f\in E_+$, and we have $1 = \mu(U)\leqslant \int f\,d\mu = \Phi(f) = f(x_0)$. Thus $x_0\in U$. As $U\in\mathscr{U}$ was arbitrary, we obtain $x_0\in\bigcap\mathscr{U}$. This proves (S10).

(S10)$\Rightarrow$(S1). Let $\Phi\in\Gamma(E)$ be given. By [4], Corollary 2, Φ is σ-smooth. Hence, by [8], 71G, there is a measure μ on the σ-algebra generated by E such that (3.1) holds. Let $(f_\alpha)_{\alpha\in A}$ be a net in E with $f_\alpha\downarrow 0$. We first prove

$$\inf_\alpha \mu(\{f_\alpha\geqslant\varepsilon\}) = 0 \quad \text{for all } \varepsilon > 0. \tag{3.6}$$

Suppose that we have $c := \inf_\alpha \mu(\{f_\alpha\geqslant\varepsilon\}) > 0$ for some $\varepsilon > 0$. Choose a sequence $(\alpha_n)\subset A$ such that $c = \inf_n \mu(\{f_{\alpha_n}\geqslant\varepsilon\})$ where w.l.o.g. $(f_{\alpha_n})_{n\in N}$ can be assumed to be decreasing. Put $F_0 := \bigcap_{n\in N}\{f_{\alpha_n}\geqslant\varepsilon\}$. Then $\mu(F_0) = c > 0$, and the family $\mathscr{D} := \{H\in\mathscr{H}(E)\colon \mu(H\cap F_0) = \mu(F_0)\}$ is an $\mathscr{H}(E)$-filter. Since $\{f_\alpha\geqslant\varepsilon\}\in\mathscr{D}$ for all $\alpha\in A$, we obtain

$$\bigcap\mathscr{D} = \emptyset. \tag{3.7}$$

Now we choose an $\mathscr{H}(E)$-ultrafilter $\mathscr{D}_0\supset\mathscr{D}$. In order to prove (3.6), it suffices to show that $\mathscr{D}_0$ has the countable intersection property, since in this case (S10) implies $\bigcap\mathscr{D}_0\neq\emptyset$ which contradicts (3.7).

Suppose that there is a sequence $(D_n) \subset \mathscr{D}_0$ with $D_n \downarrow \emptyset$. Then one can find a sequence $(h_n) \subset E$ with $h_n \downarrow 0$ and $h_n | D_n \geqslant 1$ for all $n \in N$ (cf. the proof of Theorem 1d) in [4]). Choose a function $\bar{f} \in E_+$ with $\{h_1 \neq 0\} \subset \{\bar{f} = 1\}$. Then $h := \sum_{n \in N} (h_n - \min(\bar{f}/2, h_n)) \in E$ (cf. the proof of Theorem 2 in [4]) and $h | D_n \geqslant n/2$ for all $n \in N$.

By 3.1, we have $\mu(\{h > r\}) = 0$ for some $r \in (0, \infty)$. Then $D_n \cap \{h \leqslant r\} = \emptyset$ for all $n > 2r$. Furthermore, we have

$$\{f_{\alpha_1} \geqslant \varepsilon\} \cap \{h \leqslant r\} = \{f_{\alpha_1} \geqslant \varepsilon\} \cap \{(r+1)\min(1, f_{\alpha_1}/\varepsilon) - h \geqslant 1\} \in \mathscr{H}(E),$$

$$\mu(\{f_{\alpha_1} \geqslant \varepsilon\} \cap \{h \leqslant r\} \cap F_0) = \mu(\{h \leqslant r\} \cap F_0) = \mu(F_0)$$

which implies $\{f_{\alpha_1} \geqslant \varepsilon\} \cap \{h \leqslant r\} \in \mathscr{D} \subset \mathscr{D}_0$. Thus we have, for all $n > 2r$, $\emptyset = D_n \cap \{h \leqslant r\} = D_n \cap \{f_{\alpha_1} \geqslant \varepsilon\} \cap \{h \leqslant r\} \in \mathscr{D}_0$. This contradiction proves (3.6).

Now fix some $\alpha_0 \in A$ and choose a function $\bar{f} \in E_+$ with $\bar{f} | \{f_{\alpha_0} \neq 0\} = 1$. Then, for all $\alpha \geqslant \alpha_0$ and any $\varepsilon > 0$, we have

$$\Phi(f_\alpha) = \int f_\alpha \, d\mu \leqslant \int f_{\alpha_0} 1_{\{f_\alpha \geqslant \varepsilon\}} \, d\mu + \int f_\alpha 1_{\{0 < f_\alpha \leqslant \varepsilon\}} \, d\mu \leqslant \int f_{\alpha_0} 1_{\{f_\alpha \geqslant \varepsilon\}} \, d\mu + \varepsilon \Phi(\bar{f})$$

which together with (3.6) implies $\lim_\alpha \Phi(f_\alpha) = 0$. ∎

If $\mathscr{L}$ is the family of closed subsets of a completely regular space, then the implication (S9) ⇒ (S1) of 3.2 is exactly Theorem 11.2 of [10].

For the subsequent applications of 3.2 we need the following two additional properties of the family $\mathscr{L}$ ([6]):

$\mathscr{L}$ is said to be *normal* if, for any two disjoint sets $L_1, L_2 \in \mathscr{L}$, there exist disjoint sets $K_1, K_2 \in \mathscr{L}' = \{X - L\colon L \in \mathscr{L}\}$ such that $L_i \subset K_i$ for $i = 1, 2$.

$\mathscr{L}$ is said to be *complement-generated* if every set $L \in \mathscr{L}$ is a countable intersection of $\mathscr{L}'$-sets.

3.3. Corollary. *If $\mathscr{L}$ is normal and complement-generated, then $\mathscr{C}(\mathscr{L})$ is a strong Daniell lattice iff X is $\mathscr{L}$-complete.*

Proof. By [5], Lemma 7, we have $\mathscr{L} = \mathscr{H}(\mathscr{C}(\mathscr{L}))$. Now our claim follows from 3.2. ∎

The assumptions of 3.3 are in particular satisfied if $\mathscr{L}$ is a σ-algebra. In this case $\mathscr{C}(\mathscr{L})$ is the space of all $\mathscr{L}$-measurable real-valued functions on X. Thus we obtain from 3.2 and 3.3 the following new result.

3.4. Corollary. *Let $(X, \mathscr{A})$ be a measurable space. Then the following assertions are equivalent:*

(1) *X is $\mathscr{A}$-complete.*

(2) *$\mathscr{C}(\mathscr{A})$ is a strong Daniell lattice.*

(3) *Every nonzero multiplicative linear functional on $\mathscr{C}(\mathscr{A})$ is an evaluation.*

(4) *Every real maximal ideal in $\mathscr{C}(\mathscr{A})$ is fixed.*

If X is a topological space and $E := \mathscr{C}(X)$, then $E = \mathscr{C}(\mathscr{F}(X)) = \mathscr{C}(\mathscr{Z}(X))$ and $\mathscr{H}(E) = \mathscr{Z}(X)$. Since, by [1], Corollary 2.3, $\mathscr{Z}(X)$-completeness and $\mathscr{B}_0(X)$-completeness are equivalent properties, we thus obtain from 3.2 and 3.4

3.5. Corollary. *For a topological space X, the following five statements are equivalent (if, in addition, X is completely regular, then each of these statements is equivalent to the realcompactness of X):*

(1) *X is $\mathscr{Z}(X)$-complete.*

(2) *$\mathscr{C}(X)$ is a strong Daniell lattice.*

(3) *$\mathscr{C}(\mathscr{B}_0(X))$ is a strong Daniell lattice.*

(4) *Every nonzero multiplicative linear functional on $\mathscr{C}(X)$ is an evaluation.*

(5) *Every real maximal ideal in $\mathscr{C}(X)$ is fixed.*

For completely regular X, statement (5) of 3.5 is exactly the definition of realcompactness given originally by Hewitt [12] ("Q-space" in the original terminology), whereas the statements (1) and (4) of 3.5 are classical characterizations of realcompact spaces (see [9] and [12]). Furthermore, an analysis of the proof of 3.2 reveals that every Baire measure on a realcompact space which integrates every continuous function is automatically τ-smooth (i.e. $\mathscr{Z}(X) \ni Z_\alpha \downarrow \emptyset$ implies $\inf \mu(Z_\alpha) = 0$). Note, however, that there are realcompact spaces supporting finite Baire measures that are not τ-smooth (see [19], p. 128).

In accordance with [11], $\mathscr{B}(X)$-complete topological spaces X are called *Borel-complete*. By 3.4, these spaces can be characterized in the following way.

3.6. Corollary. *A topological space X is Borel-complete iff $\mathscr{C}(\mathscr{B}(X))$ is a strong Daniell lattice.*

As in a discrete space Borel-completeness is the same as realcompactness, we obtain from 3.6 and [9], 12.2 the following characterization of nonmeasurable cardinals.

3.7. Corollary. *For a set X, the following two statements are equivalent:*

(1) card(X) *is nonmeasurable.*

(2) *The space of all real-valued functions on X is a strong Daniell lattice.*

If $(X, \mathscr{A})$ is a measurable space and $E := \mathscr{C}(\mathscr{A})$ is a strong Daniell lattice, then all elements of $\Gamma(E)$ are of the same structure, as the following result shows.

3.8. Proposition. *Let $(X, \mathscr{A})$ be a measurable space and let $E := \mathscr{C}(\mathscr{A})$ be the space of all $\mathscr{A}$-measurable real-valued functions on X. Then the following two assertions are equivalent:*

(1) *E is a strong Daniell lattice.*

(2) *Every $\Phi \in \Gamma(E)$ is elementary* ([7]), *i.e. there exist $a_1, \ldots, a_n \in \mathbf{R}_+$ and $x_1, \ldots, x_n \in X$ such that $\Phi = \sum_{i=1}^{n} a_i I_{x_i}$.*

Proof. Since any elementary $\Phi \in \Gamma(E)$ is τ-smooth, it remains to prove (1)$\Rightarrow$(2). Let $\Phi \in \Gamma(E)$ be given and put $\mu(A) := \Phi(1_A)$ for $A \in \mathcal{A}$. Then μ is a finite measure on $\mathcal{A}$ which is τ-smooth at $\emptyset$ (i.e. $\mathcal{A} \ni A_\alpha \downarrow \emptyset$ implies $\inf_\alpha \mu(A_\alpha) = 0$). As $\Phi(f) = \int f\,d\mu$ for $f \in E$, it suffices to show that μ is of the form $\sum_{i=1}^{n} a_i \delta_{x_i}$ with $a_i \in \mathbf{R}_+$ and $x_i \in X$ for $i = 1, \ldots, n$.

By [15], 2.1 and 2.2, there are measures μ_1, μ_2 on $\mathcal{A}$ with $\mu = \mu_1 + \mu_2$ such that μ_1 is nonatomic and μ_2 is purely atomic, i.e. there is a countable disjoint family $\{B_k: k \in N_0\}$ of μ_2-atoms such that

$$\mu_2(A) = \mu_2\big(A \cap \bigcup_{k \in N_0} B_k\big) \quad \text{for all } A \in \mathcal{A}.$$

It follows from [3], 2.1 that, for any $x \in X$, one can find a set $A_x \in \mathcal{A}$ with $x \in A_x$ and $\mu_1(A_x) = 0$. Denoting by $\mathcal{D}$ the collection of all finite unions of sets A_x, we obtain $\bigcup \mathcal{D} = X$, hence $\mu_1(X) = \sup\{\mu_1(D): D \in \mathcal{D}\} = 0$, since μ_1 is also τ-smooth at $\emptyset$. Thus we have $\mu_1 = 0$, i.e.

$$\mu(A) = \sum_{k \in N_0} \mu(A \cap B_k) \quad \text{for all } A \in \mathcal{A}.$$

Now $A \in \mathcal{A} \to \mu(A \cap B_k) \cdot (\mu(B_k))^{-1}$ defines a $\{0, 1\}$-valued measure, hence a Dirac measure by 3.4 and [1], 2.2. Therefore we obtain

$$\mu = \sum_{k \in N_0} \mu(B_k)\,\delta_{x_k} \quad \text{with } x_k \in B_k.$$

Now our claim follows from the fact that the family $\{B_k: k \in N_0\}$ is finite, since otherwise we would have $\Phi(f) = \infty$ for $f := \sum_{k \in N_0} (\mu(B_k))^{-1} \cdot 1_{B_k} \in E$. ∎

3.9. Remarks. (a) The implication (1)$\Rightarrow$(2) of 3.8 does not remain true for $E := \mathcal{C}(\mathcal{L})$ if $\mathcal{L}$ is not a σ-algebra: Consider $X = [0, 1]$ with the Euclidean topology and $\mathcal{L} = \mathcal{F}(X)$. By 3.5, $E := \mathcal{C}(\mathcal{L}) = \mathcal{C}(X)$ is a strong Daniell lattice. On the other hand, it follows from the Riesz representation theorem that the functional $\Phi(f) := \int f\,d\lambda$, $f \in E$, where λ denotes Lebesgue measure, is not elementary.

(b) From 3.4 and 3.8 we obtain an alternative proof of the equivalence of the statements (2) and (3) of [3], Theorem 2.6. This result also reveals that the assertions (c) and (d) of Theorem 1 in [7] are equivalent.

(c) From 3.7 and 3.8 we obtain the main theorem of [16] and, in particular, Satz 5.4.6 of [17].

4. The vector lattice $\mathcal{C}^b(\mathcal{L}, \mathcal{B})$. Under the same assumptions as in the preceding section, we are now concerned with the vector lattice $\mathcal{C}^b(\mathcal{L}, \mathcal{B}) := \{f \in \mathcal{C}(\mathcal{L}, \mathcal{B}): f \text{ bounded}\}$. As $\mathcal{C}^b(\mathcal{L}, \mathcal{B})$ satisfies the conditions (P1)–(P4), the statements (S1)–(S7) are equivalent for $\mathcal{C}^b(\mathcal{L}, \mathcal{B})$. Four further equivalent statements are given in

4.1. Theorem. *The following statements are equivalent*:

(S1) *$\mathcal{C}^b(\mathcal{L}, \mathcal{B})$ is a strong Daniell lattice.*

(S11) *Every $B \in \mathcal{B}$ is relatively $\mathcal{T}_E$-compact (with $E := \mathcal{C}^b(\mathcal{L}, \mathcal{B})$).*

(S12) *$\mathcal{C}^b(\mathcal{L}, \mathcal{B}) = \mathcal{C}(\mathcal{L}, \mathcal{B})$ and $\mathcal{C}(\mathcal{L}, \mathcal{B})$ is a strong Daniell lattice.*

(S13) *Every maximal ideal in $\mathcal{C}^b(\mathcal{L}, \mathcal{B})$ is fixed.*

(S14) *Every nonzero multiplicative linear functional on $\mathcal{C}^b(\mathcal{L}, \mathcal{B})$ is an evaluation.*

Proof. For abbreviation we put $E := \mathcal{C}^b(\mathcal{L}, \mathcal{B})$.

The equivalence of (S1) and (S11) follows from 2.5, since any $\mathcal{B}$-set is a subset of some $\mathcal{H}(E)$-set and conversely.

(S11)$\Rightarrow$(S12). It is obvious that every $f \in \mathcal{C}(\mathcal{L}, \mathcal{B})$ is bounded. Thus the second assertion follows from the equivalence of (S1) and (S11).

Since the implication (S12)$\Rightarrow$(S1) is trivial, the statements (S1), (S11) and (S12) are equivalent. Therefore the implication (S12)$\Rightarrow$(S13) can be proved in the same way as Theorem 7.2 in [10].

(S13)$\Rightarrow$(S14) can be proved in the same way as the implication (S9)$\Rightarrow$(S8) of 3.2.

(S14)$\Rightarrow$(S1). According to 2.3 (b) it suffices to show that every $\mathcal{H}(E)$-ultrafilter has nonvoid intersection. Let $\mathcal{U}$ be an $\mathcal{H}(E)$-ultrafilter and put $\mathcal{U}_0 := \mathcal{U} \cup \{X\}$. Define $\Phi(f) := \sup\{t \in [0, \|f\|]: \{f \geqslant t\} \in \mathcal{U}_0\}$ for $f \in E_+$ and $\Phi(f) := \Phi(f^+) - \Phi(f^-)$ for $f \in E$. One can easily verify that Φ is a multiplicative linear functional on E. If $U \in \mathcal{U}$, say $U = \{f \geqslant 1\}$ with $f \in E_+$, then $\Phi(f) \geqslant 1$. Thus Φ is nonzero and hence, by (S14), $\Phi = I_{x_0}$ for some $x_0 \in X$, which implies $x_0 \in \bigcap \mathcal{U}$. ∎

It follows from 4.1 and [4], Theorem 3, that $\mathcal{C}^b(\mathcal{L}, \mathcal{B})$ is a strong Daniell lattice iff $\mathcal{C}^b(\mathcal{L}, \mathcal{B})$ is a Daniell lattice and $\mathcal{C}(\mathcal{L}, \mathcal{B})$ is a strong Daniell lattice.

For the special case of a completely regular space X and the family $\mathcal{L}$ of closed subsets of X, the equivalence of the statements (S5), (S11), (S12) and (S13) for $\mathcal{C}^b(\mathcal{L}, \mathcal{B})$ has been proved by other methods in [17], Satz 5.1.4 and Satz 5.2.2.

If X is an arbitrary topological space and $E := \mathcal{C}^b(X)$, then $E = \mathcal{C}^b(\mathcal{F}(X)) = \mathcal{C}^b(\mathcal{Z}(X))$ and $\mathcal{H}(E) = \mathcal{Z}(X)$, and each of the statements (S2)–(S7), (S11)–(S14) is necessary and sufficient for $E = \mathcal{C}^b(X)$ to be a strong Daniell lattice. If, in addition, X is completely regular, then the given topology on X equals $\mathcal{T}_E$, and we obtain from 3.5 and 4.1 the following well-known characterization of compactness (see [9], 5H, and [19], 8.1).

4.2. Corollary. *For a completely regular space X the following statements are equivalent*:

(1) *X is compact.*

(2) *X is pseudocompact and realcompact.*

(3) *$\mathscr{C}^b(X)$ is a strong Daniell lattice.*

We now consider the special case where $\mathscr{L}$ is a σ-algebra and hence $\mathscr{C}^b(\mathscr{L})$ is the space of all bounded $\mathscr{L}$-measurable functions. The following result gives several necessary and sufficient conditions for $\mathscr{C}^b(\mathscr{L})$ to be a strong Daniell lattice.

4.3. Theorem. *For a measurable space $(X, \mathscr{A})$ the following statements are equivalent*:

(1) *$\mathscr{C}^b(\mathscr{A})$ is a strong Daniell lattice.*

(2) *$\mathscr{C}^b(\mathscr{A})$ is a Daniell lattice.*

(3) *$\mathscr{A}$ is finite.*

(4) *$\mathscr{A}$ is a compact class.*

(5) *$\mathscr{A}$ is a semicompact class.*

Proof. As we have $\mathscr{A} = \mathscr{K}(\mathscr{C}^b(\mathscr{A}))$, the equivalence $(1)\Leftrightarrow(4)$ follows from 2.2, whereas the equivalence $(2)\Leftrightarrow(5)$ follows from [4], Corollary 1. Since the implications $(3)\Rightarrow(4)\Rightarrow(5)$ are trivial, it remains to prove $(5)\Rightarrow(3)$. Assume that $\mathscr{A}$ is infinite. Then there exists a sequence $(A_n)_{n\in N}$ of pairwise disjoint nonvoid $\mathscr{A}$-sets with $X = \bigcup_{n\in N} A_n$. We then have

$$\bigcap_{n\in N}(X-A_n) = \emptyset \quad \text{and} \quad \bigcap_{n=1}^{k}(X-A_n) \neq \emptyset \quad \text{for all } k\in N.$$

So $\mathscr{A}$ is not semicompact. ∎

For the special case where $\mathscr{A}$ is the power set of X, we obtain from 4.3

4.4. Corollary. *A set X is finite iff the space of all bounded real-valued functions on X is a (strong) Daniell lattice.*

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