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The Mayer–Vietoris and the Puppe sequences in K-theory for C^*-algebras

by

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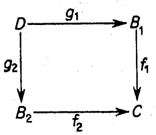
Abstract. We show the existence of a Mayer–Vietoris and a Puppe sequences in the K-theory for C^*-algebras. Both sequences generalize the respective sequences in the commutative case in the sense that they reduce to those sequences under the identification $K_\alpha(C_0(X)) = \tilde{K}^\alpha(X)$ if all algebras involved are chosen to be commutative, i.e. of the form $C_0(X)$ for a locally compact space X. The sequences are used to calculate the K-theory of certain bundle-C^*-algebras with continuous identity field.

0. Notation and preliminaries. For any C^*-algebra A call $SA := \{f\colon [0, 1] \to A \text{ continuous}, f(0) = 0 = f(1)\}$ the *suspension* of A. For two C^*-algebras A and B we say that two morphisms $\varphi_i\colon A \to B$, $i = 0, 1$, are *homotopic* if there exists a family $\Phi_t\colon A \to B$ of morphisms for $t \in [0, 1]$ such that $\Phi\colon I \times A \to B$ defined by $\Phi(t, a) = \Phi_t(a)$ is jointly continuous and $\Phi_i = \varphi_i$ for $i = 0, 1$. We write $\varphi_1 \simeq \varphi_0$. The morphism $\varphi\colon A \to B$ is called a *homotopy equivalence* if there exists a morphism $\psi\colon B \to A$ such that $\varphi \circ \psi \simeq \mathrm{id}_B$ and $\psi \circ \varphi \simeq \mathrm{id}_A$. A C^*-algebra C is called *contractible* if $\mathrm{id}_C \simeq 0\colon C \to C$. Recall (cf. [3]) that the K-functor does not distinguish homotopic morphisms. Thus homotopy equivalences induce isomorphisms and contractible C^*-algebras have vanishing K-groups.

I. Mayer–Vietoris sequence. Let B_1, B_2 and C be C^*-algebras and $f_i\colon B_i \to C$ C^*-morphisms for $i = 1, 2$. Suppose f_2 is onto. Consider the pullback

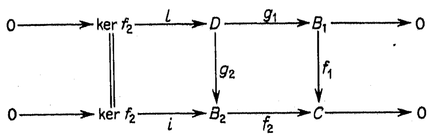


The C^*-algebra D can be written as $\{(b_1, b_2) \in B_1 \oplus B_2\colon f_1(b_1) = f_2(b_2)\}$. Then there is a natural inclusion $j\colon D \to B_1 \oplus B_2$. The map j induces group homomorphisms $j_*\colon K_*(D) \to K_*(B_1) \oplus K_*(B_2)$.

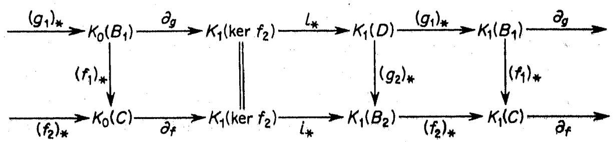
We define group homomorphisms $v_*\colon K_*(B_1) \oplus K_*(B_2) \to K_*(C)$ by $v_*\colon (f_1)_* - (f_2)_*$, where $(f_i)_*\colon K_*(B_i) \to K_*(C)$ is the group homomorphism induced by f_i for $i=1,2$. This means, for $\mathfrak{b}_i \in K_*(B_i)$, that $v_*(\mathfrak{b}_1 \oplus \mathfrak{b}_2) = (f_1)_*(\mathfrak{b}_1) - (f_2)_*(\mathfrak{b}_2)$.

There are two more maps which play an important role in the Mayer–Vietoris sequence. We show the construction of $\alpha_0\colon K_0(C) \to K_1(D)$; the map $\alpha_1\colon K_1(C) \to K_0(D)$ is constructed analogously.

Note first that there is a natural isomorphism between $\ker f_2$ and $\ker g_1$. Let $l\colon \ker f_2 \to D$ be the inclusion induced by that isomorphism. Note also that the surjectivity of f_2 implies that g_1 is onto. Thus we get the following commutative diagram with exact rows:

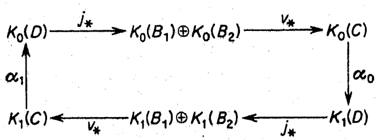


This diagram induces the following commutative diagram with exact rows:



Now we define $\alpha_0\colon K_0(C) \to K_1(D)$ by $\alpha_0 := l_* \circ \partial_f$.

Theorem (Mayer–Vietoris sequence, cf. [2], [6]). *Let B_1, B_2 and C be C^*-algebras, $f_i\colon B_i \to C$ be C^*-morphisms for $i=1,2$ and let D be the pullback over f_1 and f_2. Moreover, assume that f_2 is surjective. Then the following sequence is exact:*



Proof. First we show that $\operatorname{im} j_* \subset \ker v_*$. For $\mathfrak{d} \in K_*(D)$ we have

$$v_*(j_*(\mathfrak{d})) = v_*((g_1)_*(\mathfrak{d}) \oplus (g_2)_*(\mathfrak{d}))$$
$$= (f_1)_*((g_1)_*(\mathfrak{d})) - (f_2)_*((g_2)_*(\mathfrak{d})) = (f_1 \circ g_1)_*(\mathfrak{d}) - (f_2 \circ g_2)_*(\mathfrak{d}) = 0.$$

The reverse inclusion is obtained by a diagram chase in the above diagram. Let $\mathfrak{b}_1 \in K_*(B_1)$ and $\mathfrak{b}_2 \in K_*(B_2)$ be such that $(f_1)_*(\mathfrak{b}_1) = (f_2)_*(\mathfrak{b}_2)$. Then $\partial_g(\mathfrak{b}_1) = \partial_f \circ (f_1)_*(\mathfrak{b}_1) = \partial_f \circ (f_2)_*(\mathfrak{b}_2) = 0$ whence there exists $\mathfrak{d} \in K_*(D)$ such that $(g_1)_*(\mathfrak{d}) = \mathfrak{b}_1$. We have $(f_2)_*(\mathfrak{b}_2 - (g_2)_*(\mathfrak{d})) = 0$ so that there exists $\mathfrak{a} \in K_*(\ker f_2)$ with $i_*(\mathfrak{a}) = \mathfrak{b}_2 - (g_2)_*(\mathfrak{d})$. Now we set $\mathfrak{d}' = \mathfrak{d} + l_*(\mathfrak{a})$ and obtain

$(g_1)_*(\mathfrak{d}') = (g_1)_*(\mathfrak{d}) + 0 = \mathfrak{b}_1$ and $(g_2)_*(\mathfrak{d}') = (g_2)_*(\mathfrak{d}) + (g_2 \circ l)_*(\mathfrak{a}) = (g_2)_*(\mathfrak{d}) + i_*(\mathfrak{a}) = \mathfrak{b}_2$. Thus we have proved that $\operatorname{im} j_* = \ker v_*$.

It remains to be shown that the Mayer–Vietoris sequence is exact at the corners. We show that for the right side, the left side is proved analogously. To see that $\operatorname{im} v_* \subset \ker \alpha_*$ calculate for $\mathfrak{b}_i \in K_*(B_i)$ that $\alpha_*((f_1)_*(\mathfrak{b}_1) - (f_2)_*(\mathfrak{b}_2)) = \alpha_*((f_1)_*(\mathfrak{b}_1)) - \alpha_*((f_2)_*(\mathfrak{b}_2)) = l_* \circ \partial_g(\mathfrak{b}_1) - l_* \circ \partial_f \circ (f_2)_*(\mathfrak{b}_2) = 0$.

The reverse inclusion again requires a little diagram chase. Suppose, for $\mathfrak{c} \in K_*(C)$, that $\alpha_*(\mathfrak{c}) = 0$. Then $l_* \circ \partial_f(\mathfrak{c}) = 0$ and there exists a $\mathfrak{b}_1 \in K_*(B_1)$ with $\partial_g(\mathfrak{b}_1) = \partial_f(\mathfrak{c})$. Therefore $\partial_f((f_1)_*(\mathfrak{b}_1) - \mathfrak{c}) = \partial_g(\mathfrak{b}_1) - \partial_f(\mathfrak{c}) = 0$. This in turn implies that there exists a $\mathfrak{b}_2 \in K_*(B_2)$ with $(f_2)_*(\mathfrak{b}_2) = (f_1)_*(\mathfrak{b}_1) - \mathfrak{c}$, thus $\mathfrak{c} = v_*(\mathfrak{b}_1 \oplus \mathfrak{b}_2)$.

The inclusion $\operatorname{im} \alpha_* \subset \ker j_*$ is seen from the following calculation for $\mathfrak{c} \in K_*(C)$. We have $j_*(\alpha_*(\mathfrak{c})) = j_*(l_* \circ \partial_f(\mathfrak{c})) = (g_1)_* \circ l_* \circ \partial_f(\mathfrak{c}) \oplus (g_2)_* \circ l_* \circ \partial_f(\mathfrak{c}) = 0 \oplus i_* \circ \partial_f(\mathfrak{c}) = 0$.

Finally we get the reverse inclusion again by diagram chasing. Note that $\ker j_* = \ker (g_1)_* \cap \ker (g_2)_*$. Thus for $\mathfrak{d} \in \ker j_*$ there exists an $\mathfrak{a} \in K_*(\ker f_2)$ with $l_*(\mathfrak{a}) = \mathfrak{d}$. We get $i_*(\mathfrak{a}) = (g_2)_* \circ l_*(\mathfrak{a}) = 0$ and hence there exists a $\mathfrak{c} \in K_*(C)$ with $\partial_f(\mathfrak{c}) = \mathfrak{a}$. This implies that $\alpha_*(\mathfrak{c}) = l_* \circ \partial_f(\mathfrak{c}) = l_*(\mathfrak{a}) = \mathfrak{d}$. This concludes the proof. ∎

II. Puppe sequence.

Definition. Let A and B be C^*-algebras and $\varphi\colon B \to A$ a C^*-morphism. Define the *mapping cone*, denoted by C_φ, as follows:

$$C_\varphi = \{(b, f) \in B \oplus P(A)\colon \varphi(b) = f(0), f(1) = 0\},$$

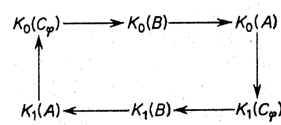
where $P(A) := \{f\colon I \to A \text{ continuous}\}$ is the algebra of paths in A.

Given the map $i\colon SA \to C_\varphi$ defined by $i(f) = (0, f)$ we get a sequence of C^*-algebras which we call the *Puppe sequence*:

$$SB \xrightarrow[S\varphi]{} SA \xrightarrow[i]{} C_\varphi \xrightarrow[v_\varphi]{} B \xrightarrow[\varphi]{} A$$

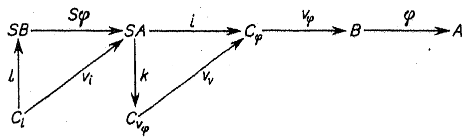
where $S\varphi(g) := \varphi \circ g$ and $v_\varphi((b, f)) := b$.

Theorem. *The Puppe sequence induces the following exact sequence in K-theory*:

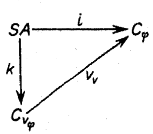


Proof. First we show that we can replace $K_*(SB)$ and $K_*(SA)$ by $K_*(C_i)$ and $K_*(C_{v_\varphi})$ respectively. In fact, we construct maps $k\colon SA \to C_{v_\varphi}$

and l: $C_i \to SB$ that induce isomorphisms in K-theory and give a diagram of the following kind that is commutative up to homotopy:

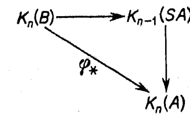


We have $C_{v_\varphi} = \{(b, f, g) \in B \oplus P(A) \oplus P(B)$: $\varphi(g(0)) = f(0)$, $f(1) = 0$, $b = g(0)$, $g(1) = 0\}$ which we can identify with $\{(f, g) \in P(A) \oplus P(B)$: $\varphi(g(0)) = f(0), f(1) = 0, g(1) = 0\}$. There is a map k: $SA \to C_{v_\varphi}$ defined by $k(f) := (f, 0)$. It is clearly injective. Now consider the cone $CB := \{g \in P(B)$: $g(1) = 0\}$ and the map μ: $C_{v_\varphi} \to CB$ defined by $\mu((f, g)) = g$. Since $((1-t)\varphi(g(0)), g) \in C_{v_\varphi}$ for any $g \in CB$ we see that μ is surjective. Clearly $\ker \mu = k(SA)$. But the cone CB is contractible and therefore the six-term sequence associated to $0 \to SA \to C_{v_\varphi} \to CB \to 0$ shows that k_*: $K_*(SA) \to K_*(C_{v_\varphi})$ is an isomorphism. Moreover, the natural map v_v: $C_{v_\varphi} \to C_\varphi$ defined by $(f, g) \mapsto (g(0), f)$ makes the following triangle commutative:



Now consider the mapping cone $C_i = \{(f, g, F) \in SA \oplus P(C_\varphi) \subset SA \oplus P(B) \oplus \oplus P(P(A))$: $i(f) = (g(0), F(\cdot, 0)), (g(1), F(\cdot, 1)) = (0, 0)\}$. We can identify C_i with the algebra $\{(g, F) \in P(B) \oplus P(P(A))$: $\varphi \circ g = F(0, \cdot)$, $F(1, \cdot) = F(\cdot, 1) = 0$, $g(0) = g(1) = 0\}$, as one easily sees, and consider the map l: $C_i \to SB$ given by $l(g, F) = g$. For a given $g \in SB$ set $F(s, t) := (\varphi \circ g(t))(1-s)$; then $(g, F) \in C_i$ and $l(g, F) = g$ whence l is surjective. The kernel of l is $\{(g, F) \in P(B) \oplus P(P(A))$: $F(0, \cdot) = F(1, \cdot) = F(\cdot, 1) = 0\}$ which is isomorphic to the cone $C(SA)$. Thus l_*: $K_*(C_i) \to K_*(SB)$ is an isomorphism. The map v_i: $C_i \to SA$ is given by $(g, F) \mapsto F(\cdot, 0)$. Consider the family of maps Φ_t: $C_i \to SA$ defined by $\Phi_t(s) := F(s(1-t), st)$; then $\Phi_0 = v_i$, $\Phi_1(s) = F(0, s)$ and Φ_t is a homotopy. Clearly $\Phi_1 = S\varphi \circ l$.

Now it suffices to prove that any sequence $C_\varphi \xrightarrow[v_\varphi]{} B \xrightarrow[\varphi]{} A$, where C_φ is the mapping cone of φ and the map v_φ is the projection onto the first factor, induces an exact sequence in K-theory. But this sequence gives rise to the short exact sequence $0 \to SA \to C_\varphi \to B \to 0$ which in turn induces the exact sequence $K_n(C_\varphi) \to K_n(B) \to K_{n-1}(SA)$. Since the triangle



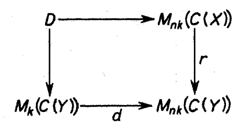
where the vertical map is the suspension isomorphism commutes we deduce that $K_*(C_\varphi) \to K_*(B) \to K_*(A)$ is exact. Thus the following sequence is exact:

$$K_*(SB) \xrightarrow{S\varphi_*} K_*(SA) \to K_*(C_\varphi) \to K_*(B) \xrightarrow{\varphi_*} K_*(A)$$

and we remark that $C_{S\varphi} = S(C_\varphi)$ so that we can close the exact sequence to obtain the diagram in the statement of the theorem. ■

III. Examples. The Mayer–Vietoris and the Puppe sequences can be applied to calculate the K-theory of C^*-algebras that are represented as section algebras of C^*-bundles. The following simple examples show how to calculate the K-groups of a section algebra from the K-groups of restrictions to smaller spaces.

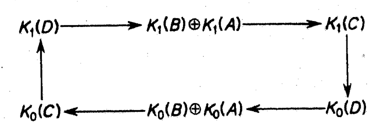
Let $Y \subset X$ be compact spaces. Define a C^*-algebra D as the following pullback:



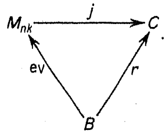
Here r simply denotes the restriction to Y and d is the map that assigns the block diagonal matrix

$$\begin{bmatrix} f & & \\ & \ddots & \\ & & f \end{bmatrix}$$

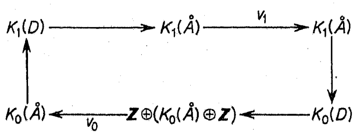
to an $f \in M_k(C(Y))$, the $k \times k$-matrix algebra over the continuous functions on Y. For the sake of brevity we define $B := M_{nk}(C(X))$, $A := M_k(C(Y))$ and $C := M_{nk}(C(Y))$. We obtain the Mayer–Vietoris sequence



The map d_*: $K_*(A) \to K_*(C)$ is, if we identify $K_*(A)$ with $K_*(C)$ under φ_*: $K_*(A) \to K_*(C)$ induced by the inclusion φ: $A \to C$ which maps $a \in A$ to the matrix in C that has a in the upper left corner and zeros elsewhere, just multiplication by n. If X is a contractible space and $y_0 \in Y$, the map ev: $B \to M_{nk} = M_{nk}(C)$ given as the evaluation at y_0 is a homotopy equivalence. Thus with the canonical embedding $j\colon M_{nk} \to C$ we get a commutative triangle up to homotopy:



Thus the triangle in K-theory induced by this one commutes, and since ev_* is an isomorphism, we can replace $K_*(B)$ by $K_*(M_{nk})$ and r_* by j_*. If we set $\mathring{A} := \{f \in A\colon f(y_0) = 0\}$ we get a split exact sequence $0 \to \mathring{A} \to A \to M_k \to 0$ and hence we get a split exact sequence in K-theory $0 \to K_*(\mathring{A}) \to K_*(A) \to K_*(M_k) \to 0$. Note that $K_1(M_{nk}) = K_1(M_k) = K_1(C) = 0$ and $K_0(M_{nk}) = K_0(M_k) = K_0(C) = \mathbf{Z}$. Hence we get the following exact sequence:



where the maps v_1 and v_0 are given as follows: $v_1(\mathfrak{a}) = -n\mathfrak{a}$ for $\mathfrak{a} \in K_1(\mathring{A})$ and

$$v_0(m \oplus (\mathfrak{a} \oplus l)) = (0 \oplus m) - n(\mathfrak{a} \oplus l) = -n\mathfrak{a} \oplus (m - nl)$$

for $m \oplus (\mathfrak{a} \oplus l) \in \mathbf{Z} \oplus (K_0(\mathring{A}) \oplus \mathbf{Z})$. If we assume that $K_1(\mathring{A})$ is torsion free, then v_1 is injective and therefore $K_1(D) \cong K_0(\mathring{A}) \oplus \mathbf{Z}/\mathrm{im}\, v_0$. But for $\mathfrak{c}, \mathfrak{d} \in K_0(\mathring{A})$ and $m_{\mathfrak{c}}, m_{\mathfrak{d}} \in \mathbf{Z}$ we have $\mathfrak{c} \oplus m_{\mathfrak{c}} - \mathfrak{d} \oplus m_{\mathfrak{d}} \in \mathrm{im}\, v_0$ if and only if there is an $\mathfrak{a} \in K_0(\mathring{A})$ and $m, l \in \mathbf{Z}$ such that $\mathfrak{c} - \mathfrak{d} = -n\mathfrak{a}$ and $m_{\mathfrak{c}} - m_{\mathfrak{d}} = m - nl$. The condition on the integers is always satisfied, thus $K_1(D) \cong K_0(\mathring{A})/nK_0(\mathring{A}) = K_0(\mathring{A}) \otimes \mathbf{Z}/n\mathbf{Z}$ if $K_0(\mathring{A})$ is torsion free, too. Further, we have the exact sequence $0 \to K_1(\mathring{A})/\mathrm{im}\, v_1 \to K_0(D) \to \ker v_0 \to 0$. We assumed $K_0(\mathring{A})$ to be torsion free, so $\ker v_0 = \{m \oplus (\mathfrak{a} \oplus l) \in \mathbf{Z} \oplus (K_0(\mathring{A}) \oplus \mathbf{Z})\colon -n\mathfrak{a} = 0,\ m = nl\} \cong \mathbf{Z}$. Thus the sequence splits and since $\mathrm{im}\, v_1 = nK_1(\mathring{A})$ we have $K_0(D) \cong (K_1(\mathring{A}) \otimes \mathbf{Z}/n\mathbf{Z}) \oplus \mathbf{Z}$. If we now observe that $K_*(\mathring{A}) \cong K_*(C_0(Y)) = \tilde{K}^*(Y)$ we get the following

EXAMPLE 1. Let $Y \subset X$ be compact spaces such that X is contractible and $\tilde{K}^*(Y)$ torsion free, and let D be the C^*-algebra of continuous functions from X into M_{nk} such that the values on Y are block diagonal matrices with identical blocks of size $k \times k$. Then $K_0(D) = (\tilde{K}^1(Y) \otimes \mathbf{Z}/n\mathbf{Z}) \oplus \mathbf{Z}$ and $K_1(D) = \tilde{K}^0(Y) \otimes \mathbf{Z}/n\mathbf{Z}$.

Similarly one calculates

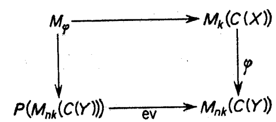
EXAMPLE 2. Let $Y \subset X$ be compact spaces with X contractible and D the algebra of continuous functions $X \to M_{nk}$ that map Y to block diagonal matrices with blocks of size $k \times k$. Then $K_0(D) = (\tilde{K}^0(Y))^{n-1} \oplus \mathbf{Z}^n$ and $K_1(D) = (\tilde{K}^1(Y))^{n-1}$.

The assumption that X be contractible has of course been made to avoid problems in calculation which arise from the fact that we do not know the map $r_*\colon K_*(B) \to K_*(C)$ in general. There are some more cases where we know this map.

EXAMPLE 3. Let $Y \subset X$ be compact spaces and Y a deformation retract of X. Let D be the C^*-algebra of continuous functions $X \to M_{nk}$ such that the values on Y are block diagonal matrices with identical blocks of size $k \times k$. Then $K_*(D) \cong K^*(Y)$. If the condition that the blocks be identical is dropped we have: $K_*(D) \cong K^*(X) \oplus (K^*(Y))^{n-1}$.

We have seen that torsion in the K-groups can cause trouble. In some cases we can get around that using the Puppe sequence.

Let X and Y be compact spaces and $f\colon Y \to X$ a continuous map. Consider the mapping cone C_f. We obtain a map $f'\colon Y \to C_f$ which is the composition of f and the canonical map $g\colon X \to C_f$. Now consider the C^*-algebras $M_k(C(X))$ and $M_{nk}(C(Y))$. We get a map $\varphi\colon M_k(C(X)) \to M_{nk}(C(Y))$ by $\varphi(a) := d(a \circ f)$ [cf. Example 1 for the definition of d]. Consider the mapping cylinder M_φ given by the pullback



where ev is the evaluation at $t = 0$. Note that $P(M_{nk}(C(Y)))$ is canonically isomorphic to $M_{nk}(C(Y \times I))$ and $M_k(C(X))$ is canonically isomorphic to the algebra of maps $X \to M_{nk}$ whose values are block diagonal matrices with identical blocks of size $k \times k$. Thus we see that C_φ is the C^*-algebra of maps from C_f into M_{nk} whose values on $g(X)$ are block diagonal matrices with identical blocks of size $k \times k$ and which vanish on $y_0 \in C_f$, the vertex of the cone. Now it is easy to get

EXAMPLE 4. Let X and Y be compact spaces and $f\colon Y \to X$ a continuous function. Let C_f be the mapping cone of f and D the C^*-algebra of continuous functions from C_f into M_{nk} whose values on the canonical image of X in C_f are block diagonal matrices with identical blocks of size $k \times k$. Let $\mathring{D}$ be the subalgebra of D consisting of those maps that vanish on $y_0 \in C_f$, the vertex of the cone. Then $K_*(D) \cong K_*(\mathring{D})$ and we get the following exact sequence:

$$\begin{array}{ccccc} K_1(\mathring{D}) & \longrightarrow & K^1(X) & \xrightarrow{nf^1} & K^1(Y) \\ \uparrow & & & & \downarrow \\ K^0(Y) & \xleftarrow[nf^0]{} & K^0(X) & \longleftarrow & K_0(\mathring{D}) \end{array}$$

Finally, if we drop the condition on the blocks, we get

EXAMPLE 5. Let X and Y be compact spaces and $f\colon Y \to X$ a continuous function. Let C_f be the mapping cone of f and D the C^*-algebra of maps from C_f into M_{nk} whose values on the canonical image of X in C_f are block diagonal matrices with blocks of size $k \times k$. Let $\mathring{D} := \ker \mathrm{ev}$, where ev is the evaluation at the vertex $y_0 \in C_f$. Then $K_1(D) \cong K_1(\mathring{D})$ and $K_0(D) \cong \mathbf{Z} \oplus K_0(\mathring{D})$. Moreover, we have the following exact sequence:

$$\begin{array}{ccccc} K_1(\mathring{D}) & \longrightarrow & \bigoplus_1^n K^1(X) & \xrightarrow{\Sigma f^1} & K^1(Y) \\ \uparrow & & & & \downarrow \\ K^0(Y) & \xleftarrow[\Sigma f^0]{} & \bigoplus_1^n K^0(X) & \longleftarrow & K_0(\mathring{D}) \end{array}$$

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τ-smooth linear functionals on vector lattices of real-valued functions

by

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Abstract. A vector lattice E of real-valued functions is said to be a strong Daniell lattice if every positive linear functional $\Phi\colon E \to \mathbf{R}$ is τ-smooth (i.e. $\lim_\alpha \Phi(f_\alpha) = 0$ for every net (f_α) in E with $f_\alpha \downarrow 0$). Under some additional assumptions which, in general, cannot be omitted, several characterizations of strong Daniell lattices are given. These results are then applied to the vector lattices $\mathscr{C}(\mathscr{L}, \mathscr{B})$ and $\mathscr{C}^b(\mathscr{L}, \mathscr{B})$ of $\mathscr{L}$-continuous (and bounded) functions with $\mathscr{B}$-bounded support, where $\mathscr{L}$ denotes a lattice of sets and $\mathscr{B}$ is an $\mathscr{L}$-bounding system.

1. Introduction. This paper is a continuation of [4]. However, whereas in [4] we are concerned with the characterization of *Daniell lattices* (i.e. vector lattices E of real-valued functions having the property that every positive linear functional on E is σ-smooth), we consider in this paper only such vector lattices on which every positive linear functional is τ-smooth. Under some additional assumptions which, in general, cannot be omitted, we give several characterizations of these so-called *strong Daniell lattices*. As application of these general characterization theorems, we can prove, among others, the following results:

(1) *For a completely regular space X the following statements are equivalent*:

(a) *X is realcompact.*

(b) *The space of all continuous functions on X is a strong Daniell lattice.*

(c) *The space of all Baire-measurable functions on X is a strong Daniell lattice.*

(2) *If $(X, \mathscr{A})$ is a measurable space, then X is $\mathscr{A}$-complete* ([1]) *iff the space of all $\mathscr{A}$-measurable functions on X is a strong Daniell lattice. In particular, a topological space X is Borel-complete* ([11]) *iff the space of all Borel-measurable functions on X is a strong Daniell lattice.*

Some special cases of our results can be found in [10] and [17]. However, the methods of proof are different. Our proceeding seems to be more direct; in contrast to [10] and [17], we do not make use of any compactification.

Throughout this paper X will denote an arbitrary nonvoid set and $E \subset \mathbf{R}^X$ a vector lattice (with respect to pointwise operations). 1_Q denotes the indicator function of a subset Q of X. For $f \in E$ we put $\|f\|$