On the reflexivity of pairs of isometries and of tensor products of some operator algebras

by

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Abstract. In the present paper we prove the reflexivity of a WOT-closed algebra generated by a pair of doubly commuting isometries. Our next result is the reflexivity of the tensor product of the algebra of all analytic Toeplitz operators on H^2 with any reflexive algebra.

1. Introduction and preliminaries. $L(K)$ denotes the algebra of all (linear bounded) operators in a complex separable Hilbert space K. I_K or I stands for the identity in K. By a subspace of K we always mean a closed subspace and by an algebra of operators on K we mean a subalgebra of $L(K)$ with unit I_K. If $\mathscr{S}$ is subset of $L(K)$, then $\mathfrak{A}(\mathscr{S})$, $\operatorname{Lat}\mathscr{S}$ stand for the WOT (= weak operator topology)-closed algebra generated by $\mathscr{S}$ and the lattice of all invariant subspaces for $\mathscr{S}$, respectively. If $T\in L(K)$ then the shorter notation $\mathfrak{A}(T)$, $\mathfrak{A}(T, \mathscr{S})$, $\operatorname{Lat} T$, $\operatorname{Lat}(T, \mathscr{S})$ will be used instead of $\mathfrak{A}(\{T\})$, $\mathfrak{A}(\{T\}\cup\mathscr{S})$, $\operatorname{Lat}\{T\}$, $\operatorname{Lat}(\{T\}\cup\mathscr{S})$. $\operatorname{Alg}\operatorname{Lat}\mathscr{S}$ stands for the algebra of all operators on K which leave invariant all subspaces from $\operatorname{Lat}\mathscr{S}$. An algebra $\mathscr{A}$ is called *reflexive* if $\mathscr{A} = \operatorname{Alg}\operatorname{Lat}\mathscr{A}$. A family $\mathscr{S}\subset L(K)$ is called *reflexive* if the algebra $\mathfrak{A}(\mathscr{S})$ is reflexive. An operator $T\in L(K)$ is called *reflexive* if so is $\mathscr{S} = \{T\}$.

Sarason [11] proved that every commutative WOT-closed algebra of normal or Toeplitz operators is reflexive. The reflexivity of an isometry was proved by Deddens [2]. Another proof of this fact was given by Wogen [14]. He also proved that quasinormal operators are reflexive. Then Olin and Thomson [9] proved the reflexivity of subnormal operators.

In this paper we study the reflexivity of a pair (two-element family) of isometries. Deddens', Wogen's and Olin and Thomson's proofs needed a sort of canonical decomposition of an isometry, of a quasinormal and of a subnormal operator respectively. But a Wold-type decomposition which was necessary there need not exist for two commuting isometries [12, Example 1]. Słociński showed [12] that the Wold-type decomposition holds for any pair $\{V_1, V_2\}$ of *doubly commuting* isometries (i.e. V_1, V_2 commute and V_1, V_2^* commute). Our first main result is:

THEOREM 1. *Every pair* $\{V_1, V_2\}$ *of doubly commuting isometries is reflexive.*

This problem is connected with the reflexivity of the tensor product of reflexive algebras. There are some partial result concerning this problem in [1], [4], [5], [7], [8]. In this paper we prove another result in this direction. Let $\mathscr{A}$, $\mathscr{B}$ be WOT-closed algebras of operators on Hilbert spaces K, H, respectively. Then, following [8], $\mathscr{B}\bar{\otimes}\mathscr{A}$ denotes the WOT-closure in $L(H\otimes K)$ of the algebraic tensor product $\mathscr{B}\otimes\mathscr{A}$ (for the definition of tensor product see [13, Chap. IV]). We denote by H^2 the classical Hardy space on the unit circle and $\mathscr{H}^\infty$ denotes the collection of all analytic Toeplitz operators on H^2. Then we can prove

THEOREM 2. *For any algebra* $\mathscr{A}_0$ *in* $L(K)$, $\mathscr{H}^\infty\bar{\otimes}\mathscr{A}_0$ *is reflexive whenever so is* $\mathscr{A}_0$.

Now, let us fix a separable Hilbert space K with the inner product $(\cdot, \cdot)$. If $\mathscr{L}\subset K$ then $[\mathscr{L}]$ denotes the smallest subspace of K containing $\mathscr{L}$. Let $T\in L(K)$ and let L belong to Lat T. Then $T|_L$ denotes the restriction of T to L. If n is a positive integer, then $K^{(n)}$ denotes the direct sum of n copies of K, $T^{(n)}$ denotes the direct sum of n copies of T acting on $K^{(n)}$. If $\mathscr{T}\subset L(K)$ then we denote $\mathscr{T}^{(n)} = \{T^{(n)}\colon T\in\mathscr{T}\}$. If $x\in K$ then $C(\mathscr{T}, x)$ denotes the smallest subspace of K containing x and invariant for $\mathscr{T}$. Following [10, Chap. 3] we recall some definitions. A function f from the unit circle C into K will be called *measurable* if, for each $x\in K$, the function $z\to(f(z), x)$ is Lebesgue measurable. $L^2(K)$ denotes the collection of all measurable functions f from C to K such that $\int\|f(z)\|^2\,dm < \infty$, where m is the normalized Lebesgue measure on C. We identify functions which are equal m-a.e. $H^2(K)$ denotes the set of functions $f\in L^2(K)$ such that $\int(f(z), x)z^n\,dm = 0$ for all $x\in K$ and $n\geqslant 1$. If K is the field of complex numbers $\mathbf{C}$, then $H^2(\mathbf{C})$ equals H^2.

We will also study operator-valued functions. Let $\mathscr{A}\subset L(K)$ be a WOT-closed algebra. A function F from C into $\mathscr{A}$ is said to be *measurable* if, for every $x\in K$, the K-valued function $z\to F(z)x$ is measurable. For such an F, let $\|F\|_\infty$ denote the essential supremum of $\|F(z)\|$ on C, i.e. $\|F\|_\infty = \inf\{\sup\{\|F(z)\|\colon z\in\sigma\}\colon \sigma$ is a Borel subset of C, $m(C-\sigma) = 0\}$. $L^\infty(\mathscr{A})$ denotes the collection of all measurable functions F from C to $\mathscr{A}$ such that $\|F\|_\infty$ is finite (we identify functions equal m-a.e.).

We can treat elements of $L^\infty(\mathscr{A})$ as operators in $L(L^2(K))$ as follows. Let $F\in L^\infty(\mathscr{A})$ and $f\in L^2(K)$. Then Ff is the function in $L^2(K)$ such that $(Ff)z = F(z)f(z)$. Since $\|F\|_\infty$ is finite, F is bounded as an operator on $L^2(K)$. $H^\infty(\mathscr{A})$ denotes the set of all elements $F\in L^\infty(\mathscr{A})$ such that $F(H^2(K))\subset H^2(K)$; such F can be treated as operators on $H^2(K)$ and $H^\infty(\mathscr{A})$ is the collection of all such operators F on $H^2(K)$.

The unilateral shift (shortly, shift) in $H^2(K)$ will play the main role in this paper. Let us recall that the *shift* in $H^2(K)$ is defined as $(Sf)z = zf(z)$ for $f\in H^2(K)$. In what follows S will always denote such a shift.

2. Properties C and C$_\infty$. A subset $\mathscr{T}_0$ of $L(K)$ has *property* C if, for each positive integer n and each y in $K^{(n)}$, there is $x\in K$ and a unitary operator $U\colon C(\mathscr{T}_0^{(n)}, y)\to C(\mathscr{T}_0, x)$ such that

$$UT^{(n)}|_{C(\mathscr{T}_0^{(n)}, y)}U^* = T|_{C(\mathscr{T}_0, x)}$$

for every T in $\mathscr{T}_0$. Property C was introduced by Wogen [14]. He applied it to prove the reflexivity of an isometry. But we will also need another property. Let $\mathscr{T}_0\subset L(K)$. If $T_0\in\mathscr{T}_0$ then T denotes the element of $H^\infty(L(K))$ defined by $T(z) = T_0$ for z in C. Put $\mathscr{T} = \{T\colon T_0\in\mathscr{T}_0\}$. We will say that $\mathscr{T}_0$ has *property* C$_\infty$ if for each f in $H^2(K)$ there is x in K and a unitary operator $U\colon C(\mathscr{T}, f)\to C(\mathscr{T}_0, x)$ such that

$$UT|_{C(\mathscr{T}, f)}U^* = T_0|_{C(\mathscr{T}_0, x)}$$

for every T_0 in $\mathscr{T}_0$.

It is clear that property C$_\infty$ implies property C and that $\mathscr{T}_0$ contained in $L(K)$ has property C (or C$_\infty$) if and only if $\mathfrak{A}(\mathscr{T}_0)$ does. Now, we have

Remark 3. (1) *Each algebra of normal operators has property* C$_\infty$.

(2) *Each shift of arbitrary multiplicity has property* C$_\infty$.

The proof that any algebra of normal operators has property C is essentially contained in the proof of the reflexivity of such an algebra [10, Theorem 9.21]. Taking $H^2(K)$ instead of $K^{(n)}$ and following the idea of the above-mentioned proof, we get property C$_\infty$ for an algebra of normal operators. (2) is easy to see because a shift of arbitrary multiplicity restricted to its cyclic invariant subspace is unitarily equivalent to the shift of multiplicity 1 [10, Theorem 3.33].

The following proposition will be needed:

PROPOSITION 4. *Let S be the shift on $H^2(K)$ and $T_0\in L(K)$. T denotes the element of $H^\infty(L(K))$ defined by $T(z) = T_0$ for z in C. If T_0 has property* C$_\infty$ *then* $\{T, S\}$ *has property* C.

Wogen [14, proof of Lemma 2] proved Proposition 4 when T_0 was a normal operator. If T_0 has only property C$_\infty$ Wogen's proof also applies with small modifications (we first apply property C$_\infty$ and next property C which results from property C$_\infty$). Theorem 1 in [14] can be generalized to

THEOREM 5. *Let $\mathscr{T} = \{T_i\}$ $(i\in I)$ be a set of operators in a separable Hilbert space K_1 and let $\mathscr{R} = \{R_i\}$ $(i\in I)$ be a set of operators in a separable Hilbert space K_2. $\mathscr{T}\oplus\mathscr{R}$ denotes the collection of all operators $T_i\oplus R_i$ on $K_1\oplus K_2$ for $i\in I$. If the algebras $\mathfrak{A}(\mathscr{T})$, $\mathfrak{A}(\mathscr{R})$ are reflexive and have property* C *then $\mathfrak{A}(\mathscr{T}\oplus\mathscr{R})$ is reflexive and has property* C.

The proof can be done in the same way as the proof of Theorem 1 in [14].

3. Reflexivity of $\mathscr{H}^\infty\bar{\otimes}\mathscr{A}_0$. The unitary isomorphism between $H^2\otimes K$ and $H^2(K)$ is well known. Hence, operators on $H^2\otimes K$ correspond unitarily to operators on $H^2(K)$. In particular, if $\varphi\in\mathscr{H}^\infty$ and $A\in L(K)$ then $\varphi\otimes A$ corresponds to the operator-valued function $\varphi(\cdot)A$ acting as an operator on $H^2(K)$ as follows: $(\varphi(\cdot)Af)(z)=\varphi(z)Af(z)$ for each $f\in H^2(K)$. For the sake of convenience, we will study operator-valued functions instead of tensor products.

Let us fix for this section a WOT-closed algebra $\mathscr{A}_0$ in $L(K)$. Then $\mathscr{A}$ denotes the collection of all $A\in H^\infty(L(K))$ such that there is $A_0\in\mathscr{A}_0$ with $A(z)=A_0$ for almost all z. It easy to see that $\mathscr{H}^\infty\bar{\otimes}\mathscr{A}_0$ regarded as an algebra of operators on $H^2\otimes K$ corresponds unitarily to $\mathfrak{A}(S,\mathscr{A})$ as an algebra on $H^2(K)$. Thus Theorem 2 becomes

THEOREM 2'. *If $\mathscr{A}_0$ is reflexive then $\mathfrak{A}(S,\mathscr{A})$ is reflexive.*

Proof. Suppose that $B\in L(H^2(K))$ and $\operatorname{Lat}(S,\mathscr{A})\subset\operatorname{Lat}B$. Let $\mathscr{A}_0^*$ denote the set $\{A_0^*\colon A_0\in\mathscr{A}_0\}$ and let $\mathscr{A}^*$ denote the set $\{A^*\colon A\in\mathscr{A}\}$. Then $\operatorname{Lat}(S^*,\mathscr{A}^*)\subset\operatorname{Lat}B^*$. For a complex number a of modulus less than 1, we define the function h_a on C as follows: $h_a(z)=(1-az)^{-1}$ for $z\in C$. The function h_a is an eigenvector of the adjoint of the shift of multiplicity 1 on H^2 corresponding to the eigenvalue a. Let $L\in\operatorname{Lat}\mathscr{A}^*$ and let $x\in L$. Thus the function $z\to h_a(z)x$, shortly denoted by $h_a x$, is an eigenvector of S^* with the same eigenvalue a. Thus $h_a L$ is invariant for S^*. Let $A^*\in\mathscr{A}^*$. Then

$$\big(A^*(h_a x)\big)(z)=A_0^*h_a(z)x=h_a(z)A_0^*x=(h_a y)(z)$$

where $y\in L$, because $L\in\operatorname{Lat}\mathscr{A}_0^*$. Thus h_aL is invariant for each element of $\mathscr{A}_0^*$. Hence, h_aL is invariant for B^*. Let $x\in K$ and let a be as above. Then there is $y\in K$ such that $B^*(h_a x)=h_a y$. Thus

$$B^*S^*(h_a x)=B^*(ah_a x)=aB^*(h_a x)=ah_a y=S^*(h_a y)=S^*B^*(h_a x).$$

The set $\{h_a x\colon x\in K,\ |a|<1\}$ is linear dense in $H^2(K)$ because $\{h_a\colon |a|<1\}$ is linear dense in H^2. Thus B^* commutes with S^*, i. e. B commutes with S. Hence, $B\in H^\infty(L(K))$ [10, Corollary 3.20].

Now, we prove that $B\in H^\infty(\mathscr{A}_0)$. Let $L\in\operatorname{Lat}\mathscr{A}_0$. Then $H^2(L)\in\operatorname{Lat}(S,\mathscr{A})\subset\operatorname{Lat}B$. Let $x\in L$ and let $\tilde{x}$ denote the function in $H^2(L)$ defined by $\tilde{x}(z)=x$ for all $z\in C$. Thus $B\tilde{x}\in H^2(L)$, so $L\ni(B\tilde{x})(z)=B(z)\tilde{x}(z)=B(z)x$ for almost all $z\in C$. Hence, $\operatorname{Lat}\mathscr{A}_0\subset\operatorname{Lat}B(z)$ for almost all $z\in C$. Since $\mathscr{A}_0$ is reflexive, $B(z)\in\mathscr{A}_0$ for almost all $z\in C$. Hence, B is an element of $H^\infty(\mathscr{A}_0)$. The following crucial lemma will finish the proof of Theorem 2'. It is separated from the whole proof, because it may be of independent interest.

LEMMA 6. *If $\mathscr{A}_0$ is reflexive then $\mathfrak{A}(S,\mathscr{A})=H^\infty(\mathscr{A}_0)$.*

Proof. Since $\{S\}\cup\mathscr{A}$ is contained in $H^\infty(\mathscr{A}_0)$, to prove the inclusion

$\subset$ it is enough to show that $H^\infty(\mathscr{A}_0)$ is WOT-closed. Let A be a WOT-limit of a sequence of elements of $H^\infty(\mathscr{A}_0)$. It is obvious that A is a measurable operator-valued function. $A\in L^\infty(L(K))$ because $\|A\|$ as norm of an operator is equal to $\|A\|_\infty$ [3, Chap. II, § 2, Proposition 2]. It is easy to see that $A\in H^\infty(L(K))$. Now $A\in H^\infty(\mathscr{A}_0)$ because $\mathscr{A}_0$ is WOT-closed.

The important part of the proof is the proof of the inclusion $\supset$. For the sake of convenience, we will study vector-valued or operator-valued functions defined on the interval $[-\pi,\pi]$ instead of C. Let B be an element of $H^\infty(\mathscr{A}_0)$ and let k be an integer. If $x,y\in K$ then

$$\Big|(2\pi)^{-1}\int_{-\pi}^{\pi}\big(B(t)x,\,y\big)e^{-ikt}\,dt\Big|\leqslant\|B\|_\infty\|x\|\,\|y\|.$$

Hence, the function $(x,y)\to(2\pi)^{-1}\int_{-\pi}^{\pi}\big(B(t)x,\,y\big)e^{-ikt}\,dt$ is a bounded sesquilinear form on K. By [13, Chap. II, Theorem 1.3], there is a bounded operator B_k on K such that

$$(B_k x,\,y)=(2\pi)^{-1}\int_{-\pi}^{\pi}\big(B(t)x,\,y\big)e^{-ikt}\,dt.$$

$H^2(K)$ is invariant for B, so it is easy to see that $B_k=0$ for $k=-1,-2,\ldots$

Now, we prove that $B_k\in\mathscr{A}_0$ for all integers k. Let $L\in\operatorname{Lat}\mathscr{A}_0$ and $x\in L$, $y\in L^\perp$ (the orthogonal complement of L in K). Then $\big(B(t)x,\,y\big)=0$ for almost all t, thus $(B_k x,\,y)=0$. This means that $B_k x\in L$. Hence L is invariant for B_k. Since $\mathscr{A}_0$ is reflexive, $B_k\in\mathscr{A}_0$.

Let us denote

$$\tilde{\sigma}_n=\sum_{k=0}^{n}B_kS^k=\sum_{k=-n}^{n}B_kS^k,\qquad n=1,2,\ldots,$$

$$\sigma_n=n^{-1}(\tilde{\sigma}_0+\ldots+\tilde{\sigma}_{n-1}).$$

If k_n denotes the nth Fejér's kernel [6, Chap. II] then as in [6, Chap. II] we can prove that

$$(*)\qquad \big(\sigma_n(s)x,\,y\big)=(2\pi)^{-1}\int_{-\pi}^{\pi}\big(B(t)x,\,y\big)k_n(s-t)\,dt=(2\pi)^{-1}\int_{-\pi}^{\pi}\big(B(s-t)x,\,y\big)k_n(t)\,dt$$

for all $x,y\in K$, $s\in[-\pi,\pi]$ and $n=1,2,\ldots$ In the second integral B is periodically extended to the function on the real line.

To finish our proof, it is enough to show that σ_n converges to B in WOT. Let $f,g\in H^2(K)$. We can periodically extend f,g to the whole real line and define for real t the functions $f_t,g_t\in H^2(K)$ as follows:

$f_t(s) = f(s+t)$, $g_t(s) = g(s+t)$ for $s \in [-\pi, \pi]$. We also define the functions $B_t \in H^\infty(\mathscr{A}_0)$ for real t by $B_t(s) = B(s-t)$ for $s \in [-\pi, \pi]$. Our first step is to show that $(B_t f, g) \to (Bf, g)$ as $t \to 0$. It is easy to prove that $(B_t f, g) = (Bf_t, g_t)$. Hence, we have

$$\begin{aligned}|(B_t f, g) - (Bf, g)| &= |(Bf_t, g_t) - (Bf, g)| \\ &\leqslant |(Bf_t, g_t) - (Bf_t, g)| + |(Bf_t, g) - (Bf, g)| \\ &\leqslant |(Bf_t, g_t - g)| + |(B(f_t - f), g)| \\ &\leqslant \|B\|_\infty \|f_t\| \|g_t - g\| + \|B\|_\infty \|f_t - f\| \|g\|.\end{aligned}$$

Because $\|f_t\| = \|f\|$ and $f_t \to f$, $g_t \to g$ as $t \to 0$ the desired result is proved.

The equality $(*)$ is satisfied for all $x, y \in K$, in particular for the vectors $f(s)$, $g(s)$. Now, using the inner product in K we can carry out the estimation in the same way as in [6, Chap. II, p. 19]. For all $\delta > 0$ we get the inequality

$$\begin{aligned}|(\sigma_n f, g) - (Bf, g)| &\leqslant \sup_{|t|<\delta} \Big|(2\pi)^{-1} \int_{-\pi}^{\pi} \big((B(s-t) - B(s)) f(s), g(s)\big) ds\Big| \\ &\quad + \sup_{|t|\geqslant\delta} k_n(t) \cdot 2\|B\|_\infty \|f\| \|g\| \\ &= \sup_{|t|<\delta} |((B_t - B) f, g)| + 2\|B\|_\infty \|f\| \|g\| \sup_{|t|\geqslant\delta} k_n(t).\end{aligned}$$

$(B_t f, g)$ converges to (Bf, g) as $t \to 0$ and $\sup_{|t|\geqslant\delta} k_n(t)$ converges to 0 as $n \to \infty$ as the Fejér's kernel. Hence $\sigma_n \to B$ as $n \to \infty$ in WOT. ∎

Hadwin and Nordgren [5] called an operator algebra *super-reflexive* if every its WOT-closed subalgebra is reflexive. From the previous results we get immediately:

PROPOSITION 7. *If $\mathscr{A}_0 \subset L(K)$ is reflexive and has property* C_∞ *then $\mathscr{H}^\infty \bar{\otimes} \mathscr{A}_0$ is super-reflexive.*

Proof. The algebra $\mathscr{H}^\infty \bar{\otimes} \mathscr{A}_0$ is reflexive by Theorem 2 and has property C by Proposition 4. Thus, by Proposition 2.5 in [5], $\mathscr{H}^\infty \bar{\otimes} \mathscr{A}_0$ is super-reflexive. ∎

Proposition 7 and Remark 3 imply immediately the following

COROLLARY 8. (1) *The algebra $\mathscr{H}^\infty \bar{\otimes} \mathscr{H}^\infty$ is super-reflexive.*

(2) *If $\mathscr{A}_0$ is an algebra of normal operators, then $\mathscr{H}^\infty \bar{\otimes} \mathscr{A}_0$ is super-reflexive.*

(3) *If $\mathscr{A} \subset L(K)$ is an algebra of normal operators then the algebra $\mathscr{H}^\infty \otimes L^\infty(\mathscr{A})$ is super-reflexive as an algebra of operators on $H^2 \otimes L^2(K)$.*

(3) was also proved in [5].

4. Reflexivity of a pair of isometries. Before we present the proof of Theorem 1, we show the following proposition concerning "acting in orthogonal directions" which will be useful.

PROPOSITION 9. *Let U be a normal operator in $L(H^2(K))$ which commutes with the shift S on $H^2(K)$. Then $U \in H^\infty(L(K))$ and there is a normal operator $U_0 \in L(K)$ such that $U(z) = U_0$ for almost all z in C.*

Proof. By Corollary 3.20 in [10], $U \in H^\infty(L(K))$ and $U(z)$ is normal for almost every z by Theorem 3.17 from [10]. For each $x \in K$ we define the function $\tilde{x} \in H^2(K)$ putting $\tilde{x}(z) = x$ for all z in C. Now, we prove that

$(**)$ *for each x in K, there is y_0 in K such that*

$$U\tilde{x} = \tilde{y}_0.$$

Let $x \in K$. Then there is $y_0 \in K$ and $f \in H^2(K)$ such that $U\tilde{x} = \tilde{y}_0 + Sf$. Thus $U\tilde{x} - \tilde{y}_0 = Sf$. Hence

$$S^*(U\tilde{x} - \tilde{y}_0) = S^* Sf = f.$$

By Putnam's theorem [10, Corollary 1.19]

$$f = US^* \tilde{x} - S^* \tilde{y}_0 = 0.$$

Thus $(**)$ is fulfilled.

For $x, y \in K$ and nonnegative integers n, m we denote by $\tilde{x}_n$, $\tilde{y}_m$ the functions defined as follows: $\tilde{x}_n(z) = z^n x$ and $\tilde{y}_m(z) = z^m y$. Since $(**)$ is fulfilled, if $m \neq n$ we have

$$(U\tilde{x}_n, \tilde{y}_m) = (US^n \tilde{x}, S^m \tilde{y}) = (S^n U\tilde{x}, S^m \tilde{y}) = 0.$$

It is obvious that the function $(x, y) \to (U\tilde{x}, \tilde{y})$ is a bounded sesquilinear form on K. By Theorem 1.3 in [13, Chap. II], there is an operator U_0 on K such that $(U_0 x, y) = (U\tilde{x}, \tilde{y})$ for all $x, y \in K$. $\tilde{U}_0$ denotes the operator in $H^\infty(L(K))$ defined by $\tilde{U}_0(z) = U_0$ for all z. To complete the proof, the equality $(\tilde{U}_0 f, g) = (Uf, g)$ for $f, g \in H^2(K)$ is needed. Because $\tilde{U}_0$, U are bounded, it is enough to show that $(\tilde{U}_0 \tilde{x}_n, \tilde{y}_m) = (U\tilde{x}_n, \tilde{y}_m)$ for all x, y in K and for all nonnegative integers n, m. Let $x, y \in K$. If $m \neq n$ then $(U\tilde{x}_n, \tilde{y}_m) = 0$ and

$$(\tilde{U}_0 \tilde{x}_n, \tilde{y}_m) = \int (U_0 x z^n, y z^m) dm = (U_0 x, y) \int z^n z^{-m} dm = 0.$$

If $m = n$ then

$$\begin{aligned}(U\tilde{x}_n, \tilde{y}_m) = (U\tilde{x}_n, \tilde{y}_n) &= \int (U(z) x z^n, y z^n) dm \\ &= \int (U(z) x, y) dm = (U\tilde{x}, \tilde{y}) \\ &= (U_0 x, y) = (U_0 x, y) \int z^n z^{-n} dm \\ &= \int (U_0 x z^n, y z^n) dm = (\tilde{U}_0 \tilde{x}_n, \tilde{y}_m).\end{aligned}$$

Hence, the above necessary equality is proved. ∎

A result like Proposition 9 concerning "acting in orthogonal directions" for two doubly commuting shifts was proved by Słociński [12, Theorem 1].

Proof of Theorem 1. Let V_1, V_2 be two doubly commuting isometries on a separable Hilbert space H. By a result of Słociński [12, Theorem 3], there exist subspaces H_{uu}, H_{us}, H_{su}, H_{ss} such that:

$$H = H_{uu} \oplus H_{us} \oplus H_{su} \oplus H_{ss},$$

H_{uu}, H_{us}, H_{su}, H_{ss} reduce V_1 and V_2,

$V_1|_{H_{uu}}$ and $V_2|_{H_{uu}}$ are unitary operators,

$V_1|_{H_{us}}$ is unitary and $V_2|_{H_{us}}$ is a shift,

$V_1|_{H_{su}}$ is a shift and $V_2|_{H_{su}}$ is unitary,

$V_1|_{H_{ss}}$ and $V_2|_{H_{ss}}$ are shifts.

This decomposition will be called the *Słociński decomposition.*

The pair $\{V_1|_{H_{uu}}, V_2|_{H_{uu}}\}$ is reflexive and has property C as a pair of commuting normal operators [10, Theorem 9.21]. The pair $\{V_1|_{H_{us}}, V_2|_{H_{us}}\}$ is reflexive by Proposition 9 and Theorem 2′ and has property C by Remark 3 (1) and Proposition 4. The pair $\{V_1|_{H_{su}}, V_2|_{H_{su}}\}$ is reflexive and has property C by the same reason. $V_1|_{H_{ss}}$, $V_2|_{H_{ss}}$ are doubly commuting shifts. Thus, they act in "orthogonal directions" by [12, Theorem 1]. This means that H_{ss} is unitarily isomorphic to $H^2(H^2(K))$. Then $V_1|_{H_{ss}}$ corresponds to the shift S in $H^2(H^2(K))$ and $V_2|_{H_{ss}}$ corresponds to the operator T defined by $T(z) = T_0$ for all z in C, where T_0 is a shift on $H^2(K)$. Hence, this pair is reflexive by Theorem 2′ and has property C by Remark 3 (2) and Proposition 4. Now, we apply Theorem 5 and we get the reflexivity of the pair $\{V_1, V_2\}$. ■

Let $\{V_1, V_2\}$ be a pair of (not necessarily doubly) commuting isometries and suppose it has the above Słociński decomposition (which need not exist in general [12, Example 1]). Putnam's theorem [10, Corollary 1.19] shows that the double commutativity of parts of V_1, V_2 can fail only on H_{ss}. Hence, the following theorem can be deduced by the same techniques as in the proof of Theorem 1.

THEOREM 10. *Let $\{V_1, V_2\}$ be a pair of commuting isometries which has the Słociński decomposition. If the pair $\{V_1|_{H_{ss}}, V_2|_{H_{ss}}\}$ is reflexive and has property C, then the pair $\{V_1, V_2\}$ is reflexive and has property C.*

We end up with the following example. Let $\{V_1, V_2\}$ be a pair of isometries having the Słociński decomposition and $H_{ss} = H^2$, $V_1|_{H_{ss}}$ is the shift on H^2, i.e. $(V_1|_{H_{ss}} f)z = zf(z)$ for all f in H^2, $V_2|_{H_{ss}}$ is the Toeplitz operator of multiplication by $\varphi \in H^\infty$, i.e. $(V_2|_{H_{ss}} f)z = \varphi(z) f(z)$, but φ is not a constant function. Then V_1, V_2 commute, but do not doubly commute. The pair $\{V_1|_{H_{ss}}, V_2|_{H_{ss}}\}$ is reflexive and has property C. Hence, the pair $\{V_1, V_2\}$ is reflexive by Theorem 10.

References

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