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Egorov's type convergence in the Dedekind completion of a C^*-algebra

by

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Abstract. The concept of convergence in Egorov's sense for nets in an abelian AW^*-algebra is introduced. We say that an abelian AW^*-algebra A has property E if every order convergent net in A also converges in Egorov's sense to the same limit. It is shown that the Dedekind completion of the hermitian part B_h of a given separable unital C^*-algebra B (regarded as an order unit vector space) satisfies property E if and only if B is abelian and its spectrum contains a dense subset of isolated points.

C^*-algebras have very nice properties as ordered vector spaces, they have not, however, the order completeness property in general. Since the hermitian part of a C^*-algebra is an Archimedean partially ordered vector space, it can be embedded, with preservation of suprema and infima, in a bounded complete vector lattice (the hermitian part of an abelian AW^*-algebra) called the *Dedekind completion* (of the C^*-algebra) (see, for example, [7] and [13]).

Our claim in this note is that this completion is very badly behaved for almost all separable C^*-algebras as far as the order convergence is concerned, in the following sense:

THEOREM. *Let A be a separable C^*-algebra and let $\mathcal{D}$ be the Dedekind completion of the hermitian part of the C^*-algebra A_1 obtained from adjunction of a unit to A, regarded as an order unit vector space. Then any bounded net $\{a_\lambda\}$ in $\mathcal{D}$ which converges to a in the order sense also converges in Egorov's sense (see below) to the same limit a if and only if A is an abelian C^*-algebra whose spectrum contains a dense set of isolated points.*

This is, however, an easy consequence of the following

PROPOSITION. *Keeping the above notations and definitions in mind, $\mathcal{D}$ is atomic (in the sense that it has sufficiently many minimal projections) if and only if A is abelian and its spectrum has a dense set of isolated points.*

Let Z be an abelian AW^*-algebra and Ω the spectrum of Z. If we denote by Z_h the set of all hermitian elements in Z, then Z_h is *-isomorphic to the set $C_r(\Omega)$ of all continuous real-valued functions on Ω. In its natural ordering, $C_r(\Omega)$ is a boundedly complete vector lattice ([2]).

In $C_r(\Omega)$ $(=Z_h)$, the concept of order convergence for bounded nets may be introduced ([8], [9]). A bounded net $\{a_\lambda\}$ in $C_r(\Omega)$ *order converges* to a (written $a_\lambda \to a$ (O)) if

$$a = \limsup a_\lambda = \liminf a_\lambda.$$

The following algebraic criterion for order convergence in Z_h was given by H. Widom ([9]).

LEMMA 1. *A bounded net $\{a_\lambda\}$ of Z_h order converges to a if and only if given a nonzero projection e in Z and an $\varepsilon > 0$ there exist a nonzero projection $f \leqslant e$ and a λ_0 such that $\lambda \geqslant \lambda_0$ implies $\|f(a_\lambda - a)\| < \varepsilon$.*

We say that a bounded net $\{a_\lambda\}$ in Z_h converges in *Egorov's sense* (*E-converges*) to a (written $a_\lambda \to a$ (E)) if there is an orthogonal family of projections $\{e_\alpha\}$ in Z such that $\sum_\alpha e_\alpha = 1$ and $\|(a_\lambda - a)e_\alpha\| \to 0$ (λ) for each α.

Clearly, "E-convergence" implies "O-convergence".

(*) Does the converse implication hold?

Unfortunately, as the following example shows, the answer to the question (*) is negative in general.

Let $\boldsymbol{R}$ be the real line and let $\mathfrak{B}(\boldsymbol{R})$ be the algebra of all bounded complex-valued Borel functions on $\boldsymbol{R}$. Let $\mathfrak{M}$ be the elements of $\mathfrak{B}(\boldsymbol{R})$ vanishing outside meagre sets of $\boldsymbol{R}$. Then $Z = \mathfrak{B}(\boldsymbol{R})/\mathfrak{M}$ (the quotient algebra) is an abelian AW^*-algebra whose spectrum has a dense meagre subset ([2]). Let q be the canonical quotient map from $\mathfrak{B}(\boldsymbol{R})$ onto Z. We know that a category analogue of Egorov's theorem is false by the following example in $\mathfrak{B}(\boldsymbol{R})$ ([4], P. 38).

Let $\varphi(x)$ be the piecewise linear continuous function defined by

$$\varphi(x) = 2x \quad \text{on } [0, 1/2],$$
$$\varphi(x) = 2-2x \quad \text{on } [1/2, 1],$$
$$\varphi(x) = 0 \quad \text{on } \boldsymbol{R}-[0, 1].$$

Let $\{r_i\}$ be a dense sequence in $\boldsymbol{R}$ (for example the set of all rational numbers) and let

$$f_n(x) = \sum_{i=1}^{\infty} 2^{-i}\varphi(2^n(x-r_i)).$$

Define $a_n = q(f_n)$. Then $\{a_n\} \subset Z_h$ is a bounded ($\|a_n\| \leqslant 1$) sequence such that $a_n \to 0$ (O) and $a_n \not\to 0$ (E). In fact, since, for each n, $|f_n(x)| \leqslant \sum_{i=1}^{\infty} 2^{-i} = 1$ for all $x \in \boldsymbol{R}$, $\|a_n\| \leqslant 1$ for all n.

Next we shall show that $a_n \to 0$ (O). Take any nonzero projection e in Z. Then there is an open interval (a, b) in $\boldsymbol{R}$ such that $e \geqslant q(\chi_{(a,b)}) \neq 0$. For any given $\varepsilon > 0$, there exists an i_0 such that $\sum_{i=i_0}^{\infty} 2^{-i} < \varepsilon$. Let $r_{k_1}, r_{k_2}, \ldots, r_{k_p}$ be $\{r_1, r_2, \ldots, r_{i_0}\} \cap (a, b)$ (possibly $= \emptyset$). Then, if we take a sufficiently large n_0, $\{[r_{k_j}, r_{k_j}+2^{-n_0}]\}_{j=1}^{p}$ are mutually disjoint subsets of (a, b). So there is an open interval $(a_1, b_1) \subset (a, b)$ which is disjoint from $\bigcup_{j=1}^{p} [r_{k_j}, r_{k_j}+2^{-n_0}]$. Let $x \in (a_1, b_1)$. Then for all $n \geqslant n_0$,

$$f_n(x) \leqslant \sum_{i=1}^{i_0-1} 2^{-i}\varphi(2^n(x-r_i)) + \varepsilon \leqslant \varepsilon,$$

because $\sum_{i=1}^{i_0-1} 2^{-i}\varphi(2^n(x-r_i)) = 0$ for all $n \geqslant n_0$. So $\|f_n \chi_{(a_1,b_1)}\| \leqslant \varepsilon$ for all $n \geqslant n_0$. By applying q, there is a nonzero subprojection f of e and a positive integer n_0 such that $\|a_n f\| \leqslant \varepsilon$ for all $n \geqslant n_0$. By applying Lemma 1, $a_n \to 0$ (O).

To prove that $a_n \not\to 0$ (E), we argue as follows. If it were true that $a_n \to 0$ (E), then it would imply that for any nonzero projection e in Z there is a nonzero subprojection f in Z of e such that $\|a_n f\| \to 0$ $(n \to \infty)$. So, if we take a subsequence if necessary, we would get

$$\|a_n f\| \leqslant 2^{-n} \quad \text{for } n = 1, 2, \ldots$$

One can choose an open interval (a, b) in $\boldsymbol{R}$ such that $q(\chi_{(a,b)}) \leqslant f$, and so $\|a_n q(\chi_{(a,b)})\| \leqslant 2^{-n}$, $n = 1, 2, \ldots$

Since $f_n \chi_{(a,b)}$ is lower semi-continuous on $\boldsymbol{R}$, this would imply that $\{x \mid f_n \chi_{(a,b)}(x) > 2^{-n}\}$ is not only meagre in $\boldsymbol{R}$ but also open in $\boldsymbol{R}$ for each n. Since $\boldsymbol{R}$ is a Baire space, $\{x \mid f_n \chi_{(a,b)}(x) > 2^{-n}\} = \emptyset$ for each n, and so $\|f_n \chi_{(a,b)}\| \leqslant 2^{-n}$ for all n, that is, $\|f_n|_{(a,b)}\| \to 0$ $(n \to \infty)$. Since $\{r_i\}$ is dense in $\boldsymbol{R}$, there is an $r_i \in (a, b)$, and hence

$$\sup_{a<x<b} f_n(x) \geqslant 2^{-i}$$

for sufficiently large n. Thus $\{f_n\}$ does not converge uniformly to 0 on (a, b) and this contradicts the claim that $f_n \chi_{(a,b)} \leqslant 2^{-n}$ for each n. Hence $a_n \not\to 0$ (E).

The next lemma gives, however, a partial positive answer to the question (*).

LEMMA 2. *Suppose that either $Z = L^\infty(\Gamma, \mu)$ for a measure space (Γ, μ) or, more generally, the spectrum Ω of Z has the property that every meagre set is rare ([2]). (Note that, under the σ-chain condition on Ω, each meagre subset of Ω is rare if and only if Z is weakly (σ, ∞)-distributive; see [7], p. 131.) Then "O-convergence" implies "E-convergence".*

Proof. If $Z = L^\infty(\Gamma, \mu)$, the proof is given by H. Widom ([8]). Next let Ω be a stonean space with the property that every meagre set is rare. Let $\{a_\lambda\}$ be any bounded net in $C_r(\Omega)$ which converges to a in the order sense. If we consider $a_\lambda - a$, we may assume that $a = 0$, without loss of generality. This means, however, that

$$\sup_\lambda \inf_{\mu \geq \lambda} a_\mu = \inf_\lambda \sup_{\mu \geq \lambda} a_\mu = 0.$$

Let $b_\lambda = \sup_{\mu \geq \lambda} a_\mu$ (resp. $c_\lambda = \inf_{\mu \geq \lambda} a_\mu$). Then $b_\lambda \downarrow 0$ (O) (resp. $c_\lambda \uparrow 0$ (O)). To prove our assertion, we may assume that $a_\lambda \downarrow 0$ (O) (consider b_λ and $-c_\lambda$). Let $b(\omega) = \inf_\lambda a_\lambda(\omega)$ for each $\omega \in \Omega$. Then b is an upper semi-continuous function on Ω.

Let, for each n,

$$\Omega_n = \{\omega |\ b(\omega) \geq n^{-1}\}.$$

Then Ω_n is a closed subset of Ω for each n. Suppose that $\Omega_n^0 \neq \emptyset$ for some n. Then, if e is the projection in $C_r(\Omega)$ corresponding to Ω_n^0, it follows that

$$a_\lambda \geq b \geq n^{-1} e \quad \text{for all } \lambda.$$

This contradicts the fact that $a_\lambda \downarrow 0$ (O) in $C_r(\Omega)$, so $\Omega_n^0 = \emptyset$ for each n and $\Omega_0 = \{\omega |\ b(\omega) > 0\} = \bigcup_{n=1}^\infty \Omega_n$ is a meagre subset of Ω, which implies, by our assumption on Ω, that Ω_0 is a rare set. It follows that $(\bar{\Omega}_0)^0 = \emptyset$, and hence $(\bar{\Omega}_0)^c$ contains a nonempty open and closed subset E, which is also contained in $\{\omega |\ b(\omega) = 0\}$. This means that $a_\lambda \downarrow 0$ pointwise on E. So, by Dini's theorem, $\|a_\lambda \chi_E\| \to 0$ (λ). If we take a nonzero projection e in $C_r(\Omega)$ corresponding to E, $\|a_\lambda e\| \to 0$ (λ) and the proof is complete.

The rest of this note is devoted to the proof of our main theorem. To do this, we need a sequence of lemmas.

Let A be a separable C^*-algebra and let A_1 be the C^*-algebra obtained from adjunction of a unit to A if A is nonunital (if A is unital, we denote A by A_1). Then the hermitian part of A_1, $(A_1)_h$, is an order unit vector space. Let V be the Dedekind (cut) completion of $(A_1)_h$ as an order unit vector space. Then V is the hermitian part of an abelian AW^*-algebra $\mathfrak{A}$ (see, for example, [13]). In fact, let X be the state space of A_1 and let $\mathfrak{B}(X)$ be the C^*-algebra of all bounded complex-valued Borel functions on X. Let $\mathfrak{M}_A = \{f \in \mathfrak{B}(X) |\ \{x \in \partial X |\ f(x) \neq 0\}$ is a meagre subset of ∂X (the set of pure states of A_1), in the relative topology of $\partial X\}$.

Then $\mathfrak{A} = \mathfrak{B}(X)/\mathfrak{M}_A$ satisfies all the requirements (see [13]).

Lemma 3. *$\mathfrak{A}$ is a Kaplansky–Rickart (KR-) algebra in the sense that $\mathfrak{A}$ has a countable order dense subset of $\mathfrak{A}_h$ $(= V)$.*

Proof. Since A_1 is norm separable, we can find a countable (norm) dense subset A_0 of $(A_1)_h$ such that for any $a \in (A_1)_h$ there is an increasing subsequence $\{a_n\}$ in A_0 such that $\|a_n - a\| \to 0$ $(n \to \infty)$ (see Theorem 4.3 in [10]). Thus $(A_1)_h$ has a countable order dense subset. Since $(A_1)_h$ is order dense in V, V has also a countable order dense subset A_0. Thus $\mathfrak{A}$ is a KR-algebra.

Lemma 4 ([12]). *Let $\mathfrak{E}$ be an abelian KR-algebra. If $\mathfrak{E}$ is not isomorphic to $\mathfrak{B}(\boldsymbol{R})/\mathfrak{M}$ or l^∞ or l_n^∞ (for any natural number n), then there is a projection $e \in \mathfrak{E}$, with $0 < e < 1$, such that $e\mathfrak{E} \cong \mathfrak{B}(\boldsymbol{R})/\mathfrak{M}$ and $(1-e)\mathfrak{E} \cong l^\infty$ or l_n^∞ for some n.*

Proof. This is a direct consequence of the fact that every atomless abelian KR-algebra is *-isomorphic to $\mathfrak{B}(\boldsymbol{R})/\mathfrak{M}$ (see [7], p. 155 and [12]).

Lemma 5. *Let $\hat{A}$ be the regular completion of A ([6], [10]). Then $\hat{A}_h$ is a σ-closed subspace of V regarded as an order unit vector space, that is, we have the following σ-normal injection from $\hat{A}_h$ into V:*

$$(A_1)_h \hookrightarrow \hat{A}_h \hookrightarrow V.$$

Proof. Let W be the Dedekind completion of $\hat{A}_h$ as an order unit vector space. Since $(A_1)_h$ is order dense in $\hat{A}_h$, the unicity of Dedekind completion tells us that W is also the Dedekind completion of $(A_1)_h$ and so $W = V$.

Next we shall show that the injection of $\hat{A}_h$ into V is σ-normal. We have $V = \mathfrak{B}(X_{\hat{A}})/\mathfrak{M}_{\hat{A}}$, where $X_{\hat{A}}$ is the state space of $\hat{A}$. Let $\{a_n\}$ be any increasing sequence in $\hat{A}_h$ which tends to a in $\hat{A}_h$ (in the order sense). Then $a_n \uparrow a$ in $\mathfrak{B}_{\hat{A}}$ (the Borel envelope of $\hat{A}$, which is canonically embedded in $\mathfrak{B}(X_{\hat{A}})$) almost everywhere, that is, $\{x \in \partial X_{\hat{A}} |\ a_n(x) \not\uparrow a(x)\}$ is a meagre subset of $\partial X_{\hat{A}}$ and so $q_{\hat{A}}(a_n) \uparrow q_{\hat{A}}(a)$ in V ([5], [14]).

Next we shall show that the Proposition implies the Theorem. Since the "if" part of the Theorem is clear, we have only to check the converse. Suppose that $\mathfrak{A}_h = V$ has the property that every bounded O-convergent net is also E-convergent to the same limit. Then, by Lemma 4 above and the counterexample cited before Lemma 2, it follows that $\mathfrak{A}$ is an atomic abelian W^*-algebra and so the Proposition implies the Theorem.

Proof of the Proposition. Suppose that V is atomic. The fact that $\hat{A}_h$ is a σ-subspace of V tells us that $\hat{A}$ has a faithful normal state and so $\hat{A}$ has a faithful representation as a von Neumann algebra ([5]). Since $\partial X_{\hat{A}}$ is $\sigma(\hat{A}^*, \hat{A})$-separable, this means that $\hat{A}$ is an atomic von Neumann algebra ([11]), so there is a sequence of Hilbert spaces $\{\mathcal{H}_n\}$ such that

$$\hat{A} = \sum_{n=1}^\infty \mathcal{L}(\mathcal{H}_n).$$

Hence there is an essential ideal I of A which is a dual C^*-algebra ([6]).

Since $(I_1)_h$ is order dense in $(A_1)_h$, V is also the Dedekind completion of $(I_1)_h$. To prove our assertion, we may assume that $A = I$. Let

$$A = \sum_{n=1}^{\infty}{}' \mathscr{C}(\mathscr{H}_n)$$

where $\sum'$ is the restricted direct sum of $\mathscr{C}(\mathscr{H}_n)$ and $\{\mathscr{H}_n\}$ is a sequence of Hilbert spaces.

Next we shall show that each nonzero $\mathscr{C}(\mathscr{H}_n)$ is one-dimensional. Since A is separable, ∂X_A is a Baire subset of X, and $\{0\}$ is a rare subset of X if A is nonunital, it follows that $\mathfrak{B}(\partial X_A)/\mathfrak{M}_A$ is a direct summand of $\mathfrak{B}(X)/\mathfrak{M}_A$ and so $\mathfrak{B}(\partial X_A)/\mathfrak{M}_A$ is also atomic (see [6], Proposition 2.1).

Let $C_b(\partial X_A)$ be the C^*-algebra of all bounded continuous functions on the completely regular space ∂X_A. Then $(C_b(\partial X_A))_h$ is order dense in $\mathfrak{B}(\partial X_A)/\mathfrak{M}_A$, because $\mathfrak{B}(\partial X_A)/\mathfrak{M}_A$ is the regular completion of $C_b(\partial X_A)$ (see [2], p. 25).

Thus the Stone–Čech compactification $\beta(\partial X_A)$ has a dense set of isolated points and so ∂X_A also has a dense set of isolated points.

Let $J_1 = \mathscr{C}(\mathscr{H}_1)$ and $J_2 = \sum_{n=2}^{\infty}{}' \mathscr{C}(\mathscr{H}_n)$. Then J_1 and J_2 are closed two-sided ideals of A such that $J_1 \oplus J_2 = A$. Let $(\partial X_A)^{J_1} = \{\varphi \in X_A \mid \varphi(J_1) \neq 0\}$. This is a nonempty open subset of ∂X_A which is homeomorphic to ∂X_{J_1} via the natural mapping $\varphi \to \varphi|J_1$. Thus it follows that ∂X_{J_1} also has a dense set of isolated points.

Note that $\partial X_{J_1} = \{\omega_\xi \mid \|\xi\| = 1,\ \xi \in \mathscr{H}_1\}$. Let φ be the mapping from $U = \{\xi \in \mathscr{H}_1 \mid \|\xi\| = 1\}$ onto ∂X_{J_1} which is defined by $\varphi(\xi) = \omega_\xi$. Then φ is a continuous mapping from U (with norm topology) onto ∂X_{J_1} (with weak*-topology). Take any isolated point ω_ξ of ∂X_{J_1} and $O = \varphi^{-1}(\{\omega_\xi\})$. Then O is a nonempty open subset of U.

Suppose that $\dim \mathscr{H}_1 \geqslant 2$. Then there is a unit vector η_0 which is orthogonal to ξ. Let

$$\xi_n = (1+10^{-n})^{-1/2}(10^{-n/2}\eta_0 + \xi).$$

Then $\|\xi_n\| = 1$ for each n and

$$(\xi_n, \xi) = (1+10^{-n})^{-1/2} \to 1 \quad (n \to \infty)$$

so $\|\xi_n - \xi\| \to 0$ $(n \to \infty)$, that is, $\omega_{\xi_n} \to \omega_\xi$ in ∂X_{J_1}. Moreover, $\xi_n \notin O$ $(n = 1, 2, \ldots)$. Since O^c is norm-closed and $\|\xi_n - \xi\| \to 0$ $(n \to \infty)$, this implies that $\xi \in O^c$. This is a contradiction, that is, $\dim \mathscr{H}_1 \leqslant 1$. It follows that $\dim \mathscr{H}_n \leqslant 1$ for all n, and so I is abelian. Since $\hat{I} = \hat{A}$ (because I is an essential ideal of A), this shows that A is abelian and its spectrum contains a dense set of isolated points. The converse assertion is clear. This completes the proof.

Remark. Let $A = M_2(C)$ (the algebra of all 2 by 2 matrices over C). Then it is classical that X_A is affinely topologically isomorphic to the 3-dimensional Euclidean ball S: $\alpha^2+\beta^2+\gamma^2 \leqslant 1/4$ ([1]). This implies that the Dedekind completion of A_h is $\mathfrak{B}(\partial S)/\mathfrak{M}_A$ which is isomorphic to $\mathfrak{B}(R)/\mathfrak{M}$ by Lemma 4.

COROLLARY. *Any separable simple C^*-algebra whose dimension is greater than one has the Dedekind completion which is isomorphic to $(\mathfrak{B}(\boldsymbol{R})/\mathfrak{M})_h$.*

Proof. Let A be any separable simple C^*-algebra whose dimension is greater than one and let V be the Dedekind completion of A_h as an order unit vector space. If V has a nontrivial atomic direct summand Ve (for some projection e in V), then, if we take $\varphi(x) = \psi(xe)$ $(x \in \hat{A})$, where ψ is any normal state on Ve, φ is a normal state on $\hat{A}$. By [12], if A were infinite-dimensional, $\hat{A}$ would not have a normal state; this implies that A is finite-dimensional and so $A \cong M_n(\boldsymbol{C})$ $(n \geqslant 2)$. If V has an atom, then ∂X_A $(= \{\omega_\xi \mid \xi \in \boldsymbol{C}^n, \|\xi\| = 1\})$ has an isolated point and V is the hermitian part of the regular completion of $C(\partial X_A)$, which tells us that the arguments in the proof of the Proposition can apply to deduce a contradiction. Thus V has no atoms. This completes the proof.

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A cheaper Swiss cheese

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Abstract. We simplify some of the computations required for McKissick's example of a normal uniform algebra.

§ 1. Introduction. McKissick [1] has constructed an elegant "Swiss cheese" K with the following property.

THEOREM 1.1. *There is a compact subset K of $\mathbf{C}$ such that $R(K)$ is normal but is not equal to $C(K)$.*

His proof depends on the following preliminary lemma.

LEMMA 1.2. *Given any $\varepsilon > 0$ we can find a sequence of open discs $\{\Delta_k\}$ and a sequence of rational functions $\{f_n\}$ such that:*

(a) *If r_k is the radius of Δ_k then $\sum_{k=1}^{\infty} r_k < \varepsilon$.*

(b) *The poles of the f_n lie in $\bigcup_{k=1}^{\infty} \Delta_k$.*

(c) *The sequence f_n tends uniformly to zero on $\{z: |z| \geqslant 1\} \setminus \bigcup_{k=1}^{\infty} \Delta_k$ and uniformly to some nowhere zero function on $\{z: |z| < 1\} \setminus \bigcup_{k=1}^{\infty} \Delta_k$.*

The standard proof of McKissick's lemma relies on a construction of Beurling and in the textbook detail of [3] fills 7 pages. The object of this note is to give a computationally simpler derivation of the lemma. Our method will have the further minor advantage of fulfilling two further conditions.

LEMMA 1.2′. *In addition to conditions* (a), (b), *and* (c) *of Lemma* 1.2 *we can demand*:

(d) $\bigcup_{k=1}^{\infty} \Delta_k \subseteq \{z: 1-\varepsilon < |z| < 1\}$.

(e) *$\Delta_1, \Delta_2, \ldots$ are disjoint.*

(Condition (d) is required for [2] but can also be obtained by modifying the original construction.)