(i) If $A \in (\mathrm{BP})$, then there exist two sequences $1 = N_0 < N_1 < \ldots$ and $0 = K_0 < K_1 < \ldots$ of integers such that

$$\|A;\, N_r,\, N_{r+1}\| \leqslant 2^{-r} \quad (r = 1, 2, \ldots),$$

$$\sup_{\varphi} \Big\{\int_0^1 \Big(\max_{1 \leqslant n \leqslant N_r} \Big|\sum_{k=K_r+1}^{K_{r+1}} a_{nk}\varphi_k(x)\Big|\Big)^2 dx\Big\}^{1/2} \leqslant 2^{-r}. \tag{4.5}$$

Setting $\mu_{nk} = r+1$ for $N_r \leqslant n < N_{r+1}$ and $K_r < k \leqslant K_{r+1}$ $(r = 0, 1, \ldots)$, it is easy to check that $\|\mu A\| < \infty$.

(ii) Similarly, if $A \notin (\mathrm{BP})$, then there exist two sequences $1 = N_0 < N_1 < \ldots$ and $0 = K_0 < K_1 < \ldots$ of integers such that

$$\|A;\, N_r,\, N_{r+1}\| \geqslant 2^r \quad (r = 1, 2, \ldots)$$

and (4.5) is satisfied. Now we set $\mu_{nk} = (r+1)^{-1}$ for $N_r \leqslant n < N_{r+1}$ and $K_r < k \leqslant K_{r+1}$ $(r = 0, 1, \ldots)$ and conclude that $\|\mu A\| = \infty$.

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Analytic functions in non-locally convex spaces and applications

by

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Abstract. The aim of this paper is to determine, for a general p-normable space X, what can in general be said about X-valued analytic functions on the disc. The results obtained are used to solve a problem raised by Turpin [17] on tensor products of quasi-Banach spaces.

1. Summary of main results. Suppose Ω is an open subset of the complex plane C and X is a quasi-Banach space. A map $f\colon \Omega \to X$ is said to be *analytic* if for every $z_0 \in \Omega$ there exists $r > 0$ such that f can be expanded in a power series for $|z - z_0| < r$, i.e.

$$f(z) = \sum_{n=0}^{\infty} x_n (z - z_0)^n$$

for $|z - z_0| < r$. This definition of analyticity is forced on us by simple examples which demonstrate that complex differentiability of f does not suffice to produce reasonable properties (cf. Aleksandrov [3], p. 39 or Turpin [16], Chapitre IX).

A key property of analytic functions is ([16], p. 195) that if $f\colon \Omega \to X$ is analytic and $\Omega_0 \subset \Omega$ is open and relatively compact in Ω then there is a Banach space B, an analytic function $g\colon \Omega_0 \to B$ and a bounded linear operator $T\colon B \to X$ so that $f(z) = T(g(z))$, $z \in \Omega_0$. From this many of the standard properties of analytic functions in a Banach space can be lifted to quasi-Banach spaces.

In this paper we will primarily be concerned with the case $\Omega = \Delta$, the open unit disc. In this case one has, for example,

$$f(z) = \sum_{n=0}^{\infty} x_n z^n, \quad |z| < 1,$$

where $\limsup \|x_n\|^{1/n} \leqslant 1$.

It seems that the main obstacle to developing the theory of analytic functions for non-locally convex spaces is the failure of the Maximum Modulus Principle. It has been observed by several authors (Etter [7], Aleksandrov [3], Peetre [14], Davis–Garling–Tomczak [5]) that some standard spaces, e.g. L_p for $0 < p < 1$, have a plurisubharmonic quasi-norm and hence if $f\colon \bar{\Delta} \to L_p$ is analytic on Δ and

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continuous on $\bar{\Delta}$ then

$$\|f(z)\| \leqslant \max_{|\zeta|=1} \|f(\zeta)\|,$$

for all $z \in \Delta$. By contrast, however, Aleksandrov notes that if we define $J_{p,0}$ to be the closed linear span in $L_p(T)$ of the Cauchy kernels $\varphi_z(w) = (1-wz)^{-1}$ and let $Q\colon L_p \to L_p/J_{p,0}$ be the quotient map then we can define

$$v(z) = Q(u(z)), \quad |z| \leqslant 1,$$

where $u(z) = (1-wz)^{-1}$. The map $v\colon \bar{\Delta} \to L_p/J_{p,0}$ is analytic on Δ, continuous on $\bar{\Delta}$ and vanishes on T.

The precise conditions on X so that such a phenomenon can occur will be investigated in a separate paper. Our aim in this paper is to determine, for a general p-normable space X, with no additional assumptions such as plurisubharmonicity of the quasi-norm, what can in general be said about X-valued analytic functions on the disc. In particular, we define, for $\sigma > 0$, $V_\sigma(X)$ to be the space of all analytic $f\colon \Delta \to X$ so that, for some constant C,

$$\|f(z)\| \leqslant C(1-|z|)^\sigma.$$

We show that (Theorem 4.6) v defined above belongs to $V_\sigma(L_p/J_{p,0})$ where $\sigma = 1/p-1$.

It turns out (Theorem 6.7) that if X is p-normable and $\sigma > 1/p-1$ then $V_\sigma(X) = \{0\}$, so that v represents extremal behaviour. In Theorem 8.3 it is shown further that if

$$\lim_{r\to 1} (1-r^2)^{1-1/p} \Big\{\int_0^{2\pi} \|f(re^{i\theta})\|^p \, d\theta\Big\}^{1/p} = 0$$

then $f = 0$.

The key result, however, is that $V_\sigma(X)$ is nontrivial if and only if there is a nontrivial linear operator $T\colon L_p/H_p \to X$ (Theorem 7.3).

These results can be used to solve a problem raised by Turpin [17] on tensor products of quasi-Banach spaces. Turpin showed that if X is a p-Banach space and Y is a q-Banach space then there is an r-convex tensor quasi-norm on $X \otimes Y$ if $1/r = 1/p+1/q-1$, and asked whether this can be improved. The author showed that in the case $p = q$ we cannot always have $r = p$ [9], by showing that there is no nonzero bilinear $B\colon L_p/H_p \times L_p/H_p \to Z$ where Z is p-normable. Here we show Turpin's result is best possible by showing that if Z is s-normable, where $1/s < 1/p + 1/q-1$, then there is no nonzero bilinear form $B\colon L_p/H_p \times L_q/H_q \to Z$ (Corollary 9.2).

The method employed is to establish a correspondence between certain classes of analytic functions and linear operators. The first such result is Theorem 5.1 which identifies the space of linear operators $\mathscr{L}(H_q, X)$ where X is p-normable and $0 < q < p$. This theorem is essentially a translation of a result of Coifman and Rochberg [4]. In Theorem 7.1 we similarly identify $\mathscr{L}(L_p, X)$ and hence $\mathscr{L}(L_p/H_p, X)$. We use these results to give an "atomic decomposition" of L_p in the spirit of Coifman–Rochberg [4] and to extend Aleksandrov's theorem ([1], [2]) that $L_p = H_p + \bar{H}_p$ for $0 < p < 1$, by showing that if $f \in L_p(T)$ then we can find $g_1, g_2 \in H_p$ so that

$$g_1(e^{i\theta}) + g_2(e^{-i\theta}) = f(e^{i\theta}),$$

$$\int_\Delta |g_j'(w)| (1-|w|^2)^{p-1} \, d\lambda(w) < \infty$$

for $j = 1, 2$ (where λ is the planar measure on Δ) (see Theorems 8.1, 8.2).

In Section 6 we also give some applications to the general theory of vector-valued analytic functions. For example, we show Liouville's theorem holds (Theorem 6.2), in the form that if $f\colon C \to X$ is entire and bounded then f is constant (see [16], [18] for a similar result when f is analytic on the Riemann sphere $C \cup \{\infty\}$). We also show (Theorem 6.3) that the uniform limit of analytic functions on Δ is again analytic.

2. Notation and plan of the paper. Throughout this paper all vector spaces are assumed complex. By definition, a *quasi-normed space* is a vector space X with a quasi-norm $x \to \|x\|$ satisfying:

(i) $\|x\| > 0$, $\quad x \neq 0$,

(ii) $\|\alpha x\| = |\alpha| \, \|x\|$, $\quad \alpha \in C$, $x \in X$,

(iii) $\|x_1 + x_2\| \leqslant C(\|x_1\| + \|x_2\|)$, $\quad x_1, x_2 \in X$,

for some C independent of x_1, x_2. In fact we will always assume that the quasi-norm is *p-subadditive* for some $p > 0$, i.e.

(iv) $\|x_1 + x_2\|^p \leqslant \|x_1\|^p + \|x_2\|^p$, $\quad x_1, x_2 \in X$.

This assumption is justified by the Aoki–Rolewicz theorem [15] that every quasi-norm is equivalent to a p-subadditive quasi-norm where $C = 2^{1/p-1}$.

If X is complete, we say it is a *quasi-Banach space*; if it has a p-subadditive quasi-norm, we say it is a *p-Banach space*. For convenience of exposition, we shall always assume that X is a p-Banach space without specifying the fact (while spaces Y, Z, etc. need not be p-Banach spaces but must be q-Banach spaces for some q).

If X and Y are quasi-Banach spaces then $\mathscr{L}(X, Y)$ denotes the space of all bounded linear operators $T\colon X \to Y$ with the usual quasi-norm $\|T\| = \sup(\|Tx\|\colon \|x\| \leqslant 1)$.

Suppose $0 < p < 1$. We let $L_p = L_p(T)$ be the space of all complex-valued Borel functions $f\colon T \to C$ satisfying

$$\|f\|_p^p = (2\pi)^{-1} \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta < \infty.$$

In general we shall use w as the independent variable when considering function spaces. Let H_p be the closed linear span in L_p of $(w^n\colon n = 0, 1, 2, \ldots)$. $\bar{H}_p$ denotes the closed linear span of $(\bar{w}^n\colon n = 0, 1, 2, \ldots)$ and $\bar{H}_{p,0}$ denotes the closed linear span of $(\bar{w}^n\colon n = 1, 2, 3, \ldots)$ or $(w^n\colon n = -1, -2, -3, \ldots)$. The intersections are denoted by $J_p = H_p \cap \bar{H}_p$ and $J_{p,0} = H_p \cap \bar{H}_{p,0}$. $J_{p,0}$ is the closed linear span of the Cauchy kernels

$$u(z) = (1-wz)^{-1}$$

for $z \in T$ (see [1]). We shall also need the spaces $J_{p,0}^{(m)}$ spanned by the functions $u^{(m)}(z) = m!\, w^m (1-wz)^{-(m+1)}$ provided $m < 1/p - 1$.

It is known that the quotient spaces L_p/H_p, $L_p/\bar{H}_p$, $L_p/\bar{H}_{p,0}$, L_p/J_p, H_p/J_p, $H_p/J_{p,0}$ are all isomorphic (cf. [3], [10]). The first three spaces in this list are isomorphic by constructing simple automorphisms of L_p which map H_p to $\bar{H}_p$ or $\bar{H}_{p,0}$. For the other spaces one needs the theorem of Aleksandrov [1] that $H_p + \bar{H}_{p,0} = L_p$.

For $p < q \leqslant 1$, the q-Banach envelope of H_p has been identified in [3] and [4]. This is the Bergman space $B_{p,q}$ of all analytic functions f defined on the unit disc Δ in the complex plane and satisfying

$$\int_\Delta |f(w)|^q (1-|w|^2)^{q/p-2}\, d\lambda(w) = \|f\|_{p,q}^q < \infty$$

where λ is the planar Lebesgue measure.

If we identify H_p in the usual way as a space of analytic functions on Δ, then $H_p \subset B_{p,q}$ and the inclusion is continuous. Furthermore, if Y is any q-Banach space and $T\colon H_p \to Y$ is a bounded linear operator then T can be extended to a bounded linear operator $\bar{T}\colon B_{p,q} \to Y$. Thus $\mathscr{L}(H_p, Y)$ and $\mathscr{L}(B_{p,q}, Y)$ are naturally isomorphic.

We now discuss the plan of the paper. In Sections 3–4 we describe a theory of integration in non-locally convex spaces originally developed by Turpin and Waelbroeck ([16], [18], [19]); roughly speaking, a function can be integrated successfully if it is sufficiently smooth. We introduce in Section 4 the class $C_\sigma(T, X)$ of "σ-differentiable" functions $f\colon T \to X$ where $\sigma > 0$. In particular, we study the function $u\colon T \to L_p$ given by

$$u(z) = (1-wz)^{-1}.$$

In Section 5, we prove our main representation theorem for operators on H_q, and apply these results in Section 6 to give results on analytic functions taking their values in an arbitrary quasi-Banach space X. In Section 7, these results and the Turpin–Waelbroeck integral are used to give a representation theorem for operators on $L_p(T)$ and on $L_p/\bar{H}_{p,0}$. In Section 8, we give some applications to the space L_p, and in Section 9 we give applications to tensor products.

Convention. Throughout the paper we adopt the convention that C is a constant which may vary from line to line and may depend on the parameters p, q, σ, ν, etc., but is independent of f, x, K, etc.

3. The class C_σ. In this section we give a self-contained treatment of an integration theory developed by Turpin and Waelbroeck ([16], [18], [19]). Our

approach goes a little further than that of Turpin and Waelbroeck as we shall need to cover the case when σ, as specified below, is an integer.

Suppose X is a p-Banach space where $0 < p \leqslant 1$. Let K be any fixed closed bounded interval in $\mathbf{R}$. Let $f\colon K \to X$ be any continuous function. Then for any closed subinterval I of K we define

$$\|f\|_I = \max_{t \in I} \|f(t)\|.$$

Now suppose $\sigma > 0$ and N is an integer with $N \geqslant \sigma - 1$; suppose further that $l > 0$. We shall say that a continuous function $f\colon K \to X$ is in $C_\sigma^{l,N}(K, X)$ if there is a constant $\gamma > 0$ with the property that for any closed subinterval I of K with length $|I| \leqslant l$ there is a polynomial $\Phi_I\colon I \to X$ of degree at most N so that

$$\|f - \Phi_I\|_I \leqslant \gamma |I|^\sigma. \tag{3.1}$$

Note that if $\sigma < 1$ and $N = 0$ then this simply implies that f is Lipschitz of order σ.

Before proceeding we observe two crucial facts. The first is a lemma due to Peck [13]: if F is any m-dimensional complex p-Banach space then there is a norm $|||\ |||$ on F satisfying

$$|||x||| \leqslant \|x\| \leqslant (2m)^{1/p-1} |||x|||, \qquad x \in F.$$

Note here that the real dimension of F is $2m$.

The second observation is that there is a constant $C = C(p, N)$ so that for any interval I and any polynomial φ of degree N we have

$$\|\varphi^{(k)}\|_I \leqslant C|I|^{-k} \|\varphi\|_I$$

for any $k \leqslant N$. This is proved by standardizing to an interval of length one and using Peck's lemma.

In the next proposition we let $\nu = [\sigma]$ be the largest integer in σ.

Proposition 3.1. (i) *The spaces $C_\sigma^{l,N}(K, X)$ are independent of $l > 0$ and $N \geqslant \sigma - 1$. Let $C_\sigma(K, X)$ denote this class.*

(ii) *If $\sigma > 1$ and $f \in C_\sigma(K, X)$ then f is continuously differentiable and $f' \in C_{\sigma-1}(K, X)$.*

(iii) *If $\sigma \notin N$ then $f \in C_\sigma(K, X)$ if and only if f is ν times continuously differentiable on K and, for some $\beta > 0$,*

$$\left\| f(t) - \sum_{k=0}^{\nu} \frac{f^{(k)}(s)}{k!}(t-s)^k \right\| \leqslant \beta |t-s|^\sigma$$

for s, $t \in K$.

(iv) *If $\sigma \in N$ (i.e. $\sigma = \nu$) and $f^{(\nu-1)}$ is Lipschitz then $f \in C_\sigma(K, X)$ if and only if, for some $\beta > 0$,*

$$\left\| f(t) - \sum_{k=0}^{\nu-1} \frac{f^{(k)}(s)}{k!}(t-s)^k \right\| \leqslant \beta |t-s|^\sigma.$$

Proof: Suppose $f \in C_\sigma^{l,N}(K, X)$. As in (3.1) we suppose that if I is a closed subinterval of K we can find a polynomial φ_I of degree at most N so that $\|f-\varphi_I\|_I \leqslant \gamma |I|^\sigma$. If I and J are intersecting intervals then

$$\|\varphi_I - \varphi_J\|_{I \cap J} \leqslant C\gamma(|I|^\sigma + |J|^\sigma),$$

and hence if $k \leqslant N$ then

$$\|\varphi_I^{(k)} - \varphi_J^{(k)}\|_{I \cap J} \leqslant C\gamma |I \cap J|^{-k}(|I|^\sigma + |J|^\sigma). \tag{3.2}$$

Fix $s \in I \cap J$. Then for $t \in I \cup J$

$$\|\varphi_I(t) - \varphi_J(t)\| = \left\| \sum_{k=0}^{N} \frac{\varphi_I^{(k)}(s) - \varphi_J^{(k)}(s)}{k!}(t-s)^k \right\| \tag{3.3}$$
$$\leqslant C\gamma(|I \cup J|/|I \cap J|)^N(|I|^\sigma + |J|^\sigma).$$

Now note that if L is an interval with $l \leqslant |L| \leqslant \frac{3}{2}l$ then we can write $L = I \cup J$ where $|I| \leqslant l$, $|J| \leqslant l$ and $|I \cap J| \geqslant \frac{1}{2}l$. Thus for $t \in L$

$$\|\varphi_I(t) - \varphi_J(t)\| \leqslant 3^N C\gamma |I|^\sigma$$

and so

$$\|f(t) - \varphi_I(t)\| \leqslant C\gamma |L|^\sigma$$

for $t \in L$. Thus $C_\sigma^{l,N} = C_\sigma^{3l/2,N}$, and it follows quickly that $C_\sigma^{l,N}$ is independent of l. Henceforward we take $l = |K|$.

Now suppose $t \in K$ and $0 < h \leqslant |K| = \delta$ say. We can find a polynomial $\varphi_h = \varphi_{t,h}$ of degree N so that $\|f(s) - \varphi_h(s)\| \leqslant C\gamma h^\sigma$ if $|s-t| \leqslant h$ and $s \in K$.

Using (3.3) we see that if $1 \leqslant \alpha \leqslant 2$ and $\alpha h \leqslant \delta$ then $\|\varphi_h(s) - \varphi_{\alpha h}(s)\| \leqslant C\gamma h^\sigma$ if $|s-t| \leqslant h$ and $s \in K$. It follows that if $0 \leqslant k \leqslant N$ then

$$\|\varphi_h^{(k)}(t) - \varphi_{\alpha h}^{(k)}(t)\| \leqslant C\gamma h^{\sigma-k}. \tag{3.4}$$

Now if $2^{-n}\delta \leqslant h < 2 \cdot 2^{-n}\delta$ where $n \in N$ then we can obtain $\varphi_h^{(k)}(t)$ as

$$\varphi_h^{(k)}(t) = \varphi_\delta^{(k)}(t) - \sum_{j=1}^{n} P_j \tag{3.5}$$

where

$$P_j = \varphi_{2 \cdot 2^{-j}\delta}^{(k)}(t) - \varphi_{2^{-j}\delta}^{(k)}(t) \quad \text{for } j < n,$$
$$P_n = \varphi_{2 \cdot 2^{-n}\delta}^{(k)}(t) - \varphi_h^{(k)}(t).$$

We note that $\|P_j\| \leqslant C\gamma(2^{-j}\delta)^{\sigma-k}$.

We consider first the case $k > \sigma$. In this case

$$\|\varphi_h^{(k)}(t) - \varphi_\delta^{(k)}(t)\| \leqslant C\gamma h^{\sigma-k}.$$

However, $\|\varphi_\delta^{(k)}(t)\| \leqslant C\delta^{-k}\|\varphi\|_K \leqslant C\delta^{-k}(\|f\|_K + \gamma\delta^\sigma)$. Hence

$$\|\varphi_h^{(k)}(t)\| \leqslant C(\delta^{-k}\|f\|_K + \gamma h^{\sigma-k}).$$

Thus

$$\left\| \sum_{k=\nu+1}^{N} \varphi_h^{(k)}(t)(s-t)^k/k! \right\| \leqslant C(\|f\|_K + \gamma) h^\sigma$$

if $|s-t| \leqslant h$, and so if $s \in K$ then

$$\left\| f(s) - \sum_{k=0}^{\nu} \varphi_h^{(k)}(t)(s-t)^k/k! \right\| \leqslant \gamma' h^\sigma$$

where γ' is independent of t and h. Thus (i) is established; we may take $N = \nu$ in the definition. Henceforward we assume $N = \nu$ so that $\deg \varphi_h \leqslant \nu$ for all t, h.

Returning to (3.5) we see that if $k < \sigma$ then $x_k = \lim_{h \to 0} \varphi_h^{(k)}(t)$ exists, and furthermore

$$\|x_k - \varphi_h^{(k)}(t)\| \leqslant C\gamma h^{\sigma-k}. \tag{3.6}$$

If $\sigma \in N$ and $k = \sigma$ then (3.4) and (3.5) yield

$$\|\varphi_h^{(k)}(t) - \varphi_\delta^{(k)}(t)\| \leqslant C\gamma\left(\log(\delta/h) + 1\right)^{1/p}$$

and hence

$$\|\varphi_h^{(k)}(t)\| \leqslant C\left(1 + \log(\delta/h)\right)^{1/p}(\|f\|_K + \gamma),$$

where $C = C(p, N, \delta)$. Let us write

$$g_t(s) = \sum_{k<\sigma} x_k(s-t)^k/k!$$

and define $\varrho_{t,h}$ by

$$\varrho_{t,h}(s) = 0 \qquad \text{if } \sigma \notin N,$$
$$\varrho_{t,h}(s) = \varphi_h^{(\nu)}(t)(s-t)^\nu/\nu! \qquad \text{if } \sigma = \nu.$$

Let $\psi_{t,h}(s) = g_t(s) + \varrho_{t,h}(s)$. Then $\|\varphi_h(s) - \psi_{t,h}(s)\| \leqslant C\gamma h^\sigma$ if $|s-t| \leqslant h$ by (3.6). Thus

$$\|f(s) - \psi_{t,h}(s)\| \leqslant C\gamma h^\sigma$$

if $|s-t| \leqslant h$ and $s \in K$.

Note first that $f(t) = \lim_{h \to 0} \psi_{t,h}(t) = g_t(t)$. Thus if $\sigma > 1$ then $f'(t)$ exists and $f'(t) = g_t'(t) = x_1$.

Suppose $s, t \in K$ and $\delta \geqslant h \geqslant |s-t|$. Then if $|\tau - t| \leqslant h$ and $\tau \in K$, we have

$$\|\psi_{t,h}(\tau) - \psi_{s,h}(\tau)\| \leqslant C\gamma h^\sigma$$

and hence

$$\|\psi_{t,h}'(\tau) - \psi_{s,h}'(\tau)\| \leqslant C\gamma h^{\sigma-1}.$$

In particular, if $\tau = s$ then

$$\|\psi_{t,h}'(s) - f'(s)\| \leqslant C\gamma h^{\sigma-1}. \tag{3.7}$$

Letting $h = |s-t|$ and $s \to t$ we see that f' is continuous. Furthermore, $f' \in C_{\sigma-1}$ since (3.7) holds if $s \in K$ and $|s-t| \leqslant h$. Thus (ii) is established.

Now for (iii) we can repeat this argument ν times to show f is ν times continuously differentiable and

$$\psi_{t,h}(s) = \sum_{k=0}^{\nu} f^{(k)}(t)(s-t)^k/k!.$$

In case (iv), we repeat the argument $\nu-1$ times to deduce that $f^{(\nu-1)}$ is in class C_1, and $\|\psi_{t,h}^{(\nu-1)}(s) - f^{(\nu-1)}(s)\| \leqslant C\gamma h$ if $|s-t| \leqslant h$. Now

$$\psi_{t,h}^{(\nu-1)}(s) = x_{\nu-1} + \varphi_h^{(\nu)}(t)(s-t) = f^{(\nu-1)}(t) + \varphi_h^{(\nu)}(t)(s-t).$$

Hence

$$\|f^{(\nu-1)}(s) - f^{(\nu-1)}(t) - \varphi_h^{(\nu)}(t)(s-t)\| \leqslant C\gamma h.$$

If $f^{(\nu-1)}$ is Lipschitz we conclude, by taking $s = t+h$ or $t-h$, that $\|\varphi_h^{(\nu)}(t)\| \leqslant C$ where C is independent of t and h. Hence

$$\|\varrho_{t,h}(s)\| \leqslant C|s-t|^{\nu}$$

and (iv) follows. The proposition is proved.

From now we shall choose $l = |K|$ and $N = \nu$ in our definition of $C_\sigma(K, X)$. We let $\gamma_\sigma(f)$ be the infimum of all possible constants γ in (3.1). Then we set

$$\|f\|_{K,\sigma} = \|f\|_K + \gamma_\sigma(f).$$

If $f \in C_\sigma(K, X)$ we say that f is of *rank* m if $f(K)$ is contained in an m-dimensional subspace.

LEMMA 3.2. *There is a constant $C = C(p, \sigma)$ so that if $f \in C_\sigma(K, X)$ and $m \geqslant 2(\nu+1)$ then there exists $g_m \in C_\sigma(K, X)$ with rank $g_m \leqslant m$ and*

$$\|f - g_m\|_K \leqslant Cm^{-\sigma}\|f\|_{K,\sigma}|K|^{\sigma} \quad \text{and} \quad \|g_m\|_{K,\sigma} \leqslant C\|f\|_{K,\sigma}.$$

Proof. We prove the statement of the lemma if $m = (N+1)(\nu+1)$ where $N \in \mathbf{N}$. The general statement then follows easily.

First suppose $\psi\colon \mathbf{R} \to \mathbf{R}$ is a C^∞-function so that $\operatorname{supp}\psi \subset [-1, 1]$, $0 \leqslant \psi \leqslant 1$, $\psi(0) = 1$ and

$$\sum_{n \in \mathbf{Z}} \psi(t-n) = 1, \qquad t \in \mathbf{R}.$$

Let $K = [a, b]$ and let $I_j = [a+(j-1)\delta/N,\ a+j\delta/N]$ for $0 \leqslant j \leqslant N+1$ where $\delta = b-a = |K|$. Define $\psi_0, \ldots, \psi_N\colon \mathbf{R} \to \mathbf{R}$ by

$$\psi_j(t) = \psi(N\delta^{-1}(t-a)-j).$$

Then

$$\sum_{j=0}^{N} \psi_j(t) = 1, \qquad t \in K,$$

and $\operatorname{supp}\psi_j \subset I_j \cup I_{j+1}$. For $l \leqslant \nu+1$,

$$|\psi_j^{(l)}(t)| \leqslant CN^l\delta^{-l}.$$

Pick polynomials $\varphi_0, \ldots, \varphi_N \in C_\sigma(K, X)$ of degree at most ν so that

$$\|f(s) - \varphi_j(s)\| \leqslant \|f\|_{K,\sigma}(2\delta/N)^{\sigma}$$

for $s \in (I_j \cup I_{j+1}) \cap K$. Let $g = g_m = \sum_{j=0}^{N} \psi_j\varphi_j$. Then $g \in C_\sigma(K, X)$ and if $s \in I_j$ then

$$\begin{aligned}\|f(s) - g(s)\| &= \|\psi_{j-1}(s)(f(s) - \varphi_{j-1}(s)) + \psi_j(s)(f(s) - \varphi_j(s))\| \\ &\leqslant C\|f\|_{K,\sigma}N^{-\sigma}\delta^{\sigma}.\end{aligned}$$

Furthermore,

$$g(s) = \psi_{j-1}(s)\varphi_{j-1}(s) + \psi_j(s)\varphi_j(s) = \varphi_{j-1}(s) + \psi_j(s)(\varphi_j(s) - \varphi_{j-1}(s)).$$

Thus

$$g^{(\nu+1)}(s) = \frac{\partial^{\nu+1}}{\partial s^{\nu+1}}(\psi_j(\varphi_j - \varphi_{j-1})).$$

By (3.2), $\|\varphi_j^{(l)}(s) - \varphi_{j-1}^{(l)}(s)\| \leqslant C\|f\|_{K,\sigma}(\delta/N)^{\sigma-l}$. Hence

$$\|g^{(\nu+1)}(s)\| \leqslant C\|f\|_{K,\sigma}(\delta N^{-1})^{\sigma-\nu-1}.$$

Let J be any closed subinterval of K. If $|J| \geqslant N^{-1}|K|$ then, since $\|f-g\|_K \leqslant C\|f\|_{K,\sigma}N^{-\sigma}\delta^{\sigma}$, there is a polynomial φ_J so that $\deg\varphi_J \leqslant \nu$ and

$$\|g - \varphi_J\| \leqslant C\|f\|_{K,\sigma}(N^{-\sigma}\delta^{\sigma} + |J|^{\sigma}) \leqslant C\|f\|_{K,\sigma}|J|^{\sigma}.$$

Now suppose $|J| \leqslant N^{-1}\delta$. Let s be the midpoint of J and define

$$\varphi(t) = \sum_{j=0}^{\nu} g^{(j)}(s)(t-s)^j/j!.$$

Then for $t \in J$

$$\|g(t) - \varphi(t)\| \leqslant C|t-s|^{\nu+1}\max_{t \in J}\|g^{(\nu+1)}(t)\|.$$

Here we use the fact that $g|J$ takes its values in a fixed $3(\nu+1)$-dimensional space, so that by Peck's lemma the quasi-norm is uniformly equivalent to a norm. Thus

$$\|g(t) - \varphi(t)\| \leqslant C\|f\|_{K,\sigma}(\delta N^{-1})^{\sigma-\nu-1}|J|^{\nu+1} \leqslant C\|f\|_{K,\sigma}|J|^{\sigma}.$$

Now note that rank $g_m \leqslant (N+1)(\nu+1)$.

Remark. Suppose, as we will later, that $K = [-2\pi, 2\pi]$ and that f is 2π-periodic. Then if N is even in the above argument, then g_m is also 2π-periodic.

Now suppose μ is a regular Borel measure on K. If $g \in C_\sigma(K, X)$ and rank $g < \infty$ then we may define the finite-dimensional integral $\int_K g\,d\mu$. By Peck's lemma we clearly obtain

$$\Big\|\int_K g\,d\mu\Big\| \leqslant (2m)^{1/p-1}\|g\|_K\|\|\mu\|$$

if rank $g = m$.

If $f\in C_\sigma(K, X)$ we can define g_m for $m \geqslant 2(\nu+1)$ with rank $g_m \leqslant m$ and so that

$$\|f-g_m\|_K \leqslant Cm^{-\sigma}|K|^\sigma\|f\|_{K,\sigma}.$$

Then, for $m\leqslant n\leqslant 2m$, $\|g_m-g_n\|_K \leqslant Cm^{-\sigma}|K|^\sigma\|f\|_{K,\sigma}$, and hence

$$\Big\|\int_K g_m\,d\mu-\int_K g_n\,d\mu\Big\| \leqslant Cm^{1/p-1-\sigma}|K|^\sigma\|f\|_{K,\sigma}.$$

Now it is easy to show that $\lim_{n\to\infty}\int g_{2^n}\,d\mu$ exists. It follows that $\lim_{m\to\infty}\int_K g_m\,d\mu$ exists and is independent of the choice of the approximating sequence g_m as long as $\|f-g_m\|\leqslant Cm^{-\sigma}$. Furthermore,

$$\Big\|\int_K f\,d\mu-\int_K g_n\,d\mu\Big\| \leqslant Cn^{1/p-1-\sigma}\|f\|_{K,\sigma}|K|^\sigma\|\mu\|. \tag{3.7}$$

However, $\|g_n\|_K \leqslant C(\|f\|_K+|K|^\sigma n^{-\sigma}\|f\|_{K,\sigma})$, and hence for all $n\geqslant 2(\nu+1)$ we have

$$\Big\|\int_K f\,d\mu\Big\| \leqslant C(n^{1/p-1}\|f\|_K+n^{1/p-1-\sigma}|K|^\sigma\|f\|_{K,\sigma})\|\mu\|. \tag{3.8}$$

Taking n to be fixed, say $2(\nu+1)$, we obtain:

LEMMA 3.3.

$$\Big\|\int_K f\,d\mu\Big\| \leqslant C(\|f\|_K+|K|^\sigma\|f\|_{K,\sigma})\|\mu\|,$$

where $C = C(p,\sigma)$ is independent of f, μ and K.

We can now state the main properties of the Turpin–Waelbroeck integral.

THEOREM 3.4. (i) *Suppose $f_n\in C_\sigma(K, X)$ where $\sigma > 1/p-1$. Suppose $\|f_n-f\|_K\to 0$ and $\sup_n\|f_n\|_{K,\sigma}<\infty$. Then*

$$\lim_{n\to\infty}\int_K f_n\,d\mu = \int_K f\,d\mu.$$

(ii) *Suppose $\mu_n\in M(K)$ and $\mu_n\to\mu$ weak*. Then if $f\in C_\sigma(K, X)$ where $\sigma>1/p-1$ then*

$$\lim_{n\to\infty}\int_K f\,d\mu_n = \int_K f\,d\mu.$$

Proof. (i) We omit the simple proof that $f\in C_\sigma(K, X)$. By 3.8 we note that if $m, n\in N$ then

$$\Big\|\int_K (f-f_m)\,d\mu\Big\| \leqslant C(n^{1/p-1}\|f-f_m\|_K+n^{1/p-1-\sigma}\|f-f_m\|_{K,\sigma}).$$

However, it is easily seen from the definition that $\sup_m\|f-f_m\|_{K,\sigma}<\infty$. Letting $m\to\infty$ we obtain

$$\limsup_{m\to\infty}\Big\|\int_K (f-f_m)\,d\mu\Big\| \leqslant Cn^{1/p-1-\sigma}$$

for all $n\in N$ and the result follows.

(ii) Here we use (3.7). Note that $\sup_n\|\mu_n\|<\infty$. Then for any m, n

$$\Big\|\int f\,d\mu_m-\int g_n\,d\mu_m\Big\| \leqslant Cn^{1/p-1-\sigma}\|f\|_{K,\sigma}.$$

Now $\lim_{m\to\infty}\int g_n\,d\mu_m = 0$ for each n. Hence

$$\limsup_{m\to\infty}\Big\|\int f\,d\mu_m\Big\| \leqslant Cn^{1/p-1-\sigma}\|f\|_{K,\sigma}$$

for all $n\in N$, and the result follows.

4. The class $C_\sigma(T, X)$. Suppose $f\in C(T, X)$. We say $f\in C_\sigma(T, X)$ if $\tilde f\in C_\sigma(K, X)$ for any closed bounded subinterval K of $\mathbf{R}$ where $\tilde f(\theta) = f(e^{i\theta})$. We set

$$\|f\|_{T,\sigma} = \|\tilde f\|_{K,\sigma}$$

where $K = [-2\pi, 2\pi]$. This interval has length greater than 2π to ensure smoothness at the end points.

Our first lemma translates the definition of C_σ into a statement about trigonometric polynomial approximation.

LEMMA 4.1. *Suppose $f\in C(T, X)$ and suppose there exist $N\in N$, $\sigma>0$, $l>0$ and $\gamma>0$ so that for any subinterval I of $[-2\pi, 2\pi]$ with $|I|<l$ there is a trigonometric polynomial*

$$\varphi_I(\theta) = \sum_{k=-N}^{N} x_k e^{ik\theta} \quad \textit{with} \quad \|\tilde f-\varphi_I\|_I \leqslant \gamma|I|^\sigma.$$

Then $f\in C_\sigma(T, X)$.

Proof. First we note the existence of a constant $C = C(p, N)$ so that for α satisfying $0<\alpha\leqslant 1$ and all $x_0,\ldots,x_{2N}\in X$ we have

$$\max_{0\leqslant k\leqslant 2N}\|x_k\| \leqslant C\max_{|t|\leqslant 1}\Big\|\sum_{k=0}^{2N} x_k\Big(\frac{e^{i\alpha t}-1}{\alpha}\Big)^k\Big\|.$$

This is a simple consequence of the fact that $(2N+1)$-dimensional subspaces of X

are uniformly normable and $\lim_{\alpha\to 0}(e^{i\alpha t}-1)/\alpha = it$. Thus

$$\max_{0\leqslant k\leqslant 2N} \alpha^k\|x_k\| \leqslant C\max_{|t|\leqslant\alpha}\Big\|\sum_{k=0}^{2N} x_k(e^{it}-1)^k\Big\|.$$

In the remainder of the argument we take $\varrho = \gamma+\|f\|$ for convenience. Let I be any interval of length at most l. Then let s be the midpoint of I. For $0<\alpha\leqslant l/2$ there is a trigonometric polynomial g_α of degree at most N so that

$$\|\tilde{f}(s+t)-g_\alpha(t)\| \leqslant C\varrho\alpha^\sigma$$

for $|t|\leqslant\alpha$. Let

$$g_\alpha(t) = \sum_{k=0}^{2N} x_{k,\alpha}e^{-iNt}(e^{it}-1)^k.$$

For $k\leqslant 2N$, $\|x_{k,l/2}\|\leqslant C\varrho$. Now arguing as in Proposition 3.1 we obtain

$$\|g_\alpha(t)-g_\beta(t)\|\leqslant C\varrho\alpha^\sigma,\qquad |t|\leqslant\beta,$$

provided $\frac{1}{2}\alpha\leqslant\beta\leqslant\alpha$. Hence

$$\|x_{k,\alpha}-x_{k,\beta}\|\leqslant C\varrho\alpha^{\sigma-k}.$$

We conclude that if $k>\sigma$ then $\|x_{k,\alpha}\|\leqslant C\varrho\alpha^{\sigma-k}$, while if $k<\sigma$ then $\|x_{k,\alpha}\|\leqslant C\varrho$.

If $\sigma\in N$ and $k=\sigma$ then arguing as in Proposition 3.1 we get

$$\|x_{k,\sigma}\|\leqslant C(\log(l/\alpha)+1)^{1/p}\varrho.$$

Thus

$$\Big\|\sum_{k=\nu+1}^{2N} x_{k,\alpha}e^{-iNt}(e^{it}-1)^k\Big\|\leqslant C\varrho\alpha^\sigma.$$

Select polynomials λ_k of degree ν so that

$$|e^{-iNt}(e^{it}-1)^k-\lambda_k(t)|\leqslant C|t|^{\nu+1}$$

for $|t|\leqslant 2\pi$. Then

$$\Big\|\sum_{k=0}^{\nu} x_{k,\alpha}e^{-iNt}(e^{it}-1)^k-\sum_{k=0}^{\nu}x_{k,\alpha}\lambda_k(t)\Big\|\leqslant C\varrho\alpha^\sigma$$

if $|t|\leqslant\alpha$. Here if σ is an integer we need to observe that $\alpha^{\nu+1}(\log(l/\alpha)+1)^{1/p}\leqslant C\alpha^\nu = C\alpha^\sigma$. Thus

$$\Big\|\tilde{f}(s+t)-\sum_{k=0}^{\nu}x_{k,\alpha}\lambda_k(t)\Big\|\leqslant C\varrho\alpha^\sigma$$

provided $|t|\leqslant\alpha$. Taking $\alpha=\frac{1}{2}|I|$ we obtain the lemma.

We will now turn to consideration of a specific example. We shall need the following general lemma.

LEMMA 4.2. *There is a constant C depending only on n, $a_1,\ldots,a_n$ so that if $z_j\in\bar{\Delta}$, $1\leqslant j\leqslant n$, and $0<a_j<1$ but $a_1+\ldots+a_n>1$ then*

$$\int_{-\pi}^{\pi}\prod_{j=1}^{n}|1-z_je^{i\theta}|^{-a_j}d\theta\leqslant C\varrho^{1-(a_1+\ldots+a_n)}$$

where $\varrho=\min_{j\neq k}|z_j-z_k|$.

Proof. Let A_j be the arc in T described by $|1-z_je^{i\theta}|<\frac{1}{2}\varrho$. Then

$$\int_{A_j}\prod_{j=1}^{n}|1-z_je^{i\theta}|^{-a_j}d\theta\leqslant(\tfrac{1}{2}\varrho)^{-\beta_j}\int_{A_j}|1-z_je^{i\theta}|^{-a_j}d\theta$$

where $\beta_j=\sum_{k\neq j}a_j$. Now using the estimate

$$|1-re^{i\theta}|^{-1}\leqslant C((1-r)^2+\theta^2)^{-1/2}\leqslant C|\theta|^{-1}$$

we see that

$$\int_{A_j}|1-z_je^{i\theta}|^{-a_j}d\theta\leqslant C^{a_j}\int_{C|\theta|^{-1}>\varrho/2}|\theta|^{-a_j}d\theta\leqslant C\varrho^{1-a_j}.$$

Thus

$$\sum_j\int_{A_j}\prod_{j=1}^{n}|1-z_je^{i\theta}|d\theta\leqslant C\varrho^{1-(a_1+\ldots+a_n)}.$$

Let B be the complement of $A_1\cup\ldots\cup A_n$ in T. Then on B

$$\prod_{j=1}^{n}|1-z_je^{i\theta}|^{-a_j}\leqslant(\min_j|1-z_je^{i\theta}|)^{-(a_1+\ldots+a_n)}$$
$$\leqslant(\sum_j|1-z_je^{i\theta}|^{-1})^{a_1+\ldots+a_n}.$$

Let $\alpha=a_1+\ldots+a_n>1$. Then

$$\{\int_B\prod_{j=1}^{n}|1-z_je^{i\theta}|^{-a_j}d\theta\}^{1/\alpha}\leqslant\sum_j\{\int_B|1-z_je^{i\theta}|^{-\alpha}d\theta\}^{1/\alpha}$$
$$\leqslant\sum_j\{\int_{|1-z_je^{i\theta}|>\varrho/2}|1-z_je^{i\theta}|^{-\alpha}d\theta\}^{1/\alpha}$$
$$\leqslant\sum_j\{\int\min(|1-z_je^{i\theta}|^{-\alpha},(\tfrac{1}{2}\varrho)^{-\alpha})d\theta\}^{1/\alpha}$$
$$\leqslant C\{\int\min(\theta^{-\alpha},(\tfrac{1}{2}\varrho)^{-\alpha})d\theta\}^{1/\alpha}\leqslant C\varrho^{1/\alpha-1}$$

and the lemma follows.

Now we let

$$u(z)=(1-wz)^{-1}$$

so that $u: \Delta \to L_p(T)$ is an analytic function. We note that

$$u^{(m)}(z) = m!\, w^m (1-wz)^{-(m+1)}.$$

If $0 \leqslant m < 1/p-1$ then $u^{(m)}$ extends continuously to $\bar{\Delta}$.

We now compare $u^{(m)}$ with its Taylor series. For $z, z+\zeta \in \bar{\Delta}$ let

$$\varrho_l^{(m)}(z, \zeta) = u^{(m)}(z+\zeta) - \sum_{j=0}^{l} u^{(m+j)}(z)\zeta^j/j!.$$

LEMMA 4.3. *Suppose $1/p \notin N$. Let $\sigma = 1/p-m-1$ and $\nu = [\sigma]$. Then*

$$\|\varrho_\nu^{(m)}(z, \zeta)\|_p \leqslant C|\zeta|^\sigma$$

where C is independent of z, ζ.

Proof. By direct calculation

$$\varrho_l^{(0)}(z, \zeta) = \zeta^{l+1} w^{l+1}(1-wz)^{-(l+1)}(1-w(z+\zeta))^{-1}$$

and $\varrho_\nu^{(m)} = (\partial^m/\partial z^m)\varrho_\nu^{(0)}$. Every term in $\varrho_\nu^{(m)}$ is thus of the type

$$\zeta^{\nu+1} w^{m+\nu+1}(1-wz)^{-(\nu+j+1)}(1-w(z+\zeta))^{-(m+1-j)}$$

for $0 \leqslant j \leqslant m$ and is thus $O(|\zeta|^\sigma)$ by Lemma 4.4.

If $1/p \in N$ the situation is more complicated. For $0 < \varphi < \pi$ we write $h(\zeta, z)$ for the polynomial

$$\sum_{j=0}^{\nu-1} (1-wz)^{-(j+1)} w^j \zeta^j + w^\nu \zeta^\nu (1-wz)^{1-\nu}(1-wze^{-i\varphi})^{-1}(1-wze^{i\varphi})^{-1}.$$

LEMMA 4.4. *If $1/p \in N$ then there is a constant C so that if $\frac{1}{2}\sin\frac{1}{2}\varphi \leqslant |\zeta| \leqslant \sin\frac{1}{2}\varphi$ we have*

$$\|u^{(m)}(z+\zeta) - (\partial^m h/\partial z^m)(\zeta, z)\| \leqslant C\varphi^\sigma.$$

Proof. For convenience we write $f_{a,b,c,d}$ for the function

$$(1-wz)^{-a}(1-w(z+\zeta))^{-b}(1-wze^{i\varphi})^{-c}(1-wze^{-i\varphi})^{-d}.$$

Now

$$u(z+\zeta) - h(\zeta, z) = \zeta^\nu w^\nu f_{\nu,1,1,1}((1-\cos\varphi)wz + w\zeta - w^2\zeta z).$$

Thus

$$u(z+\zeta) - h(\zeta, z) = ((1-\cos\varphi)wzf_{\nu,1,1,1} + w\zeta f_{\nu-1,1,1,1})\zeta^\nu w^\nu.$$

Since $1-\cos\varphi = O(\varphi^2)$ and $|\zeta| = O(\varphi)$ we can use Lemma 4.2 to deduce

$$\|u^{(m)}(z+\zeta) - \partial^m h/\partial z^m\| \leqslant C\varphi^\sigma$$

provided $\frac{1}{2}\sin\frac{1}{2}\varphi \leqslant |\zeta| \leqslant \sin\frac{1}{2}\varphi$. This follows on checking each term in the derivative and noting that since $\nu+m < 1/p$ each term of the form $f_{a,b,c,d}$ satisfies $\|f_{a,b,c,d}\| \leqslant C\varphi^{(1/p)-a-b-c-d}$.

THEOREM 4.5. *Let $\sigma = 1/p-m-1$. Then $u^{(m)} \in C_\sigma(T, X)$.*

Proof. For $1/p \notin N$, this is immediate from Lemma 4.1 and Lemma 4.3. If $1/p \in N$, we use instead Lemma 4.4, which shows that by appropriate choice of z we can approximate $u^{(m)}(e^{i\theta})$ on any interval I with $|I| \leqslant \frac{1}{2}$ by a trigonometric polynomial $\varphi(\theta)$ of degree at most ν so that

$$\|u^{(m)} - \varphi\|_I \leqslant C|I|^\sigma.$$

Now let $J_{p,0}^{(m)}$ be the closed linear span of the functions $u^{(m)}(z)$ for $|z| = 1$. We note that if $m \leqslant \beta \leqslant n < 1/p-1$ then $w^\beta(1-wz)^{-(n+1)}$ is in $J_{p,0}^{(m)}$. In fact this can be proved simply by induction. If $v(z) = w^\beta(1-wz)^{-(n+1)}$ is in $J_{p,0}^{(m)}$ and $n+1 < 1/p-1$ then $v'(z) \in J_{p,0}^{(m)}$, i.e. $w^{\beta+1}(1-wz)^{-(n+2)} \in J_{p,0}^{(m)}$. But then this also implies $w^\beta(1-wz)^{-(n+1)} + w^{\beta+1}z(1-wz)^{-(n+2)} = w^\beta(1-wz)^{-(n+2)} \in J_{p,0}^{(m)}$.

Now if

$$h(z, w) = \sum_{j=1}^{N} a_j(1-\alpha_j wz)^{-(k_j+1)}$$

where $|\alpha_j| = 1$, $0 \leqslant k_j < 1/p-m-1$ and $a_j \in C$, then $\partial^m h/\partial z^m \in J_{p,0}^{(m)}$. In particular, if h is the function defined before Lemma 4.4 then $\partial^m h/\partial z^m$ is in $J_{p,0}^{(m)}$ for all z. To see this simply write the last term in partial fractions.

Now by Lemmas 4.3 and 4.4 we immediately obtain:

THEOREM 4.6. *Suppose $0 < p < 1$ and $0 \leqslant m < 1/p-1$. Let $\sigma = 1/p-1-m$. Then for $0 < r < 1$*

$$d(u^{(m)}(re^{i\theta}), J_{p,0}^{(m)}) \leqslant C(1-r)^\sigma$$

uniformly in θ.

5. Analytic functions and linear operators. As usual X will denote a p-Banach space where $0 < p < 1$. We denote by $A_0(X)$ the space of continuous functions $f: \bar{\Delta} \to X$ which are analytic in the open unit disc Δ. $A_0(X)$ is quasi-normed by

$$\|f\|_0 = \max_{|z| \leqslant 1} \|f(z)\|.$$

For $\sigma > 0$ and $\sigma \notin N$, let $\nu = [\sigma]$. We let $A_\sigma(X)$ denote the space of analytic functions on Δ such that

$$\sup_{|z|<1} \|f^{(\nu+1)}(z)\|(1-|z|^2)^{\nu+1-\sigma} < \infty.$$

On $A_\sigma(X)$ we impose the quasi-norm

(5.1) $$\|f\|_\sigma = \sup_{|z|<1} \|f^{(\nu+1)}(z)\|(1-|z|^2)^{\nu+1-\sigma} + \sum_{k=0}^{\nu} \|f^{(k)}(0)\|.$$

If σ is an integer we let $\nu = \sigma$ and define $A_\sigma(X)$ to be the space of analytic functions

f defined on Δ so that

$$\sup_{|z|<1} (1-|z|^2)\,\|f^{(\nu+1)}(z)\| < \infty .$$

The quasi-norm on $A_\sigma(X)$ is again defined by (5.1), with $\nu = \sigma$ of course.

Note that we have not asserted that the spaces $A_\sigma(X)$, $\sigma \geqslant 0$, are complete. This fact will, however, be established later.

Now suppose E is any quasi-Banach space of scalar-valued analytic functions on Δ containing the disc algebra $A(\Delta)$ and so that the inclusion $A(\Delta) \to E$ is bounded and has dense range. If $T \in \mathscr{L}(E, X)$ we define the *analytic transform* f_T to be the function $f_T\colon \Delta \to X$ given by

$$f_T(z) = T(u(z))$$

where $u(z) = (1-wz)^{-1}$. It is clear that f_T is analytic on X and has the power series expansion

$$f_T(z) = \sum_{n=0}^{\infty} x_n z^n$$

where $T(w^n) = x_n$.

The analytic transform induces a one-one correspondence between $\mathscr{L}(E, X)$ and a certain space of X-valued analytic functions on Δ.

THEOREM 5.1. *Suppose $0 < q < p \leqslant 1$. Then the map $T \to f_T$ induces a linear isomorphism between the spaces $\mathscr{L}(H_q, X)$ and $A_\sigma(X)$ where $\sigma = 1/q - 1$.*

Proof. If $T \in \mathscr{L}(H_q, X)$ then $f_T(z) = T(u(z))$ where $u(z) = (1-wz)^{-1}$. It follows that $f_T^{(\nu+1)}(z) = T(u^{(\nu+1)}(z))$ and the fact that $f_T \in A_\sigma(X)$ with $\|f_T\|_\sigma \leqslant C\|T\|$ follows easily from the fact that $u \in A_\sigma(H_q)$.

Conversely, let us suppose $f \in A_\sigma(X)$. Then f has a Maclaurin expansion

$$f(z) = \sum_{n=0}^{\infty} x_n z^n, \qquad |z| < 1 .$$

For $r < 1$ let $T_r \in \mathscr{L}(H_q, X)$ be defined by

$$T_r \varphi = \sum_{n=0}^{\infty} a_n r^n x_n$$

where $\varphi(w) = \sum_{n=0}^{\infty} a_n w^n \in H_q$. Since $|a_n| \leqslant C(n+1)^{1/q-1}$ it is clear that $T_r\varphi$ is well defined and T_r is bounded.

If $\varphi(w) = \sum_{n=0}^{\infty} a_n w^n$ in H_q define $\psi(w) \in H_q$ by

$$\psi(w) = \sum_{n=\nu+1}^{\infty} a_n w^{n-\nu-1} .$$

Then $\|\psi\|_q \leqslant C\|\varphi\|_q$ where $C = C(\nu, q)$. Now $\psi \in B_{q,p}$ and $\|\psi\|_{q,p} \leqslant C\|\varphi\|_q$. Hence by a theorem of Coifman and Rochberg [4] we can write in $B_{q,p}$

$$\psi(w) = \sum_{k=1}^{\infty} \alpha_k (1-|z_k|^2)^{\nu+1-\sigma} (1-wz_k)^{-(\nu+2)}$$

where $z_k \in \Delta$ and (α_k) are so that

$$\left(\sum |\alpha_k|^p\right)^{1/p} \leqslant C\|\varphi\|_q .$$

Now T_r is bounded also on $B_{q,p}$ since $B_{q,p}$ is the containing p-Banach space of H_q. Thus if we write

$$w^{\nu+1}\psi(w) = ((\nu+1)!)^{-1} \sum_{k=1}^{\infty} \alpha_k (1-|z_k|^2)^{\nu+1-\sigma} u^{(\nu+1)}(z_k)$$

then

$$T_r(w^{\nu+1}\psi(w)) = ((\nu+1)!)^{-1} \sum_{k=1}^{\infty} \alpha_k (1-|z_k|^2)^{\nu+1-\sigma} f^{(\nu+1)}(rz_k).$$

Thus $\|T_r(w^{\nu+1}\psi(w))\| \leqslant C\|f\|_\sigma \|\varphi\|_q$. Now

$$\Big\|T_r\Big(\sum_{k=0}^{\nu} a_k w^k\Big)\Big\| \leqslant C\|f\|_\sigma \|\varphi\|_q$$

and hence

$$\|T_r\varphi\| \leqslant C\|f\|_\sigma \|\varphi\|_q .$$

Thus $\|T_r\| \leqslant C\|f\|_\sigma$. As $\lim_{r\to 1} T_r(w^n)$ exists for all $n \geqslant 0$, we can define a bounded linear operator T so that $T(w^n) = x_n$ and $\|T\| \leqslant C\|f\|_\sigma$. Clearly $f_T = f$.

COROLLARY 5.2. *If $\sigma > 1/p - 1$, then $A_\sigma(X)$ is complete.*

We now use the identification of Theorem 5.1 to derive some important facts about the class $A_\sigma(X)$ with no restriction on σ.

THEOREM 5.3. *Suppose $\sigma > 0$ and $f \in A_\sigma(X)$. Then*

(i) *If m is an integer and $0 \leqslant m < \sigma$ then $f^{(m)} \in A_0(X)$, i.e., $f^{(m)}$ extends continuously to Δ. The map $f \to f^{(m)}$ is continuous from $A_\sigma(X)$ into $A_0(X)$.*

(ii) *If m is an integer and $m = \sigma$, then for some constant $B = B(f)$ and exponent $\alpha > 0$,*

$$\|f^{(m)}(z)\| \leqslant B\left(1+(\log(1-|z|)^{-1})^\alpha\right).$$

(iii) *For $0 \leqslant r \leqslant 1$, the functions f_r given by*

$$f_r(e^{i\theta}) = f(re^{i\theta})$$

are in $C_\sigma(\mathbf{T}, X)$ and $\sup_{1/2 \leqslant r \leqslant 1} \|f_r\|_{\mathbf{T},\sigma} < \infty$.

(iv) *If* $\sigma > 1$ *then*

$$(d/d\theta) f(e^{i\theta}) = ie^{i\theta} f'(e^{i\theta}).$$

(v) *Each of the spaces* $A_\sigma(X)$ *is complete.*

Proof. First select $n \in N$ so that $\sigma + n > 1/p - 1$. Let $1/q = \sigma + n + 1$. Integrate f n times to produce $F \in A_{\sigma+n}$ with $F(0) = \ldots = F^{(n-1)}(0) = 0$. Then $\|F\|_{\sigma+n} = \|f\|_\sigma$ and so there is a bounded linear operator $T\colon H_q \to X$ with $T(u(z)) = F(z)$ for $|z| < 1$, and $\|T\| \leqslant C\|f\|_\sigma$. Let $Q_r\colon H_q \to H_q$ be the map

$$Q_r \varphi(w) = \varphi(rw)$$

for $0 \leqslant r \leqslant 1$. Then $TQ_r(u(z)) = F(rz)$ and hence

$$TQ_r(u^{(n)}(z)) = r^n f(rz).$$

For (i) we observe, taking $r = 1$, that $u^{(m+n)}(z) \in A_0(H_q)$ and hence $T(u^{(m+n)}(z)) = f^{(m)}(z) \in A_0(X)$, and of course

$$\max \|f^{(m)}(z)\| \leqslant C\|T\| \leqslant C\|f\|_\sigma.$$

For (ii), note that $m + n = 1/q - 1$. Then in H_q

$$\|u^{(m+n)}(z)\| \leqslant C\left(1 + \left(\log\frac{1}{1-|z|}\right)^{1/q}\right)$$

and the result follows.

To prove (iii) note that $u^{(n)}(e^{i\theta}) \in C_\sigma(T, H_q)$ and hence, since $\|TQ_r\| \leqslant \|T\|$, $r^n f(re^{i\theta}) \in C_\sigma(T, X)$ and

$$\sup_{0 \leqslant r \leqslant 1} \|r^n f(re^{i\theta})\|_{T,\sigma} \leqslant C\|f\|_\sigma.$$

(iii) now follows.

To prove (iv) note that

$$\frac{d}{d\theta} f(e^{i\theta}) = T\left(\frac{d}{d\theta} u^{(n)}(e^{i\theta})\right) = ie^{i\theta}\, T(u^{(n+1)}(e^{i\theta})) = ie^{i\theta} f'(e^{i\theta}).$$

For (v) suppose f_l is a Cauchy sequence in $A_\sigma(X)$. Let F_l be the corresponding n-fold integrals in $A_{\sigma+n}(X)$. Then F_l is a Cauchy sequence and converges in $A_{\sigma+n}(X)$ to a limit F by Corollary 5.2. Now $F_l^{(n)} \to F^{(n)}$ in $A_\sigma(X)$ and (v) is proved.

LEMMA 5.4. *Suppose* $\sigma > 0$. *Then* $f \in A_\sigma(X)$ *if and only if* $g \in A_\sigma(X)$ *where* $g(z) = zf(z)$.

Proof. For $n \in N$

$$g^{(n)}(z) = zf^{(n)}(z) + nf^{(n-1)}(z).$$

Suppose $f \in A_\sigma(X)$. Pick $n = \nu + 1$. Then

$$\|g^{(\nu+1)}(z)\| \leqslant C(1-|z|^2)^{\sigma-\nu-1}$$

by applying Theorem 5.3 (i) or (ii) depending on whether $\sigma \notin N$ or $\sigma \in N$.

Conversely, suppose $g \in A_\sigma(X)$. Then, by induction, $f^{(m)}$ is bounded if $0 \leqslant m < \sigma$. If $\sigma \notin N$, $f^{(\nu)}$ is bounded, and hence, since

$$zf^{(\nu+1)}(z) = g^{(\nu+1)}(z) - (\nu+1) f^{(\nu)}(z),$$

we see that $f \in A_\sigma(X)$.

If $\sigma \in N$, then $f^{(\nu-1)}$ is bounded and in this case

$$\|f^{(\nu)}(z)\| \leqslant C\left(1 + \left(\log\frac{1}{1-|z|}\right)^\alpha\right)$$

and hence again

$$\|f^{(\nu+1)}(z)\| \leqslant C(1-|z|)^{-1}$$

so that $f \in A_\sigma(X)$.

We are now ready to prove a limiting case of Theorem 5.3 when $\sigma = 1/p - 1$. For this we define C_p to be the space of functions f analytic on the open unit disc so that

$$\|f\| = |f(0)| + \{\int_\Delta |f'(w)|^p (1-|w|^2)^{p-1}\, d\lambda(w)\}^{1/p} < \infty$$

(where λ denotes the planar measure $dx\,dy$). Then $f \in C_p$ if and only if $f' \in B_{r,p}$ where $1/r = 1/p + 1$. The natural integration operator J maps $B_{r,p}$ isomorphically onto the subspace $C_{p,0}$ of functions vanishing at 0. It is well known (Duren [6], p. 88) that J maps $B_{r,p}$ into H_p. Thus $C_p \subset H_p$. However, $C_p \cong B_{r,p} \oplus C \cong l_p$ (cf. [12], [20]) and hence $C_p \neq H_p$.

THEOREM 5.5. *Suppose* $0 < p < 1$. *Then the mapping* $T \to f_T$ *defines a linear isomorphism between the spaces* $\mathscr{L}(C_p, X)$ *and* $A_\sigma(X)$ *where* $\sigma = 1/p - 1$.

Proof. If $T \in \mathscr{L}(C_p, X)$ then $TJ \in \mathscr{L}(H_r, X)$. Let g be the analytic transform of TJ so that $g \in A_{\sigma+1}(X)$ and $\|g\|_{\sigma+1} \leqslant C\|T\|$. Now if $f = f_T$ is the analytic transform of T then

$$f(z) = z\frac{d}{dz}(zg(z))$$

and hence $f \in A_\sigma(X)$ by Lemma 5.4. By an application of the Uniform Boundedness Principle it may be seen that

$$\sup_{\|T\| \leqslant 1} \sup_{z \in \Delta} (1-|z|^2)^{\nu+1-\sigma} \|f_T^{(\nu+1)}(z)\| < \infty$$

and for $k \leqslant \nu$

$$\sup_{\|T\| \leqslant 1} \|f_T^{(k)}(0)\| < \infty$$

so that $\|f_T\|_\sigma \leqslant C\|T\|$.

Conversely, suppose $f\in A_\sigma(X)$. Then $z^{-1}(f(z)-f(0))\in A_\sigma(X)$ and hence if

$$g(z)=\sum_{n=0}^{\infty}\frac{f^{(n+1)}(0)}{(n+1)(n+1)!}z^n$$

then $g\in A_{\sigma+1}(X)$. Thus there is an operator $S\colon B_{r,p}\to X$ so that $S(u(z))=g(z)$. Now define $T\colon C_p\to X$ by $T\varphi=S\varphi'+f(0)\varphi(0)$. (Note $\varphi'=d\varphi/dw$.) Then

$$\begin{aligned}T(u(z))&=S(z/(1-wz)^2)+f(0)\\&=zS(1/(1-wz))+z^2S(w/(1-wz)^2)+f(0)\\&=zg(z)+z^2g'(z)+f(0)\\&=\sum_{n=0}^{\infty}\frac{f^{(n+1)}(0)}{(n+1)!}\left(\frac{1}{n+1}+\frac{n}{n+1}\right)z^{n+1}+f(0)\\&=\sum_{n=0}^{\infty}\frac{f^{(n+1)}(0)}{(n+1)!}z^{n+1}+f(0)=f(z).\end{aligned}$$

This shows that the map $T\to f_T$ is a surjection and by the Open Mapping Theorem it is also an isomorphism.

6. Remarks on analytic functions. We now give some applications of the results of Section 5 to the general theory of analytic functions in a quasi-Banach space.

THEOREM 6.1. *Suppose $n\in N$ and $n>1/p$. Then for any $f\in A_0(X)$ we have*

$$\|f^{(m)}(0)\|\leqslant C(m+n)!\,\|f\|$$

where C is independent of m and f.

Proof. Pick q so that $n+1>1/q>n$. Let

$$F(z)=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{(n+k)!}z^{n+k}.$$

Then $F^{(n)}\in A_0(X)$ and so $F\in A_\sigma(X)$ where $\sigma=1/q-1$. Define $T\in\mathscr{L}(H_q,X)$ to be the operator with analytic transform F. Note that $\|F\|_\sigma\leqslant\|f\|$. Then

$$\left\|\frac{f^{(m)}(0)}{(m+n)!}\right\|=\|T(w^{(k+m)})\|\leqslant\|T\|\leqslant C\|f\|.$$

THEOREM 6.2 (Liouville). *Let $f\colon \boldsymbol{C}\to X$ be a bounded entire function. Then f is constant* (compare [16], [21]).

Proof. Simply apply Theorem 6.1 to f_r where $f_r(z)=f(rz)$ for $0<r<\infty$. One concludes that $f'(0)=f''(0)=\ldots=0$.

THEOREM 6.3. *$A_0(X)$ is complete (i.e. it is a closed subspace of $C(\bar{\Delta},X)$).*

Proof. Observe that the linear maps $f\to f^{(m)}(0)$ $(m\in N)$ extend to the closure of $A_0(X)$ and that, for fixed $z\in\Delta$, the series $\sum f^{(m)}(0)\,z^m/m!$ converges to $f(z)$ uniformly on the set $\{f\in A_0(X)\colon \|f\|\leqslant 1\}$. Hence if $g\in\overline{A_0(X)}$ then

$$g(z)=\sum_{m=0}^{\infty}x_mz^m,\qquad |z|<1,$$

for some $x_m\in X$.

THEOREM 6.4. *Suppose $\sigma>1/p-1$ and $f\in A_\sigma(X)$. Then for $m\geqslant 0$*

$$f^{(m)}(0)=\frac{m!}{2\pi}\int_0^{2\pi}f(e^{i\theta})e^{-im\theta}d\theta,$$

while for $m<0$

$$0=\int_0^{2\pi}f(e^{i\theta})e^{-im\theta}d\theta.$$

Here the integral is the Turpin–Waelbroeck integral as described in Section 3, of f with respect to $d\mu=e^{-im\theta}d\theta$.

Proof. For $r<1$, let

$$g_n(e^{i\theta})=\sum_{k=0}^{n}f^{(k)}(0)\,r^k e^{ik\theta}/k!.$$

Then $\operatorname{rank} g_n\leqslant n+1$ and if $f_r(e^{i\theta})=f(re^{i\theta})$ then

$$\begin{aligned}\|g_n-f_r\|_T&\leqslant\Big\{\sum_{k=n+1}^{\infty}\|f^{(k)}(0)\|^p r^{kp}/(k!)^p\Big\}^{1/p}\\&\leqslant C\Big\{\sum_{k=n+1}^{\infty}((s+k)!/k!)^p r^{kp}\Big\}^{1/p}\end{aligned}$$

where $s\in N$ and $s>1/p$, by Theorem 6.1. Hence

$$\|g_n-f_r\|_T=O\left(\left(\frac{1+r}{2}\right)^n\right)=O(n^{-\sigma})$$

so that

$$\begin{aligned}&\frac{m!}{2\pi}\int_0^{2\pi}f(re^{i\theta})e^{-im\theta}d\theta=r^mf^{(m)}(0), && m\geqslant 0,\\&\frac{m!}{2\pi}\int_0^{2\pi}f(re^{i\theta})e^{-im\theta}d\theta=0, && m<0.\end{aligned}$$

Now we use Theorem 3.4 (i) and Theorem 5.3 (i) and (iii) to let $r\to 1$, and obtain the result.

Now recall that for $\sigma\in\boldsymbol{R}$, $V_\sigma(X)$ consists of analytic $f\colon\Delta\to X$ so that for some C

$$\|f(z)\| \leqslant C(1-|z|)^{\sigma}, \quad z\in\Delta.$$

LEMMA 6.5. *If* $f\in V_\sigma(X)$ *then* $f'\in V_{\sigma-1}(X)$.

Proof. If $z_0\in\Delta$, let $\alpha=\frac{1}{2}(1-|z_0|)$. Define $g(z)=f(z_0+\alpha z)$ for $|z|\leqslant 1$. Then $g\in A_0(X)$ and so

$$\|g'(0)\|\leqslant C\sup_{|z|\leqslant 1}\|g(z)\|, \quad \text{i.e.} \quad \|\alpha f'(z_0)\|\leqslant C\alpha^{\sigma}.$$

Hence $\|f'(z_0)\|\leqslant C(1-|z_0|)^{\sigma-1}$.

THEOREM 6.6. *For* $\sigma>0$, $f\in V_\sigma(X)$ *if and only if* $f\in A_\sigma(X)$ *and* $f(z)=0$ *for* $z\in T$.

Proof. Let $\nu=[\sigma]$. Then repeated application of Lemma 6.5 shows that if $f\in V_\sigma(X)$, then $f\in A_\sigma(X)$.

For the other direction, select n so that $\sigma+n>1/p-1$. Pick q so that $1/q=\sigma+n+1$. Integrate f n times to produce $F\colon \Delta\to X$ with $F(0)=\ldots=F^{(n-1)}(0)=0$ and $\|F\|_{\sigma+n}=\|f\|_\sigma$. Then there is a bounded linear operator $T\colon H_q\to X$ with $T(u(z))=F(z)$.

Note $T(u^{(n)}(z))=f(z)$. If $f=0$ on T then $T(J^{(n)}_{q,0})=0$ and so we can apply Theorem 4.6 to deduce $\|f(z)\|\leqslant C(1-|z|)^{\sigma}$.

THEOREM 6.7. *If* $\sigma>1/p-1$ *then* $V_\sigma(X)=\{0\}$.

Proof. By Theorem 6.4 if $f\in V_\sigma(X)$ then $f^{(m)}(0)=0$ for all m.

Remark. We have seen that if $\sigma=1/p-1$ then $V_\sigma(X)$ can be nontrivial.

7. Operators on L_p. We now introduce the space $E_\sigma(X)$. This will consist of all functions $f\colon \boldsymbol{C}\to X$ with the properties:

(a) f continuously extends to the Riemann sphere $\boldsymbol{C}^*=\boldsymbol{C}\cup\{\infty\}$ if we set $f(\infty)=0$.

(b) f is analytic on $\boldsymbol{C}^*\setminus T$.

(c) $f\in A_\sigma(X)$ on the disc Δ.

(d) $f(1/z)\in A_\sigma(X)$ on the disc Δ.

$E_\sigma(X)$ is quasi-normed by

$$\|f\|_{\sigma,E}=\max(\|f\|_\sigma, \|f(1/z)\|_\sigma).$$

We shall consider only the case $\sigma=1/p-1$. In this instance we have seen that $A_\sigma(X)$ is complete and the injection $A_\sigma(X)\hookrightarrow A_0(X)$ is bounded. From these observations we see that $E_\sigma(X)$ is complete.

If $T\in\mathscr{L}(L_p, X)$ where $0<p<1$ then its analytic transform f_T is defined by

$$f_T(z)=T((1-wz)^{-1}), \quad z\in\boldsymbol{C}.$$

THEOREM 7.1. *Suppose* $0<p<1$. *Then the analytic transform* $T\to f_T$ *induces a linear isomorphism between* $\mathscr{L}(L_p, X)$ *and* $E_\sigma(X)$ *where* $\sigma=1/p-1$.

Remark. As usual X is a p-normable space.

Proof. One direction is very easy. If $T\in\mathscr{L}(L_p, X)$ then it is immediate that $f_T\in E_\sigma(X)$ and $\|f_T\|_{\sigma,E}\leqslant C\|T\|$. We now show that the map $T\to f_T$ is a surjection. Since it is trivially an injection the conclusion follows from the Open Mapping Theorem.

Let us suppose $f\in E_\sigma(X)$. Then f has a Taylor series expansion around the origin,

$$f(z)=\sum_{n=0}^{\infty} x_n z^n, \quad |z|<1,$$

and a Laurent series expansion around ∞,

$$f(z)=\sum_{n=1}^{\infty} y_n z^{-n}, \quad |z|>1.$$

It is readily seen that $f=f_T$ if and only if $T(w^n)=x_n$ for $n\geqslant 0$ and $T(w^{-n})=-y_n$ for $n>0$. We therefore need to show the existence of such an operator T.

Let us first suppose $x_0=f(0)=0$. Then we define two analytic functions on Δ by

$$F_1(z)=\sum_{n=1}^{\infty}\frac{x_n}{n}z^n, \quad F_2(z)=\sum_{n=1}^{\infty}\frac{y_n}{n}z^n.$$

Then $F_1'(z)=z^{-1}f(z)$ and hence, by Lemma 5.4, $F_1'\in A_\sigma$ and so $F_1\in A_{\sigma+1}$. Similarly $F_2'(z)=z^{-1}f(z^{-1})$ and so $F_2\in A_{\sigma+1}$. In particular, F_1 and F_2 extend continuously to $\bar{\Delta}$ (Theorem 5.3). Since $\sigma+1=1/p>1/p-1$ we can utilize Theorem 6.4:

$$x_n/n=(2\pi)^{-1}\int_0^{2\pi}F_1(e^{i\theta})e^{-in\theta}\,d\theta, \quad n\geqslant 1,$$
$$y_n/n=(2\pi)^{-1}\int_0^{2\pi}F_2(e^{i\theta})e^{-in\theta}\,d\theta, \quad n\geqslant 1.$$

Also

$$\int_0^{2\pi}F_1(e^{i\theta})e^{in\theta}\,d\theta=\int_0^{2\pi}F_2(e^{i\theta})e^{in\theta}\,d\theta=0.$$

Next we note, by Theorem 5.3 again, that

$$(d/d\theta)F_1(e^{-i\theta})=-ie^{-i\theta}F_1'(e^{-i\theta})=-if(e^{-i\theta}),$$
$$(d/d\theta)F_2(e^{i\theta})=ie^{i\theta}F_2'(e^{i\theta})=if(e^{-i\theta}).$$

Now define $G\in C_{\sigma+1}(T, X)$ by $G(e^{i\theta})=-F_1(e^{i\theta})-F_2(e^{i\theta})$. Then $(d/d\theta)G(e^{i\theta})=0$. Now we use Proposition 3.1 (iii) or (iv), depending on whether $\sigma \notin N$ or $\sigma \in N$, to deduce that

$$\|G(e^{i\varphi}) - G(e^{i\theta})\| \leqslant C|\varphi - \theta|^{1/p}.$$

Now it follows that there is an operator T: $L_p \to X$ with

$$T\chi_{(\theta,\varphi)} = (2\pi i)^{-1}\big(G(e^{i\varphi}) - G(e^{i\theta})\big)$$

where $\chi_{(\theta,\varphi)}(e^{it}) = 1$ if $\theta \leqslant t \leqslant \varphi$ and zero elsewhere (when $0 \leqslant \theta < \varphi \leqslant 2\pi$).

We now compute $T(w^n)$. To do this we introduce simple functions h_N ($N = 1, 2, \ldots$). Let

$$h_N(e^{it}) = e^{in\theta_k}, \qquad \theta_{k-1} < t \leqslant \theta_k,$$

where $\theta_k = 2k\pi i/N$, for $k = 0, 1, \ldots, N$. Then $Th_N \to Tw^n$. However,

$$Th_N = (2\pi i)^{-1} \sum_{k=1}^{N} e^{in\theta_k}\big(G(e^{i\theta_k}) - G(e^{i\theta_{k-1}})\big)$$

$$= (2\pi i)^{-1} \sum_{k=1}^{N} G(e^{i\theta_k})(e^{in\theta_k} - e^{in\theta_{k+1}})$$

where $\theta_{N+1} = \theta_1$. Hence

$$Th_N = (2\pi i)^{-1} \int G(e^{i\theta})\, d\mu_N(\theta)$$

where μ_N is the measure on $[0, 2\pi]$ given by

$$\mu_N = \sum_{k=1}^{N} (e^{in\theta_k} - e^{in\theta_{k+1}})\,\delta(\theta_k).$$

In the weak*-topology $\mu_N \to \mu$ where $d\mu = -ine^{in\theta}\,d\theta$. At this stage we appeal to Theorem 3.4 (ii) to deduce that

$$Tw^n = \frac{-n}{2\pi} \int_0^{2\pi} G(e^{i\theta})\, e^{in\theta}\, d\theta$$

$$= \frac{n}{2\pi} \int_0^{2\pi} \big(F_1(e^{-i\theta}) + F_2(e^{i\theta})\big) e^{in\theta}\, d\theta.$$

If $n \geqslant 0$ then $Tw^n = x_n$, while for $n > 0$, $Tw^{-n} = -y_n$ as required. This settles the special case when $f(0) = 0$.

If $f(0) = x_0 \neq 0$, let X_0 be the one-dimensional space spanned by x_0 and let Q: $X \to X/X_0$ be the quotient map. Then $Qf(0) = 0$ and so there is a bounded linear operator S: $L_p(\boldsymbol{T}) \to X/X_0$ with $S(w^n) = Qx_n$ for $n \geqslant 0$ and $S(w^{-n}) = -Qy_n$ for $n > 0$. By results in [11], S has a unique lift T: $L_p \to X$ with $QT = S$. Let

$$T\big((1 - wz)^{-1}\big) = g(z).$$

Then $f - g$ has range in X_0, i.e. $f(z) - g(z) = h(z)x_0$ where $h \in E_\sigma(\boldsymbol{C}) = \{0\}$ by Liouville's Theorem. Hence T has analytic transform f.

The isomorphism between $\mathscr{L}(L_p, X)$ and $E_\sigma(X)$ follows from the Open Mapping Theorem.

An operator T on L_p vanishes on $\bar{H}_{p,0}$ if its analytic transform f_T vanishes for $|z| \geqslant 1$. Then we must have $f_T \in V_\sigma(X)$ on the disc. Conversely, if $f \in V_\sigma(X)$ then f can be continued over $\boldsymbol{C}$ to be zero outside the disc and hence there is an operator T on L_p so that $T(\bar{H}_{p,0}) = 0$ and $f_T = f$ in the open unit disc. Summarizing:

Theorem 7.2. *There is a natural linear isomorphism between $\mathscr{L}(L_p/\bar{H}_{p,0}, X)$ (or $\mathscr{L}(H_p/J_{p,0}, X)$) and $V_\sigma(X)$ implemented by*

$$T(qu(z)) = f_T(z), \qquad |z| < 1,$$

where q: $L_p \to L_p/\bar{H}_{p,0}$ is the quotient map.

Theorem 7.3. *In order that $V_\sigma(X) \neq \{0\}$ it is necessary and sufficient that there exists a nonzero linear operator T: $L_p/H_p \to X$.*

8. Applications to L_p. We first apply our main theorem to extend a theorem due to Aleksandrov [1] that $L_p = H_p + \bar{H}_p$. Our extension uses the space C_p introduced in Section 5. As noted there, C_p is strictly contained in H_p.

Theorem 8.1 *There is a constant C so that if $f \in L_p(\boldsymbol{T})$ then there exist $g_1, g_2 \in C_p$ with $\|g_1\|_{C_p} \leqslant C\|f\|_p$, $\|g_2\|_{C_p} \leqslant C\|f\|_p$ and*

$$f(e^{i\theta}) = g_1(e^{i\theta}) + g_2(e^{-i\theta})$$

a.e. on $\boldsymbol{T}$.

Proof. We define a linear operator W: $C_p \oplus C_p \to L_p(\boldsymbol{T})$ by

$$W(h_1, h_2) = h_1(e^{i\theta}) + h_2(e^{-i\theta}).$$

(Note that each $h_i \in H_p$ and so has boundary values a.e. on $\boldsymbol{T}$).

Let $N = W^{-1}(0)$, and let Q: $C_p \oplus C_p \to Y = C_p \oplus C_p/N$ be the quotient map. Define f: $\boldsymbol{C} \to Y$ by

$$f(z) = \begin{cases} Q(u(z), 0), & |z| \leqslant 1, \\ Q(0, 1 - u(1/z)), & |z| \geqslant 1. \end{cases}$$

Then f is continuous on $\boldsymbol{C} \cup \{\infty\}$ and it is readily verified to be in $E_\sigma(Y)$. Hence there is an operator S: $L_p(\boldsymbol{T}) \to Y$ with

$$S\big((1 - wz)^{-1}\big) = f(z), \qquad z \in \boldsymbol{C}.$$

It is easily seen that $SW = Q$ and hence Y is isomorphic to $L_p(\boldsymbol{T})$ and W is a surjection as required.

An immediate corollary of Theorem 8.1 is an atomic decomposition for $L_p(\boldsymbol{T})$ in the spirit of [4] which may also be regarded as a strengthening of Aleksandrov's theorem.

THEOREM 8.2. *Suppose* $0<p<1$ *and* $\beta>1/p$. *Then there exists* $\eta_0 = \eta_0(p, \beta)$ *so that if* $\eta<\eta_0$ *and* $(\zeta_n)_{n=1}^{\infty}$ *is an* η*-lattice in* Δ *for the Bergman metric and* $\zeta_n \neq 0$ *then there is a constant* C *so that if* $f\in L_p(T)$ *then*

$$f(w) = \sum_{n=1}^{\infty} a_n(1-\zeta_n w)^{-\beta}(1-|\zeta_n|^2)^{\beta-1/p}+\sum_{n=1}^{\infty} b_n(1-\bar{\zeta}_n \bar{w})^{-\beta}(1-|\zeta_n|^2)^{\beta-1/p}$$

where $\sum|a_n|^p+\sum|b_n|^p \leqslant C\|f\|_p^p$.

Proof. First we note that if $C_{p,0}$ is the set of $g\in C_p$ so that $g(0)=0$ then the decomposition in Theorem 8.1 can be achieved with $g_1, g_2\in C_{p,0}$. Indeed, if not, there is a linear functional $\tau\neq 0$ on L_p so that $\tau(W(h_1, h_2))=0$ for $h_1, h_2\in C_{p,0}$. Hence

$$\tau\circ W(h_1, h_2) = ah_1(0)+bh_2(0)$$

where $a, b\in C$. By the openness of W, τ is continuous on L_p and we have a contradiction.

Now we can write

$$f(w) = g_1(w)+g_2(\bar{w})$$

where $g_1, g_2\in C_{p,0}$ and $\|g_i\|\leqslant C\|f\|_p$ $(i=1,2)$. Then $g_1', g_2'\in B_{p,r}$ where $1/r = 1/p+1$. Hence by Theorem 2 of [4],

$$g_1'(w) = \sum_{n=1}^{\infty} c_n(1-\zeta_n w)^{-(\beta+1)}(1-|\zeta_n|^2)^{\beta-1/p},$$

$$g_2'(w) = \sum_{n=1}^{\infty} d_n(1-\bar{\zeta}_n w)^{-(\beta+1)}(1-|\zeta_n|^2)^{\beta-1/p},$$

where $\sum|c_n|^p+\sum|d_n|^p\leqslant C\|f\|_p^p$. Thus

$$g_1(w) = \sum_{n=1}^{\infty} c_n\beta^{-1}\zeta_n^{-1}(1-\zeta_n w)^{-\beta}(1-|\zeta_n|^2)^{\beta-1/p},$$

$$g_2(w) = \sum_{n=1}^{\infty} d_n\beta^{-1}\bar{\zeta}_n^{-1}(1-\bar{\zeta}_n w)^{-\beta}(1-|\zeta_n|^2)^{\beta-1/p}.$$

Now let $a_n = c_n\zeta_n^{-1}\beta^{-1}$, $b_n = d_n\bar{\zeta}_n^{-1}\beta^{-1}$. Then

$$\sum|a_n|^p+\sum|b_n|^p\leqslant(\min|\zeta_n|)^{-p}\beta^{-p}\sum(|c_n|^p+|d_n|^p)\leqslant C\|f\|^p.$$

The result now follows easily.

THEOREM 8.3. *Suppose* $f\colon \Delta\to X$ *is analytic and*

$$\lim_{r\to 1}(1-r^2)^{1-1/p}\Big\{\int_0^{2\pi}\|f(re^{i\theta})\|^p\,d\theta\Big\}^{1/p} = 0$$

Then $f=0$.

Proof. Define $g\colon \Delta\to L_p(T, X)$ by

$$g(z)(w) = f(wz).$$

Then g is analytic and $g\in V_\sigma(L_p(T,X))$ where $\sigma = 1/p-1$. In fact, there is a monotone decreasing function $\varrho\colon [0,1]\to \boldsymbol{R}$ so that $\lim_{r\to 1}\varrho(r)=0$ and

$$\|g(z)\|\leqslant \varrho(|z|)(1-|z|^2)^{1/p-1}.$$

By Theorem 6.1 we conclude that if $|z|\geqslant\frac{1}{2}$ then

$$\|g'(z)\|\leqslant C\varrho(2|z|-1)(1-|z|^2)^{1/p-2},$$

and differentiating $\nu+1$ times where $\nu=[\sigma]$,

$$\|g^{(\nu+1)}(z)\|\leqslant C\varrho(2^{\nu+1}|z|-2^{\nu+1}+1)(1-|z|^2)^{1/p-\nu-2}$$

for $|z|\geqslant 1-2^{-\nu+1}$.

Now by Theorem 8.2 pick a suitable η-lattice (ζ_n) and define a bounded linear operator $W\colon l_p\to L_p$ so that

$$We_{2n-1} = (1-\zeta_n w)^{-(\nu+2)}(1-|\zeta_n|^2)^{\nu+2-1/p},$$
$$We_{2n} = (1-\bar{\zeta}_n\bar{w})^{-(\nu+2)}(1-|\zeta_n|^2)^{\nu+2-1/p},$$

where (e_n) are the basic vectors of l_p. Then W maps l_p onto L_p. Define $W'\colon l_p\to L_p$ by $W'(a) = (\nu+1)!\,w^{\nu+1}\,W(a)$; then W' is also onto.

By Theorem 7.1 there is an operator $T\colon L_p\to L_p(T,X)$ so that

$$T(u(z)) = g(z),\quad |z|<1,$$
$$T(u(z)) = 0,\quad |z|\geqslant 1.$$

Now

$$TW'(e_{2n-1}) = g^{(\nu+1)}(\zeta_n)(1-|\zeta_n|^2)^{\nu+2-1/p}$$

and so $\lim_{n\to\infty}\|TW'(e_{2n-1})\| = 0$. Also

$$TW'(e_{2n}) = (\nu+1)!\,T\big(w^{\nu+1}(1-\bar{\zeta}_n\bar{w})^{-(\nu+2)}\big)(1-|\zeta_n|^2)^{\nu+2-1/p}.$$

However,

$$(1-\bar{\zeta}_n\bar{w})^{-(\nu+2)} = \sum_{k=0}^{\infty}\frac{(\nu+k+1)!}{k!(\nu+1)!}\bar{\zeta}_n^k w^{-k}$$

in L_p and hence

$$TW'(e_{2n}) = (1-|\zeta_n|^2)^{\nu+2-1/p}\sum_{k=0}^{\nu+1}\frac{(\nu+k+1)!}{k!}\bar{\zeta}_n^k\,T(w^{\nu+1-k}).$$

Thus $\lim_{n\to\infty}\|TW'(e_{2n})\| = 0$ since $\nu+2>1/p$ and $|\zeta_n|\to 1$.

It follows that TW' is compact on l_p and hence that T is compact on L_p. Now this means $T=0$ (cf. [8]). Thus $g=f=0$.

9. Applications to tensor products. Turpin [17] has shown that if X is p-normable and Y is q-normable then there is an r-convex quasi-norm on $X\otimes Y$ so that

$$\|x\otimes y\| = \|x\|\cdot\|y\|, \quad x\in X,\ y\in Y,$$

where $1/r = 1/p+1/q-1$. We now show this is best possible.

THEOREM 9.1. *Suppose $0<p,q\leqslant 1$ and suppose $1/r=1/p+1/q-1$. Let Z be an r-Banach space and suppose $B\colon L_p/H_p\times L_q/H_q\to Z$ is a nonzero bounded bilinear form. Then there is a nonzero linear operator $T\colon L_r/H_r\to Z$.*

Proof. We identify L_p/H_p with $L_p/\bar{H}_{p,0}$. Let $v_p\colon \Delta\to L_p/\bar{H}_{p,0}$ be defined by

$$v_p(z) = Q\big(u(z)\big)$$

where Q is the quotient map. Then

$$\|v_p(z)\| \leqslant C(1-|z|^2)^{1/p-1}$$

in $L_p/\bar{H}_{p,0}$ and similarly

$$\|v_q(z)\| \leqslant C(1-|z|^2)^{1/q-1}.$$

Thus if $|\zeta|=1$ then

$$\|B(v_p(z),\ v_q(\zeta z))\| \leqslant C(1-|z|^2)^{1/r-1}.$$

Now by Theorem 7.3 if there is *no* nontrivial operator in $\mathscr{L}(L_r/H_r, Z)$ then

$$B(v_p(z),\ v_q(\zeta z)) = 0$$

for $z\in\Delta$ and $|\zeta|=1$. If $|z|<1$ the function $\zeta\to B(v_p(z), v_q(\zeta z))$ is analytic for $|\zeta|<|z|^{-1}$ and is zero for $|\zeta|=1$. Since zeros of nontrivial analytic functions are isolated we have $B(v_p(z), v_q(\zeta z))=0$ for $|\zeta|<|z|^{-1}$ and hence $B(v_p(z_1), v_q(z_2))=0$ for $z_1, z_2\in\Delta$. This implies $B=0$, contrary to assumption.

COROLLARY 9.2. *If Z is s-normable where $1/s<1/p+1/q-1$ then there is no nonzero bilinear form $B\colon L_p/H_q\times L_q/H_q\to Z$.*

COROLLARY 9.3. *Let B be a nontrivial n-linear form on $\prod_{j=1}^{n} L_{p_j}/H_{p_j}$, where $0<p_j\leqslant 1$, whose range is contained in an r-normable space Z. Then*

$$\frac{1}{r} \geqslant \sum_{j=1}^{n}\frac{1}{p_j} - n.$$

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