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Good λ inequalities for the area integral and the nontangential maximal function

by

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Abstract. We refine the constants of the good λ inequalities for the area integral $A(x)$ and the nontangential maximal function $N(x)$. As an application we refine the inequalities concerning $A(x)/N(x)$ and $N(x)/A(x)$ which were obtained by R. Fefferman, Gundy, Silverstein and Stein.

1. Introduction. Throughout the paper, functions considered are real-valued. Let $d \geqslant 1$ be an integer. Let $u(y, t)$ be a harmonic function in the $(d+1)$-dimensional Euclidean half-space

$$\boldsymbol{R}_+^{d+1} = \{(y, t):\ y \in \boldsymbol{R}^d,\ t > 0\}.$$

For $\alpha > 0$ and $x \in \boldsymbol{R}^d$, let

$$N(x, \alpha) = \sup\{|u(y, t)|:\ (y, t) \in \Gamma(x, \alpha)\},$$

$$A(x, \alpha) = \Big\{ \iint_{\Gamma(x,\alpha)} |\nabla u(y, t)|^2 t^{1-d}\, dy\, dt \Big\}^{1/2},$$

where

$$\Gamma(x, \alpha) = \{(y, t) \in \boldsymbol{R}_+^{d+1}:\ |x-y| < \alpha t\}.$$

These functions N and A are usually called the *nontangential maximal function* and the *area integral*, respectively.

In [4], R. Fefferman, Gundy, Silverstein and Stein showed that if $\lambda > 0$, $\gamma > 2$, $k > 1$ and if β is sufficiently large, then

(1.1) $$|\{x \in \boldsymbol{R}^d:\ A(x, 1) > \gamma\lambda,\ N(x, \beta) \leqslant \lambda\}| \leqslant C_1 \gamma^{-k} |\{x \in \boldsymbol{R}^d:\ A(x, 1) > \lambda\}|,$$

(1.2) $$|\{x \in \boldsymbol{R}^d:\ N(x, 1) > \gamma\lambda,\ A(x, \beta) \leqslant \lambda\}| \leqslant C_1 \gamma^{-k} |\{x \in \boldsymbol{R}^d:\ N(x, 1) > \lambda\}|,$$

where C_1 is a positive constant depending only on β, k and d and where $|\{\cdot\}|$ denotes the Lebesgue measure of the set $\{\cdot\}$. Their argument is a refinement of Burkholder and Gundy [1]. Distribution function inequalities of this kind are called *good λ inequalities*.

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From (1.1) and (1.2), it was shown in [4] that if $k > 1$, β is sufficiently large and if $0 < p < \infty$, then

$$\int_{\mathbf{R}^d} \{(A(x,1)/N(x,\beta))^k A(x,1)\}^p dx \leqslant C_2 \int_{\mathbf{R}^d} A(x,1)^p dx, \tag{1.3}$$

$$\int_{\mathbf{R}^d} \{(N(x,1)/A(x,\beta))^k N(x,1)\}^p dx \leqslant C_2 \int_{\mathbf{R}^d} N(x,1)^p dx, \tag{1.4}$$

where C_2 is a positive constant depending only on β, k, p and d. These inequalities seem to give certain estimates on the growth of the ratios A/N and N/A.

In this paper we refine γ^{-k} in (1.1) and (1.2) into the forms of $\exp(-c\gamma^2)$ and $\exp(-c\gamma)$ respectively, where $\exp(\lambda) = e^\lambda$.

THEOREM 1. *Let $u(y, t)$ be a harmonic function defined on $\mathbf{R}_+^{d+1}$. Let*

$$0 < \alpha < \beta, \tag{1.5}$$

$\lambda > 0$, and $\gamma > 1$. Then

$$|\{x \in \mathbf{R}^d\colon A(x,\alpha) > \gamma\lambda,\ N(x,\beta) \leqslant \lambda\}| \leqslant C_3 \exp(-c_4\gamma^2)|\{x \in \mathbf{R}^d\colon A(x,\alpha) > \lambda\}|, \tag{1.6}$$

$$|\{x \in \mathbf{R}^d\colon N(x,\alpha) > \gamma\lambda,\ A(x,\beta) \leqslant \lambda\}| \leqslant C_3 \exp(-c_4\gamma)|\{x \in \mathbf{R}^d\colon N(x,\alpha) > \lambda\}|, \tag{1.7}$$

where C_3 and c_4 are positive constants depending only on α, β and d.

As a consequence of the above theorem we can refine (1.3) and (1.4) as follows.

COROLLARY 1. *Assume all the conditions in Theorem 1. Let $p \in (0, +\infty)$. Then*

$$\int_{\mathbf{R}^d} \exp\{c_5 A(x,\alpha)^2/N(x,\beta)^2\} A(x,\alpha)^p dx \leqslant C_6 \int_{\mathbf{R}^d} A(x,\alpha)^p dx, \tag{1.8}$$

$$\int_{\mathbf{R}^d} \exp\{c_5 N(x,\alpha)/A(x,\beta)\} N(x,\alpha)^p dx \leqslant C_6 \int_{\mathbf{R}^d} N(x,\alpha)^p dx, \tag{1.9}$$

where $c_5 = c_4/32$ and C_6 is a positive constant depending only on α, β, p and d.

Remark. As for the maximal singular integral operator and the Hardy–Littlewood maximal function, a good λ inequality with the constant of exponential type was obtained by R. Coifman [2]. See also R. Hunt [7].

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2. Preliminaries.

Notation. For a measurable set E, χ_E denotes its characteristic function. For $x \in \mathbf{R}^d$, $r > 0$ and $\alpha > 0$, let

$$B(x,r) = \{y \in \mathbf{R}^d\colon |x-y| < r\},$$
$$Q(B(x,r)) = \{(y,t) \in \mathbf{R}_+^{d+1}\colon y \in B(x,r),\ t \in (0,r)\},$$
$$\Gamma(x,\alpha,r) = \{(y,t) \in \Gamma(x,\alpha)\colon t \in (0,r)\},$$
$$\Gamma(x,\alpha,r)' = \{(y,t) \in \Gamma(x,\alpha)\colon t \geqslant r\}.$$

For the sake of convenience we define the supremum of the empty set to be zero (not $-\infty$). Let $\mathbf{N} = \{1, 2, 3, \ldots\}$ and $\mathbf{Z} = \{0, \pm 1, \pm 2, \ldots\}$. The letter C denotes various positive constants that depend only on α, β and d unless otherwise explicitly stated.

DEFINITION 2.1. For $f \in L^1_{\mathrm{loc}}(\mathbf{R}^d)$ and for a positive measure ν defined on $\mathbf{R}_+^{d+1}$ let

$$\|f\|_{\mathrm{BMO}} = \sup_B \inf_{a \in \mathbf{R}} \int_B |f(x) - a|\, dx/|B|, \qquad \|\nu\|_c = \sup_B \nu(Q(B))/|B|,$$

where the supremum is taken over all balls B in $\mathbf{R}^d$.

LEMMA 2.1. *Let $f \in L^1_{\mathrm{loc}}(\mathbf{R}^d)$,*

$$\|f\|_{\mathrm{BMO}} \leqslant 1 \tag{2.1}$$

and $\gamma > 1$. Then

$$|\{x \in \mathbf{R}^d\colon f(x) > \gamma\}| \leqslant Ce^{-c\gamma}|\{x \in \mathbf{R}^d\colon f(x) > 1\}|, \tag{2.2}$$

where C and c are positive constants depending only on d.

Proof. We may assume

$$|\{x \in \mathbf{R}^d\colon f(x) > 1\}| < +\infty. \tag{2.3}$$

Let $\{I_i\}_{i \in \mathbf{N}}$ be the maximal dyadic cubes in $\mathbf{R}^d$ such that

$$|\{x \in I_i\colon f(x) > 1\}|/|I_i| > 1/2. \tag{2.4}$$

Condition (2.3) implies

$$\bigcup_{i \in \mathbf{N}} I_i \supset \{x \in \mathbf{R}^d\colon f(x) > 1\} \quad \text{a.e.} \tag{2.5}$$

Let I_i^* be the dyadic double of I_i. Then the maximality of I_i implies

$$|\{x \in I_i^*\colon f(x) \leqslant 1\}|/|I_i^*| \geqslant 1/2,$$

which combined with (2.1) implies

$$\int_{I_i^*} f(x)\, dx/|I_i^*| \leqslant C,$$

which again combined with (2.1) implies

$$\int_{I_i} f(x)\,dx/|I_i| \leqslant C, \tag{2.6}$$

where the two C's are not the same. Applying the result of John–Nirenberg [8] to each I_i, we get

$$\begin{aligned} |\{x\in \mathbf{R}^d:\ f(x) > \gamma\}| &= \sum_{i\in N} |\{x\in I_i:\ f(x) > \gamma\}| \quad \text{by (2.5)} \\ &\leqslant \sum_{i\in N} Ce^{-c\gamma}|I_i| \quad \text{by (2.6) and [8]} \\ &\leqslant Ce^{-c\gamma}2|\{x\in \mathbf{R}^d:\ f(x) > 1\}| \quad \text{by (2.4).}\ \blacksquare \end{aligned}$$

3. Proof of (1.6). Let $r > 0$,

$$\Omega = \{x\in \mathbf{R}^d:\ N(x, \beta) > 1\},$$

$$W = \bigcup_{x\in\Omega^c} \Gamma(x, \alpha),$$

$$\mathscr{A}(x)^2 = \iint_{W\cap\Gamma(x,\alpha)} |\nabla u(y, t)|^2 t^{1-d}\,dy\,dt,$$

$$\mathscr{A}_r(x)^2 = \iint_{W\cap\Gamma(x,\alpha,r)} |\nabla u(y, t)|^2 t^{1-d}\,dy\,dt,$$

$$\mathscr{A}'_r(x)^2 = \iint_{W\cap\Gamma(x,\alpha,r)'} |\nabla u(y, t)|^2 t^{1-d}\,dy\,dt.$$

In all the lemmas in this section the above notation is used.

LEMMA 3.A. *Assume all the conditions in Theorem* 1. *Let* $(y, t)\in W$. *Then*

$$|u(y, t)| \leqslant 1, \tag{3.1}$$

$$|\nabla u(y, t)\,t| \leqslant C, \tag{3.2}$$

where C is a constant depending only on α, β and d.

Inequality (3.1) is clear. Inequality (3.2) follows from [9, p. 207, Lemma].

LEMMA 3.B. *Assume all the conditions in Theorem* 1. *Then*

$$\big\||\nabla u(y, t)|^2 \chi_W(y, t)\,t\,dy\,dt\big\|_c \leqslant C, \tag{3.3}$$

where C is a constant depending only on α, β and d.

This is essentially proved in [1].

Proof. Take any ball $B = B(x_0, r_0)$. Let $\varepsilon\in(0, r_0)$. Applying Green's theorem to each connected component of the open set

$$\mathscr{R} = \mathscr{R}_\varepsilon = W\cap\{(y, t):\ |x_0-y| < r_0+\alpha t,\ \varepsilon < t < r_0\},$$

we have

$$\begin{aligned} 2\iint_{\mathscr{R}} |\nabla u(y, t)|^2 t\,dy\,dt &\leqslant \int_{\partial\mathscr{R}} \{|u|^2+|u|\,|\nabla u|\,t\}\,d\sigma \\ &\leqslant \int_{\partial\mathscr{R}} C\,d\sigma \quad \text{by (3.1) and (3.2)} \\ &\leqslant C|B| \end{aligned} \tag{3.4}$$

by the Lipschitz continuity of ∂W, where $d\sigma$ denotes the surface element in $\mathbf{R}^{d+1}$ and $\partial\mathscr{R}$ the boundary of $\mathscr{R}$. (If $\partial\mathscr{R}$ is not smooth enough to apply Green's theorem, then we approximate each connected component of $\mathscr{R}$ by subregions with very smooth boundaries. See [9, p. 206, Lemma]. Then a limiting argument gives $\iint_{\mathscr{R}} |\nabla u|^2 t\,dy\,dt \leqslant C|B|$.) Letting $\varepsilon\to+0$ in (3.4), we have

$$\iint_{W\cap Q(B)} |\nabla u|^2 t\,dy\,dt \leqslant \lim_{\varepsilon\to+0} \iint_{\mathscr{R}_\varepsilon} |\nabla u|^2 t\,dy\,dt \leqslant C|B|,$$

which implies the desired result. ∎

LEMMA 3.1. *Assume all the conditions in Theorem* 1. *Assume that $\mathscr{A}(x) \not\equiv +\infty$. Let $r > 0$. Then $\mathscr{A}'_r(x) < +\infty$ for any $x\in\mathbf{R}^d$ and*

$$|\mathscr{A}'_r(x)^2-\mathscr{A}'_r(z)^2| \leqslant C|x-z|/r \tag{3.5}$$

for any $x, z\in\mathbf{R}^d$, where C is a constant depending only on α, β and d.

Proof. Note that

$$\begin{aligned} &\iint_{W\cap(\Gamma(x,\alpha,r)'\sim\Gamma(z,\alpha,r)')} |\nabla u(y, t)|^2 t^{1-d}\,dy\,dt \\ &\leqslant \int_r^{+\infty} Ct^{-1-d}|\{y\in\mathbf{R}^d:\ (y, t)\in\Gamma(x, \alpha, r)'\sim\Gamma(z, \alpha, r)'\}|\,dt \quad \text{by (3.2)} \\ &\leqslant \int_r^{+\infty} Ct^{-1-d}|x-z|\,t^{d-1}\,dt \leqslant C|x-z|/r \end{aligned}$$

for any $x, z\in\mathbf{R}^d$, where $\sim$ denotes the symmetric difference. The desired conclusion follows easily from the above inequality. ∎

LEMMA 3.2. *Assume all the conditions in Theorem* 1. *If $\mathscr{A}(x) \not\equiv +\infty$, then*

$$\|\mathscr{A}^2\|_{\mathrm{BMO}} \leqslant C,$$

where C is a constant depending only on α, β and d.

Proof. Take any ball $B = B(x_0, r_0)$. Since $\mathscr{A}^2 = \mathscr{A}^2_{r_0}+\mathscr{A}'^2_{r_0}$ and since

$$|\mathscr{A}'_{r_0}(x_0)^2-\mathscr{A}'_{r_0}(x)^2| \leqslant C$$

for any $x \in B$ by (3.5), we get

$$\inf_{a \in \mathbf{R}} \int_B |\mathscr{A}(x)^2 - a|\, dx \leqslant \int_B \{\mathscr{A}_{r_0}(x)^2 + C\}\, dx$$
$$\leqslant C \iint_{W \cap Q(B(x_0,(1+\alpha)r_0))} |\nabla u|^2\, t\, dy\, dt + C|B|$$
$$\leqslant C|B| \quad \text{by (3.3).} \ \blacksquare$$

Proof of (1.6). We give the proof in the case $\lambda = 1$; consider $u(y, t)/\lambda$ if necessary. We have

$$(3.6) \quad |\{x \in \mathbf{R}^d\colon A(x, \alpha) > \gamma,\ N(x, \beta) \leqslant 1\}|$$
$$= |\{x \in \Omega^c\colon A(x, \alpha) > \gamma\}|$$
$$\leqslant |\{x \in \mathbf{R}^d\colon \mathscr{A}(x)^2 > \gamma^2\}| \quad \text{since } A(x, \alpha) = \mathscr{A}(x) \text{ for } x \in \Omega^c$$
$$\leqslant C \exp(-c\gamma^2)|\{x \in \mathbf{R}^d\colon \mathscr{A}(x)^2 > 1\}| \quad \text{by Lemmas 2.1 and 3.2}$$
$$\leqslant C \exp(-c\gamma^2)|\{x \in \mathbf{R}^d\colon A(x, \alpha) > 1\}| \quad \text{by } \mathscr{A}(x) \leqslant A(x, \alpha). \ \blacksquare$$

4. Proof of (1.7). Let $r > 0$, $B = B(x_0, r_0)$,

$$\omega = \{x \in \mathbf{R}^d\colon A(x, \beta) > 1\},$$
$$w = \bigcup_{x \in \omega^c} \Gamma(x, \alpha),$$
$$n(x) = \sup\{|u(y, t)|\colon (y, t) \in w \cap \Gamma(x, \alpha)\},$$
$$n_r(x) = \sup\{|u(y, t)|\colon (y, t) \in w \cap \Gamma(x, \alpha, r)\},$$
$$n'_r(x) = \sup\{|u(y, t)|\colon (y, t) \in w \cap \Gamma(x, \alpha, r)'\},$$
$$u_B(y, t) = u(y, t) - u(x_0, (1+2\alpha) r_0/\alpha),$$
$$n_B(x) = \sup\{|u_B(y, t)|\colon (y, t) \in w \cap \Gamma(x, \alpha, r_0)\}.$$

In all the lemmas in this section the above notation is used.

LEMMA 4.1. *If $w \neq \emptyset$, then there exists a nonnegative function $\theta_0(x)$ defined on $\mathbf{R}^d$ such that*

$$(4.1) \qquad w = \{(y, t) \in \mathbf{R}_+^{d+1}\colon \theta_0(y) < t\},$$

and that

$$(4.2) \qquad |\theta_0(x) - \theta_0(z)| \leqslant |x-z|/\alpha$$

for any $x, z \in \mathbf{R}^d$.

This is an easy geometrical property of the region w.

LEMMA 4.2. *Let $x_0 \in \mathbf{R}^d$ and $r_0 > 0$. Let $w \cap \Gamma(x, \alpha, r_0) \neq \emptyset$ for some $x \in B(x_0, r_0)$. Then*

$$(4.3) \qquad (x_0, (1+2\alpha) r_0/\alpha) \in w.$$

Proof. Let $(y_0, t_0) \in w \cap \Gamma(x, \alpha, r_0)$. Then

$$|x_0 - y_0| \leqslant |x_0 - x| + |x - y_0| < r_0 + \alpha t_0 \leqslant \alpha\{(1+2\alpha) r_0/\alpha - t_0\}.$$

Thus the point $(x_0, (1+2\alpha) r_0/\alpha)$ is contained in the cone

$$\{(y, t) \in \mathbf{R}_+^{d+1}\colon t > t_0,\ |y - y_0| \leqslant \alpha(t - t_0)\},$$

which is contained in w by the geometrical property of w. $\blacksquare$

LEMMA 4.A. *Assume all the conditions in Theorem 1. Let $(y, t) \in w$. Then*

$$(4.4) \qquad |\nabla u(y, t)\, t| \leqslant C,$$

where C is a constant depending only on α, β and d.

This is an easy consequence of the harmonicity of ∇u (cf. [9, p. 207]).

LEMMA 4.B. *Assume all the conditions in Theorem 1. Then*

$$(4.5) \qquad \big\||\nabla u(y, t)|^2 \chi_w(y, t)\, t\, dy\, dt\big\|_c \leqslant C,$$

where C is a constant depending only on α, β and d.

This is essentially shown in [4].

Proof. We put

$$a(x)^2 = \iint_{w \cap \Gamma(x, (\beta-\alpha)/2)} |\nabla u(y, t)|^2\, t^{1-d}\, dy\, dt.$$

Let $x \in \mathbf{R}^d$ and let z be the point of ω^c closest to x. Since $\Gamma(x, (\beta - \alpha)/2) \cap w \subset \Gamma(z, \beta)$, we get

$$(4.6) \qquad a(x)^2 \leqslant 1.$$

(This geometrical observation is pointed out in [4, p. 7959].) Let B be any ball. Then, by (4.6), we get

$$\iint_{w \cap Q(B)} |\nabla u(y, t)|^2 t\, dy\, dt \leqslant C \int_B a(x)^2\, dx \leqslant C|B|$$

which implies (4.5). $\blacksquare$

LEMMA 4.C. *Assume all the conditions in Theorem 1. Let $B = B(x_0, r_0)$ and $\gamma > 0$. Then*

$$(4.7) \qquad |\{x \in B\colon n_B(x) > \gamma\}| \leqslant C\gamma^{-2}|B|,$$

where C is a constant depending only on α, β and d.

This is essentially proved in [1] and [4].

Proof. We may assume that $\gamma > 0$ is large enough. We may also assume that $w \cap \Gamma(x, \alpha, r_0) \neq \emptyset$ for some $x \in B$. Then (4.3) holds by Lemma 4.2.

In the following, $\theta_0(x)$ denotes the function obtained by Lemma 4.1. Let $x \in B$ and let $n_B(x) > \gamma$. Then there exists

(4.8) $$(z, s) \in w \cap \Gamma(x, \alpha, r_0)$$

such that

(4.9) $$|u_B(z, s)| > \gamma.$$

Since the line segment joining (z, s) and $(x_0, (1+2\alpha)r_0/\alpha)$ is contained in w, (4.4) implies

$$C \log\{((1+2\alpha)r_0/\alpha)/s\} > \gamma.$$

Since γ is large enough, this implies

(4.10) $$3s < r_0.$$

The geometrical property of w and (4.8) imply $B(x, \alpha s) \times \{3s\} \subset w$, i.e.,

(4.11) $$\theta_0(y) < 3s \quad \text{for any } y \in B(x, \alpha s).$$

Since the line segment joining (z, s) and any point of $B(x, \alpha s) \times \{3s\}$ is contained in w, (4.4) and (4.9) imply that

(4.12) $$|u_B(y, 3s)| > \gamma - C \quad \text{for any } y \in B(x, \alpha s).$$

Namely, if $x \in B$ and if $n_B(x) > \gamma$, then we can find a ball $B(x, \alpha s)$ that satisfies (4.10)–(4.12). Hence, by Stein [9, p. 9, Lemma], there exists a finite sequence of balls $\{B(x_i, \alpha s_i)\}_{i=1}^m$ such that $x_i \in B$,

(4.10)′ $$3s_i < r_0,$$

(4.11)′ $$\theta_0(y) < 3s_i \quad \text{for any } y \in B(x_i, \alpha s_i),$$

(4.12)′ $$|u_B(y, 3s_i)| > \gamma - C \quad \text{for any } y \in B(x_i, \alpha s_i),$$

(4.13) $$B(x_i, 2\alpha s_i) \cap B(x_j, 2\alpha s_j) = \emptyset, \quad i \neq j,$$

(4.14) $$|\{x \in B: n_B(x) > \gamma\}| \leqslant C\Big|B \cap \bigcup_{i=1}^m B(x_i, \alpha s_i)\Big|.$$

For $1 \leqslant i \leqslant m$ let

$$\theta_i(y) = \begin{cases} 3s_i & \text{if } y \in B(x_i, \alpha s_i), \\ 6s_i - 3|y - x_i|/\alpha & \text{if } y \in B(x_i, 2\alpha s_i) \setminus B(x_i, \alpha s), \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\tilde\theta(y) = \begin{cases} 0 & \text{if } y \in B, \\ |y - x_0|/\alpha - r_0/\alpha & \text{otherwise,} \end{cases}$$

and let

$$\theta(y) = \max\{\varepsilon, \tilde\theta(y), \theta_0(y), \theta_1(y), \theta_2(y), \ldots, \theta_m(y)\},$$

where

$$\varepsilon = \min\{3s_i: 1 \leqslant i \leqslant m\}.$$

Note that if $1 \leqslant i \leqslant m$ and if $y \in B(x_i, \alpha s_i) \cap B$, then

(4.15) $$\theta(y) = \theta_i(y) = 3s_i$$

by (4.11)′ and by the disjointness of {the support of $\theta_i\}_{1 \leqslant i \leqslant m}$ (recall (4.13)). Note that

(4.16) $$\begin{aligned} |\{x \in B: n_B(x) > \gamma\}| &\leqslant C\Big|B \cap \bigcup_{i=1}^m B(x_i, \alpha s_i)\Big| \quad \text{by (4.14)} \\ &\leqslant C\big|\{x \in B: \theta(x) < r_0, |u_B(x, \theta(x))| > \gamma - C\}\big| \\ &\qquad \text{by (4.15), (4.10)' and (4.12)'.} \end{aligned}$$

Let $\mathscr{R} = \{(y, t) \in \mathbf{R}_+^{d+1}: \theta(y) < t < r_0\}$. Then

(4.17) $$\mathscr{R} \subset w \cap \{B(x_0, (1+\alpha)r_0) \times (0, r_0)\}$$

by $\theta(y) \geqslant \max\{\theta_0(y), \tilde\theta(y)\}$. Let

$$\partial^+ = \{(y, r_0) \in \partial\mathscr{R}: y \in \mathbf{R}^d\},$$

$$\partial^- = \{(y, t) \in \partial\mathscr{R}: y \in \mathbf{R}^d, t < r_0\} = \{(y, \theta(y)): y \in \mathbf{R}^d, \theta(y) < r_0\}.$$

Since the line segment joining $(x_0, (1+2\alpha)r_0/\alpha)$ and any point of ∂^+ is contained in w, (4.4) implies

(4.18) $$|u_B(y, r_0)| \leqslant C \quad \text{for any } (y, r_0) \in \partial^+.$$

Therefore, applying Green's theorem to each connected component of the open set $\mathscr{R}$, we have

$$\begin{aligned} \int_{\partial^-} |u_B|^2\, d\sigma &\leqslant C \iint_{\mathscr{R}} |\nabla u|^2 t\, dy\, dt \\ &\quad + C\int_{\partial^+} \{|u_B|^2 + |u_B|\,|\nabla u|\, t\}\, d\sigma + C\int_{\partial^-} |u_B|\,|\nabla u|\, t\, d\sigma \\ &\leqslant C \iint_{w \cap Q(B(x_0, (1+\alpha)r_0))} |\nabla u|^2 t\, dy\, dt \\ &\quad + C\int_{\partial^+} d\sigma + C\int_{\partial^-} |u_B|\, d\sigma \quad \text{by (4.17), (4.18) and (4.4)} \\ &\leqslant C|B| + C|B|^{1/2}\Big\{\int_{\partial^-} |u_B|^2\, d\sigma\Big\}^{1/2} \end{aligned}$$

by (4.5), the Lipschitz continuity of θ and by Hölder's inequality. Thus

$$\int_{\partial^-} |u_B|^2\, d\sigma \leqslant C|B|,$$

which, combined with (4.16), implies (4.7). (If $\partial\mathscr{R}$ is not smooth enough to apply Green's theorem, then we approximate each connected component of $\mathscr{R}$ by subregions with very smooth boundaries. See [9, p. 206, Lemma]. Then a limiting argument gives the desired result.) ■

LEMMA 4.3. *Assume all the conditions in Theorem* 1. *Assume that* $n(x) \not\equiv +\infty$. *Let* $r > 0$. *Then* $n'_r(x) < +\infty$ *for any* $x \in \mathbf{R}^d$ *and*

$$|n'_r(x) - n'_r(z)| \leqslant C|x-z|/r \tag{4.19}$$

for any x, $z \in \mathbf{R}^d$, *where* C *is a constant depending only on* α, β *and* d.

Proof. If $(y, t) \in w \cap \Gamma(x, \alpha, r)'$, then we get

$$(y+(z-x)/2,\ t+|z-x|/2\alpha) \in w \cap \Gamma(z, \alpha, r)'$$

by the geometrical property of w, and

$$|u(y, t) - u(y+(z-x)/2,\ t+|z-x|/2\alpha)| \leqslant C|x-z|/r$$

by (4.4) and by $t \geqslant r$. The desired conclusion follows easily from this observation. ■

LEMMA 4.4. *Assume all the conditions in Theorem* 1. *If* $n(x) \not\equiv +\infty$, *then*

$$\|n\|_{\mathrm{BMO}} \leqslant C,$$

where C *is a constant depending only on* α, β *and* d.

Proof. Take any ball $B = B(x_0, r_0)$. By Lemma 4.2 we can take $r_1 \in \{r_0, (1+2\alpha)r_0/\alpha\}$ so that

$$w \cap \Gamma(x, \alpha, r_1) = \emptyset \quad \text{for any } x \in B \tag{4.20}$$

or that

$$w \cap \Gamma(x, \alpha, r_1) \neq \emptyset \quad \text{for any } x \in B. \tag{4.21}$$

Since

$$n(x) = \max\{n_{r_1}(x), n'_{r_1}(x)\}$$

and since

$$|n'_{r_1}(x_0) - n'_{r_1}(x)| \leqslant C$$

for any $x \in B$ by (4.19), we get

$$\inf_{a \in \mathbf{R}} \int_B |n(x) - a|\, dx \leqslant \inf_{a \in \mathbf{R}} \int_B |\max\{n_{r_1}(x), n'_{r_1}(x_0)\} - a|\, dx + C|B|$$

$$\leqslant \inf_{a \in \mathbf{R}} \int_B |n_{r_1}(x) - a|\, dx + C|B| = (4.22) + C|B|.$$

In the case (4.20), we get $(4.22) = 0$. In the case (4.21), we get

$$\begin{aligned}(4.22)\quad &\leqslant \int_B \big|n_{r_1}(x) - |u(x_0, (1+2\alpha)r_1/\alpha)|\big|\, dx \\ &= \int_B \big|\sup\{|u(y, t)| - |u(x_0, (1+2\alpha)r_1/\alpha)|: (y, t) \in w \cap \Gamma(x, \alpha, r_1)\}\big|\, dx \\ &\leqslant \int_B n_{B(x_0, r_1)}(x)\, dx \\ &\leqslant C|B| \quad \text{by (4.7)},\end{aligned}$$

which implies the desired result. ■

Proof of (1.7). We give the proof only in the case $\lambda = 1$. We have

$$\begin{aligned}(4.23)\quad &|\{x \in \mathbf{R}^d:\ N(x, \alpha) > \gamma,\ A(x, \beta) \leqslant 1\}| \\ &= |\{x \in \omega^c:\ N(x, \alpha) > \gamma\}| \\ &\leqslant |\{x \in \mathbf{R}^d:\ n(x) > \gamma\}| \quad \text{since } N(x, \alpha) = n(x) \text{ for } x \in \omega^c \\ &\leqslant C\exp(-c\gamma)|\{x \in \mathbf{R}^d:\ n(x) > 1\}| \quad \text{by Lemmas 2.1 and 4.4} \\ &\leqslant C\exp(-c\gamma)|\{x \in \mathbf{R}^d:\ N(x, \alpha) > 1\}| \quad \text{by } n(x) \leqslant N(x, \alpha). \ ■\end{aligned}$$

5. Proofs of (1.8) and (1.9). Since the proofs of (1.8) and (1.9) are very similar, we prove only (1.8). We follow the argument in [4]. Then

$$\begin{aligned}(5.1)\quad &\int_{\mathbf{R}^d} \exp\{\tfrac{1}{32}c_4 A(x, \alpha)^2/N(x, \beta)^2\} A(x, \alpha)^p\, dx \\ &\leqslant \sum_{l \in \mathbf{Z}} \sum_{j \in \mathbf{N}} \int_{E_{l,j}} \exp\{\tfrac{1}{32}c_4 A(x, \alpha)^2/N(x, \beta)^2\} A(x, \alpha)^p\, dx + C\int_{\mathbf{R}^d} A(x, \alpha)^p\, dx,\end{aligned}$$

where

$$E_{l,j} = \{x \in \mathbf{R}^d:\ 2^l < A(x, \alpha) \leqslant 2^{l+1},\ 2^{l-j-1} < N(x, \beta) \leqslant 2^{l-j}\}.$$

By (1.6), we have with $\tau(\lambda) = |\{x \in \mathbf{R}^d:\ A(x, \alpha) > \lambda\}|$,

$$|E_{l,j}| \leqslant C_3 \exp(-c_4 2^{2j})\tau(2^{l-j}),$$

so that the first quantity in the right-hand side of (5.1) is dominated by:

$$\begin{aligned}&\leqslant \sum_{l \in \mathbf{Z}} \sum_{j \in \mathbf{N}} \exp(\tfrac{1}{32}c_4 2^{2(j+2)}) 2^{(l+1)p} C_3 \exp(-c_4 2^{2j})\tau(2^{l-j}) \\ &= C_3 2^p \sum_{l \in \mathbf{Z}} \sum_{j \in \mathbf{N}} 2^{lp} \exp(-\tfrac{1}{2}c_4 2^{2j})\tau(2^{l-j}) \\ &= C_3 2^p \sum_{l \in \mathbf{Z}} \sum_{j \in \mathbf{N}} 2^{(l-j)p}\tau(2^{l-j}) 2^{jp} \exp(-\tfrac{1}{2}c_4 2^{2j}) \\ &= C_3 2^p \sum_{m \in \mathbf{Z}} 2^{mp}\tau(2^m) \sum_{j \in \mathbf{N}} 2^{jp} \exp(-\tfrac{1}{2}c_4 2^{2j}) \\ &= C_p \sum_{m \in \mathbf{Z}} 2^{mp}\tau(2^m) \leqslant C_p \int_{\mathbf{R}^d} A(x, \alpha)^p\, dx,\end{aligned}$$

where C_p denotes positive constants depending only on α, β, p and d. This gives (1.8).

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A characterization of the Banach property for summability matrices

by

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Dedicated to Prof. Z. Ciesielski on his 50th birthday

Abstract. A doubly infinite matrix $A = \{a_{nk}: n, k = 1, 2, \ldots\}$ of real numbers is said to have the *Banach property* if for every orthonormal system $\{\varphi_k(x): k = 1, 2, \ldots\}$ in (0, 1) we have

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}\varphi_k(x) = 0 \quad \text{a.e.}$$

We define a norm $\|A\|$ in such a way that a matrix A has the Banach property if and only if $\|A\| < \infty$. Some consequences of this characterization are also included.

1. Introduction. Let $\varphi = \{\varphi_k(x): k = 1, 2, \ldots\}$ be an orthonormal system (in abbreviation: ONS) in the unit interval (0,1) and let $A = \{a_{nk}: n, k = 1, 2, \ldots\}$ be a doubly infinite matrix of real numbers. Following Banach (see e.g. [2]) we say that the matrix A has the *Banach property* (shortly, $A \in (\mathrm{BP})$) if for every ONS φ in (0,1) we have

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}\varphi_k(x) = 0 \quad \text{a.e.} \tag{1.1}$$

Taking φ to be the Rademacher ONS $r = \{r_k(x) = \operatorname{sign} \sin 2^k \pi x: k = 1, 2, \ldots\}$ (see e.g. [5, p. 212]), one can easily deduce that if $A \in (\mathrm{BP})$, then

$$\lim_{n\to\infty} a_{nk} = 0 \quad (k = 1, 2, \ldots). \tag{1.2}$$

In fact, since

$$r_1(x+\tfrac{1}{2}) = -r_1(x), \qquad r_k(x+\tfrac{1}{2}) = r_k(x) \quad (k = 2, 3, \ldots),$$

one can write

$$\sum_{k=1}^{\infty} a_{nk} r_k(x+\tfrac{1}{2}) = -2a_{n1} r_1(x) + \sum_{k=1}^{\infty} a_{nk} r_k(x).$$

If (1.1) with $\varphi = r$ holds for both x and $x+1/2$ (which happens for almost every x in (0,1)), then letting $n \to \infty$ in the last equality yields (1.2) for $k = 1$. The proof for $k = 2, 3, \ldots$ is quite similar.