STUDIA MATHEMATICA, T. LXXXIII. (1986)

Eberlein compacts in $L_1(X)$

by

JÜRGEN BATT and GEORG SCHLÜCHTERMANN (München)

Abstract. We prove that every compact subset of the space $L_1(X)$ of Bochner integrable functions with values in a Banach space X, endowed with the topology $\sigma' := \sigma(L_1(X), L_\infty(X'))$, is an Eberlein compact, and moreover that the space $(L_1(X), \sigma')$ is angelic. For this purpose we show that the σ'-closure L of the span of a σ'-compact subset of $L_1(X)$ can be continuously embedded into some $c_0(\Gamma)$ with weak topology (in analogy to the well-known result of D. Amir and J. Lindenstrauss for weakly compactly generated Banach spaces). The spaces L (the σ'-compactly generated subspaces) are further investigated and results concerning the norm and σ'-closures of convex subsets in $L_1(X)$ are derived.

1. Introduction. If $L_1(X)$ is the space of Bochner integrable functions on a positive finite measure space (S, Σ, μ) with values in the Banach space X, $\langle L_1(X), L_\infty(X')\rangle$ is a dual pairing and it is known that the topology $\sigma' := \sigma(L_1(X), L_\infty(X'))$ is strictly coarser than the weak topology $\sigma(L_1(X), L_1(X)')$ if and only if X' does not have the Radon–Nikodým property. The σ'-compact subsets of $L_1(X)$ were investigated in [3]. In particular, it was proved that the notions "relatively compact", "countably compact" and "sequentially compact" are equivalent in the topology σ'.

In the main theorem of the present paper we show that for every σ'-compact subset K of $L_1(X)$ there exists a linear bounded injective mapping T from $L := \overline{\operatorname{span} K}^{\sigma'}$ into some $c_0(\Gamma)$ which is σ'-weakly continuous (Section 3). This result implies that σ'-compact subsets of $L_1(X)$ are even homeomorphic to weakly compact subsets of a Banach space, that is, they are Eberlein compacts. It implies further that the space $L_1(X)$ endowed with the topology σ' has countably determined compactness [11, p. 30], i.e. (following D.H. Fremlin) is "angelic". A separated topological space is called *angelic* if every countably compact subset A is relatively compact and each point in $\bar{A}$ is the limit of a sequence in A. The rich structure of Eberlein compacts and angelic spaces was investigated in the work of D. Amir and J. Lindenstrauss [1], of Y. Benyamini, T. Starbird, M.E. Rudin, M. Wage and E. Michael [5, 4, 16] and in the book of K. Floret [11] (see also the papers of W. Govaerts [12] and of R. J. Hunter and J. Lloyd [15] for more recent developments).

In [1] D. Amir and J. Lindenstrauss have shown that the norm closed span of a weakly compact subset of a Banach space (a weakly compactly generated space) can be mapped into some $c_0(\Gamma)$ by some injective bounded linear mapping. This fact is used for the proof of our main result together with the construction of a "long" sequence of conditional expectation operators (similar to the one of a "long" sequence of projections in the proof of that result). There exists a relationship between the weakly compactly generated spaces and the subspaces L of $L_1(X)$ in our main theorem (termed σ'-compactly generated), which is reflected in comparable characterizations (Section 4).

In Section 5 we show that for a σ'-compact subset K of $L_1(X)$ the σ'- and norm closures of the (absolutely) convex hull coincide, but that these closures of the linear span are in general different. The equality of the closures of aco K makes it possible to show – in connection with a well-known result of M. Talagrand for weakly compactly generated spaces [18] – that the norm closed span of a σ'-compact set K is $\mathscr{K}$-analytic and hence a Lindelöf space in the σ'-topology.

2. Preliminaries. Throughout the paper, (S, Σ, μ) denotes a positive finite measure space and $L_1(\Sigma, X) = L_1(X)$ and $L_\infty(\Sigma, X) = L_\infty(X)$ are the corresponding spaces of Bochner integrable and essentially bounded measurable functions with values in a Banach space X. The symbol X will be omitted for the spaces of scalar functions. The topology

$$\sigma\big(L_1(\Sigma, X), L_\infty(\Sigma, X')\big) = \sigma\big(L_1(X), L_\infty(X')\big)$$

will be denoted by σ'. If for some index β, Σ_β is a sub-σ-algebra of Σ we write

$$\sigma'_\beta := \sigma'\big(L_1(\Sigma_\beta, X), L_\infty(\Sigma_\beta, X')\big) = \sigma' \cap L_1(\Sigma_\beta, X)$$

(with the measure $\mu|_{\Sigma_\beta}$) and let E_β: $L_1(\Sigma, X) \to L_1(\Sigma_\beta, X)$ be the conditional expectation operator, which is σ'–σ'_β-continuous. If L is a linear subspace of $L_1(\Sigma_\beta, X)$ we remark that $\bar{L}^{\sigma'_\beta} = \bar{L}^{\sigma'}$ (in fact, if $f \in \bar{L}^{\sigma'}$, then $x' \cdot f \in L_1(\Sigma_\beta)$ for all $x' \in X'$ because $L_1(\Sigma_\beta)$ is weakly closed in $L_1(\Sigma)$, hence $f \in L_1(\Sigma_\beta, X)$ by Pettis' measurability criterion [7, p. 42] and so $f \in \bar{L}^{\sigma'_\beta}$).

For subspaces $L \subset L_1(\Sigma, X)$ we let $L^\times := L_\infty(\Sigma, X')/L^\circ$ (the polar L° taken in $L_\infty(\Sigma, X')$). If we let $\sigma'_L := \sigma(L, L^\times)$, (L, σ'_L) is a subspace of $(L_1(\Sigma, X), \sigma')$ [14, p. 163], that is, $\sigma'_L = \sigma' \cap L$. In general, $L^\times$ is not the norm dual of L. We consider $L^\times$ endowed with the quotient norm. It follows from the fact that $L_\infty(\Sigma, X')$ is norming for $L_1(\Sigma, X)$ [9, p. 232] that L is canonically isometrically embedded in $L^{\times\prime}$ with the canonical norm. By $B(Y)$ we denote the unit ball of a Banach space Y; σ is always the weak topology of a Banach space, and co A [aco A] the convex [absolutely convex] hull of a subset A in a linear space.

3. The main result.

THEOREM. *If (S, Σ, μ) is a positive finite measure space, let $K \subset L_1(\Sigma, X)$ be a σ'-compact subset and $L := \overline{\text{span}\, K}^{\sigma'}$. Then there exists a set Γ and a linear bounded injective operator T: $L \to c_0(\Gamma)$, $\|T\| \leqslant 1$, which is σ'–σ-continuous.*

Proof. We shall first prove the theorem for countably generated σ-algebras Σ and then proceed by transfinite induction.

(1) Let there exist a countable algebra $\Sigma_0 \subset \Sigma$ which generates Σ. Then there exists a countable subsystem $\Delta \subset \Sigma$, with $\emptyset, S \in \Delta$ and closed under finite intersections, which generates Σ and which has the following property: For all $\delta > 0$ the number of sets $A \in \Delta$ for which $\mu(A) \geqslant \delta$ is at most finite. In fact, this is clear if (S, Σ, μ) is purely atomic or is the interval $[0, 1]$ with Lebesgue measure; the case of a nonatomic separable positive finite measure space is reduced to the latter case by using the isomorphism established in [13, p. 173]. For all $A \in \Delta$ the mapping

$$I_A\colon L_1(\Sigma, X) \to X, \quad f \mapsto \int_A f\,d\mu$$

is σ'–σ-continuous, hence $I_A(K)$ is a weakly compact subset of X. By the result of D. Amir and J. Lindenstrauss [1, p. 35] there exists a set Γ_A and a linear bounded injective operator

$$T_A\colon \overline{\text{span}\, I_A(K)}^{\|\cdot\|} \to c_0(\Gamma_A), \quad \|T_A\| \leqslant 1.$$

Let $\Gamma := \bigcup_{A \in \Delta} \Gamma_A$ (disjoint union). We can define

$$T\colon L \to c_0(\Gamma), \quad f \mapsto \big((T_A \circ I_A f)_\gamma\big)_{\gamma \in \Gamma} \quad (\text{for } \gamma \in \Gamma_A).$$

In fact, the range of T lies in $c_0(\Gamma)$: Since for $f \in L$ the measure m_f: $\Sigma \to X$, $A \mapsto I_A f$ is absolutely continuous with respect to μ, for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $A \in \Sigma$ with $\mu(A) < \delta$ we have $\|I_A f\| < \varepsilon$. By the property of Δ, the number of sets $A \in \Delta$ for which $\|I_A f\| \geqslant \varepsilon$ is at most finite and hence in view of $\|T_A\| \leqslant 1$ the number of $\gamma \in \Gamma$ for which $|(Tf)_\gamma| \geqslant \varepsilon$ is at most finite. T is linear with $\|T\| \leqslant 1$ and injective because all T_A are injective and Δ generates Σ. For $\xi \in l_1(\Gamma)$ and $f \in L$ we have $\xi|_A \in l_1(\Gamma_A)$ and

$$\begin{aligned}\langle \xi, Tf\rangle &= \sum_{\gamma \in \Gamma} \xi_\gamma \cdot (Tf)_\gamma = \sum_{A \in \Delta} \langle \xi|_A, T_A \circ I_A f\rangle \\ &= \sum_{A \in \Delta} \langle (T_A \circ I_A)'(\xi|_A), f\rangle \\ &= \Big\langle \sum_{A \in \Delta} (T_A \circ I_A)'(\xi|_A), f\Big\rangle,\end{aligned}$$

where the first component of the last pair is an element in $L_\infty(\Sigma, X')$

because $T_A \circ I_A$: $L_1(\Sigma, X) \to c_0(\Gamma_A)$ is σ'-σ-continuous. This shows that T is σ'–σ-continuous.

(2) Now we prove the following assertion $P(\eta)$:

For all positive finite measure spaces (S, Σ, μ) with the property that there exists a subsystem $\Delta \subset \Sigma$ generating Σ and having the cardinal number $|\Delta| = \eta$ and for all σ'-compact subsets $K \subset L_1(\Sigma, X)$ there exist a set $\Gamma = \Gamma_{\Sigma,K}$ and a linear bounded injective operator

$$T = T_{\Sigma,K}\colon\ L := \overline{\operatorname{span} K}^{\sigma'} \to c_0(\Gamma), \qquad \|T\| \leqslant 1,$$

which is σ'–σ-continuous,

by transfinite induction with respect to $\eta \geqslant \omega$.

$P(\omega)$ is true by part (1) of the proof. Now let $\lambda > \omega$ be a cardinal number and assume that

(1) $P(\eta)$ is true for all cardinal numbers η with $\omega \leqslant \eta < \lambda$.

Let (S, Σ, μ), Δ and K be as in $P(\lambda)$. Let $\Delta = \{A_\alpha, \alpha < \lambda\}$ and for all ordinals β with $\omega \leqslant \beta \leqslant \lambda$ let $\Delta_\beta := \{A_\alpha, \alpha < \beta\}$ and Σ_β be the σ-algebra generated by Δ_β. For an ordinal β with $\omega \leqslant \beta < \lambda$ the σ-algebra $\Sigma_{\beta+1}$ is generated by the system $\Delta_{\beta+1}$ which has the cardinal number $|\Delta_{\beta+1}| = |\beta+1| = |\beta| \leqslant \beta < \lambda$; furthermore, $L_1(\Sigma_\beta, X) \cap \sigma'_{\beta+1} = \sigma'_\beta$,

$$E_{\beta+1} - E_\beta\colon\ L_1(\Sigma, X) \to L_1(\Sigma_{\beta+1}, X)$$

is σ'–$\sigma'_{\beta+1}$-continuous and $K_{\beta+1} := (E_{\beta+1} - E_\beta)K$ is $\sigma'_{\beta+1}$-compact. By (1), $P(|\beta+1|)$ is true, and there exist a set $\Gamma_{\beta+1} := \Gamma_{\Sigma_{\beta+1}, K_{\beta+1}}$ and a linear bounded injective operator

$$T_{\beta+1} := T_{\Sigma_{\beta+1}, K_{\beta+1}}\colon\ \overline{\operatorname{span} K_{\beta+1}}^{\sigma'_{\beta+1}} \to c_0(\Gamma_{\beta+1}), \qquad \|T_{\beta+1}\| \leqslant 1,$$

which is $\sigma'_{\beta+1}$–σ-continuous. Similarly, there exist a set $\Gamma_\omega := \Gamma_{\Sigma_\omega, E_\omega K}$ and a linear bounded injective operator

$$T_\omega := T_{\Sigma_\omega, E_\omega K}\colon\ \overline{\operatorname{span} E_\omega K}^{\sigma'} \to c_0(\Gamma_\omega), \qquad \|T_\omega\| \leqslant 1,$$

which is σ'_ω–σ-continuous. Let us define

$$\Gamma := \Gamma_\omega \cup \bigcup_{\omega \leqslant \beta < \lambda} \Gamma_{\beta+1} \qquad \text{(disjoint union)},$$

$$T\colon\ \overline{\operatorname{span} K}^{\sigma'} \to c_0(\Gamma), \qquad f \mapsto ((Tf)_\gamma)_{\gamma\in\Gamma},$$

where

$$(Tf)_\gamma := \begin{cases} (T_\omega E_\omega f)_\gamma, & \gamma \in \Gamma_\omega, \\ \frac{1}{2}(T_{\beta+1}(E_{\beta+1} - E_\beta) f)_\gamma, & \gamma \in \Gamma_{\beta+1} \end{cases}$$

(note that $(E_{\beta+1} - E_\beta)\overline{\operatorname{span} K}^{\sigma'} \subset \overline{(E_{\beta+1} - E_\beta)\operatorname{span} K}^{\sigma'_{\beta+1}} \subset \overline{\operatorname{span} K_{\beta+1}}^{\sigma'_{\beta+1}}$).

The range of T lies in $c_0(\Gamma)$; in fact, let $f \in \overline{\operatorname{span} K}^{\sigma'}$ and $\varepsilon > 0$. If the set of ordinals

(2) $$\{\beta;\ \omega \leqslant \beta < \lambda;\ \|E_{\beta+1}f - E_\beta f\|_1 \geqslant \varepsilon\}$$

were not finite, there would exist a sequence $(\beta_i)_{i\in N}$ of ordinals such that

(3) $$\omega \leqslant \beta_1 < \beta_2 < \ldots < \lambda \quad \text{and} \quad \|E_{\beta_i+1}f - E_{\beta_i}f\|_1 \geqslant \varepsilon, \quad i \in N.$$

For the limit ordinal $\alpha := \sup_{i\in N} \beta_i$ we have

$$\Delta_\alpha = \bigcup_{i\in N} \Delta_{\beta_i}.$$

Therefore Σ_α is the smallest σ-algebra containing all Σ_{β_i}, and $E_{\beta_i}f \to E_\alpha f$ by the martingale convergence theorem. This contradicts (3) and the set (2) is at most finite. Hence the subset

$$\{\beta;\ \omega \leqslant \beta < \lambda;\ \tfrac{1}{2}\|T_{\beta+1}(E_{\beta+1} - E_\beta)f\| \geqslant \varepsilon\}$$

is at most finite and $|(Tf)_\gamma| \geqslant \varepsilon$ for at most finitely many $\gamma \in \Gamma$. T being linear with $\|T\| \leqslant 1$, we have to show that T is injective. Let $f \in \overline{\operatorname{span} K}^{\sigma'}$ and $Tf = 0$. By transfinite induction with respect to the ordinals β satisfying $\omega \leqslant \beta < \lambda$ we prove

(4) $$E_\beta f = 0.$$

$Tf = 0$ implies $T_\omega(E_\omega f) = 0$ in $c_0(\Gamma_\omega)$ and $E_\omega f = 0$ by the injectivity of T_ω. Now let α, $\omega \leqslant \alpha < \lambda$, be an ordinal and assume $E_\beta f = 0$ for $\omega \leqslant \beta < \alpha$. If α is a limit ordinal, then $\Delta_\alpha = \bigcup_{\beta<\alpha} \Delta_\beta$ and

$$E_\alpha f = \lim_{\beta<\alpha} E_\beta f = 0.$$

If $\alpha = \beta+1$ for some β with $\omega \leqslant \beta < \alpha$ then $Tf = 0$ implies

$$T_{\beta+1}(E_{\beta+1}f - E_\beta f) = 0 \quad \text{in } c_0(\Gamma_{\beta+1})$$

and hence by the injectivity of $T_{\beta+1}$

$$E_\alpha f = E_{\beta+1} f = E_\beta f = 0.$$

This shows (4). Since λ is a limit ordinal it follows that

$$f = \lim_{\beta<\lambda} E_\beta f = 0,$$

hence T is injective. By using the σ'–σ-continuity of the components of T with values in $c_0(\Gamma_\omega)$ and $c_0(\Gamma_{\beta+1})$, the σ'–σ-continuity of T follows as in part (1) of the proof. This shows that $P(\lambda)$ is true, and the proof of the theorem is complete.

4. Corollaries of the main result.

Corollary 1. *Every σ'-compact subset of $L_1(\Sigma, X)$ is an Eberlein compact.*

Corollary 2. *$(L_1(\Sigma, X), \sigma')$ is angelic.*

Let us call a linear subspace $L \subset L_1(\Sigma, X)$ *σ'-compactly generated* if there exists a σ'-compact subset $K \subset L_1(\Sigma, X)$ such that $L = \overline{\operatorname{span} K}^{\sigma'}$. The result of D. Amir and J. Lindenstrauss and our main result show that weakly compactly generated spaces and σ'-compactly generated subspaces of $L_1(\Sigma, X)$ have an important property in common. We now show that they can be similarly characterized.

Corollary 3. *For a σ'-closed linear subspace L of $L_1(\Sigma, X)$ the following conditions are equivalent:*

(i) *L is σ'-compactly generated.*

(ii) *There exist a set Γ and a linear bounded operator U: $l_1(\Gamma) \to L$ which is $\sigma(l_1(\Gamma), c_0(\Gamma))$–$\sigma'$-continuous and whose range is σ'-dense in L.*

(iii) *There exist a set Γ and a linear bounded injective operator V: $L^\times \to c_0(\Gamma)$ which is $\sigma(L^\times, L)$–σ-continuous.*

(iv) *There exists a norm bounded set $\{f_\gamma, \gamma \in \Gamma\}$ in L such that*

$$L = \overline{\operatorname{span}\{f_\gamma, \gamma \in \Gamma\}}^{\sigma'} \quad \text{and} \quad \langle F, f_\cdot \rangle \in c_0(\Gamma) \quad \text{for all } F \in L^\times.$$

Proof. (i)$\Rightarrow$(iii): Let $L = \overline{\operatorname{span} K}^{\sigma'}$ for a σ'-compact set K. Corollary 1 implies that K is an Eberlein compact. Hence [1, p. 37] $C(K, \sigma')$ is weakly compactly generated and there exists a linear bounded injective operator T: $C(K, \sigma') \to c_0(\Gamma)$ for some set Γ. V is obtained as the composition $T \circ R$, where R is the restriction operator R: $L^\times \to C(K, \sigma')$. To get the $\sigma(L^\times, L)$–σ'-continuity of R from Choquet's theorem one has to know that one can assume K to be convex. But this follows from the fact proved in [3, p. 416] that $\overline{\operatorname{aco} K}^{\sigma'}$ is σ'-compact if K is.

(iii)$\Rightarrow$(ii): U is obtained as the adjoint of V with respect to the dual systems $\langle L^\times, L\rangle$ and $\langle c_0(\Gamma), l_1(\Gamma)\rangle$; the boundedness of U follows from the fact that L is isometrically embedded in $L^{\times\prime}$.

(ii)$\Rightarrow$(iv): If $\{e_\gamma, \gamma \in \Gamma\}$ are the unit vectors in $l_1(\Gamma)$, the set $\{Ue_\gamma, \gamma \in \Gamma\}$ has the desired properties.

(iv)$\Rightarrow$(iii): V is obtained as the operator $F \mapsto \langle F, f_\cdot\rangle$, $F \in L^\times$. If $\xi \in l_1(\Gamma)$ and $F \in L^\times$ then by Lebesgue's theorem

$$\langle \xi, VF\rangle = \sum_{\gamma \in \Gamma} \xi_\gamma \cdot \langle F, f_\gamma\rangle = \langle F, \sum_{\gamma \in \Gamma} \xi_\gamma f_\gamma\rangle,$$

where $\sum_{\gamma \in \Gamma} \xi_\gamma f_\gamma \in L$ (because L is σ'-closed and hence norm closed). Hence V is $\sigma(L^\times, L)$–σ-continuous. For further details, we refer the reader to the theory of weakly compactly generated Banach spaces (e.g., [8, p. 147–153]).

5. Norm closure and σ'-closure in $L_1(X)$.

Proposition 1. *If K is a σ'-compact subset of $L_1(\Sigma, X)$, then*

$$\overline{\operatorname{co} K}^{\|\cdot\|} = \overline{\operatorname{co} K}^{\sigma'} \quad \text{and} \quad \overline{\operatorname{aco} K}^{\|\cdot\|} = \overline{\operatorname{aco} K}^{\sigma'}.$$

Proof. For the first equality, we only need to show

$$\overline{\operatorname{co} K}^{\sigma'} \subset \overline{\operatorname{co} K}^{\|\cdot\|}.$$

(1) We assume there exists a norm separable subspace $L \subset L_1(\Sigma, X)$ such that $C := \overline{\operatorname{co} K}^{\sigma'} \subset L$. In this case we show that the σ'-compact set (C, σ') [3, p. 416] is metrizable. There exists a countable subalgebra $\mathscr{A} \subset \Sigma$ and a closed separable subspace $X_0 \subset X$ such that $L \subset L_1(\Sigma_0, X_0)$, where Σ_0 is the σ-algebra generated by $\mathscr{A}$ [10, p. 168]. We construct a countable set $F \subset L_\infty(\Sigma_0, X_0')$, which is $\sigma(L_\infty(\Sigma_0, X_0'), L_1(\Sigma_0, X_0))$-dense in $L_\infty(\Sigma_0, X_0')$, as follows. The unit ball $B(X_0')$ is compact and metrizable [10, p. 426] and hence separable in $\sigma(X_0', X_0)$, and the same holds for all scalar multiples of $B(X_0')$. Hence there exists a countable $\sigma(X_0', X_0)$-dense subset H_0 of X_0'. Furthermore, let $\mathscr{A} = \{A_1, A_2, A_3, \ldots\}$ be an enumeration of $\mathscr{A}$, $\pi_n := \{E_1^{(n)}, \ldots, E_{k_n}^{(n)}\}$ a finite partition of S which contains $A_1, \ldots, A_n$, and $\mathscr{A}_n$ the algebra generated by π_n. We let

$$F := \bigcup_{n \in N} \Big\{\sum_{i=1}^{k_n} z_i' \cdot \chi_{E_i^{(n)}},\ z_1', \ldots, z_{k_n}' \in H_0,\ E_1^{(n)}, \ldots, E_{k_n}^{(n)} \in \pi_n\Big\}.$$

To show that this countable set is dense, let $g \in L_\infty(\Sigma_0, X_0')$, $f_1, \ldots, f_m \in L_1(\Sigma_0, X_0)$ and $\varepsilon > 0$ be given. There exists $n \in N$ such that for the conditional expectation operator $E_{\pi_n}^{X_0}$: $L_1(\Sigma_0, X_0) \to L_1(\mathscr{A}_n, X_0)$ we have

$$\|E_{\pi_n}^{X_0} f_j - f_j\|_1 \leqslant \varepsilon/[4(\|g\|_\infty + 1)], \quad j = 1, \ldots, m.$$

We note that

$$\langle g, E_{\pi_n}^{X_0} f_j\rangle = \langle g, E_{\pi_n}^{X_0}(E_{\pi_n}^{X_0} f_j)\rangle = \langle E_{\pi_n}^{X_0'} g, E_{\pi_n}^{X_0} f_j\rangle.$$

There exist elements $x_1', \ldots, x_{k_n}' \in X_0'$ and $x_{1,j}, \ldots, x_{k_n,j} \in X_0$ such that

$$E_{\pi_n}^{X_0'} g = \sum_{i=1}^{k_n} x_i' \cdot \chi_{E_i^{(n)}}, \qquad E_{\pi_n}^{X_0} f_j = \sum_{i=1}^{k_n} x_{i,j} \cdot \chi_{E_i^{(n)}}, \quad j = 1, \ldots, m.$$

For $i = 1, \ldots, k_n$ there exist elements $y_i' \in H_0$, $\|y_i'\| \leqslant \|g\|_\infty$, such that

$$|\langle x_i' - y_i', x_{i,j}\rangle| \leqslant \varepsilon/[2(\mu(E_i^{(n)}) + 1)], \quad j = 1, \ldots, m.$$

For $g_0 := \sum_{i=1}^{k_n} y_i' \cdot \chi_{E_i^{(n)}} \in F$ it follows that

$$|\langle g - g_0, f_j\rangle| \leqslant |\langle g - g_0, E_{\pi_n}^{X_0} f_j\rangle| + \tfrac{1}{2}\varepsilon$$
$$\leqslant |\langle E_{\pi_n}^{X_0'} g - g_0, E_{\pi_n}^{X_0} f_j\rangle| + \tfrac{1}{2}\varepsilon \leqslant \varepsilon.$$

Hence F is dense. If $F = \{g_1, g_2, \ldots\}$ is an enumeration of F, then

$$d(f, h) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|\langle g_n, f-h\rangle|}{1+|\langle g_n, f-h\rangle|}$$

is a metric on C which defines a topology equivalent to σ'. Now let $f_0 \in \overline{\text{co } K}^{\sigma'}$. We want to show $f_0 \in \overline{\text{co } K}^{\|\cdot\|}$. According to Choquet's theorem [6 II, p. 140] there exists a nonnegative Borel measure ν on the set ex C of extreme points of C with $\nu(\text{ex } C) = 1$ such that for all

$$g \in \big(L_1(\Sigma_0, X_0), \sigma(L_1(\Sigma_0, X_0), L_\infty(\Sigma_0, X_0'))\big)' = L_\infty(\Sigma_0, X_0')$$

we have

(5) $$\langle g, f_0\rangle = \int_{\text{ex}C} \langle g, f\rangle \, d\nu(f).$$

We consider the function

$$\Phi\colon \text{ex } C \to L_1(\Sigma_0, X_0), \quad f \mapsto f.$$

Since Φ takes its values in the separable subspace L and since $g \circ \Phi$ is continuous on C and hence ν-measurable for all $g \in L_\infty(\Sigma_0, X_0')$ (which is a norming subspace of $(L_1(\Sigma_0, X_0), \|\cdot\|)'$), Φ is ν-measurable by Pettis' measurability criterion [7, p. 42]. Since Φ is also bounded, it is Bochner integrable and we have from (5)

$$f_0 = \int_{\text{ex}C} \Phi(f)\, d\nu(f).$$

This implies [7, p. 48]

$$f_0 = \frac{1}{\nu(\text{ex } C)} \int_{\text{ex}C} \Phi(f)\, d\nu(f) \in \overline{\text{co}(\Phi(\text{ex } C))}^{\|\cdot\|} = \overline{\text{co}(\text{ex } C)}^{\|\cdot\|}.$$

On the other hand, we have ex $C \subset K$ [10, p. 440]. Hence $f_0 \in \overline{\text{co } K}^{\|\cdot\|}$.

(2) The general case is reduced to case (1) as follows. Let $f_0 \in \overline{\text{co } K}^{\sigma'}$. By Corollary 2, there exists a sequence $(f_n) \subset \text{co } K$ such that $f_n \to f_0$ in σ'. Let K_0 be the countable set of those elements in K which are used for the convex representation of the f_n, $n \in N$. We note that $f_0 \in \overline{\text{co } K_0}^{\sigma'}$. There exists a closed separable subspace $X_1 \subset X$ and a σ-algebra $\Sigma_1 \subset \Sigma$ such that $L_0 := L_1(\Sigma_1, X_1)$ is separable and $K_0 \subset L_0$. Hence also co $K_0 \subset L_0$ and $C_0 := \overline{\text{co } K_0}^{\sigma'}$ is σ'-compact. We show $C_0 \subset L_0$. For each $h \in C_0$ there exists again a sequence $(h_n) \subset \text{co } K_0$ with $h_n \to h$ in σ'. Since the closed subspace X_1 is also weakly closed, we have $\int_A h\, d\mu \in X_1$ for all $A \in \Sigma$ and hence $h \in L_1(\Sigma, X_1)$. Similarly, since $L_1(\Sigma_1)$ is weakly closed in $L_1(\Sigma)$ we have $x' \circ h \in L_1(\Sigma_1)$. By Pettis' measurability criterion it follows that

$h \in L_1(\Sigma_1, X_1) = L_0$. Hence $C_0 \subset L_0$ and by case (1)

$$f_0 \in \overline{\text{co } K_0}^{\sigma'} = \overline{\text{co } K_0}^{\|\cdot\|} \subset \overline{\text{co } K}^{\|\cdot\|}.$$

The equality of the closures of the absolutely convex hull is proved in a similar way.

An example. Let $X = l_1$ and $S = [0, 1]$ with Lebesgue measure μ. There exist norm compact, absolutely convex sets K_0 and K_1 in $L_1(X)$ such that

$$\overline{\text{span } K_1}^{\|\cdot\|} \subsetneqq \overline{\text{span } K_1}^{\sigma'} = \overline{\text{span } K_0}^{\|\cdot\|} = \overline{\text{span } K_0}^{\sigma'}.$$

In fact, let (e_n) be the unit vector basis in l_1 and (r_n) the sequence of Rademacher functions on $[0, 1]$. For a sequence (z_k) with $z_k > 0$, $k \in N$ and $\sum_{k=1}^{\infty} z_k = 1$ consider the sequence (f_n) in $L_1(X)$ given by

$$f_n := \frac{1}{n}\Big(\sum_{k=1}^{n} z_k \cdot e_{2k-1} + r_n \cdot e_{2n}\Big), \qquad n \in N.$$

Then $f_n \to 0$ in norm and since $r_n \cdot e_{2n} \to 0$ in σ' [2, p. 300] we have

$$n \cdot f_n \to \sum_{k=1}^{\infty} z_k \cdot e_{2k-1} =: f_0 \qquad \text{in } \sigma'.$$

For any scalar sequence $\lambda := (\lambda_k)$ with $\|\lambda\|_{l_1(\omega)} := \sum_{k=1}^{\infty} |\lambda_k|/k < \infty$ and any scalar λ_0 we have

$$|\lambda_0| \leqslant \sum_{k=1}^{\infty} \Big|\lambda_0 + \sum_{n=k}^{\infty} \frac{\lambda_n}{n}\Big| z_k + \|\lambda\|_{l_1(\omega)} = \Big\|\sum_{n=0}^{\infty} \lambda_n f_n\Big\|_1$$

$$= \Big\|\sum_{k=1}^{\infty} \Big(\lambda_0 + \sum_{n=k}^{\infty} \frac{\lambda_n}{n}\Big) z_k \cdot e_{2k-1} + \sum_{k=1}^{\infty} \frac{\lambda_k}{k} \cdot r_k \cdot e_{2k}\Big\|_1$$

$$\leqslant |\lambda_0| + 2\|\lambda\|_{l_1(\omega)}.$$

This shows

$$K_i := \overline{\text{aco}\{f_n,\ n = i, i+1, \ldots\}}^{\|\cdot\|} = \Big\{\sum_{n=i}^{\infty} \lambda_n f_n;\ \sum_{n=i}^{\infty} |\lambda_n| \leqslant 1\Big\},$$

$$\overline{\text{span } K_i}^{\|\cdot\|} = \Big\{\sum_{n=i}^{\infty} \lambda_n f_n;\ (1-i)|\lambda_0| + \sum_{n=1}^{\infty} \frac{|\lambda_n|}{n} < \infty\Big\}, \qquad i = 0, 1,$$

and hence $f_0 \notin \overline{\text{span } K_1}^{\|\cdot\|}$. To prove $\overline{\text{span } K_1}^{\sigma'} = \overline{\text{span } K_0}^{\|\cdot\|}$ it is sufficient to show that $\overline{\text{span } K_0}^{\|\cdot\|}$ is σ'-closed. Any function g in the σ'-closure of

$\overline{\operatorname{span} K_0}^{\|\cdot\|}$ has the form

$$g = \sum_{k=1}^{\infty} (\alpha_k z_k)\cdot e_{2k-1} + \sum_{k=1}^{\infty} \frac{\lambda_k}{k}\cdot r_k \cdot e_{2k}$$

with $\sum_{k=1}^{\infty} |\lambda_k|/k < \infty$ and

$$\alpha_1 = \frac{\lambda_1}{1} + \ldots + \frac{\lambda_{k-1}}{k-1} + \alpha_k, \quad k = 2, 3, \ldots,$$

because this holds for any element in $\overline{\operatorname{span} K_0}^{\|\cdot\|}$. If we let $\lambda_0 := \lim_{k\to\infty} \alpha_k$, then

$$\|g - \sum_{n=0}^{m} \lambda_n f_n\|_1 = \sum_{k=1}^{\infty} \left|\alpha_k - \sum_{n=k}^{m} \frac{\lambda_n}{n} - \lambda_0\right| \cdot z_k + \sum_{k=m+1}^{\infty} \frac{|\lambda_k|}{k}$$

$$\leqslant 2 \sum_{k=m+1}^{\infty} \frac{|\lambda_k|}{k} \to 0 \quad (m \to \infty),$$

so that $g \in \overline{\operatorname{span} K_0}^{\|\cdot\|}$.

PROPOSITION 2. *If K is a σ'-compact subset of $L_1(\Sigma, X)$, then $L := \overline{\operatorname{span} K}^{\|\cdot\|}$ is $\mathscr{K}$-analytic (and hence Lindelöf) in its σ'-topology.*

Proof. Since $\overline{\operatorname{aco} K}^{\sigma'}$ is σ'-compact [3, p. 416] and coincides with $\overline{\operatorname{aco} K}^{\|\cdot\|}$ by Proposition 1, we may assume that K is absolutely convex; so

$$L = \overline{\bigcup_{n=1}^{\infty} n\cdot K}^{\|\cdot\|}.$$

Let $\varkappa\colon L \to L^{\times\prime}$ be the canonical isometric embedding. Since $\varkappa(L)$ is norm closed in $L^{\times\prime}$ we have

$$\varkappa(L) = \bigcap_{m\in N} \bigcup_{n\in N} \left(n\cdot\varkappa(K) + \frac{1}{m} B(L^{\times\prime})\right).$$

Since the set $n\cdot\varkappa(K) + (1/m) B(L^{\times\prime})$ is compact, $\varkappa(L)$ is $\mathscr{K}$-analytic in the topology $\sigma(L^{\times\prime}, L)$ and hence L is $\mathscr{K}$-analytic in the topology $\sigma(L, L^{\times})$ [6 I, p. 142] and a Lindelöf space [17].

6. Remarks.

1. In case Σ is generated by a countable algebra $\mathscr{A} = \{A_1, A_2, \ldots\}$ let $\mathscr{T}$ be the locally convex topology of $L_1(\Sigma, X)$ generated by the base of 0-neighbourhoods

$$\left\{\frac{1}{m} \bigcap_{i=1}^{n} G_i^{\circ},\ n \in N,\ m \in N\right\}$$

with the $\sigma(L_\infty(\Sigma, X'), L_1(\Sigma, X))$-compact sets

$$G_i := \left\{\sum_{j=1}^{i} x_j' \cdot \chi_{A_j};\ x_1', \ldots, x_i' \in B(X')\right\}, \quad i \in N.$$

Then $(L_1(\Sigma, X), \mathscr{T})' = E(\mathscr{A}, X')$ (space of X'-valued $\mathscr{A}$-simple functions), $\mathscr{T}$ is metrizable and $\mathscr{T} \subset \tau(L_1(\Sigma, X), E(\mathscr{A}, X'))$. By a theorem of J. Dieudonné and L. Schwartz [11, p. 39 (2)], $(L_1(\Sigma, X), \sigma(L_1(\Sigma, X), E(\mathscr{A}, X')))$ is angelic and hence [11, p. 31 (2)] $(L_1(\Sigma, X), \sigma')$ is angelic. This follows also from the representation

$$L_\infty(\Sigma, X') = \overline{\bigcup_{n\in N} \bigcup_{i\in N} n\cdot G_i}^{\sigma(L_\infty(\Sigma, X'), L_1(\Sigma, X))}$$

and a theorem of W. F. Eberlein and Yu. L. Shmul'yan [11, p. 38]. These arguments, however, seem to be restricted to the case of a countably generated σ-algebra Σ.

2. One of us (G. S.) has recently constructed a σ'-compactly generated space L for which there does not exist a σ'-compact subset K_0 such that $L = \overline{\operatorname{span} K_0}^{\|\cdot\|}$

References

[1] D. Amir and J. Lindenstrauss, *The structure of weakly compact sets in Banach spaces*, Ann. of Math. 88 (1968), 34–46.
[2] J. Batt, *On weak compactness in spaces of vector-valued measures and Bochner integrable functions in connection with the Radon–Nikodým property of Banach spaces*, Rev. Roumaine Math. Pures Appl. 19 (1974), 285–304.
[3] J. Batt and W. Hiermeyer, *On compactness in $L_p(\mu, X)$ in the weak topology and in the topology $\sigma(L_p(\mu, X), L_q(\mu, X'))$*, Math. Z. 182 (1983), 409–423.
[4] Y. Benyamini, M. E. Rudin and M. Wage, *Continuous images of weakly compact subsets of Banach spaces*, Pacific J. Math. 70 (1977), 309–324.
[5] Y. Benyamini and T. Starbird, *Embedding weakly compact sets into Hilbert space*, Israel J. Math. 23 (1976), 137–141.
[6] G. Choquet, *Lectures on Analysis I, II*, W. A. Benjamin, Amsterdam 1969.
[7] J. Diestel and J. Uhl jr., *Vector Measures*, Math. Surveys 15, Amer. Math. Soc., Providence, R.I. 1977.
[8] J. Diestel, *Geometry of Banach Spaces – Selected Topics*, Lecture Notes in Math. 485, Springer, Berlin–Heidelberg–New York 1975.
[9] N. Dinculeanu, *Vector Measures*, Pergamon, New York 1967.
[10] N. Dunford and J. T. Schwartz, *Linear Operators, Part I: General Theory*, Interscience, New York 1966.
[11] K. Floret, *Weakly Compact Sets*, Lecture Notes in Math. 801, Springer, Berlin–Heidelberg–New York 1980.
[12] W. Govaerts, *A productive class of angelic spaces*, J. London Math. Soc. (2) 22 (1980), 355–364.
[13] P. R. Halmos, *Measure Theory*, Van Nostrand, New York 1950.
[14] H. Jarchow, *Locally Convex Spaces*, B. G. Teubner, Stuttgart 1981.

[15] R. J. Hunter and J. Lloyd, *Weakly compactly generated locally convex spaces*, Math. Proc. Cambridge Philos. Soc. 82 (1977), 85–98.
[16] E. Michael and M. E. Rudin, *A note on Eberlein compacts*, Pacific J. Math. 72 (1977), 487–495.
[17] M. Sion, *On analytic sets in topological spaces*, Trans. Amer. Math. Soc. 96 (1960), 341–353.
[18] M. Talagrand, *Sur une conjecture de H. H. Corson*, Bull. Sci. Math. 99 (1975), 211–212.

MATHEMATISCHES INSTITUT DER UNIVERSITÄT MÜNCHEN
Theresienstr. 39, D 8000 München 2, Federal Republic of Germany

Received August 13, 1984 (1992)

STUDIA MATHEMATICA, T. LXXXIII. (1986)

Good λ inequalities for the area integral and the nontangential maximal function

by

TAKAFUMI MURAI (Nagoya) and AKIHITO UCHIYAMA* (Sendai)

Abstract. We refine the constants of the good λ inequalities for the area integral $A(x)$ and the nontangential maximal function $N(x)$. As an application we refine the inequalities concerning $A(x)/N(x)$ and $N(x)/A(x)$ which were obtained by R. Fefferman, Gundy, Silverstein and Stein.

1. Introduction. Throughout the paper, functions considered are real-valued. Let $d \geqslant 1$ be an integer. Let $u(y, t)$ be a harmonic function in the $(d+1)$-dimensional Euclidean half-space

$$\mathbf{R}_+^{d+1} = \{(y, t):\ y \in \mathbf{R}^d,\ t > 0\}.$$

For $\alpha > 0$ and $x \in \mathbf{R}^d$, let

$$N(x, \alpha) = \sup\{|u(y, t)|:\ (y, t) \in \Gamma(x, \alpha)\},$$

$$A(x, \alpha) = \Big\{\iint_{\Gamma(x,\alpha)} |\nabla u(y, t)|^2 t^{1-d}\, dy\, dt\Big\}^{1/2},$$

where

$$\Gamma(x, \alpha) = \{(y, t) \in \mathbf{R}_+^{d+1}:\ |x-y| < \alpha t\}.$$

These functions N and A are usually called the *nontangential maximal function* and the *area integral*, respectively.

In [4], R. Fefferman, Gundy, Silverstein and Stein showed that if $\lambda > 0$, $\gamma > 2$, $k > 1$ and if β is sufficiently large, then

$$(1.1)\quad |\{x \in \mathbf{R}^d:\ A(x, 1) > \gamma\lambda,\ N(x, \beta) \leqslant \lambda\}| \leqslant C_1 \gamma^{-k} |\{x \in \mathbf{R}^d:\ A(x, 1) > \lambda\}|,$$

$$(1.2)\quad |\{x \in \mathbf{R}^d:\ N(x, 1) > \gamma\lambda,\ A(x, \beta) \leqslant \lambda\}| \leqslant C_1 \gamma^{-k} |\{x \in \mathbf{R}^d:\ N(x, 1) > \lambda\}|,$$

where C_1 is a positive constant depending only on β, k and d and where $|\{\cdot\}|$ denotes the Lebesgue measure of the set $\{\cdot\}$. Their argument is a refinement of Burkholder and Gundy [1]. Distribution function inequalities of this kind are called *good λ inequalities.*

---

* Both authors supported in part by Grant-in-Aid for Scientific Research (No. 59540109, No. 59740063 and No. 59740056), Japan.
AMS (MOS) subject classification (1980): 42 B 25.