As an application of this inequality we assume, without loss, that φ is also in $U(\Omega:\zeta)$, as in (6.1). Then in view of (4.5) and (6.2), the coefficient $a_\varphi(\zeta)$ is $-S_\varphi(\zeta,\zeta)$, and thus

$$|a_\varphi(\zeta)-l_\Omega^{(s)}(\zeta,\zeta)| \leqslant \frac{\sqrt{b}+\varkappa}{1+\varkappa\sqrt{b}} K_\Omega^{(s)}(\zeta,\zeta).$$

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A note on the spectral mapping theorem

by

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Abstract. We prove that if σ is a semispectrum defined on the commutative subsets of a Banach algebra that satisfies one inclusion of the spectral mapping theorem $P(\sigma(A)) = \sigma(P(A))$ then it also satisfies the other.

1. Introduction. Let $\mathscr{A}$ be a complex unital Banach algebra with unit e. The family of all nonvoid subsets of $\mathscr{A}$ consisting of pairwise commuting elements will be designated by $c(\mathscr{A})$. We shall write $c_0(\mathscr{A})$ for the family of all finite elements of $c(\mathscr{A})$. The elements of $c(\mathscr{A})$ will be denoted by A_I, where I is a nonvoid set of indices, so $A_I = \{a_i\}_{i\in I}$. If the set I is finite, we can identify A_I with $A = (a_1, \ldots, a_n) \in \mathscr{A}^n$ for some $n \in N$. If $A = (a_1, \ldots, a_n) \in \mathscr{A}^n$ and $B = (b_1, \ldots, b_m) \in \mathscr{A}^m$, then (A, B) will denote the element $(a_1, \ldots, a_n, b_1, \ldots, b_m) \in \mathscr{A}^{n+m}$. If I, J are nonvoid sets of indices, then $P_J(T_I)$ will stand for a family of polynomials $\{p_j(T_I)\}_{j\in J}$, with complex coefficients, in indeterminates $T_I = \{t_i\}_{i\in I}$. Of course, each p_j depends only upon a finite number of indeterminates $t_{i_1}, \ldots, t_{i_n}$.

Each such system of polynomials induces a map, denoted by the same symbol P_J: $C^I \to C^J$, given by $Z_I \to (p_j(Z_I))_{j\in J} \in C^J$, where $Z_I = (z_i)_{i\in I}$. Such a map is called a *polynomial map*. Also if $A_I \in c(\mathscr{A})$, we can evaluate P_J on A_I obtaining an element $P_J(A_I) \in c(\mathscr{A})$.

In [4], Żelazko gave the following axioms and definitions.

Suppose that to each $A_I \in c(\mathscr{A})$ there corresponds a nonvoid compact subset of C^I:

(1) $$A_I \to \sigma(A_I) \subset C^I.$$

1.1. Axioms.

(I) $$\sigma(A_I) \subset \prod_{i\in I} \sigma(a_i)$$

where $A_I = \{a_i\}_{i\in I} \in c(\mathscr{A})$ and $\sigma(a_i)$ is the usual spectrum of an element $a_i \in \mathscr{A}$.

(II) $\sigma(\{a\})$ is the usual spectrum, $\sigma(a)$, of a for $\{a\} \in c(\mathscr{A})$.

(III) *Spectral mapping property*:

$$\sigma(P_J(A_I)) = P_J(\sigma(A_I))$$

where $A_I \in c(\mathscr{A})$, P_J is a polynomial map.

(IV) *Projection property*:

$$\sigma(A_J) = \pi(\sigma(A_I))$$

where $A_I \in c(\mathscr{A})$, J is a nonvoid subset of I and π is the projection of C^I onto C^J given by $\pi(Z_I) = Z_J$.

(V) $$\sigma(A_I + \Lambda_I e) = \sigma(A_I) + \Lambda_I$$

where $A_I \in c(\mathscr{A})$, $\Lambda_I = \{\lambda_i\}_{i\in I}$, $\lambda_i \in C$, $A_I + \Lambda_I e = \{a_i + \lambda_i e\}_{i\in I}$.

1.2. Definitions. A map (1) is called:

a *spectroid* if it satisfies (I) and (V);
a *semispectrum* if it satisfies (I), (IV), (V);
a *subspectrum* if it satisfies (I), (III);
a *spectrum* if it satisfies (II), (III).

Let us note that (II), (III) $\Rightarrow$ (I), (III) $\Rightarrow$ (IV) and (III) $\Rightarrow$ (V), so every spectrum is a subspectrum, every subspectrum is a semispectrum and every semispectrum is a spectroid.

In [3], Słodkowski and Żelazko prove that if σ is a semispectrum defined on elements of $c_0(\mathscr{A})$, there exists a unique semispectrum on $c(\mathscr{A})$ which, restricted to $c_0(\mathscr{A})$, equals σ, and if σ is a subspectrum or a spectrum, then its extension is also a subspectrum or a spectrum, respectively (see also [4, 3.4]). Therefore, from now on we will work only with semispectra defined on $c_0(\mathscr{A})$.

1.3. Definition [1]. Let $A \in \mathscr{A}^n$, $B \in \mathscr{A}^m$, and $\lambda \in C^n$. We shall write

(2) $$\sigma_{A=\lambda}(B) = \{\mu \in C^m\colon (\lambda, \mu) \in \sigma(A, B)\}.$$

In [1, 4.2], Harte proves that the joint spectrum has the following property.

Let $A \in \mathscr{A}^n$ be a commuting system that commutes with $B \in \mathscr{A}^m$. Then

(3) $$\sigma(B) = \bigcup_{\lambda\in C^n} \sigma_{A=\lambda}(B).$$

Harte uses this result to prove that the joint spectrum of a commuting system of elements of $\mathscr{A}$ satisfies the spectral mapping property.

On the other hand, in [3] Słodkowski and Żelazko prove that if X is a Banach space and σ is a semispectrum defined on $c(\mathscr{L}(X))$, then the spectral mapping property (III) is equivalent to

(4) $$\sigma(P_J(A_I)) \subset P_J(\sigma(A_I)).$$

The principal goal of this note is to prove, for a semispectrum, that

(5) $$P_J(\sigma(A_I)) \subset \sigma(P_J(A_I))$$

is also equivalent to the spectral mapping theorem (III).

2. Spectral mapping property.

2.1. Lemma. *If σ is a map (1) defined on $c_0(\mathscr{A})$ which satisfies axiom* (I), *then the following properties are equivalent*:

(i) $$\sigma(\pi(S)) \subset \pi(\sigma(S))$$

where $S \in \mathscr{A}^r$ for some $r \in N$ and π is a projection.

(ii) *If $A \in \mathscr{A}^n$, $B \in \mathscr{A}^m$ for some $n, m \in N$, are commuting systems, and the elements of A commute with the elements of B then*

$$\sigma(B) = \bigcup_{\lambda\in C^n} \sigma_{A=\lambda}(B).$$

Proof. (i) $\Rightarrow$ (ii). Let $S = (A, B) \in \mathscr{A}^{n+m}$, and let π be the projection with $\pi(S) = B$. Then

$$\sigma(B) = \sigma(\pi(A, B)) \subset \pi(\sigma(A, B)).$$

Let $\mu \in \sigma(B)$. Then there exists $\lambda \in C^n$ such that

$$(\lambda, \mu) \in \sigma(A, B) \quad \text{and} \quad \pi(\lambda, \mu) = \mu$$

so $\mu \in \sigma_{A=\lambda}(B)$.

(ii) $\Rightarrow$ (i). Let $S \in \mathscr{A}^r$ and let π be a projection. Reordering the coordinates, we can assume that π is the projection onto the last m coordinates. We take $n = r - m$ and write

$$S = (A, B) \quad \text{with } A \in \mathscr{A}^n,\ B \in \mathscr{A}^m \text{ and } B = \pi(S).$$

If $\mu \in \sigma(\pi(S)) = \sigma(B)$, then by hypothesis there exists $\lambda \in C^n$ such that $\mu \in \sigma_{A=\lambda}(B)$. Thus,

$$(\lambda, \mu) \in \sigma(A, B) = \sigma(S),$$

so $\mu = \pi(\lambda, \mu) \in \pi(\sigma(S))$.

In order to prove our main theorem, we need the following result due to Harte [1, 3.3]:

2.2. Lemma. *If*

$$P(\sigma(A)) \subseteq \sigma(P(A))$$

for every polynomial map $P\colon \mathscr{A}^n \to \mathscr{A}^m$, then

$$P(\sigma(A)) = \sigma(P(A))$$

for every polynomial map of the form $P(z) = (z, Q(z))$ where Q is a polynomial map from $\mathscr{A}^n$ to $\mathscr{A}^{m-n}$.

2.3. THEOREM. *If σ is a semispectrum defined on $c_0(\mathscr{A})$ then the following properties are equivalent*:

(i) *For every $A \in c_0(\mathscr{A})$ and every polynomial map P*

$$P(\sigma(A)) \subset \sigma(P(A)).$$

(ii) *For every $A \in c_0(\mathscr{A})$ and every polynomial map P*

$$P(\sigma(A)) \supset \sigma(P(A)).$$

(iii) *For every $A \in c_0(\mathscr{A})$ and every polynomial map P*

$$P(\sigma(A)) = \sigma(P(A)).$$

Proof. (i)$\Rightarrow$(ii) and (i)$\Rightarrow$(iii). Let $\mu \in \sigma(P(A))$. By hypothesis, σ satisfies the projection property (IV), so by Lemma 2.1 there exists $\lambda \in C^n$ such that $\mu \in \sigma_{A=\lambda}(P(A))$. Thus, $(\lambda, \mu) \in \sigma(A, P(A)) = \sigma(\tilde{P}(A))$ where $\tilde{P}(z) = (z, P(z))$.

By Lemma 2.2, $(\lambda, \mu) \in \tilde{P}(\sigma(A)) = \{(\lambda, P(\lambda)) \colon \lambda \in \sigma(A)\}$ so $\mu \in P(\lambda)$ with $\lambda \in \sigma(A)$. Hence

$$\sigma(P(A)) \subset P(\sigma(A))$$

and thus

$$\sigma(P(A)) = P(\sigma(A)).$$

On the other hand, (ii)$\Rightarrow$(i) and (ii)$\Rightarrow$(iii) are part of Theorem 3.3 of [3].

Note. There is a misprint in formula (3.4) of [3]: it must be read as $\sigma^*(PS) \subset P\sigma^*(S)$.

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