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Fredholm spectrum and Grunsky inequalities in general domains

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Abstract. We discuss the Fredholm spectrum for general domains and study its applications to conformal and quasi-conformal mappings. In particular, we establish an improvement of the Grunsky inequalities which is valid for general domains. This improvement constitutes an extension of a recent result of Schiffer concerning the sharpening of Grunsky inequalities for the unit disk by a factor smaller than 1, and which is the reciprocal of the least Fredholm eigenvalue of the smooth simply connected image domain.

§ 1. Introduction. Let φ be a univalent holomorphic function of the unit disk Δ onto a simply connected domain $\Omega^* = \varphi(\Delta)$ whose boundary $\partial\Omega^*$ is of class $C^{2,\varepsilon}$ $(0 < \varepsilon < 1)$ and consider the Grunsky coefficients (c_{mn}) of φ, defined by

$$\log\frac{\varphi(z)-\varphi(\zeta)}{z-\zeta} = \sum_{m,n=0}^{\infty} c_{mn} z^m \zeta^n, \qquad z, \zeta \in \Delta.$$

In a recent paper [13], Schiffer has established the following improved Grunsky inequality:

$$\Big|\sum_{m,n=1}^{\infty} \sqrt{mn}\, c_{mn}\alpha_m\alpha_n\Big| \leqslant (\lambda_1^*)^{-1} \sum_{n=1}^{\infty} |\alpha_n|^2 \tag{1.1}$$

where $\lambda_1^* = \lambda_1(\Omega^*)$ is the least Fredholm eigenvalue of Ω^* and $\{\alpha_n\}$ is an arbitrary sequence of complex numbers. The customary Grunsky inequality is inequality (1.1) with $(\lambda_1^*)^{-1}$ replaced by 1, and as $\lambda_1^* > 1$ because of the smoothness assumptions on $\partial\Omega^*$, the present inequality (1.1) constitutes an improvement on it.

The symmetric matrix (g_{mn}) with $g_{mn} = \sqrt{mn}\, c_{mn}$ is known as the *Grunsky operator* $\mathscr{G}_\varphi$. Its domain of definition is l_2, the space of complex sequences $\alpha = (\alpha_1, \alpha_2, \ldots)$ with the norm

$$\|\alpha\|_2 = \Big\{\sum_{n=1}^{\infty} |\alpha_n|^2\Big\}^{1/2} < \infty.$$

In this way, inequality (1.1) may be formulated as

$$|(\mathcal{G}_\varphi \alpha, \bar{\alpha})_2| \leqslant (\lambda_1^*)^{-1} \|\alpha\|_2^2, \quad \alpha \in l_2. \tag{1.2}$$

As is well known such an inequality is equivalent to

$$\|\mathcal{G}_\varphi \alpha\|_2 \leqslant (\lambda_1^*)^{-1} \|\alpha\|_2, \quad \alpha \in l_2 \tag{1.3}$$

which means that $\mathcal{G}_\varphi$ is a strict contraction operator on l_2 with norm $\|\mathcal{G}_\varphi\| \leqslant (\lambda_1^*)^{-1} < 1$.

Schiffer's result is based on an elegant theory due to Bergman and Schiffer [2] which relates the Bergman kernel function of a regular domain with the Hilbert transform, and hence with the classical Fredholm eigenvalues of the regular domain. An extension of the Bergman–Schiffer theory to general domains can be found in Ozawa [8], Suita [16], Sakai [10] and Burbea [4]. In this paper we give an operator-theoretic extension of this theory which is different than that found in [2, 8, 10, 12, 16] and in so doing we shall also extend inequalities (1.2) and (1.3) to general domains. In some sense this paper may be viewed as exhibiting a unified approach to the subject matter with the emphasis on the general validity of the results. However, because of the unified nature of this paper, some overlap with previously established results seems to be unavoidable. On the other hand, the approach taken here has apparently the advantage of rendering this paper as virtually self-contained.

§ 2. The Hilbert transform and the Bergman projector. In this section we give a brief description of the theory of the Hilbert transform and the Bergman projector in general domains.

We consider the Hilbert space $L_2(C)$ with the induced norm

$$\|f\| = \{\pi^{-1} \int_C |f(z)|^2 \, dm(z)\}^{1/2},$$

where dm is the area Lebesgue measure. The subspace of C^∞-functions with compact supports is denoted by $C_0^\infty(C)$. We also consider the Sobolev space $W_2^1(C)$, the space of all $L_2(C)$ functions f whose first partial derivatives ∂f and $\bar{\partial} f$ (taken in the distributional sense) also belong to $L_2(C)$. It is a Hilbert space under the norm

$$\|f\|_1 = \{\|f\|^2 + \|\partial f\|^2 + \|\bar{\partial} f\|^2\}^{1/2}.$$

The closure of $C_0^\infty(C)$ in $W_2^1(C)$ is denoted by $\mathring{W}_2^1(C)$. On $L_2(C)$ we consider the identity operator I, the conjugation operator J (i.e. $Jf = \bar{f}$) and certain transforms (integral operators). For a kernel $h(\cdot, \cdot)$ on $C \times C$, we define

$$\{Hf\}(\zeta) = (f, h(\cdot, \zeta))$$

and its (formal) adjoint

$$\{H^* f\}(\zeta) = (f, Jh(\zeta, \cdot)) = (f, \overline{h(\zeta, \cdot)}),$$

where $(\cdot, \cdot)$ is the induced inner product of $L_2(C)$. When the kernel $h(\cdot, \cdot)$ is singular, the integrals shall always be taken in the principal value sense. The (formal) transpose of H is defined by $H^t = JH^*J$.

Given any open subset Ω of C, we view $L_2(\Omega)$ as naturally embedded in $L_2(C)$ by regarding any $f \in L_2(\Omega)$ as zero outside Ω and by setting $I_\Omega = I \cdot \chi_\Omega$ as the identity operator of $L_2(\Omega)$, where χ_Ω is the characteristic function of Ω. This induces the spaces $C_0^\infty(\Omega)$, $W_2^1(\Omega)$, and $\mathring{W}_2^1(\Omega)$, and we use the natural notation of $(\cdot, \cdot)_\Omega$ and $\|\cdot\|_\Omega$, and $\|\cdot\|_{1,\Omega}$ to denote the restricted inner product and norm of $L_2(\Omega)$, and norm of $W_2^1(\Omega)$, respectively. In particular, we use the convention of $J_\Omega = J \cdot \chi_\Omega$ and $H_\Omega = H \cdot \chi_\Omega$ and, hence, $H_\Omega^* = H^* \cdot \chi_\Omega$ and $H_\Omega^t = H^t \cdot \chi_\Omega$.

The kernels $(z-\zeta)^{-1}$ and $(z-\zeta)^{-2}$ give rise to the familiar transforms: the Cauchy transform

$$\{Sf\}(\zeta) = \frac{1}{\pi} \int_C \frac{1}{\overline{z - \zeta}} f(z) \, dm(z)$$

and the Hilbert transform

$$\{Tf\}(\zeta) = \frac{1}{\pi} \int_C \frac{1}{\overline{(z - \zeta)^2}} f(z) \, dm(z).$$

Thus $S^t = -S$, $T^t = T$ and

$$\partial S = -I, \quad \bar{\partial} S = T. \tag{2.1}$$

Moreover, for any $f \in \mathring{W}_2^1(C)$, $S(\partial f) = -S^*(\bar{\partial} f) = -f$ and, hence, $\partial f = -T^*(\bar{\partial} f)$ and $\bar{\partial} f = -T(\partial f)$. In particular,

$$\partial f = -T_\Omega^*(\bar{\partial} f), \quad \bar{\partial} f = -T_\Omega(\partial f) \quad (f \in \mathring{W}_2^1(\Omega)). \tag{2.2}$$

The most important property, however, is the fact that T is a unitary operator on $L_2(C)$, namely $T^*T = TT^* = I$. In particular,

$$T^* T_\Omega = T T_\Omega^* = I_\Omega,$$

and thus T_Ω is a contraction operator. In fact, for any $f \in L_2(\Omega)$,

$$\|T_\Omega f\|_\Omega \leqslant \|T_\Omega f\| = \|T\chi_\Omega f\| = \|\chi_\Omega f\| = \|f\|_\Omega.$$

It therefore follows that $T_\Omega^* T_\Omega$ and

$$\Lambda_\Omega = I_\Omega - T_\Omega^* T_\Omega \tag{2.3}$$

are both positive and contractive operators on $L_2(\Omega)$. The related operator $T_\Omega T_\Omega^*$ will not be considered here because its properties can be completely read off through the relationship $T_\Omega T_\Omega^* = (T_\Omega^* T_\Omega)^t$.

In order to have a clear understanding of the significance of the operator Λ_Ω we shall assume here and throughout the remaining parts of the paper that Ω is a domain in C. Let $H_2(\Omega)$ be the subspace of $L_2(\Omega)$ consisting of functions in $L_2(\Omega)$ which are also holomorphic in Ω. The domain Ω is said to belong to 0_G if $H_2(\Omega)=\{0\}$. We decompose $L_2(\Omega)$ as the direct sum

$$L_2(\Omega)=H_2(\Omega)\oplus H_2(\Omega)^{\perp}.$$

In view of Weyl's lemma (see, for example, [4]), the annihilator $H_2(\Omega)^{\perp}$ may be identified as

$$H_2(\Omega)^{\perp}=\{\partial h\in L_2(\Omega)\colon\ h\in \mathring{W}_2^1(\Omega)\}.$$

On the other hand, in view of (2.2), $T_\Omega^* T_\Omega(\partial h)=\partial h$ for any $h\in \mathring{W}_2^1(\Omega)$, and hence $T_\Omega^* T_\Omega$ and Λ_Ω reduce to the identity and the zero operators, respectively, on $H_2(\Omega)^{\perp}$. Moreover, since also Λ_Ω maps $H_2(\Omega)$ into itself, Λ_Ω may serve as a "quasi-projection" of $L_2(\Omega)$ into $H_2(\Omega)$. The true orthogonal projection P_Ω of $L_2(\Omega)$ onto $H_2(\Omega)$ is known as the *Bergman projector* and the deviation

$$B_\Omega=P_\Omega-\Lambda_\Omega \tag{2.4}$$

will be called the *Fredholm transform*. Clearly, B_Ω is a self-adjoint operator of $L_2(\Omega)$ into $H_2(\Omega)$ which, in view of (2.3), admits the alternative expression

$$B_\Omega=T_\Omega^* T_\Omega-(I_\Omega-P_\Omega)$$

and hence $B_\Omega f=T_\Omega^* T_\Omega f$ for all $f\in H_2(\Omega)$. It follows that B_Ω is also a positive and contractive operator on $L_2(\Omega)$, i.e. $0\leqslant B_\Omega\leqslant I_\Omega$.

We now consider the spectrum $\sigma(B_\Omega\colon L_2(\Omega))$ of the Fredholm operator B_Ω. However, since B_Ω is trivial on $H_2(\Omega)^{\perp}$ we may restrict our attention to only $\sigma(\Omega)\equiv\sigma(B_\Omega\colon H_2(\Omega))$; thus $\sigma(B_\Omega\colon L_2(\Omega))=\sigma(\Omega)\cup\{0\}$. The spectrum $\sigma(\Omega)$ is then the set of all numbers μ such that $B_\Omega-\mu I_\Omega$ has no bounded inverse in $H_2(\Omega)$. Since B_Ω is positive and contractive, $\sigma(\Omega)$ is a closed subset of [0, 1]. In accordance with the spectral theory of self-adjoint operators, we have a unique spectral decomposition

$$B_\Omega=\int_{\sigma(\Omega)}\mu\,dE(\mu)$$

where E is the (unique) resolution of identity corresponding to B_Ω. It also follows that for any $f\in H_2(\Omega)$, and hence for any $f\in L_2(\Omega)$,

$$\|B_\Omega f\|_\Omega^2=\int_{\sigma(\Omega)}\mu^2\,d(E(\mu)f,f)_\Omega<\infty.$$

Moreover, for

$$a\equiv a(\Omega)=\operatorname{Inf}\{(B_\Omega f,f)_\Omega\colon f\in H_2(\Omega),\ \|f\|_\Omega=1\},$$
$$b\equiv b(\Omega)=\operatorname{Sup}\{(B_\Omega f,f)_\Omega\colon f\in H_2(\Omega),\ \|f\|_\Omega=1\},$$

we have $\sigma(\Omega)\subseteq[a,b]\subseteq[0,1]$ and

$$b=\|B_\Omega^{1/2}\|^2=\|B_\Omega\| \tag{2.5}$$

where $B_\Omega^{1/2}$ is the square root of B_Ω and $\|B_\Omega\|$ is the operator norm of B_Ω on $H_2(\Omega)$. Evidently, $\|B_\Omega\|$ coincides with the full operator norm of B_Ω on $L_2(\Omega)$.

The spectrum $\sigma(\Omega)$ is divided into two disjoint parts $\sigma(\Omega)=\sigma_p(\Omega)\cup\sigma_c(\Omega)$; the *point spectrum* $\sigma_p(\Omega)$ consisting of all μ such that $B_\Omega-\mu I_\Omega$ is not injective on $H_2(\Omega)$ and the *continuous spectrum* $\sigma_c(\Omega)$ consisting of all μ such that $B_\Omega-\mu I_\Omega$ is an injective mapping of $H_2(\Omega)$ onto a dense proper subspace of $H_2(\Omega)$. The point spectrum consists of eigenvalues of B_Ω, thus for $\Omega\notin 0_G$, $\mu\in\sigma_p(\Omega)$ implies the existence of an eigenfunction $f\in H_2(\Omega)\setminus\{0\}$ such that $B_\Omega f=\mu f$. To conform with classical tradition [2, 12, 13, 15] we shall also use the numbers $\lambda=\mu^{-1/2}$; they are called the *Fredholm eigenvalues*. In this paper, however, we refer to the spectrum $\{\sigma(\Omega)\}^{1/2}$ of $B_\Omega^{1/2}$ as the *Fredholm spectrum* and to $\{\sigma_p(\Omega)\}^{1/2}=\{\mu^{1/2}\colon\ \mu\in\sigma_p(\Omega)\}$ as the *Fredholm point spectrum*. More about the justification for this terminology will be given later (see § 3). At any rate the lack of consistency in the symbols of λ and μ will not cause a serious confusion. When there are only point spectra, namely when $\sigma(\Omega)=\sigma_p(\Omega)$, we have an orthonormal basis $\{\psi_k\}$ of eigenfunctions in $H_2(\Omega)$ of B_Ω and corresponding eigenvalues $\{\mu_k\}=\{\lambda_k^{-2}\}$ with

$$1\geqslant b=\mu_1\geqslant\mu_2\geqslant\ldots\geqslant\mu_n\geqslant\mu_{n+1}\geqslant\ldots\geqslant\mu_\infty=a\geqslant 0$$

or

$$1\leqslant b^{-1/2}=\lambda_1\leqslant\lambda_2\leqslant\ldots\leqslant\lambda_n\leqslant\lambda_{n+1}\leqslant\ldots\leqslant\lambda_\infty=a^{-1/2}\leqslant\infty.$$

§3. Operators and kernels in $L_2(\Omega)$. The operator Λ_Ω in (2.3) admits the alternative representation

$$\{\Lambda_\Omega f\}(\zeta)=(f,\Gamma_\Omega(\cdot,\zeta))_\Omega \tag{3.1}$$

where $\Gamma_\Omega(\cdot,\cdot)$ is the hermitian kernel

$$\Gamma_\Omega(z,\zeta)=\frac{1}{\pi}\int_{C\setminus\Omega}\frac{dm(t)}{(t-z)^2\overline{(t-\zeta)^2}}. \tag{3.2}$$

This kernel is positive-definite and sesqui-holomorphic on $\Omega\times\Omega$. We also note that for $h(z,\zeta)=\overline{(z-\zeta)^2}$ we have

$$\Gamma_\Omega(z,\zeta)=\{T_{C\setminus\Omega}^*[h(\cdot,\zeta)]\}(z)$$

and hence for a fixed $\zeta \in \Omega$,

$$\|\Gamma_\Omega(\cdot, \zeta)\|_\Omega^2 \leqslant \|\Gamma_\Omega(\cdot, \zeta)\|^2 = \|T^*_{C\setminus\Omega}[h(\cdot, \zeta)]\|^2$$
$$= \|T^*[\chi_{C\setminus\Omega} h(\cdot, \zeta)]\|^2 = \|\chi_{C\setminus\Omega} h(\cdot, \zeta)\|^2$$
$$= \frac{1}{\pi}\int_{C\setminus\Omega} \frac{1}{|t-\zeta|^4}\, dm(t) = \Gamma_\Omega(\zeta, \zeta),$$

which shows that

$$\|\Gamma_\Omega(\cdot, \zeta)\|_\Omega \leqslant \|\Gamma_\Omega(\cdot, \zeta)\| = \sqrt{\Gamma_\Omega(\zeta, \zeta)}, \qquad \zeta \in \Omega.$$

In a similar manner, the Bergman projector P_Ω is given by

(3.3) $$\{P_\Omega f\}(\zeta) = (f, K_\Omega(\cdot, \zeta))_\Omega$$

where $K_\Omega(\cdot, \cdot)$ is the standard Bergman kernel of Ω. Let $G_\Omega(\cdot, \cdot)$ be the customary Green function of Ω. Thus

$$G_\Omega(z, \zeta) = R_\Omega(z, \zeta) - \log|z-\zeta|$$

where $R_\Omega(\cdot, \cdot)$ is symmetric and harmonic on $\Omega \times \Omega$. As is well known,

$$K_\Omega(z, \zeta) = -2\partial_z \bar{\partial}_\zeta G_\Omega(z, \zeta)$$

and hence $K_\Omega(z, \zeta) = -2\partial_z \bar{\partial}_\zeta R_\Omega(z, \zeta)$. This suggests the introduction of the *adjoint kernel* [2]:

$$L_\Omega(z, \zeta) = -2\partial_z \partial_\zeta G_\Omega(z, \zeta)$$

and thus

(3.4) $$L_\Omega(z, \zeta) = (z-\zeta)^{-2} - l_\Omega(z, \zeta)$$

where

$$l_\Omega(z, \zeta) = 2\partial_z \partial_\zeta R_\Omega(z, \zeta)$$

is symmetric and holomorphic on $\Omega \times \Omega$. These two symmetric kernels induce the *Bergman–Schiffer transforms* [4]:

$$\{Q_\Omega f\}(\zeta) = (f, L_\Omega(\cdot, \zeta))_\Omega,$$
(3.5) $$\{A_\Omega f\}(\zeta) = (f, l_\Omega(\cdot, \zeta))_\Omega.$$

It follows that

(3.6) $$T_\Omega = Q_\Omega + A_\Omega$$

on $L_2(\Omega)$ and, of course, $T_\Omega^t = T_\Omega$, $Q_\Omega^t = Q_\Omega$ and $A_\Omega^t = A_\Omega$. Now, as P_Ω is the orthogonal projection of $L_2(\Omega)$ onto $H_2(\Omega)$, we clearly have $P_\Omega^2 = P_\Omega$. On the other hand, by a judicious use of Green's formula one shows that $Q_\Omega P_\Omega = 0$ on $L_2(\Omega)$ which means, in accordance with (3.6), that $T_\Omega = A_\Omega$ on $H_2(\Omega)$. In particular, for $z, \zeta \in \Omega$,

$$(l_\Omega(\cdot, z), l_\Omega(\cdot, \zeta))_\Omega = \frac{1}{\pi}\int_\Omega \frac{1}{(t-\zeta)^2}\, l_\Omega(t, z)\, dm(t).$$

Another use of Green's formula shows that the value of the last singular integral is precisely $K_\Omega(z, \zeta) - \Gamma_\Omega(z, \zeta)$ and thus [2]:

(3.7) $$(l_\Omega(\cdot, z), l_\Omega(\cdot, \zeta))_\Omega = K_\Omega(z, \zeta) - \Gamma_\Omega(z, \zeta).$$

In other words, $A_\Omega^* A_\Omega = P_\Omega - \Lambda_\Omega$ or, by (2.4), $B_\Omega = A_\Omega^* A_\Omega$ on $L_2(\Omega)$.

For convenience we record the above and other relations amongst the operators as a proposition (see also [4]):

PROPOSITION 1. *On $L_2(\Omega)$ we have*

(i) $\Lambda_\Omega = \Lambda_\Omega P_\Omega = I_\Omega - T_\Omega^* T_\Omega$,

(ii) $B_\Omega = A_\Omega^* A_\Omega = P_\Omega - \Lambda_\Omega$,

(iii) $A_\Omega = T_\Omega - Q_\Omega = T_\Omega P_\Omega = A_\Omega P_\Omega$,

(iv) $I_\Omega - P_\Omega = Q_\Omega^* Q_\Omega = T_\Omega^* Q_\Omega$,

(v) $Q_\Omega P_\Omega = A_\Omega^* Q_\Omega = 0$, $P_\Omega^2 = P_\Omega$.

Item (iii) of this proposition shows that A_Ω is trivial on $H_2(\Omega)^\perp$ while item (ii) shows that the operator norm of A_Ω on $H_2(\Omega)$ satisfies

(3.8) $$\|A_\Omega\|^2 = \|B_\Omega\|.$$

Again, $\|A_\Omega\|$ coincides with the full operator norm of A_Ω on $L_2(\Omega)$. Moreover, from item (ii) we also obtain the (unique) polar decomposition

(3.9) $$A_\Omega = U_\Omega B_\Omega^{1/2}$$

where U_Ω is a partial isometry relative to the range $\mathscr{R}(B_\Omega^{1/2})$ of $B_\Omega^{1/2}$. This means that U_Ω is a linear isometry of the closure of $\mathscr{R}(B_\Omega^{1/2})$ onto the closure of $\mathscr{R}(A_\Omega)$ and U_Ω is zero on $\mathscr{R}(B_\Omega^{1/2})^\perp$. Now, the conjugate-linear operator JA_Ω has its range inside $H_2(\Omega)$, is trivial on $H_2(\Omega)^\perp$ and on $H_2(\Omega)$ is equal to JT_Ω. Its spectrum on $L_2(\Omega)$, in view of (3.9), is $e^{i\theta}\{\sigma(\Omega)\}^{1/2} \cup \{0\}$, $0 \leqslant \theta < 2\pi$. It follows that the spectrum of $JT_\Omega = JA_\Omega$ on $H_2(\Omega)$ is $e^{i\theta}\{\sigma(\Omega)\}^{1/2}$, which is essentially the Fredholm spectrum. In particular, when $\sigma(\Omega) = \sigma_p(\Omega)$, the orthonormal basis $\{\psi_k\}$ in $H_2(\Omega)$ of eigenfunctions of B_Ω, with corresponding eigenvalues $\sigma_p(\Omega) = \{\lambda_k^{-2}\}$, can also be assumed (by multiplying each ψ_k by a suitable factor $e^{i\theta_k}$, $0 \leqslant \theta_k < 2\pi$) to satisfy

$$\lambda_k J A_\Omega \psi_k = \lambda_k J T_\Omega \psi_k = \lambda_k J U_\Omega B_\Omega^{1/2} \psi_k = \psi_k.$$

It follows that [2]

(3.10) $$l_\Omega(z,\zeta)=\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\psi_k(z)\psi_k(\zeta),$$

$$K_\Omega(z,\zeta)-\Gamma_\Omega(z,\zeta)=\sum_{k=1}^{\infty}\frac{1}{\lambda_k^2}\psi_k(z)\overline{\psi_k(\zeta)},$$

and since $K_\Omega(z,\zeta)=\sum_{k=1}^{\infty}\psi_k(z)\overline{\psi_k(\zeta)}$, also

$$\Gamma_\Omega(z,\zeta)=\sum_{k=1}^{\infty}\left(1-\frac{1}{\lambda_k^2}\right)\psi_k(z)\overline{\psi_k(\zeta)}.$$

In the particular case that the boundary $\partial\Omega$ of Ω is of class $C^{2,\varepsilon}$ $(0<\varepsilon<1)$ the operators under consideration are compact. Moreover, if also Ω is of connectivity p $(1\leqslant p<\infty)$ then the eigenvalue $\lambda_k=1$ is of degeneracy $p-2$. In fact, let C_k $(1\leqslant k\leqslant p)$ be the boundary components of $\partial\Omega$ and let ω_k be the (real) harmonic measure with respect to C_k. Then $i\partial\omega_k$ belongs to $H_2(\Omega)$ and $JT_\Omega(i\partial\omega_k)=i\partial\omega_k$, and hence, $T_\Omega^*T_\Omega(i\partial\omega_k)=i\partial\omega_k$. In particular, let $H_2^{(s)}(\Omega)$ be the subspace of $H_2(\Omega)$ consisting of all functions in $H_2(\Omega)$ with single-valued integrals. Then the linear span of the $p-1$ linearly independent functions $i\partial\omega_1,\ldots,i\partial\omega_{p-1}$ is precisely $N_2^{(s)}(\Omega)\equiv H_2(\Omega)\ominus H_2^{(s)}(\Omega)$. It follows that $\lambda_1=\ldots=\lambda_{p-1}=1$ and their corresponding eigenfunctions are $\psi_k=i\partial\omega_k$, $k=1,\ldots,p-1$, which belong to $N_2^{(s)}(\Omega)$. Moreover, $\lambda_k>1$ for every $k\geqslant p$ and the corresponding eigenfunctions ψ_k, $k\geqslant p$, belong to $H_2^{(s)}(\Omega)$ (see also [2, 12, 13, 15]). The eigenvalues λ_k occurred early in potential theory as the *Poincaré–Fredholm eigenvalues* associated with double-layer potentials in the plane [2, 12, 13, 15]. This connection may be seen at once from the following considerations: Let $z\in\partial\Omega$ and consider the kernel

$$k_\Omega(z,\zeta)=\partial_{n_z}\log|z-\zeta|$$

where n_z is the outward normal to $\partial\Omega$ at z. For any continuous function f on $\partial\Omega$, the double-layer potential

$$\{D_\Omega f\}(\zeta)=\pi^{-1}\int_{\partial\Omega}f(z)k_\Omega(z,\zeta)|dz|$$

represents a harmonic function in Ω and $C\backslash\bar{\Omega}$ with a boundary jump, expressed as

(3.11) $$\{D_\Omega^{(b)}f\}(\zeta)=\{D_\Omega f\}(\zeta)-f(\zeta),\qquad \zeta\in\partial\Omega.$$

Using Green's formula we have the identity $D_\Omega=2I_\Omega+S_\Omega\partial-S_\Omega^*\bar{\partial}$, where S_Ω is the previously mentioned restricted Cauchy transform. This identity is valid on $C^1(\bar{\Omega})$. Therefore, in view of (2.1),

(3.12) $$\partial D_\Omega=T_\Omega^*\bar{\partial}+\partial.$$

Now, the Poincaré–Fredholm eigenvalue problem is given by

(3.13) $$\hat{\lambda}_k D_\Omega^{(b)}h_k=h_k,\qquad k=0,1,2,\ldots$$

where the eigenfunctions $\{h_k\}$ are (real) harmonic in Ω as well as in $C\backslash\bar{\Omega}$. It could be shown that $\hat{\lambda}=-1$ is not an eigenvalue $\hat{\lambda}_k$ of (3.13), that $|\hat{\lambda}_k|\geqslant 1$ and that $\hat{\lambda}_k=1$ if and only if $h_k\equiv\text{const}$. We may therefore assume that $\hat{\lambda}_0=1$ and $h_0=1$, and $\hat{\lambda}_k>1$ for $k\geqslant 1$. From (3.11) and (3.13) we conclude that

$$\frac{\hat{\lambda}_k}{1+\hat{\lambda}_k}D_\Omega h_k=h_k,$$

and hence, by (3.12), $\hat{\lambda}_kJT_\Omega\partial h_k=\partial h_k$. This shows that $\lambda_{k+p-1}=\hat{\lambda}_k$ and $\psi_{k+p-1}=\partial h_k$ for $k=1,2,\ldots$, which is the desired connection.

§ 4. Extreme spectra. The spectrum $\sigma(\Omega)$ can reach two extreme cases namely, $\sigma(\Omega)=\{0\}$ or $\sigma(\Omega)=\{1\}$.

THEOREM 1. *The following statements are equivalent*:

(1) $\sigma(\Omega)=\{0\}$; (2) $B_\Omega=0$ *on* $H_2(\Omega)$; (3) $\Lambda_\Omega=I_\Omega$ *on* $H_2(\Omega)$; (4) $\Gamma_\Omega(z,\zeta)=K_\Omega(z,\zeta)$ *for every* $z,\zeta\in\Omega$; (5) $\Gamma_\Omega(z,\zeta)=K_\Omega(z,\zeta)$ *for every* $z\in\Omega$ *and some* $\zeta\in\Omega$; (6) $\Gamma_\Omega(\zeta,\zeta)=K_\Omega(\zeta,\zeta)$ *for some* $\zeta\in\Omega$; (7) *either* $\Omega\in 0_G$ *or* Ω *is a disk (in the extended sense) less (possibly) a closed subset of zero inner capacity.*

Proof. The implications $(1)\Rightarrow(2)\Rightarrow(3)\Rightarrow(4)\Rightarrow(5)\Rightarrow(6)$ and $(7)\Rightarrow(1)$ are straightforward. As for the implication $(6)\Rightarrow(7)$, we argue as follows: Since $K_\Omega(\zeta,\zeta)-\Gamma_\Omega(\zeta,\zeta)=\|l_\Omega(\cdot,\zeta)\|_\Omega^2=0$ we conclude that $L_\Omega(z,\zeta)=(z-\zeta)^{-2}$ for some $\zeta\in\Omega$ and every $z\in\Omega$. It follows that $L_\Omega(\cdot,\zeta)$ has a single-valued indefinite integral in Ω. This, as is well known (see also [11, p. 104]), means that for $\Omega\notin 0_G$ there exists a conformal mapping φ of Ω onto $\Delta\backslash E$, where E is a closed subset of the unit disk Δ whose inner capacity is zero. The conformal invariance of the Green function implies that

$$L_\Omega(z,\zeta)=L_{\Delta\backslash E}(\varphi(z),\varphi(\zeta))\varphi'(z)\varphi'(\zeta)$$

for any $z,\zeta\in\Omega$. But $L_{\Delta\backslash E}(\omega,\tau)=(\omega-\tau)^{-2}$ for every $\omega,\tau\in\Delta\backslash E$. In particular, for the above fixed $\zeta\in\Omega$ we have

$$(z-\zeta)^{-2}=[\varphi(z)-\varphi(\zeta)]^{-2}\varphi'(z)\varphi'(\zeta)$$

for every $z\in\Omega$. This means that φ is a Möbius transformation and the proof is complete.

THEOREM 2. *The following statements are equivalent*:

(1) $\sigma(\Omega)=\{1\}$; (2) $B_\Omega=I_\Omega$ *on* $H_2(\Omega)$; (3) $\Lambda_\Omega=0$ *on* $H_2(\Omega)$; (4) $\Gamma_\Omega(z,\zeta)=0$ *for every* $z,\zeta\in\Omega$; (5) $\Gamma_\Omega(z,\zeta)=0$ *for every* $z\in\Omega$ *and some* $\zeta\in\Omega$; (6) $\Gamma_\Omega(\zeta,\zeta)=0$ *for some* $\zeta\in\Omega$; (7) $m(C\backslash\Omega)=0$.

Proof. The implications $(1)\Rightarrow(2)\Rightarrow(3)\Rightarrow(4)\Rightarrow(5)\Rightarrow(6)\Rightarrow(7)\Rightarrow(1)$ are straightforward and trivial.

Let $\varphi\colon \Omega \to \Omega^*$ be a univalent holomorphic mapping of Ω onto $\Omega^* = \varphi(\Omega)$. Such a mapping is also called *biholomorphic*. Then the transformation T_φ given by

$$T_\varphi f = (f \circ \varphi) \cdot \varphi' \tag{4.1}$$

constitutes an isometry of $L_2(\Omega^*)$ and $H_2(\Omega^*)$ onto $L_2(\Omega)$ and $H_2(\Omega)$, respectively. In particular, $(T_\varphi)^* = T_{\varphi^{-1}}$, $T_\varphi T_{\varphi^{-1}} = I_\Omega$ and $T_{\varphi^{-1}} T_\varphi = I_{\Omega^*}$. Also, as $G_{\Omega^*}(\varphi(z), \varphi(\zeta)) = G_\Omega(z, \zeta)$ for all $z, \zeta \in \Omega$, we have

$$L_\Omega(z, \zeta) = L_{\Omega^*}(\varphi(z), \varphi(\zeta))\varphi'(z)\varphi'(\zeta)$$

and clearly,

$$K_\Omega(z, \zeta) = K_{\Omega^*}(\varphi(z), \varphi(\zeta))\varphi'(z)\overline{\varphi'(\zeta)}. \tag{4.2}$$

In particular, using (3.4),

$$l_{\Omega^*}(\varphi(z), \varphi(\zeta))\varphi'(z)\varphi'(\zeta) = l_\Omega(z, \zeta) + S_\varphi(z, \zeta) \tag{4.3}$$

where

$$S_\varphi(z, \zeta) = \partial_z \partial_\zeta \log \frac{\varphi(z) - \varphi(\zeta)}{z - \zeta}, \tag{4.4}$$

and we note that

$$6S_\varphi(z, z) = \left(\frac{\varphi''}{\varphi'}\right)' - \frac{1}{2}\left(\frac{\varphi''}{\varphi'}\right)^2, \quad \varphi = \varphi(z), \tag{4.5}$$

is the familiar Schwarzian derivative.

Of particular importance is the condition guaranteeing that $B_\Omega = T_\varphi B_{\Omega^*} T_{\varphi^{-1}}$ for in that case the spectra $\sigma(\Omega)$ and $\sigma(\Omega^*)$ coincide.

THEOREM 3. *Let $\varphi\colon \Omega \to \Omega^*$ be a biholomorphic mapping of Ω onto Ω^*. Then $B_\Omega = T_\varphi B_{\Omega^*} T_{\varphi^{-1}}$ if and only if*

$$\|S_\varphi(\cdot, z)\|_\Omega^2 = -2\operatorname{Re}(l_\Omega(\cdot, z), S_\varphi(\cdot, z))_\Omega \tag{4.6}$$

for every $z \in \Omega$.

Proof. In view of (2.4), (3.1), (3.3) and (4.2), $B_\Omega = T_\varphi B_{\Omega^*} T_{\varphi^{-1}}$ is equivalent to $\Lambda_\Omega = T_\varphi \Lambda_{\Omega^*} T_{\varphi^{-1}}$ which in turn is equivalent to

$$\Gamma_\Omega(z, \zeta) = \Gamma_{\Omega^*}(\varphi(z), \varphi(\zeta))\varphi'(z)\overline{\varphi'(\zeta)}$$

for every $z, \zeta \in \Omega$. However, this is equivalent to

$$\Gamma_\Omega(z, z) = \Gamma_{\Omega^*}(\varphi(z), \varphi(z))|\varphi'(z)|^2$$

for every $z \in \Omega$. This condition, in view of (4.2), may also be written as

$$K_\Omega(z, z) - \Gamma_\Omega(z, z) = \{K_{\Omega^*}(\varphi(z), \varphi(z)) - \Gamma_{\Omega^*}(\varphi(z), \varphi(z))\}|\varphi'(z)|^2,$$

for every $z \in \Omega$, which means, in view of (3.7) and (4.3), that

$$\|l_\Omega(\cdot, z)\|_\Omega^2 = \|l_\Omega(\cdot, z) + S_\varphi(\cdot, z)\|_\Omega^2$$

for every $z \in \Omega$. The theorem follows now at once.

COROLLARY 1. *Let $\varphi\colon \Omega \to \Omega^*$ be a Möbius transformation of Ω onto Ω^*. Then $\sigma(\Omega) = \sigma(\Omega^*)$.*

Proof. In this case $S_\varphi(\cdot, z) \equiv 0$ for any $z \in \Omega$ and thus condition (4.6) is satisfied trivially.

The converse of this corollary does not hold in general. For example, let $\Omega = \hat{C} \setminus [0, 1]$ and $\Omega^* = \hat{C} \setminus \gamma$, where γ is a continuum which is not an arc of a circle or a segment of a straight line. Then there is a Riemann biholomorphic mapping $\varphi\colon \Omega \to \Omega^*$ of Ω onto Ω^*. Any such mapping is, of course, not a Möbius transformation. On the other hand $\sigma(\Omega) = \sigma(\Omega^*) = \{1\}$ by Theorem 2, since $m(C \setminus \Omega) = m(C \setminus \Omega^*) = 0$.

§ 5. **Norm inequalities.** The following norm inequalities are crucial:

THEOREM 4. *For every $f \in L_2(\Omega)$, we have*

$$\|A_\Omega f\|_\Omega \leqslant \sqrt{b}\,\|f\|_\Omega, \tag{5.1}$$

$$|(A_\Omega f, \bar{f})_\Omega| \leqslant \sqrt{b}\,\|P_\Omega f\|_\Omega^2, \tag{5.2}$$

where

$$b \equiv b(\Omega) = \|A_\Omega\|^2 = \|B_\Omega\|. \tag{5.3}$$

More generally,

$$|(A_\Omega f, \bar{g})_\Omega| \leqslant \sqrt{b}\,\|P_\Omega f\|_\Omega \|P_\Omega g\|_\Omega \tag{5.4}$$

for every $f, g \in L_2(\Omega)$.

Proof. The relation (5.3) follows at once from (2.5) and (3.8). Inequality (5.1) is easily deduced from Proposition 1 (ii). Indeed,

$$\|A_\Omega f\|_\Omega^2 = (B_\Omega f, f)_\Omega = \|B_\Omega^{1/2} f\|_\Omega^2 \leqslant \|B_\Omega\|\,\|f\|_\Omega^2.$$

We now prove (5.4). According to Proposition 1 (iii),

$$(A_\Omega f, \bar{g})_\Omega = (A_\Omega P_\Omega f, \bar{g})_\Omega = (P_\Omega f, A_\Omega^* \bar{g})_\Omega.$$

Therefore, using Proposition 1 (ii),

$$\begin{aligned}
|(A_\Omega f, \bar{g})_\Omega|^2 &\leqslant \|P_\Omega f\|_\Omega^2 \|A_\Omega^* \bar{g}\|_\Omega^2 = \|P_\Omega f\|_\Omega^2 (A_\Omega A_\Omega^* \bar{g}, \bar{g})_\Omega \\
&= \|P_\Omega f\|_\Omega^2 (A_\Omega^* A_\Omega g, g)_\Omega = \|P_\Omega f\|_\Omega^2 (B_\Omega g, g)_\Omega \\
&= \|P_\Omega f\|_\Omega^2 \|P_\Omega g\|_\Omega^2 (B_\Omega g, g)_\Omega / \|P_\Omega g\|_\Omega^2 \\
&\leqslant \|P_\Omega f\|_\Omega^2 \|P_\Omega g\|_\Omega^2 b.
\end{aligned}$$

The last inequality follows from

$$\operatorname*{Sup}_{g\in L_2(\Omega)} \frac{(B_\Omega g, g)_\Omega}{\|P_\Omega g\|_\Omega^2} = \operatorname*{Sup}_{h\in H_2(\Omega)} \frac{(B_\Omega h, h)_\Omega}{\|h\|_\Omega^2} = b,$$

because B_Ω is zero on $H_2(\Omega)^\perp$. This proves (5.4) and hence (5.2). This concludes the proof.

Let φ: $\Omega \to \Omega^*$ be a biholomorphic mapping of Ω onto $\Omega^* = \varphi(\Omega)$. We introduce the *Grunsky transform*

(5.5) $$G_\varphi \equiv T_{\varphi^{-1}}^t A_{\Omega^*} T_{\varphi^{-1}}$$

where $T_{\varphi^{-1}}^t$ is the transpose of the isometry $T_{\varphi^{-1}}$, as given in (4.1), of $L_2(\Omega)$ onto $L_2(\Omega^*)$. Clearly, G_φ is a linear operator on $L_2(\Omega)$ into itself, with $G_\varphi^t = G_\varphi$, and whose $L_2(\Omega)$-norm is

(5.6) $$\|G_\varphi\| = \|A_{\Omega^*}\| = \sqrt{\|B_{\Omega^*}\|} = \sqrt{b(\Omega^*)}.$$

Moreover, in view of (3.5) and (4.1)–(4.3), G_φ admits the alternative expression

(5.7) $$\{G_\varphi f\}(\zeta) = (f, l_\Omega(\cdot, \zeta) + S_\varphi(\cdot, \zeta))_\Omega, \quad \zeta \in \Omega,$$

for every $f \in L_2(\Omega)$. Under these circumstances, Theorem 4 has the following corollaries:

Corollary 2. *Let φ: $\Omega \to \Omega^*$ be a biholomorphic mapping of Ω onto Ω^*. Then, for every $f, g \in L_2(\Omega)$,*

(5.8) $$\|G_\varphi f\|_\Omega \leqslant \sqrt{b^*}\, \|f\|_\Omega,$$

(5.9) $$|(G_\varphi f, \bar{g})_\Omega| \leqslant \sqrt{b^*}\, \|P_\Omega f\|_\Omega \|P_\Omega g\|_\Omega,$$

where $b^* \equiv b(\Omega^*) = \|A_{\Omega^*}\|^2 = \|G_\varphi\|^2 = \|B_{\Omega^*}\|$.

Moreover, $G_\varphi = A_\Omega$ if and only if φ is a Möbius transformation.

Proof. By (5.1), (5.5) and (5.6),

$$\|G_\varphi f\|_\Omega = \|A_{\Omega^*} T_{\varphi^{-1}} f\|_{\Omega^*} \leqslant \sqrt{b^*}\, \|T_{\varphi^{-1}} f\|_{\Omega^*} = \sqrt{b^*}\, \|f\|_\Omega$$

and (5.8) follows. Similarly, using (5.4) and (5.5),

$$\begin{aligned} |(G_\varphi f, \bar{g})_\Omega| &= |(A_{\Omega^*} T_{\varphi^{-1}} f, \overline{T_{\varphi^{-1}} g})_{\Omega^*}| \\ &\leqslant \sqrt{b^*}\, \|P_{\Omega^*} T_{\varphi^{-1}} f\|_{\Omega^*} \|P_{\Omega^*} T_{\varphi^{-1}} g\|_{\Omega^*} \\ &= \sqrt{b^*}\, \|P_\Omega f\|_\Omega \|P_\Omega g\|_\Omega \end{aligned}$$

and (5.9) follows. Finally, from (3.5) and (5.7), we deduce that $G_\varphi = A_\Omega$ if and only if $S_\varphi(\cdot, \zeta) \equiv 0$ for every $\zeta \in \Omega$. But this is equivalent to φ being a Möbius transformation, and the proof is complete.

Corollary 3. *Let φ: $\Omega \to \Omega^*$ be a biholomorphic mapping of Ω onto Ω^*. Then, for every $f \in L_2(\Omega)$,*

(5.10) $$\|G_\varphi f\|_\Omega \leqslant \sqrt{b^*}\, \|P_\Omega f\|_\Omega,$$

an inequality sharper than (5.8).

Proof. This follows from (5.9) by letting $g = \overline{G_\varphi f}$.

Corollary 4. *Let φ: $\Omega \to \Omega^*$ be a biholomorphic mapping of Ω onto Ω^*. Then for any system of points $z_1, \ldots, z_n, \zeta_1, \ldots, \zeta_m$ of Ω and any corresponding scalars $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$ of C we have*

$$\Big|\sum_{i=1}^n \sum_{j=1}^m \alpha_i \beta_j [l_\Omega(z_i, \zeta_j) + S_\varphi(z_i, \zeta_j)]\Big|^2$$
$$\leqslant b^* \Big(\sum_{i,j=1}^n \alpha_i \bar{\alpha}_j K_\Omega(z_i, z_j)\Big)\Big(\sum_{i,j=1}^m \beta_i \bar{\beta}_j K_\Omega(\zeta_i, \zeta_j)\Big)$$

where $b^* = b(\Omega^*)$.

Proof. In (5.9), we set

$$f = \sum_{i=1}^n \bar{\alpha}_i K_\Omega(\cdot, z_i), \quad g = \sum_{j=1}^m \bar{\beta}_i K_\Omega(\cdot, \zeta_j).$$

Then

$$\overline{G_\varphi f} = \sum_{i=1}^n \alpha_i [l_\Omega(z_i, \cdot) + S_\varphi(z_i, \cdot)]$$

and

$$(G_\varphi f, \bar{g})_\Omega = (g, \overline{G_\varphi f})_\Omega = \sum_{i=1}^n \sum_{j=1}^m \bar{\alpha}_i \bar{\beta}_j [\overline{l_\Omega(z_i, \zeta_j)} + \overline{S_\varphi(z_i, \zeta_j)}].$$

Since, for example,

$$\|P_\Omega f\|_\Omega^2 = \sum_{i,j=1}^n \alpha_i \bar{\alpha}_j K_\Omega(z_i, z_j),$$

the corollary follows at once.

The matrix representations of the $L_2(\Omega)$-operators P_Ω and G_φ translate the $L_2(\Omega)$-norm inequalities (5.9) and (5.10) into l_2-norm inequalities of the type (1.2) and (1.3), mentioned in the introduction. Specifically, let φ: $\Omega \to \Omega^*$ be a biholomorphic mapping as before. Fix a point z_0 in Ω and define

$$r \equiv r(z_0; \Omega) = \operatorname{dist}(z_0, \partial\Omega).$$

Then the disk $\Delta_r \equiv \Delta_r(z_0) = \{z \in C: |z - z_0| < r\}$ is contained in Ω and the function $\log([\varphi(z) - \varphi(\zeta)]/(z - \zeta))$ is holomorphic on $\Delta_r \times \Delta_r$. We assume, without any loss of generality, that $z_0 = 0 \in \Omega$, and hence we may have the power series development

$$\log\frac{\varphi(z)-\varphi(\zeta)}{z-\zeta} = \sum_{m,n=0}^{\infty} \frac{c_{mn}}{r^{m+n}} z^m \zeta^n \tag{5.11}$$

on $\Delta_r \times \Delta_r$. In particular, by (4.4),

$$S_\varphi(z, \zeta) = \sum_{m,n=1}^{\infty} mn \frac{c_{mn}}{r^{m+n}} z^{m-1} \zeta^{n-1}, \quad z, \zeta \in \Delta_r. \tag{5.12}$$

Similarly, since $l_\Omega(\cdot, \cdot)$ and $K_\Omega(\cdot, \cdot)$ are holomorphic and sesqui-holomorphic, respectively, on $\Omega \times \Omega$, we have

$$l_\Omega(z, \zeta) = \sum_{m,n=1}^{\infty} mn \frac{l_{mn}}{r^{m+n}} z^{m-1} \zeta^{n-1}, \quad z, \zeta \in \Delta_r; \tag{5.13}$$

$$K_\Omega(z, \zeta) = \sum_{m,n=1}^{\infty} \sqrt{mn} \frac{k_{mn}}{r^{m+n}} z^{m-1} \bar{\zeta}^{n-1}, \quad z, \zeta \in \Delta_r. \tag{5.14}$$

Evidently,

$$c_{mn} = c_{nm}, \quad l_{mn} = l_{nm}, \quad k_{mn} = \bar{k}_{nm}.$$

The hermitian matrix $\mathscr{K} = (k_{mn})$ is known as the *Bergman operator* while the symmetric matrix $\mathscr{G}_\varphi = (g_{mn})$ with $g_{mn} = \sqrt{mn}(c_{mn}+l_{mn})$ is called the *Grunsky operator*.

We now apply Corollary 2 by choosing $f \in L_2(\Omega)$ to be of the form

$$f = f_N \chi_{\Delta_r}, \quad f_N(z) = \sum_{n=1}^{N} \alpha_n \frac{\sqrt{n}}{r^n} z^{n-1}$$

with scalars $\alpha_1, \ldots, \alpha_N$. Similarly, we let $g = g_N \chi_{\Delta_r}$ with g_N being as f_N with the scalars $\alpha_1, \ldots, \alpha_N$ replaced by $\beta_1, \ldots, \beta_N$. Then

$$\|f\|_\Omega^2 = \sum_{n=1}^{N} |\alpha_n|^2, \quad \|g\|_\Omega^2 = \sum_{n=1}^{N} |\beta_n|^2.$$

Moreover,

$$(P_\Omega f, g)_\Omega = \sum_{n,m=1}^{N} k_{nm} \alpha_m \bar{\beta}_n \quad \text{and} \quad (G_\varphi f, \bar{g})_\Omega = \sum_{n,m=1}^{N} \bar{g}_{nm} \alpha_m \beta_n.$$

Since $\|P_\Omega f\|_\Omega^2 = (P_\Omega f, f)_\Omega$, Corollary 2 gives

$$\Big| \sum_{n,m=1}^{N} \bar{g}_{nm} \alpha_m \beta_n \Big|^2 \leqslant b^* \Big(\sum_{n,m=1}^{N} k_{nm} \alpha_m \bar{\alpha}_n \Big) \Big(\sum_{n,m=1}^{N} k_{nm} \beta_m \bar{\beta}_n \Big)$$

and, of course,

$$\sum_{n,m=1}^{N} k_{nm} \alpha_m \bar{\alpha}_n \leqslant \sum_{n=1}^{N} |\alpha_n|^2.$$

This permits the translation of Corollaries 2 and 3 into the l_2-setting. Accordingly, we consider vectors $\alpha = (\alpha_1, \alpha_2, \ldots)$ of l_2 with the norm

$$\|\alpha\|_2 = \Big\{ \sum_{n=1}^{\infty} |\alpha_n|^2 \Big\}^{1/2}$$

and we view l_2 as the domain of definition of both $\mathscr{K}$ and $\mathscr{G}_\varphi$. This leads to the following theorem, the proof of which is contained already in the foregoing discussion.

THEOREM 5. *The Bergman operator $\mathscr{K}$ is a bounded hermitian projection operator on l_2, i.e. $\mathscr{K} = \mathscr{K}^* = \mathscr{K}^2$ and*

$$(\mathscr{K}\alpha, \alpha)_2 = \|\mathscr{K}\alpha\|_2^2 \leqslant \|\alpha\|_2^2, \quad \alpha \in l_2.$$

Let $\varphi: \Omega \to \Omega^*$ be a biholomorphic mapping of Ω onto Ω^*. Then the Grunsky operator $\mathscr{G}_\varphi$ is a bounded symmetric operator on l_2 with

$$\|\mathscr{G}_\varphi \alpha\|_2 \leqslant \sqrt{b^*} \|\mathscr{K}\alpha\|_2 \leqslant \sqrt{b^*} \|\alpha\|_2, \quad \alpha \in l_2, \tag{5.15}$$

where $b^* = b(\Omega^*)$. Moreover,

$$|(\mathscr{G}_\varphi \alpha, \bar{\beta})_2|^2 \leqslant b^* (\mathscr{K}\bar{\alpha}, \bar{\alpha})_2 (\mathscr{K}\bar{\beta}, \bar{\beta})_2, \quad \alpha, \beta \in l_2, \tag{5.16}$$

and, in particular,

$$|(\mathscr{G}_\varphi \alpha, \bar{\alpha})_2| \leqslant \sqrt{b^*} \|\mathscr{K}\bar{\alpha}\|_2^2, \quad \alpha \in l_2. \tag{5.17}$$

When Ω is the unit disk Δ, we have $K_\Delta(z, \zeta) = (1-z\bar{\zeta})^{-2}$ and $l_\Delta(z, \zeta) \equiv 0$ for every $z, \zeta \in \Delta$. It follows from (5.12)–(5.13), as $r = 1$, that $l_{mn} = 0$ and $\mathscr{K} = (k_{mn}) = I$, the identity operator of l_2. Let $\varphi: \Delta \to \Omega^*$ be a biholomorphic mapping of Δ onto $\Omega^* = \varphi(\Delta)$. Then $\mathscr{G}_\varphi = (g_{mn})$ with $g_{mn} = \sqrt{mn}\, c_{mn}$ and by the last theorem, $|(\mathscr{G}_\varphi \alpha, \bar{\alpha})_2| \leqslant \sqrt{b^*} \|\alpha\|_2^2$ and $\|\mathscr{G}_\varphi \alpha\|_2 \leqslant \sqrt{b^*} \|\alpha\|_2$ for all $\alpha \in l_2$. If, in addition, $\partial\Omega$ is of class $C^{2,\varepsilon}$ $(0 < \varepsilon < 1)$ then, as Ω^* is simply connected, $\lambda_1^* = \lambda_1(\Omega^*) = 1/\sqrt{b^*} > 1$, and we recover the inequalities (1.1)–(1.3) mentioned in the introduction.

We note that in view of the last theorem, $\mathscr{K}^* = \mathscr{K}$, $\mathscr{G}_\varphi^* = \mathscr{G}_\varphi$, $\|\mathscr{K}\| = 1$ and $\|\mathscr{G}_\varphi\| \leqslant \sqrt{b^*} \leqslant 1$. We also note that condition (5.16) is equivalent to

$$|(\mathscr{G}_\varphi \alpha, \beta)_2|^2 \leqslant b^* (\bar{\mathscr{K}}\alpha, \alpha)_2 (\mathscr{K}\beta, \beta)_2, \quad \alpha, \beta \in l_2. \tag{5.18}$$

This condition is completely equivalent to the statement that the matrix operator

$$\mathscr{A} = \begin{bmatrix} \sqrt{b^*}\, \bar{\mathscr{K}} & \bar{\mathscr{G}}_\varphi \\ \mathscr{G}_\varphi & \sqrt{b^*}\, \mathscr{K} \end{bmatrix}$$

is a positive operator on $l_2 \times l_2$ (of course, $\mathscr{A}$ is bounded on $l_2 \times l_2$). Indeed, we write the vectors γ of $l_2 \times l_2$ in column form with the norm $\|\gamma\|_{2\times 2}$

$=\sqrt{\gamma^*\gamma}$; $\gamma=[\alpha,\beta]^t$, $\alpha,\beta\in l_2$. Then

$$\gamma^*\mathscr{A}\gamma=\sqrt{b^*}(\mathscr{K}\alpha,\alpha)_2+2\mathrm{Re}(\mathscr{G}_\varphi\alpha,\beta)_2+\sqrt{b^*}(\mathscr{K}\beta,\beta)_2.$$

From (5.18) we deduce that

$$\gamma^*\mathscr{A}\gamma\geqslant\sqrt{b^*}\{\sqrt{(\mathscr{K}\alpha,\alpha)_2}-\sqrt{(\mathscr{K}\beta,\beta)_2}\}^2$$

and hence $\mathscr{A}$ is positive on $l_2\times l_2$. Conversely, if $\mathscr{A}$ is positive on $l_2\times l_2$ then

$$\sqrt{b^*}(\mathscr{K}\alpha,\alpha)_2+2\mathrm{Re}(\mathscr{G}_\varphi\alpha,\beta)_2+\sqrt{b^*}(\mathscr{K}\beta,\beta)_2\geqslant 0$$

for every $\alpha,\beta\in l_2$. Replacing α by $x\alpha$, where x is an arbitrary real number, we obtain

$$\sqrt{b^*}(\mathscr{K}\alpha,\alpha)_2x^2+2[\mathrm{Re}(\mathscr{G}_\varphi\alpha,\beta)_2]x+\sqrt{b^*}(\mathscr{K}\beta,\beta)_2\geqslant 0$$

and hence

$$[\mathrm{Re}(\mathscr{G}_\varphi\alpha,\beta)_2]^2\leqslant b^*(\mathscr{K}\alpha,\alpha)_2(\mathscr{K}\beta,\beta)_2,\qquad \alpha,\beta\in l_2,$$

which, evidently, is equivalent to (5.18). Similarly, (5.17) is equivalent to (5.18).

The weaker form of inequality (5.17), namely

(5.19) $$|(\mathscr{G}_\varphi\alpha,\bar{\alpha})_2|\leqslant(\mathscr{K}\alpha,\alpha)_2,\qquad \alpha\in l_2,$$

is an example of symmetric-hermitian quadratic inequalities that occur in the literature (see, for example, [2, 5]) concerned with holomorphic or biholomorphic continuation. Assume, without loss of generality, that $0\in\Omega$ and that φ is holomorphic in the neighborhood of 0 with $\varphi'(0)\neq 0$. Then there exists a small disk $\Delta_r=\{z\in C: |z|<r\}$, contained in Ω, such that φ is biholomorphic on Δ_r. With this disk the developments in (5.11)–(5.14) are still in force, and thus the operators $\mathscr{K}$ and $\mathscr{G}_\varphi$ are well defined. If, in addition, (5.19) is satisfied, then in view of [2, p. 240] the function $l_\Omega(\cdot,\cdot)+$ $+S_\varphi(\cdot,\cdot)$, and hence also the function $S_\varphi(\cdot,\cdot)$, is holomorphic on all of $\Omega\times\Omega$. This shows that φ is biholomorphic on all Ω.

§ 6. The reduced spectrum. The spectrum of B_Ω on $L_2(\Omega)$, $\sigma(B_\Omega: L_2(\Omega))$, is a closed subset of [0, 1] and it contains 0. It may contain the point 1, too. By restricting B_Ω on $H_2(\Omega)$, the spectrum was made "thinner" at 0. Indeed, $\sigma(B_\Omega: L_2(\Omega))=\sigma(\Omega)\cup\{0\}$ where $\sigma(\Omega)\equiv\sigma(B_\Omega: H_2(\Omega))$. We now apply another restriction on B_Ω which at this time will correspond to "thinning" the spectrum at 1.

Let $H_2^{(s)}(\Omega)$ be the subspace of $H_2(\Omega)$ consisting of all functions in $H_2(\Omega)$ with single-valued indefinite integrals in Ω. This gives the direct sum decomposition $L_2(\Omega)=H_2^{(s)}(\Omega)\oplus H_2^{(s)}(\Omega)^\perp$ with $H_2(\Omega)=H_2^{(s)}(\Omega)$ if and only if either (i) $\Omega\in 0_G$ or (ii) Ω is conformally equivalent to the unit disk less (possibly) a closed subset of zero inner capacity. As mentioned before, when Ω is a regular domain of connectivity p $(1\leqslant p<\infty)$, $N_2^{(s)}(\Omega)$ $=H_2(\Omega)\ominus H_2^{(s)}(\Omega)$ is a finite-dimensional Hilbert space of dimension $p-1$ on which B_Ω reduces to the identity operator when $p>1$. We shall show, more generally, that for any domain Ω, B_Ω reduces to the identity operator on $N_2^{(s)}(\Omega)$, and we may, without any essential loss, restrict B_Ω to $H_2^{(s)}(\Omega)$.

On $H_2^{(s)}(\Omega)$ we have the *reduced Bergman kernel* $K_\Omega^{(s)}(\cdot,\cdot)$ and the *reduced Bergman projector* $P_\Omega^{(s)}$ given by

$$\{P_\Omega^{(s)}f\}(\zeta)=(f,K_\Omega^{(s)}(\cdot,\zeta)_\Omega).$$

The theory for the space $H_2^{(s)}(\Omega)$ proceeds along lines similar to those in $H_2(\Omega)$. There are, however, some significant changes for which we provide a brief description.

We say that $\Omega\in 0_{AD}$ if $H_2^{(s)}(\Omega)=\{0\}$, and a compact set E in C is said to be of class N_D if $C\setminus E\in 0_{AD}$. The domain Ω is said to belong to class $\mathscr{N}_D$ if Ω is conformally equivalent to $\Delta\setminus E$, where E is a set satisfying $E\cap K\in N_D$ for every compact subset K of the unit disk Δ. The subclass $\mathscr{N}_D^d$ of $\mathscr{N}_D$ consists of those domains Ω which are Möbius images of $\Delta\setminus E$, with $E\cap K\in N_D$ for every compact subset K of Δ. The subset of Ω consisting of all points $z\in\Omega$ such that $K_\Omega^{(s)}(z,z)=0$ is denoted by N_Ω. Clearly, $N_\Omega=\Omega$ if and only if $\Omega\in 0_{AD}$. If $\Omega\notin 0_{AD}$ then N_Ω is a closed discrete subset of Ω which is empty when Ω is of finite connectivity. For a fixed $\zeta\in\Omega$, assumed, without loss of generality, to be $\zeta\neq\infty$, we consider the class $U(\Omega:\zeta)$ of all univalent meromorphic functions f in Ω which are normalized by

(6.1) $$f(z)=(z-\zeta)^{-1}+a(z-\zeta)+\ldots$$

about ζ with

(6.2) $$a\equiv a_f(\zeta)=\lim_{z\to\zeta}\{f'(z)+(z-\zeta)^{-2}\}.$$

The *parallel* and *vertical slit mappings* of Ω, with respect to $\zeta\in\Omega$, are the unique functions $p_\Omega(\cdot:\zeta)$ and $q_\Omega(\cdot:\zeta)$ with the largest and smallest, respectively, real parts of the coefficient $a_f(\zeta)$ among all functions f of $U(\Omega:\zeta)$. Introduce the functions

(6.3) $$\Phi(z)\equiv\Phi_\Omega(z:\zeta)=\tfrac{1}{2}[p_\Omega(z:\zeta)-q_\Omega(z:\zeta)],$$

(6.4) $$\Psi(z)\equiv\Psi_\Omega(z:\zeta)=\tfrac{1}{2}[p_\Omega(z:\zeta)+q_\Omega(z:\zeta)].$$

As is well known, Ψ is the unique function with the largest outer area $A_e(\Omega:\zeta)$ among all functions in $U(\Omega:\zeta)$, that is

$$A_e(\Omega:\zeta)=m(\hat{C}\setminus\Psi(\Omega))=\max\{m(\hat{C}\setminus f(\Omega)): f\in U(\Omega:\zeta)\}.$$

Also, $\Psi(\Omega)$ is a canonical domain for all the domains Ω^* which are obtained from Ω by means of biholomorphic functions φ with $\varphi(\zeta)=\zeta$, $\varphi'(\zeta)=1$. It is also well known that

(6.5) $$K_{\Omega}^{(s)}(z,\zeta) = \Phi'_{\Omega}(z:\zeta)$$

and its "adjoint" is given by

(6.6) $$L_{\Omega}^{(s)}(z,\zeta) = -\Psi'_{\Omega}(z:\zeta).$$

It follows that

(6.7) $$L_{\Omega}^{(s)}(z,\zeta) = (z-\zeta)^{-2} - l_{\Omega}^{(s)}(z,\zeta)$$

where $l_{\Omega}^{(s)}(\cdot,\cdot)$ is symmetric and holomorphic on $\Omega\times\Omega$. The following relations are also well known:

$$\|\Phi'_{\Omega}(\cdot:\zeta)\|_{\Omega}^2 = K_{\Omega}^{(s)}(\zeta,\zeta) = \frac{1}{\pi}A_e(\Omega:\zeta),$$

and $\Phi_{\Omega}(\cdot:\zeta)/\Phi'_{\Omega}(\zeta:\zeta)$ is the unique function with the smallest inner area $A_i(\Omega:\zeta)$ among all holomorphic functions f on $\Omega\notin 0_{AD}$ for which $f'(\zeta)=1$. Moreover,

$$A_i(\Omega:\zeta)A_e(\Omega:\zeta) \equiv \pi^2$$

(cf. Sario and Oikawa [11, pp. 125–144]).

We may now introduce the *reduced Bergman–Schiffer transforms*

$$\{Q_{\Omega}^{(s)}f\}(\zeta) = (f, L_{\Omega}^{(s)}(\cdot,\zeta))_{\Omega} \quad\text{and}\quad \{A_{\Omega}^{(s)}f\}(\zeta) = (f, l_{\Omega}^{(s)}(\cdot,\zeta))_{\Omega}.$$

When Ω is a regular domain, we have

$$\overline{K_{\Omega}^{(s)}(z,\zeta)\,dz} = -L_{\Omega}^{(s)}(z,\zeta)\,dz, \qquad z\in\partial\Omega$$

and hence, by Green's formula,

(6.8) $$(l_{\Omega}^{(s)}(\cdot,z), l_{\Omega}^{(s)}(\cdot,\zeta))_{\Omega} = K_{\Omega}^{(s)}(z,\zeta) - \Gamma_{\Omega}(z,\zeta).$$

With these relations, Proposition 1 may now be supplemented by the following

PROPOSITION 2. *On $L_2(\Omega)$ we have*

(i) $\Lambda_{\Omega} = \Lambda_{\Omega}P_{\Omega}^{(s)}$,

(ii) $A_{\Omega}^{(s)*}A_{\Omega}^{(s)} = P_{\Omega}^{(s)} - \Lambda_{\Omega}$,

(iii) $A_{\Omega}^{(s)} = T_{\Omega} - Q_{\Omega}^{(s)} = T_{\Omega}P_{\Omega}^{(s)} = A_{\Omega}^{(s)}P_{\Omega}^{(s)}$,

(iv) $I_{\Omega} - P_{\Omega}^{(s)} = Q_{\Omega}^{(s)*}Q_{\Omega}^{(s)} = T_{\Omega}^{*}Q_{\Omega}^{(s)}$,

(v) $Q_{\Omega}^{(s)}P_{\Omega}^{(s)} = A_{\Omega}^{(s)*}Q_{\Omega}^{(s)} = 0, \quad P_{\Omega}^{(s)2} = P_{\Omega}^{(s)} = P_{\Omega}P_{\Omega}^{(s)}$.

From this proposition it follows that the *reduced Fredholm transform*

$$B_{\Omega}^{(s)} = P_{\Omega}^{(s)} - \Lambda_{\Omega}$$

has all the previously mentioned properties of the Fredholm transform B_{Ω}. In fact

(6.9) $$B_{\Omega} = B_{\Omega}^{(s)} + (P_{\Omega} - P_{\Omega}^{(s)})$$

and the obvious positivity relations

$$I_{\Omega} \geqslant B_{\Omega} \geqslant B_{\Omega}^{(s)} \geqslant 0$$

hold. We also deduce, using Propositions 1 and 2, that

$$B_{\Omega}(P_{\Omega}-P_{\Omega}^{(s)}) = (P_{\Omega}-\Lambda_{\Omega})(P_{\Omega}-P_{\Omega}^{(s)}) = P_{\Omega}^2 - P_{\Omega}P_{\Omega}^{(s)} - \Lambda_{\Omega}P_{\Omega} + \Lambda_{\Omega}P_{\Omega}^{(s)}$$
$$= P_{\Omega} - P_{\Omega}^{(s)} - \Lambda_{\Omega} + \Lambda_{\Omega},$$

and hence

$$B_{\Omega}(P_{\Omega}-P_{\Omega}^{(s)}) = P_{\Omega} - P_{\Omega}^{(s)}$$

which means that B_{Ω} reduces to the identity operator on $N_2^{(s)}(\Omega)$. We may therefore restrict B_{Ω} to $H_2^{(s)}(\Omega)$, in which case B_{Ω} reduces to $B_{\Omega}^{(s)}$ by virtue of (6.9). In particular, $B_{\Omega}^{(s)}$ is trivial on $H_2^{(s)}(\Omega)^{\perp}$ and the spectrum $\sigma(B_{\Omega}^{(s)}:L_2(\Omega))$ is $\sigma^{(s)}(\Omega)\cup\{0\}$ with $\sigma^{(s)}(\Omega)\equiv\sigma(B_{\Omega}^{(s)}:H_2^{(s)}(\Omega))$. We also note the relation $\sigma(\Omega) = \sigma^{(s)}(\Omega)\cup\{1\}$ if $N_2^{(s)}(\Omega)\neq\{0\}$. It is now clear that almost all theorems established previously for the space $H_2(\Omega)$ are also valid, with minor modifications involving the insertion of the superscript "s", for the space $H_2^{(s)}(\Omega)$. This includes the *reduced spectrum* $\sigma^{(s)}(\Omega)$ and its extreme points $a^{(s)}(\Omega)$ and $b^{(s)}(\Omega)$. Here

$$0 \leqslant a(\Omega) \leqslant a^{(s)}(\Omega) \leqslant b^{(s)}(\Omega) \leqslant b(\Omega) \leqslant 1$$

with

$$a^{(s)}(\Omega) = \|B_{\Omega}^{(s)}\| = \|A_{\Omega}^{(s)}\|^2$$

where the operator norms are taken over $H_2^{(s)}(\Omega)$, and, again, they coincide with the full operator norms over $L_2(\Omega)$. In particular, when $\sigma^{(s)}(\Omega)$ reduces to only point spectrum we have an orthonormal basis $\{\psi_k\}$ of eigenfunctions in $H_2^{(s)}(\Omega)$ of $B_{\Omega}^{(s)}$ and corresponding eigenvalues $\{\mu_k^{(s)}\} = \{[\lambda_k^{(s)}]^{-2}\}$ with

$$1 \geqslant b^{(s)}(\Omega) = \mu_1^{(s)} \geqslant \mu_2^{(s)} \geqslant \ldots \geqslant \mu_n^{(s)} \geqslant \mu_{n+1}^{(s)} \geqslant \ldots \geqslant \mu_{\infty}^{(s)} = a^{(s)}(\Omega) \geqslant 0.$$

Moreover, if $\partial\Omega$ is of class $C^{2,\varepsilon}$ $(0<\varepsilon<1)$ and Ω is of connectivity p $(1\leqslant p<\infty)$ then $\lambda_1 = \ldots = \lambda_{p-1} = 1$ and $\lambda_{k+p-1} = \lambda_k^{(s)} > 1$ for $k = 1, 2, \ldots$

Theorems 2, 3, 4, 5 and their corollaries are now also valid, with the obvious modifications, in our present setting of $H_2^{(s)}(\Omega)$. The analogue for Theorem 1, however, is different and requires the introduction of some other concepts. The domain Ω is said to be *canonical with respect to* ζ if $\zeta\in\Omega$ and $l_{\Omega}^{(s)}(z,\zeta) = 0$ for every $z\in\Omega$. In view of (6.6) and (6.7), this condition is equivalent to $-\Psi'_\Omega(z:\zeta) = L^{(s)}_\Omega(z,\zeta) = (z-\zeta)^{-2}$, and since the function $F(z:\zeta) \equiv (z-\zeta)\Psi_\Omega(z:\zeta)$ is holomorphic with $F(\zeta:\zeta) = 1$ and $F'(\zeta:\zeta) = 0$, the condition is equivalent to $\Psi_\Omega(z:\zeta) = (z-\zeta)^{-1}$. The obvious modification of the relation (4.3) leads to the following proposition:

PROPOSITION 3. *If Ω is canonical with respect to ζ and φ is a Möbius transformation then $\varphi(\Omega)$ is canonical with respect to $\varphi(\zeta)$. Conversely, if $\varphi\colon \Omega \to \Omega^*$ is a biholomorphic mapping of Ω onto $\Omega^* = \varphi(\Omega)$, and Ω and Ω^* are canonical with respect to ζ and $\varphi(\zeta)$, respectively, then φ is a Möbius transformation. In particular, a domain $\Omega \notin 0_{AD}$ is canonical with respect to any point $\zeta \in \Omega$ if and only if Ω is of class $\mathcal{N}^{\mathrm{d}}_{\mathrm{D}}$.*

For a fixed $\zeta \in \Omega$ $(\zeta \neq \infty)$, the function

$$\psi_\Omega(z:\zeta) = \zeta + \frac{1}{\Psi_\Omega(z:\zeta)}, \quad z \in \Omega$$

maps conformally Ω onto a domain $D \equiv \psi_\Omega(\Omega:\zeta)$ with $\zeta \in D$ and $\psi_\Omega(\zeta:\zeta) = \zeta$, $\psi'_\Omega(\zeta:\zeta) = 1$. In particular, the Schwarzian operator, introduced in (4.4), for the function $\varphi(z) \equiv \psi_\Omega(z:\zeta)$, $z \in \Omega$, admits the form

$$S_\varphi(z,\zeta) = -\Psi'_\Omega(z:\zeta) - (z-\zeta)^{-2}$$

and thus, in view of (6.6)–(6.7),

$$S_\varphi(z,\zeta) = -l^{(s)}_\Omega(z,\zeta), \quad z \in \Omega.$$

Moreover, for $v = \varphi(z) = \psi_\Omega(z:\zeta)$, we have

$$p_D(v:\zeta) = p_\Omega(z:\zeta), \quad q_D(v:\zeta) = q_\Omega(z:\zeta)$$

and hence, by (6.3)–(6.4),

$$\Phi_D(v:\zeta) = \Phi_\Omega(z:\zeta), \quad \Psi_D(v:\zeta) = \Psi_\Omega(z:\zeta).$$

In particular, $\Psi_D(v:\zeta) = (v-\zeta)^{-1}$, $v \in D$, which means that D is canonical with respect to $\zeta \in D$. This, with Proposition 3, gives:

PROPOSITION 4. *Let $\zeta \in \Omega$ and τ be an arbitrary point in the plane. Then there exists a canonical domain Ω^* with respect to τ which is conformally equivalent to Ω with $\tau \in \Omega^*$ corresponding to $\zeta \in \Omega$. Moreover, such a correspondence $g\colon \Omega \to \Omega^*$ may be exhibited via $g = h \circ \varphi$ where h is a Möbius transformation with $h(\zeta) = \tau$ and $\varphi(z) = \psi_\Omega(z:\zeta)$, $z \in \Omega$.*

We now prove:

PROPOSITION 5. *Assume that the spectrum $\sigma^{(s)}(\Omega)$ of $\Omega \notin 0_{AD}$ reduces to only point spectrum $\{\mu^{(s)}_k\}$. Then Ω is canonical with respect to ζ if and only if all eigenfunctions ψ_n corresponding to positive eigenvalues $\mu^{(s)}_n > 0$ vanish at $\zeta \in \Omega$. In that case, if also $\zeta \in \Omega \setminus N_\Omega$, then there exists at least one zero eigenvalue, i.e. $\mu^{(s)}_\infty = d^{(s)}(\Omega) = 0$. Here $\{\psi_k\}$ is an orthonormal basis of eigenfunctions in $H^{(s)}_2(\Omega)$ of $B^{(s)}_\Omega$, corresponding to $\{\mu^{(s)}_k\}$.*

Proof. For a fixed $\zeta \in \Omega$, we have

$$K^{(s)}_\Omega(z,\zeta) - \Gamma_\Omega(z,\zeta) = \sum_{k=1}^{\infty} \mu^{(s)}_k \psi_k(z)\overline{\psi_k(\zeta)}.$$

In view of (6.8), Ω is canonical with respect to ζ if and only if $K^{(s)}_\Omega(z,\zeta) = \Gamma_\Omega(z,\zeta)$ for every $z \in \Omega$. The latter is equivalent to $K^{(s)}_\Omega(\zeta,\zeta) = \Gamma_\Omega(\zeta,\zeta)$ which means

$$\sum_{k=1}^{\infty} \mu^{(s)}_k |\psi_k(\zeta)|^2 = 0.$$

This is equivalent to $\psi_n(\zeta) = 0$ for all $\mu^{(s)}_n > 0$ and the first part of the proposition follows. For the second part, if also $\zeta \in \Omega \setminus N_\Omega$ then $K^{(s)}_\Omega(\zeta,\zeta) > 0$ or $\sum_{k=1}^{\infty} |\psi_k(\zeta)|^2 > 0$ which means that there exists at least one eigenfunction ψ_n with $\psi_n(\zeta) \neq 0$ and hence some $\mu^{(s)}_n = 0$. In particular $\mu^{(s)}_\infty = d^{(s)}(\Omega) = 0$, concluding the proof.

The following theorem is the analogue of Theorem 1 in the $H^{(s)}_2(\Omega)$-setting. In this theorem, the implications (1)⇔(7) are due to Suita [16] (see also Ozawa [8]) and (7)⇔(8)⇔(10) are due to Sakai [10].

THEOREM 6. *The following statements are equivalent:*

(1) $\sigma^{(s)}(\Omega) = \{0\}$; (2) $B^{(s)}_\Omega = 0$ on $H^{(s)}_2(\Omega)$; (3) $\Lambda_\Omega = I_\Omega$ on $H^{(s)}_2(\Omega)$; (4) $\Gamma_\Omega(z,\zeta) = K^{(s)}_\Omega(z,\zeta)$ for every $z, \zeta \in \Omega$; (5) Ω is canonical with respect to every $\zeta \in \Omega$; (6) $l^{(s)}_\Omega(z,\zeta) = 0$ for every $z, \zeta \in \Omega$; (7) $\Psi_\Omega(\cdot:\zeta)$ is linear for every $\zeta \in \Omega$; (8) $\Phi_\Omega(\cdot:\zeta)$ is linear for some $\zeta \in \Omega$; (9) either $\Omega \in 0_{AD}$ or $\Omega \in \mathcal{N}_{\mathrm{D}}$ and is canonical with respect to some point $\zeta \in \Omega$; (10) either $\Omega \in 0_{AD}$ or $\Omega \in \mathcal{N}^{\mathrm{d}}_{\mathrm{D}}$.

Proof. The implications (1)⇒(2)⇒(3)⇒(4)⇒(5)⇒(6)⇒(7) are straightforward while (7)⇔(8)⇔(10)⇔(1) are the above mentioned results of Ozawa, Sakai and Suita [8, 10, 16]. The implication (10)⇒(9) is also straightforward. We now prove (9)⇒(10). Let $\Omega \notin 0_{AD}$. Then $\Omega \in \mathcal{N}_{\mathrm{D}}$ and there exists a biholomorphic mapping φ of Ω onto $\varphi(\Omega) = \Delta \setminus E$, where E is a set satisfying $E \cap K \in N_{\mathrm{D}}$ for every compact subset K of the unit disk Δ. Clearly, $l^{(s)}_{\Delta\setminus E}(\omega,\tau) = 0$ for every $\omega, \tau \in \Delta \setminus E = \varphi(\Omega)$. However, by the analogue of (4.3),

$$l^{(s)}_{\varphi(\Omega)}(\varphi(z),\varphi(\zeta))\varphi'(z)\varphi'(\zeta) = l^{(s)}_\Omega(z,\zeta) + S_\varphi(z,\zeta)$$

and thus $S_\varphi(z,\zeta) = 0$ for every $z \in \Omega$. This condition is equivalent to φ being a Möbius transformation and thus $\Omega \in \mathcal{N}^{\mathrm{d}}_{\mathrm{D}}$. The proof is now complete.

§ 7. Quasi-conformal mappings. Let φ be a homeomorphism of the complex plane which is conformal on Ω and $\varkappa$-quasi-conformal on its complement $\hat{C}\setminus\bar{\Omega}$, that is $\varphi \leqslant \varkappa < 1$ and $|\bar{\partial}\varphi| \leqslant \varkappa|\partial\varphi|$ on $\hat{C}\setminus\bar{\Omega}$, where the partial derivatives are taken in the distributional sense. We write

$$b \equiv b^{(s)}(\Omega), \qquad b^* = b^{(s)}(\Omega^*), \qquad \Omega^* = \varphi(\Omega).$$

According to a result of Springer [15]

$$\frac{1+\sqrt{b}}{1-\sqrt{b}}\,\frac{1-\varkappa}{1+\varkappa} \leqslant \frac{1+\sqrt{b^*}}{1-\sqrt{b^*}} \leqslant \frac{1+\sqrt{b}}{1-\sqrt{b}}\,\frac{1+\varkappa}{1-\varkappa}$$

in which case $b < 1$ if and only if $b^* < 1$. It follows that

$$\frac{\sqrt{b}-\varkappa}{1-\varkappa\sqrt{b}} \leqslant \sqrt{b^*} \leqslant \frac{\sqrt{b}+\varkappa}{1+\varkappa\sqrt{b}}. \tag{7.1}$$

The right-hand side of this inequality may therefore replace $\sqrt{b^*}$ in the $H_2^{(s)}(\Omega)$-setting of Corollaries 2, 3, 4 and Theorem 5. In particular, using the $H_2^{(s)}(\Omega)$-version of (5.17), we have

$$|(\mathscr{G}_\varphi \alpha, \bar{\alpha})_2| \leqslant \frac{\sqrt{b}+\varkappa}{1+\varkappa\sqrt{b}} \|\mathscr{K}\bar{\alpha}\|_2^2, \qquad \alpha \in l_2. \tag{7.2}$$

A special case of this very general result is when $\Omega \in \mathscr{N}_{\mathrm{D}}^{\mathrm{d}}$. In that case, in view of Theorem 6, $\sigma^{(s)}(\Omega) = \{0\}$ which means $b = b^{(s)}(\Omega) = 0$ and thus

$$|(\mathscr{G}_\varphi \alpha, \bar{\alpha})_2| \leqslant \varkappa \|\mathscr{K}\bar{\alpha}\|_2^2 \leqslant \varkappa \|\alpha\|_2^2, \qquad \alpha \in l_2.$$

A particular case of this result, namely when Ω is the unit disk Δ, was first proved by Kühnau [7] (see also Schiffer and Schober [14] and Schiffer [13]). We note that in this case $\mathscr{K}$ is the identity matrix and $\mathscr{G}_\varphi = (g_{mn})$ where $g_{mn} = \sqrt{mn}\, c_{mn}$ with c_{mn} being the Grunsky coefficients of φ. The result of Kühnau is then

$$\Big|\sum_{m,n=1}^{\infty} \sqrt{mn}\, c_{mn} \alpha_m \alpha_n\Big| \leqslant \varkappa \sum_{n=1}^{\infty} |\alpha_n|^2.$$

Inequality (7.2) is a considerable extension of it to general domains Ω.

Let $\mathscr{D}$ be the class of domains Ω for which $b = b^{(s)}(\Omega) < 1$. Another result of Springer [15] states that if $\Omega \in \mathscr{D}$ and $\varphi\colon \Omega \to \Omega^*$ is a conformal mapping of Ω onto $\Omega^* = \varphi(\Omega)$, then φ has a $\varkappa$-quasi-conformal $(0 \leqslant \varkappa < 1)$ homeomorphic extension to $\boldsymbol{C}$ if and only if $\Omega^* \in \mathscr{D}$. Now, the expression $(\sqrt{b}+\varkappa)/(1+\varkappa\sqrt{b})$ in (7.1) is always dominated by 1 and it is equal to 1 if and only if $b = 1$. This observation, coupled with Springer's result and following an argument similar to that found, for example, in Pommerenke [9, pp. 286–292], leads to the following characterization of quasi-conformal extensions.

THEOREM 7. *Let Ω be a domain of class $\mathscr{D}$, i.e. $b = b^{(s)}(\Omega) < 1$, and assume that $\varphi\colon \Omega \to \Omega^*$ is a conformal mapping of Ω onto $\Omega^* = \varphi(\Omega)$. Then, the following statements are equivalent:*

(i) *φ has a $\varkappa'$-quasi-conformal $(0 \leqslant \varkappa' < 1)$ homeomorphic extension to $\boldsymbol{C}$;*

(ii) $|(\mathscr{G}_\varphi \alpha, \bar{\alpha})_2| \leqslant [(\sqrt{b}+\varkappa)/(1+\varkappa\sqrt{b})] \|\mathscr{K}\bar{\alpha}\|_2^2$ *for some $0 \leqslant \varkappa < 1$ and all $\alpha \in l_2$.*

§ 8. Fredholm determinant. We let Δ_Ω and $\Delta_\Omega^{(s)}$ denote the disks $\{z \in \boldsymbol{C}\colon |z| < \|B_\Omega\|^{-1}\}$ and $\{z \in \boldsymbol{C}\colon |z| < \|B_\Omega^{(s)}\|^{-1}\}$, respectively, thus $\Delta \subseteq \Delta_\Omega \subseteq \Delta_\Omega^{(s)}$. We may then introduce the operator

$$L_\Omega(z) = \log(I_\Omega - zB_\Omega)$$

given by

$$L_\Omega(z) = -\sum_{n=1}^{\infty} \frac{1}{n}(zB_\Omega)^n,$$

which converges absolutely for $z \in \Delta_\Omega$. This is a bounded operator on $L_2(\Omega)$ for any $z \in \Delta_\Omega$ and, moreover, $L_\Omega(\cdot)$ is an operator-valued holomorphic function on Δ_Ω. Clearly, $L_\Omega(z)$ is zero on $H_2(\Omega)^\perp$ and, in view of the spectral theorem,

$$L_\Omega(z) = \int_{\sigma(\Omega)} \log(1-z\mu)\, dE(\mu), \qquad z \in \Delta_\Omega, \tag{8.1}$$

$$\|L_\Omega(z)\| = \sup\{|\log(1-z\mu)|\colon \mu \in \sigma(\Omega)\}. \tag{8.2}$$

Therefore

$$\|L_\Omega(z)\| \leqslant \|L_\Omega(|z|)\| = -\log(1-|z|\,\|B_\Omega\|), \qquad z \in \Delta_\Omega,$$

and $L_\Omega(z)^* = L_\Omega(\bar{z})$, while $-L_\Omega(z)$ is a positive operator on $L_2(\Omega)$ for all real z in Δ_Ω.

The domain Ω is said to be of *Fredholm trace class*, in short $\Omega \in \mathscr{F}$, if the nonnegative function

$$g_\Omega(z) \equiv \|l_\Omega(\cdot, z)\|_\Omega^2 = K_\Omega(z, z) - \Gamma_\Omega(z, z), \qquad z \in \Omega,$$

is in $L_2(\Omega)$. In this case, the operator A_Ω belongs to the Hilbert–Schmidt class $\mathscr{I}_2$ and hence $B_\Omega = A_\Omega^* A_\Omega$ belongs to the trace class $\mathscr{I}_1$. In particular, A_Ω and B_Ω are compact, and

$$\|g_\Omega\|_\Omega^2 = \sum_{k=1}^{\infty} \mu_k < \infty. \tag{8.3}$$

In terms of trace norms

$$\|B_\Omega\|_n = \{\operatorname{Tr} B_\Omega^n\}^{1/n} = \Big\{\sum_{k=1}^{\infty} \mu_k^n\Big\}^{1/n} \qquad (n = 1, 2, \ldots) \tag{8.4}$$

we have

$$\|B_\Omega\|_n \leqslant \|B_\Omega\|_1^{1-m/n} \|B_\Omega\|_m^{m/n}, \qquad n \geqslant m,$$

and thus

(8.5) $$\mu_1 = \|B_\Omega\| \leqslant \|B_\Omega\|_n \leqslant \|B_\Omega\|_1 = \sum_{k=1}^{\infty} \mu_k \quad (n = 1, 2, \ldots),$$

(8.6) $$\sqrt{\mu_1} = \|A_\Omega\| \leqslant \|A_\Omega\|_n = \|B_\Omega\|_{n/2}^{1/2} = \Big\{\sum_{k=1}^{\infty} \mu_k^{n/2}\Big\}^{1/n} \quad (n = 2, 3, \ldots).$$

The condition $\Omega \in \mathscr{F}$ is also equivalent to the requirement that the operator $L_\Omega(z)$ belongs to the trace class $\mathscr{I}_1$ for every $z \in \Delta_\Omega$. In that case

(8.7) $$\operatorname{Tr}\{L_\Omega(z)\} = \sum_{k=1}^{\infty} \log(1 - z\mu_k), \quad z \in \Delta_\Omega,$$

(8.8) $$\|L_\Omega(z)\|_1 = \sum_{k=1}^{\infty} |\log(1 - z\mu_k)|.$$

In particular,

(8.9) $$\|L_\Omega(z)\|_1 \leqslant \|L_\Omega(|z|)\|_1 = -\sum_{k=1}^{\infty} \log(1 - |z|\,\mu_k), \quad z \in \Delta_\Omega,$$

and we note that the convergence of the sum in (8.9) is equivalent to (8.3).

The *Fredholm determinant* for a domain $\Omega \in \mathscr{F}$ is defined by

(8.10) $$D_\Omega(z) = \exp \operatorname{Tr}\{L_\Omega(z)\} = \exp \operatorname{Tr}\{\log(I_\Omega - zB_\Omega)\}.$$

Initially this function is defined only for $z \in \Delta_\Omega$. However, using (8.7), we have the alternative expression

(8.11) $$D_\Omega(z) = \prod_{k=1}^{\infty} (1 - z\mu_k)$$

for the Fredholm determinant, showing that, in fact, D_Ω is an entire function. Moreover, as B_Ω is compact and self-adjoint, the multiplicity of $\mu_k \neq 0$ is finite and thus, for $\mu_k \neq 0$, $z = \mu_k^{-1}$ is a zero of finite order of D_Ω. In particular, there exists an integer $n_\Omega \geqslant 0$ so that

(8.12) $$D_\Omega(z) = (1-z)^{n_\Omega} D_{\Omega,1}(z)$$

where $D_{\Omega,1}$ is an entire function satisfying $D_{\Omega,1}(1) \neq 0$.

A very parallel theory for the reduced Fredholm transform $B_\Omega^{(s)}$ can be given in the $H_2^{(s)}(\Omega)$-setting. Thus,

$$L_\Omega^{(s)}(z) = \log(I_\Omega - zB_\Omega^{(s)}), \quad z \in \Delta_\Omega^{(s)},$$

is an operator-valued holomorphic function on $\Delta_\Omega^{(s)}$, and $L_\Omega^{(s)}(z) = 0$ on $H_2^{(s)}(\Omega)^\perp$. Moreover, $L_\Omega^{(s)}(z)^* = L_\Omega^{(s)}(\bar{z})$ and $-L_\Omega^{(s)}(z)$ is a positive operator on $L_2(\Omega)$ for all real $z \in \Delta_\Omega^{(s)}$. Also,

(8.13) $$L_\Omega^{(s)}(z) = \int_{\sigma^{(s)}(\Omega)} \log(1 - z\mu)\, dE(\mu), \quad z \in \Delta_\Omega^{(s)},$$

$$\|L_\Omega^{(s)}(z)\| = \sup\{|\log(1 - z\mu)|:\ \mu \in \sigma^{(s)}(\Omega)\},$$

$$\|L_\Omega^{(s)}(z)\| \leqslant \|L_\Omega^{(s)}(|z|)\| = -\log(1 - |z|\,\|B_\Omega^{(s)}\|), \quad z \in \Delta_\Omega^{(s)}.$$

The theories for $H_2(\Omega)$ and $H_2^{(s)}(\Omega)$ are trivially identical when $N_2^{(s)}(\Omega) = H_2(\Omega) \ominus H_2^{(s)}(\Omega)$ is trivial. On the other hand, $\sigma(\Omega) = \sigma^{(s)}(\Omega) \cup \{1\}$ when $N_2^{(s)}(\Omega) \neq \{0\}$ and hence, comparing the expressions (8.1) and (8.13), we obtain

$$L_\Omega(z) = L_\Omega^{(s)}(z) + \log(1-z)\, E(1), \quad z \in \Delta_\Omega$$

where $E(1) = E(1)\, E(1)$ is a bounded projection on $L_2(\Omega)$. In particular,

$$\|L_\Omega(z)\| = \max\{\|L_\Omega^{(s)}(z)\|, |\log(1-z)|\}, \quad z \in \Delta_\Omega,$$

$$(L_\Omega(z) f, f)_\Omega = (L_\Omega^{(s)}(z) f, f)_\Omega + \log(1-z)\, \|E(1) f\|_\Omega^2, \quad z \in \Delta,\ f \in L_2(\Omega),$$

whenever $N_2^{(s)}(\Omega) \neq \{0\}$. It follows that always $\|L_\Omega^{(s)}(z)\| \leqslant \|L_\Omega(z)\|$ for every $z \in \Delta_\Omega \subseteq \Delta_\Omega^{(s)}$ and that $-L_\Omega(z) \geqslant -L_\Omega^{(s)}(z) \geqslant 0$ for all real z in $\Delta_\Omega^{(s)}$.

The domain Ω is said to be of *reduced Fredholm trace class*, in short $\Omega \in \mathscr{F}^{(s)}$, if the nonnegative function

$$g_\Omega^{(s)}(z) \equiv \|l_\Omega^{(s)}(\cdot, z)\|_\Omega^2 = K_\Omega^{(s)}(z, z) - \Gamma_\Omega(z, z), \quad z \in \Omega,$$

is in $L_2(\Omega)$. Since $g_\Omega^{(s)} \leqslant g_\Omega$ on Ω we deduce that $\mathscr{F} \subseteq \mathscr{F}^{(s)}$ and, moreover, a domain Ω in $\mathscr{F}^{(s)}$ is also in $\mathscr{F}$ if and only if $N_2^{(s)}(\Omega)$ is of finite dimension. Recall that B_Ω and $B_\Omega^{(s)}$ always reduce to the identity and zero operators, respectively, on $N_2^{(s)}(\Omega)$. For $\Omega \in \mathscr{F}$, we may therefore define the *deficiency index* $d_\Omega \geqslant 0$ as the integer $d_\Omega = \dim\{N_2^{(s)}(\Omega)\}$. For example, if $\partial\Omega$ is of class $C^{2,\varepsilon}$ $(0 < \varepsilon < 1)$ and Ω is of connectivity p $(1 \leqslant p < \infty)$ then $\Omega \in \mathscr{F}$ and $d_\Omega = p-1$.

The relations (8.3)–(8.12) are also valid when $\Omega \in \mathscr{F}^{(s)}$, provided the superscript "s" is inserted in the quantities g_Ω, A_Ω, B_Ω, $L_\Omega(\cdot)$, $D_\Omega(\cdot)$, Δ_Ω, μ_k and n_Ω. In particular, the *reduced Fredholm determinant*

(8.14) $$D_\Omega^{(s)}(z) = \prod_{k=1}^{\infty} (1 - z\mu_k^{(s)}), \quad \Omega \in \mathscr{F}^{(s)},$$

is an entire function with

(8.15) $$D_\Omega^{(s)}(z) = (1-z)^{n_\Omega^{(s)}} D_{\Omega,1}^{(s)}(z)$$

where $D_{\Omega,1}^{(s)}$ is an entire function satisfying $D_{\Omega,1}^{(s)}(1) \neq 0$. If, in addition, Ω is also in $\mathscr{F}$, then, assuming without loss that $d_\Omega = \dim\{N_2^{(s)}(\Omega)\} \geqslant 1$,

$$\mu_1 = \ldots = \mu_{d_\Omega} = 1, \quad \mu_{d_\Omega + k} = \mu_k^{(s)} \quad (k = 1, 2, \ldots).$$

It follows that for any $\Omega \in \mathscr{F}$

$$\|B_\Omega\|_n^n = d_\Omega + \|B_\Omega^{(s)}\|_n^n \quad (n = 1, 2, \ldots),$$

$$D_\Omega(z) = (1-z)^{d_\Omega} D_\Omega^{(s)}(z). \tag{8.16}$$

A comparison of (8.16) with (8.12) and (8.15) also shows that $d_\Omega = n_\Omega - n_\Omega^{(s)}$ and $D_{\Omega,1} \equiv D_{\Omega,1}^{(s)}$ with

$$D_{\Omega,1}(z) = \prod_{k=n_\Omega+1}^{\infty} (1 - z\mu_k), \qquad \Omega \in \mathscr{F}.$$

Here

$$1 \geqslant \|B_\Omega\| = \mu_1 \geqslant \ldots \geqslant \mu_{d_\Omega+1} = \|B_\Omega^{(s)}\| \geqslant \ldots \geqslant \mu_{n_\Omega+1} \geqslant \ldots,$$

$$\mu_k < 1, \quad k = n_\Omega+1, n_\Omega+2, \ldots$$

Clearly, $d_\Omega = n_\Omega$ if and only if $\|B_\Omega^{(s)}\| < 1$, in which case $D_{\Omega,1} = D_\Omega^{(s)}$.

We return to the more general case of $\Omega \in \mathscr{F}^{(s)}$. In this case $D_\Omega^{(s)} = D_{\Omega,1}^{(s)}$ if and only if $\|B_\Omega^{(s)}\| < 1$ or if and only if $n_\Omega^{(s)} = 0$. Let $\Omega \in \mathscr{F}^{(s)}$ with $\|B_\Omega^{(s)}\| < 1$, or $n_\Omega^{(s)} = 0$. By (8.14) we deduce that $0 < D_\Omega^{(s)}(1) \leqslant 1$ and, in view of Theorem 6, $D_\Omega^{(s)}(1) = 1$ if and only if $\Omega \in \mathscr{N}_D^d$. We now consider a circular domain Δ_p whose boundary consists of p disjoint circles. The conformal type of Δ_p is determined by $\tau(p)$ real parameters (Riemann moduli) $m_1, \ldots, m_{\tau(p)}$, where $\tau(1) = 0$ (i.e. zero number of parameters), $\tau(2) = 1$ and $\tau(p) = 3p-6$, $p \geqslant 3$. The circular domain Δ_p will be denoted by $\Delta_p(m_1, \ldots, m_{\tau(p)})$ once the $\tau(p)$ moduli are specified; thus $\Delta_1(\cdot)$ is a disk, $\Delta_2(m_1)$ is an annulus with modulus m_1, $m_1 > 1$, $\Delta_3(m_1, m_2, m_3)$ is a concentric annulus with modulus m_1, $m_1 > 1$, minus a disk whose center and radius are determined by m_1, m_2, and m_3, and so on. We shall now extend the classes $\mathscr{N}_D$ and $\mathscr{N}_D^d$, introduced previously in § 6. A domain Ω is said to belong to class $\mathscr{N}_{D,p}(m_1, \ldots, m_{\tau(p)})$ if Ω is conformally equivalent to $\Delta_p(m_1, \ldots, m_{\tau(p)}) \setminus E$, where E is a set satisfying $E \cap K \in N_D$ for every compact subset K of the circular domain $\Delta_p(m_1, \ldots, m_{\tau(p)})$. The subclass $\mathscr{N}_{D,p}^d(m_1, \ldots, m_{\tau(p)})$ of $\mathscr{N}_{D,p}(m_1, \ldots, m_{\tau(p)})$ consists of those domains Ω which are Möbius images of $\Delta_p(m_1, \ldots, m_{\tau(p)}) \setminus E$, with $E \cap K \in N_D$ for every compact subset K of $\Delta_p(m_1, \ldots, m_{\tau(p)})$. Evidently,

$$\mathscr{N}_{D,1}(\cdot) = \mathscr{N}_D, \qquad \mathscr{N}_{D,1}^d(\cdot) = \mathscr{N}_D^d.$$

We are now in a position to state the following theorem, first proved by Schiffer [12] under some additional smoothness assumptions on the domains in question. As the original proof of Schiffer [12] is also applicable to the present more general case, the proof of the following theorem is omitted.

THEOREM 8. *Let*

$$\mathscr{F}^{(s)}(m_1, \ldots, m_{\tau(p)}) \equiv \mathscr{N}_{D,p}(m_1, \ldots, m_{\tau(p)}) \cap \{\Omega \in \mathscr{F}^{(s)}: \|B_\Omega^{(s)}\| < 1\}.$$

Then

$$\mathscr{N}_{D,p}^d(m_1, \ldots, m_{\tau(p)}) \subseteq \mathscr{F}^{(s)}(m_1, \ldots, m_{\tau(p)})$$

and the reduced Fredholm determinant $D_\Omega^{(s)}$ *is an invariant of* $\mathscr{N}_{D,p}^d(m_1, \ldots, m_{\tau(p)})$, *i.e.,*

$$D_\Omega^{(s)}(z) \equiv D(z: m_1, \ldots, m_{\tau(p)}), \qquad z \in C,$$

for all $\Omega \in \mathscr{N}_{D,p}^d(m_1, \ldots, m_{\tau(p)})$. *Moreover,*

$$D(1: m_1, \ldots, m_{\tau(p)}) = \max\{D_\Omega^{(s)}(1): \Omega \in \mathscr{F}^{(s)}(m_1, \ldots, m_{\tau(p)})\}$$

and the maximum is attained only by (any) member Ω *of* $\mathscr{N}_{D,p}^d(m_1, \ldots, m_{\tau(p)})$. *Here,* $0 < D(1: m_1, \ldots, m_{\tau(p)}) \leqslant 1$ *and* $D(1: m_1, \ldots, m_{\tau(p)}) = 1$ *if and only if* $p = 1$.

§ 9. **Area-excess function.** For any $\zeta \in \Omega \setminus N_\Omega$, we consider the function

$$\beta(\Omega:\zeta) \equiv \left\{1 - \frac{\Gamma_\Omega(\zeta, \zeta)}{K_\Omega^{(s)}(\zeta, \zeta)}\right\}^{1/2}.$$

The analogous function, where $K_\Omega^{(s)}(\zeta, \zeta)$ is replaced by $K_\Omega(\zeta, \zeta)$, will not be treated here as it is less interesting than $\beta(\Omega:\zeta)$. Since

$$0 \leqslant K_\Omega^{(s)}(\zeta, \zeta) - \Gamma_\Omega(\zeta, \zeta) \leqslant K_\Omega^{(s)}(\zeta, \zeta)$$

we deduce that $0 \leqslant \beta(\Omega:\zeta) \leqslant 1$. In the case that $\zeta \in N_\Omega$, we set, in consistency with $K_\Omega^{(s)}(\zeta, \zeta) = \Gamma_\Omega(\zeta, \zeta) = 0$, $\beta(\Omega:\zeta) = 0$ and thus $0 \leqslant \beta(\Omega:z) \leqslant 1$ for all z in Ω.

An alternative expression for $\beta(\Omega:\zeta)$ may also be given through the notion of area, described in § 6. Recall that for $\zeta \in \Omega$, the outer area is $A_e(\Omega:\zeta) = \pi K_\Omega^{(s)}(\zeta, \zeta)$ while for $\zeta \in \Omega \setminus N_\Omega$ the inner area $A_i(\Omega:\zeta)$ satisfies $A_i(\Omega:\zeta) A_e(\Omega:\zeta) = \pi^2$. We now define

$$A_0(\Omega:\zeta) \equiv \int_{C\setminus\Omega} |t-\zeta|^{-4}\, dm(t), \qquad \zeta \in \Omega.$$

Thus $A_0(\Omega:\zeta)$ represents the area of the image of $C\setminus\Omega$ under the mapping $(t-\zeta)^{-1}$. In view of (3.2) we also have $A_0(\Omega:\zeta) = \pi \Gamma_\Omega(\zeta, \zeta)$. It follows that

$$\beta(\Omega:\zeta) = \left\{1 - \frac{A_0(\Omega:\zeta)}{A_e(\Omega:\zeta)}\right\}^{1/2} = \{1 - \pi^{-2} A_0(\Omega:\zeta) A_i(\Omega:\zeta)\}^{1/2}.$$

For this reason, $\beta(\Omega:\zeta)$ is called the *area-excess function* at $\zeta \in \Omega$. Its properties are summarized in the following theorem:

THEOREM 9. *The function* $\beta(\Omega:\cdot)$ *maps* Ω *into* $[0, 1]$, *it is continuous on* $\Omega \setminus N_\Omega$ *and satisfies the following properties*:

1°. *For all* $\zeta \in \Omega \setminus N_\Omega$, $0 \leqslant \sqrt{a} \leqslant \beta(\Omega:\zeta) \leqslant \sqrt{b} \leqslant 1$ *where* $a = a^{(s)}(\Omega)$ *and* $b = b^{(s)}(\Omega)$.

2°. *For* $\zeta \in \Omega \setminus N_\Omega$, $\beta(\Omega:\zeta) = 1$ *if and only if* $m(C\setminus\Omega) = 0$.

3°. *For* $\zeta \in \Omega$, $\beta(\Omega:\zeta) = 0$ *if and only if* Ω *is canonical with respect to* ζ. *In particular, if* $\zeta \in \Omega \setminus N_\Omega$ *and if the spectrum* $\sigma^{(s)}(\Omega)$ *of* $\Omega \notin 0_{AD}$ *reduces to only point spectrum* $\{\mu_k^{(s)}\}$, *then* $\beta(\Omega:\zeta) = 0$ *if and only if all eigenfunctions* ψ_n *corresponding to positive eigenvalues* $\mu_n^{(s)} > 0$ *vanish at* ζ, *in which case there exists at least one zero eigenvalue, i.e.* $\mu_\infty^{(s)} = a^{(s)}(\Omega) = 0$.

4°. $\beta(\Omega) \equiv \operatorname{Sup}\{\beta(\Omega:z):\ z \in \Omega\}$ *is zero if and only if either* $\Omega \in 0_{AD}$ *or* $\Omega \in \mathcal{N}_D^d$.

5°. *Let* $\varphi:\ \Omega \to \Omega^*$ *be a biholomorphic mapping of* Ω *onto* $\Omega^* = \varphi(\Omega)$. *Then for* $\zeta \in \Omega$, $\beta(\Omega:\zeta) = \beta(\Omega^*:\varphi(\zeta))$ *if and only if*

$$\|S_\varphi(\cdot,\zeta)\|_\Omega^2 = -2\operatorname{Re}\big(l_\Omega^{(s)}(\cdot,\zeta),\, S_\varphi(\cdot,\zeta)\big)_\Omega. \tag{9.1}$$

In particular, if φ *is a Möbius transformation then* $\beta(\Omega:z) = \beta(\Omega^*:\varphi(z))$ *for every* $z \in \Omega$.

Proof. Item 2° is straightforward and follows directly from the definition of $\beta(\Omega:\zeta)$. The first part of item 3° is also straightforward and follows from the identity

$$\|l_\Omega^{(s)}(\cdot,\zeta)\|_\Omega^2 = K_\Omega^{(s)}(\zeta,\zeta) - \Gamma_\Omega(\zeta,\zeta) \tag{9.2}$$

which is a special case of (6.8). The second part of item 3° follows immediately from Proposition 5. Item 4° is a consequence of the equivalence of the statements (4) and (10) of Theorem 6. As for item 5°, we use the identities (4.2)–(4.4) in their $H_2^{(s)}(\Omega)$-setting. It follows that $\zeta \in N_\Omega$ if and only if $\varphi(\zeta) \in N_{\Omega^*}$, in which case $\beta(\Omega:\zeta) = \beta(\Omega^*:\varphi(\zeta)) = 0$ and $l_\Omega^{(s)}(\cdot,\zeta) = l_{\Omega^*}^{(s)}(\cdot,\varphi(\zeta)) = S_\varphi(\cdot,\zeta) = 0$. We may therefore assume that $\zeta \in \Omega \setminus N_\Omega$ and thus $\varphi(\zeta) \in \Omega^* \setminus N_{\Omega^*}$ with $\Omega, \Omega^* \notin 0_{AD}$. In this case $\beta(\Omega:\zeta) = \beta(\Omega^*:\varphi(\zeta))$ is equivalent to $\Gamma_\Omega(\zeta,\zeta) = \Gamma_{\Omega^*}(\varphi(\zeta),\varphi(\zeta))|\varphi'(\zeta)|^2$ and hence to

$$K_\Omega^{(s)}(\zeta,\zeta) - \Gamma_\Omega(\zeta,\zeta) = \{K_{\Omega^*}^{(s)}(\varphi(\zeta),\varphi(\zeta)) - \Gamma_{\Omega^*}(\varphi(\zeta),\varphi(\zeta))\}|\varphi'(\zeta)|^2$$

which means, in view of (9.2), that

$$\|l_\Omega^{(s)}(\cdot,\zeta)\|_\Omega^2 = |\varphi'(\zeta)|^2\, \|l_{\Omega^*}^{(s)}(\cdot,\varphi(\zeta))\|_{\Omega^*}^2$$

or, by (4.3),

$$\|l_\Omega^{(s)}(\cdot,\zeta)\|_\Omega^2 = \|l_\Omega^{(s)}(\cdot,\zeta) + S_\varphi(\cdot,\zeta)\|_\Omega^2.$$

The latter is equivalent to (9.1). If φ is a Möbius transformation, then $S_\varphi(\cdot,z) \equiv 0$ for any $z \in \Omega$ and hence (9.1) is trivially satisfied. It follows that $\beta(\Omega:z) = \beta(\Omega^*:\varphi(z))$ for any $z \in \Omega$. This proves item 5°. We now prove item 1°. By definition

$$a = a^{(s)}(\Omega) = \operatorname{Inf}\{(B_\Omega^{(s)} f, f)_\Omega:\ f \in H_2^{(s)}(\Omega),\ \|f\|_\Omega = 1\},$$

$$b = b^{(s)}(\Omega) = \operatorname{Sup}\{(B_\Omega^{(s)} f, f)_\Omega:\ f \in H_2^{(s)}(\Omega),\ \|f\|_\Omega = 1\}$$

and $0 \leqslant a \leqslant b = \|B_\Omega^{(s)}\| \leqslant 1$. Fix $\zeta \in \Omega \setminus N_\Omega$ and choose

$$f_0 \equiv K_\Omega^{(s)}(\cdot,\zeta)/\sqrt{K_\Omega^{(s)}(\zeta,\zeta)}.$$

Then $f_0 \in H_2^{(s)}(\Omega)$ and $\|f_0\|_\Omega = 1$. Moreover, $(B_\Omega^{(s)} f_0, f_0)_\Omega = [\beta(\Omega:\zeta)]^2$ and item 1° follows. This concludes the proof.

Item 5° could also be established by using the identities

$$\|l_\Omega^{(s)}(\cdot,\zeta)\|_\Omega = \sqrt{K_\Omega^{(s)}(\zeta,\zeta)}\,\beta(\Omega:\zeta), \tag{9.3}$$

$$\|l_\Omega^{(s)}(\cdot,\zeta) + S_\varphi(\cdot,\zeta)\|_\Omega = \sqrt{K_\Omega^{(s)}(\zeta,\zeta)}\,\beta(\Omega^*:\varphi(\zeta)). \tag{9.4}$$

Also, the example at the end of § 4 shows that $\beta(\Omega:z) = \beta(\Omega^*:\varphi(z))$ for all $z \in \Omega$ does not, in general, imply that φ is a Möbius transformation.

We now consider the *maximum area excess*

$$\beta(\Omega) = \operatorname{Sup}\{\beta(\Omega:z):\ z \in \Omega\} \tag{9.5}$$

which, in view of 1° of Theorem 9, satisfies

$$\beta(\Omega) \leqslant \sqrt{b^{(s)}(\Omega)}. \tag{9.6}$$

In the following analysis it is convenient to assume that $\infty \in \Omega$. Since, by 5° of Theorem 9, $\beta(\Omega:z)$ is invariant under Möbius transformations, this assumption amounts to no loss of any generality. Some minor modifications are required in this case. Using the previous definitions, one easily shows that

$$\pi \lim_{z\to\infty} |z|^4 K_\Omega^{(s)}(z,z) = A_e(\Omega:\infty),$$

$$\pi \lim_{z\to\infty} |z|^4 \Gamma_\Omega(z,z) = A_0(\Omega:\infty) = m(C\setminus\Omega)$$

and thus

$$\beta(\Omega:\infty) = \left\{1 - \frac{A_e(\Omega:\infty)}{A_0(\Omega:\infty)}\right\}^{1/2}, \qquad \infty \in \Omega.$$

These are, of course, consistent with the quantities for $\zeta \neq \infty$. In particular, $A_e(\Omega:\infty) = m(C\setminus\Omega)$ if and only if Ω is canonical with respect to $\infty \in \Omega$. The latter is equivalent to $\Psi_\Omega(z:\infty) \equiv z$ on Ω.

Let $\Delta_e = \hat{C}\setminus\bar{\Delta}$ denote the exterior of the unit disk Δ and consider the subclasses $\mathcal{N}_D(\infty)$ and $\mathcal{N}_D^d(\infty)$ of $\mathcal{N}_D$ and $\mathcal{N}_D^d$, respectively, consisting of those domains Ω for which $\infty \in \Omega$. Clearly, $\mathcal{N}_D^d(\infty) \subseteq \mathcal{N}_D(\infty)$ and $\Delta_e \in \mathcal{N}_D^d(\infty)$. Any $\Omega \in \mathcal{N}_D(\infty)$ may be mapped conformally onto $\Delta_e\setminus E$, where E is a set satisfying $E \cap K \in N_D$ for every compact subset K of Δ_e, with infinity going to infinity. Let φ denote the inverse of this mapping. Near infinity this function admits the expansion

$$z = \varphi(\omega) = r(\Omega)\Big[\omega + \sum_{n=0}^{\infty} c_n \omega^{-n}\Big], \qquad \omega \in \Delta_e\setminus E,\ z \in \Omega,$$

where $r(\Omega) > 0$ is the *mapping radius* of $\Omega \in \mathcal{N}_D(\infty)$. It follows easily that

$$\pi[r(\Omega)]^2 = A_c(\Omega:\infty), \quad \Omega \in \mathcal{N}_D(\infty)$$

and hence

$$\beta(\Omega:\infty) = \left\{1 - \frac{m(C\setminus\Omega)}{\pi[r(\Omega)]^2}\right\}^{1/2}, \quad \Omega \in \mathcal{N}_D(\infty). \tag{9.7}$$

In particular, for any $\Omega \in \mathcal{N}_D(\infty)$

$$r(\Omega) \geq \sqrt{m(C\setminus\Omega)/\pi}$$

with equality if and only if $\Omega \in \mathcal{N}_D^d(\infty)$, in which case Ω is the exterior Ω_1 of a disk of radius $r(\Omega)$ less (possibly) a set E, satisfying $E \cap K \in N_D$ for every compact subset K of Ω_1.

As an example, we consider the mapping

$$z = \varphi_c(\omega) = \omega + c\omega^{-1}, \quad 0 \leq c < 1.$$

This mapping maps Δ_e onto Ω_c, the exterior of an ellipse with the major axes $1+c$ and $1-c$ (any ellipse is similar to such an ellipse). Since Ω_c is simply connected, $H_2^{(s)}(\Omega_c) = H_2(\Omega_c)$ and, moreover, the spectrum $\sigma(\Omega_c)$ consists of only eigenvalues and $\sigma(\Omega_c) = \sigma(\hat{C}\setminus\Omega_c)$. In fact, any two complementary simply connected domains have the same set of eigenvalues $\{\mu_n\}$ and the corresponding eigenfunctions $\{\psi_n\}$ and $\{\varphi_n\}$ are related by $\varphi_n = i(1-\mu_n)^{-1/2} T\psi_n$, where T is the (extended) Hilbert transform of $L_2(C)$. This result may be easily deduced from the discussion of § 3 (see also Bergman and Schiffer [2]). The spectrum $\sigma(\Omega_c)$ can be easily determined, as is done in Bergman and Schiffer [2]. In fact, since $l_{\Delta_e}(\omega, \tau) \equiv 0$ we deduce, using (4.3), that

$$l_{\Omega_c}(z, \zeta) = -c\frac{[\varphi_c'(\omega)\varphi_c'(\tau)]^{-1}}{(\omega\tau - c)^2} = \sum_{n=1}^{\infty} c^n \psi_n(z)\psi_n(\zeta) \tag{9.8}$$

where

$$\psi_n(z) = i\sqrt{n}[\psi(z)]^{-(n+1)}\psi'(z), \quad \psi = \varphi_c^{-1}, \; z = \varphi_c(\omega) \in \Omega_c.$$

Since $\{i\sqrt{n}\,\omega^{-(n+1)}\}$ is an orthonormal basis of $H_2(\Delta_e)$, $\{\psi_n\}$ is an orthonormal basis of $H_2(\Omega_c)$. Moreover, a comparison of (9.8) with (3.10) shows that the ψ_n are the eigenfunctions of B_{Ω_c} and that their corresponding eigenvalues are

$$\mu_n = \lambda_n^{-2} = c^{2n} \quad (n = 1, 2, \ldots).$$

The domain Ω_c is clearly a member of the class $\mathcal{N}_D(\infty)$. Furthermore, $r(\Omega) = 1$, $m(C\setminus\Omega_c) = \pi(1-c^2)$ and $b^{(s)}(\Omega) = \mu_1 = c^2$. It follows from (9.7) that $\beta(\Omega_c:\infty) = c$, and hence equality holds in (9.6), that is

$$\beta(\Omega_c:\infty) = \max\{\beta(\Omega_c:z):\; z \in \Omega_c\} = \beta(\Omega_c) = \sqrt{b^{(s)}(\Omega_c)} = c.$$

This observation, coupled with the preceding discussion, yields the following result (compare Bergman and Schiffer [2]):

THEOREM 10. *Let Ω_c be the exterior of an ellipse, $0 \leq c < 1$, as before. Then*

$$\operatorname{Sup}\{\beta(\Omega)/\sqrt{b^{(s)}(\Omega)}:\; \Omega \notin 0_{AD}\} = 1$$

and the supremum is attained by $\Omega = \varphi(\Omega_c)$, where φ is any Möbius transformation. Moreover,

$$\beta(\Omega:\varphi(\infty)) = \max\{\beta(\Omega:z):\; z \in \Omega\} = \beta(\Omega) = \sqrt{b^{(s)}(\Omega)} = c.$$

The spectrum $\sigma(\Omega)$ consists of eigenvalues $\{c^{2n}\}$ with corresponding eigenfunctions $\{\psi_n\}$ given by

$$\psi_n(z) = i\sqrt{n}[\psi(z)]^{-(n+1)}\psi'(z), \quad \psi = \varphi^{-1}\circ\varphi_c^{-1}, \; z \in \Omega.$$

We now consider the quasi-conformal aspect of Theorem 10. Let $Q_\varkappa$ be the family of all pairs (Ω, φ), where $\Omega \notin 0_{AD}$ is a domain and φ is a homeomorphism of the (extended) plane which is conformal on Ω and $\varkappa$-quasi-conformal, $0 \leq \varkappa < 1$, on its complement. From (7.1) and (9.6) we deduce that

$$\beta(\varphi(\Omega)) \leq \frac{\sqrt{b}+\varkappa}{1+\varkappa\sqrt{b}}, \quad b \equiv b^{(s)}(\Omega), \tag{9.9}$$

for any pair $(\Omega, \varphi) \in Q_\varkappa$. Let

$$\varphi_\varkappa(z) \equiv \begin{cases} z + \varkappa z^{-1}, & z \in \bar{\Delta}_e, \\ z + \varkappa\bar{z}, & z \in \Delta. \end{cases}$$

Clearly, $(\Delta_e, \varphi_\varkappa) \in Q_\varkappa$. As before,

$$\beta(\varphi_\varkappa(\Delta_e):\; \infty) = \max\{\beta(\varphi_\varkappa(\Delta_e):\; \varphi_\varkappa(z)):\; z \in \Delta_e\} = \beta(\varphi_\varkappa(\Delta_e)) = \varkappa,$$

and since $b^{(s)}(\Delta_e) = 0$ we conclude that equality holds in (9.9) for the pair $(\Delta_e, \varphi_\varkappa)$. This gives:

THEOREM 11. *For any $0 \leq \varkappa < 1$,*

$$\operatorname{Sup}\left\{\frac{1+\varkappa\sqrt{b}}{\sqrt{b}+\varkappa}\beta(\varphi(\Omega)):\; (\Omega, \varphi) \in Q_\varkappa,\; b \equiv b^{(s)}(\Omega)\right\} = 1$$

and the supremum is attained by the pair $(\Delta_e, \varphi_\varkappa)$, in which case

$$\beta(\varphi_\varkappa(\Delta_e)) = \beta(\varphi_\varkappa(\Delta_e):\infty) = \varkappa, \quad b^{(s)}(\Delta_e) = 0.$$

The identities (9.3)–(9.4) imply, by using the triangle inequality,

$$|\beta(\Omega:\zeta) - \beta(\Omega^*:\varphi(\zeta))| \leq \{K_\Omega^{(s)}(\zeta, \zeta)\}^{-1/2}\|S_\varphi(\cdot, \zeta)\|_\Omega \leq \beta(\Omega:\zeta) + \beta(\Omega^*:\varphi(\zeta))$$

for any biholomorphic mapping φ of Ω onto $\Omega^* = \varphi(\Omega)$ and any $\zeta \in \Omega$. These inequalities have been observed previously by Harmelin [6]. Here, however, by using (9.5), (9.6) and (9.9), we obtain the following improvement on the result of Harmelin [6]:

THEOREM 12. *Let φ be biholomorphic on Ω. Then*

$$\sup_{z\in\Omega} \{K_\Omega^{(s)}(z, z)\}^{-1/2} \|S_\varphi(\cdot, z)\|_\Omega \leqslant \beta(\Omega)+\beta(\varphi(\Omega))$$
$$\leqslant \sqrt{b^{(s)}(\Omega)}+\sqrt{b^{(s)}(\varphi(\Omega))}$$
$$\leqslant 1+\sqrt{b^{(s)}(\Omega)} \leqslant 2.$$

In particular, if also $(\Omega, \varphi) \in Q_\varkappa$, $0 \leqslant \varkappa < 1$, then

$$\sup_{z\in\Omega} \{K_\Omega^{(s)}(z, z)\}^{-1/2} \|S_\varphi(\cdot, z)\|_\Omega \leqslant \frac{2\sqrt{b}+\varkappa(1+b)}{1+\varkappa\sqrt{b}}, \qquad b \equiv b^{(s)}(\Omega).$$

Moreover, the constants in these inequalities are sharp.

These inequalities can be localized to give bounds for the Schwarzian derivative $S_\varphi(\cdot, \cdot)$. This was done earlier in Burbea [3] (compare also Beardon and Gehring [1] and Harmelin [6]). In order to understand the meaning of the results, it is useful to introduce certain Banach spaces $B_{\varrho,\infty}(\Omega)$, $B_{K,\infty}(\Omega)$, called *Bers spaces*, of measurable functions on Ω. These spaces are defined as follows: Let $\Omega \notin 0_G$ and consider its Poincaré metric $\varrho_\Omega(\cdot)$; the space $B_{\varrho,\infty}(\Omega)$ is defined as the space of all measurable functions f on Ω with the finite norm

$$\|f\|_{\Omega,\varrho,\infty} \equiv \sup_{z\in\Omega} \{\varrho_\Omega(z)\}^{-1} |f(z)|.$$

If also $\Omega \notin 0_{AD}$, then $B_{K,\infty}(\Omega)$ is defined as the space of all measurable functions f on Ω with the finite norm

$$\|f\|_{\Omega,K,\infty} \equiv \sup_{z\in\Omega} \{K_\Omega^{(s)}(z, z)\}^{-1/2} |f(z)|.$$

The (closed) subspaces of $B_{\varrho,\infty}(\Omega)$ and $B_{K,\infty}(\Omega)$ whose elements are also holomorphic on Ω are denoted by $H_{\varrho,\infty}(\Omega)$ and $H_{K,\infty}(\Omega)$, respectively. It is easily seen that all these spaces are conformally invariant in the sense that the linear transformation T_φ in (4.1) is also an isometry of $B_{\varrho,\infty}(\Omega^*)$, $H_{\varrho,\infty}(\Omega^*)$, $B_{K,\infty}(\Omega^*)$ and $H_{K,\infty}(\Omega^*)$ onto $B_{\varrho,\infty}(\Omega)$, $H_{\varrho,\infty}(\Omega)$, $B_{K,\infty}(\Omega)$ and $H_{K,\infty}(\Omega)$, respectively. Moreover, since $0_G \subset 0_{AD}$ and $\{K_\Omega^{(s)}(z, z)\}^{1/2} \leqslant \varrho_\Omega(z)$ for every $z \in \Omega$, $\Omega \notin 0_{AD}$ (see Burbea [3]), we conclude that

$$\|f\|_{\Omega,\varrho,\infty} \leqslant \|f\|_{\Omega,K,\infty}, \qquad f \in B_{K,\infty}(\Omega),\ \Omega \notin 0_{AD},$$

and thus $B_{K,\infty}(\Omega) \subseteq B_{\varrho,\infty}(\Omega)$ and $H_{K,\infty}(\Omega) \subseteq H_{\varrho,\infty}(\Omega)$, the injections being continuous and contractive in both cases. In a similar fashion, one defines

the Banach spaces $B_{\varrho,\infty}(\Omega\times\Omega)$, $H_{\varrho,\infty}(\Omega\times\Omega)$, $B_{K,\infty}(\Omega\times\Omega)$ and $H_{K,\infty}(\Omega\times\Omega)$. For example, $H_{K,\infty}(\Omega\times\Omega)$ is a closed subspace of $B_{K,\infty}(\Omega\times\Omega)$ and it consists of all holomorphic functions $f(\cdot,\cdot)$ on $\Omega\times\Omega$ with the finite norm

$$\|f\|_{\Omega\times\Omega,K,\infty} \equiv \sup_{z,\zeta\in\Omega} \{K_\Omega^{(s)}(z, z) K_\Omega^{(s)}(\zeta, \zeta)\}^{-1/2} |f(z, \zeta)|.$$

Using the above notation, Theorem 12 and (9.3)–(9.4) admit alternative geometric interpretation in terms of spheres of $B_{K,\infty}(\Omega)$ and $B_{K,\infty}(\Omega^*)$, and hence of $B_{\varrho,\infty}(\Omega)$ and $B_{\varrho,\infty}(\Omega^*)$. Moreover, since for any $\zeta \in \Omega$

$$f(\zeta) = (f, K_\Omega^{(s)}(\cdot, \zeta))_\Omega, \qquad f \in H_2^{(s)}(\Omega),$$

we deduce, using the Cauchy–Schwarz inequality, that

$$\|f\|_{\Omega,K,\infty} \leqslant \|f\|_\Omega, \qquad f \in H_2^{(s)}(\Omega),\ \Omega \notin 0_{AD}$$

and hence $H_2^{(s)}(\Omega) \subseteq H_{K,\infty}(\Omega)$, with the injection being continuous and contractive. In particular, using (9.4),

$$\|l_\Omega^{(s)}(\cdot, \cdot)+S_\varphi(\cdot, \cdot)\|_{\Omega\times\Omega,K,\infty} \leqslant \beta(\varphi(\Omega)) \tag{9.10}$$

and thus $l_\Omega^{(s)}(\cdot, \cdot)+S_\varphi(\cdot, \cdot)$ is in $H_{K,\infty}(\Omega\times\Omega) \subseteq H_{\varrho,\infty}(\Omega\times\Omega)$. Moreover, if also $(\Omega, \varphi) \in Q_\varkappa$, $0 \leqslant \varkappa < 1$, then, by (9.9),

$$\|l_\Omega^{(s)}(\cdot, \cdot)+S_\varphi(\cdot, \cdot)\|_{\Omega\times\Omega,K,\infty} \leqslant \frac{\sqrt{b}+\varkappa}{1+\varkappa\sqrt{b}}, \qquad b = b^{(s)}(\Omega).$$

Similarly, Theorem 12 shows that $S_\varphi(\cdot, \cdot) \in H_{K,\infty}(\Omega\times\Omega)$ whenever φ is biholomorphic on Ω, and

$$\|S_\varphi(\cdot, \cdot)\|_{\Omega\times\Omega,K,\infty} \leqslant \beta(\Omega)+\beta(\varphi(\Omega)) \leqslant \sqrt{b}+\sqrt{b^*} \leqslant 1+\sqrt{b} \leqslant 2,$$

where $b \equiv b^{(s)}(\Omega)$, $b^* = b^{(s)}(\varphi(\Omega))$ $(\Omega \notin 0_{AD})$, and if also $(\Omega, \varphi) \in Q_\varkappa$, $0 \leqslant \varkappa < 1$, then

$$\|S_\varphi(\cdot, \cdot)\|_{\Omega\times\Omega,K,\infty} \leqslant \frac{2\sqrt{b}+\varkappa(1+b)}{1+\varkappa\sqrt{b}}.$$

We remark that if φ is meromorphic on Ω and $S_\varphi(\cdot, \cdot) \in B_{\varrho,\infty}(\Omega\times\Omega)$, i.e. $\|S_\varphi(\cdot, \cdot)\|_{\Omega\times\Omega,\varrho,\infty} < \infty$, then φ is biholomorphic on Ω. Indeed, in this case $S_\varphi(\cdot, \cdot)$ is free of poles and thus $S_\varphi(\cdot, \cdot)$ is holomorphic on $\Omega\times\Omega$, a condition equivalent to φ being biholomorphic on Ω. Finally, (9.10) implies that for any biholomorphic mapping φ on Ω and any $z, \zeta \in \Omega$,

$$|l_\Omega^{(s)}(z, \zeta)+S_\varphi(z, \zeta)| \leqslant \sqrt{b^*}\{K_\Omega^{(s)}(z, z) K_\Omega^{(s)}(\zeta, \zeta)\}^{1/2}, \qquad b^* = b^{(s)}(\varphi(\Omega)).$$

This inequality may also be deduced from the $H_2^{(s)}(\Omega)$-version of Corollary 4, and constitutes an improvement on the well-known inequality for which b^* is replaced by 1. In particular, if also $(\Omega, \varphi) \in Q_\varkappa$, $0 \leqslant \varkappa < 1$, then for $\zeta \in \Omega$,

$$|l_\Omega^{(s)}(\zeta, \zeta)+S_\varphi(\zeta, \zeta)| \leqslant \frac{\sqrt{b}+\varkappa}{1+\varkappa\sqrt{b}} K_\Omega^{(s)}(\zeta, \zeta), \qquad b = b^{(s)}(\Omega).$$

As an application of this inequality we assume, without loss, that φ is also in $U(\Omega:\zeta)$, as in (6.1). Then in view of (4.5) and (6.2), the coefficient $a_\varphi(\zeta)$ is $-S_\varphi(\zeta,\zeta)$, and thus

$$|a_\varphi(\zeta)-l_\Omega^{(s)}(\zeta,\zeta)| \leqslant \frac{\sqrt{b}+\varkappa}{1+\varkappa\sqrt{b}} K_\Omega^{(s)}(\zeta,\zeta).$$

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A note on the spectral mapping theorem

by

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Abstract. We prove that if σ is a semispectrum defined on the commutative subsets of a Banach algebra that satisfies one inclusion of the spectral mapping theorem $P(\sigma(A)) = \sigma(P(A))$ then it also satisfies the other.

1. Introduction. Let $\mathscr{A}$ be a complex unital Banach algebra with unit e. The family of all nonvoid subsets of $\mathscr{A}$ consisting of pairwise commuting elements will be designated by $c(\mathscr{A})$. We shall write $c_0(\mathscr{A})$ for the family of all finite elements of $c(\mathscr{A})$. The elements of $c(\mathscr{A})$ will be denoted by A_I, where I is a nonvoid set of indices, so $A_I = \{a_i\}_{i\in I}$. If the set I is finite, we can identify A_I with $A = (a_1, \ldots, a_n) \in \mathscr{A}^n$ for some $n \in N$. If $A = (a_1, \ldots, a_n) \in \mathscr{A}^n$ and $B = (b_1, \ldots, b_m) \in \mathscr{A}^m$, then (A, B) will denote the element $(a_1, \ldots, a_n, b_1, \ldots, b_m) \in \mathscr{A}^{n+m}$. If I, J are nonvoid sets of indices, then $P_J(T_I)$ will stand for a family of polynomials $\{p_j(T_I)\}_{j\in J}$, with complex coefficients, in indeterminates $T_I = \{t_i\}_{i\in I}$. Of course, each p_j depends only upon a finite number of indeterminates $t_{i_1}, \ldots, t_{i_n}$.

Each such system of polynomials induces a map, denoted by the same symbol P_J: $C^I \to C^J$, given by $Z_I \to (p_j(Z_I))_{j\in J} \in C^J$, where $Z_I = (z_i)_{i\in I}$. Such a map is called a *polynomial map*. Also if $A_I \in c(\mathscr{A})$, we can evaluate P_J on A_I obtaining an element $P_J(A_I) \in c(\mathscr{A})$.

In [4], Żelazko gave the following axioms and definitions.

Suppose that to each $A_I \in c(\mathscr{A})$ there corresponds a nonvoid compact subset of C^I:

$$A_I \to \sigma(A_I) \subset C^I. \tag{1}$$

1.1. Axioms.

$$\sigma(A_I) \subset \prod_{i\in I} \sigma(a_i) \tag{I}$$

where $A_I = \{a_i\}_{i\in I} \in c(\mathscr{A})$ and $\sigma(a_i)$ is the usual spectrum of an element $a_i \in \mathscr{A}$.

(II) $\sigma(\{a\})$ is the usual spectrum, $\sigma(a)$, of a for $\{a\} \in c(\mathscr{A})$.