a weak* limit of some subnet of $\{\mu_1^\alpha\}_{\alpha\in\mathscr{A}}$ in the Banach space of finite Borel measures on X_1 regarded as the dual space of the space of continuous functions on X_1.

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Holomorphic functions of uniformly bounded type on nuclear Fréchet spaces

by

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Dedicated to Professor Dr. H.-G. Tillmann on his sixtieth birthday

Abstract. It is studied under what conditions every entire function on a given nuclear Fréchet space E (resp. every holomorphic function on an open polycylindrical set $P \subset E$) is of uniformly bounded type. Necessary as well as sufficient conditions (resp. a characterization) are given in terms of the invariants (LB^∞), $(\bar{\Omega})$, $(\tilde{\Omega})$ known from the theory of linear operators between Fréchet spaces. A holomorphic characterization of nuclear Fréchet spaces with $(\tilde{\Omega})$ is presented and also examples and applications.

For a complex locally convex space E we denote by $H(E)$ the vector space of all entire functions on E, i.e. of all continuous complex functions on E which are Gâteaux-analytic. An entire function on E is called of uniformly bounded type if it is bounded on all multiples of some zero neighbourhood in E. By $H_{\mathrm{ub}}(E)$ we denote the linear space of all entire functions on E which are of uniformly bounded type. Colombeau and Mujica [4] have shown $H(E) = H_{\mathrm{ub}}(E)$ for each (DFM)-space E, while a classical example of Nachbin [16] gives $H_{\mathrm{ub}}(E) \subsetneqq H(E)$ for the nuclear Fréchet space $E = H(C)$. In [14] we have shown that a nuclear locally convex space E satisfies $H(E) = H_{\mathrm{ub}}(E)$ if and only if the entire functions on E are universally extendable in the following sense: Whenever E is a topological linear subspace of a locally convex space F with a fundamental system of continuous semi-norms induced by semi-inner products, then each $f \in H(E)$ has a holomorphic extension to F.

In the present article we investigate necessary as well as sufficient conditions for nuclear Fréchet spaces E to satisfy the relation $H(E) = H_{\mathrm{ub}}(E)$. We prove that this relation defines a subclass which contains all spaces with property $(\tilde{\Omega})$ and which is contained in the subclass of spaces with property (LB^∞). The properties $(\tilde{\Omega})$ and (LB^∞) have been introduced and investigated in Vogt [25], where it has been shown that $(\tilde{\Omega})$ is strictly stronger than (LB^∞). It remains open whether the relation $H(E) = H_{\mathrm{ub}}(E)$ defines a new linear topological invariant which is inherited by quotient spaces or equals one of the invariants (LB^∞) or $(\tilde{\Omega})$.

Furthermore we obtain a holomorphic characterization of nuclear Fréchet spaces E with $(\tilde{\Omega})$ which is related to the one given in Dineen, Meise and Vogt [7], [8], where it is shown that E has $(\tilde{\Omega})$ if and only if E contains a bounded subset which is not uniformly polar. We also give new examples of nuclear Fréchet spaces of holomorphic functions which satisfy $(\tilde{\Omega})$ and which are not quotients of power series spaces of finite type. From a result of Colombeau and Matos [3] we deduce that every nonzero convolution operator on $(H(E), \tau_0)$ is surjective provided that E is a nuclear Fréchet space with $(\tilde{\Omega})$.

Moreover, using an appropriate definition of $H_{ub}(U)$ for open subsets U of a nuclear Fréchet space E, we give the following characterization: The identity $H(P) = H_{ub}(P)$ holds for some (resp. all) polycylindrical open subsets P of E if and only if E has property $(\tilde{\Omega})$ introduced by Vogt [25], where P is called polycylindrical if it is a finite intersection of sets of the form $\{x \in E \mid |y(x)| < 1\}$, y a nonzero continuous linear functional on E.

The main tools to obtain these results are an interpolation argument used in Section 3 as well as methods and results from the theory of nuclear Fréchet spaces, in particular from Vogt [24], [25] and from Dineen, Meise and Vogt [8].

The article contains four sections. In the first one we recall some definitions and results and fix the notation. In Section 2 we introduce holomorphic functions of uniformly bounded type and derive necessary conditions for the identity $H(E) = H_{ub}(E)$ (resp. $H(P) = H_{ub}(P)$, P a polycylindrical set in E). Sufficient conditions for these identities are obtained in Section 3 by means of an interpolation argument. The last section contains some applications and new examples of nuclear Fréchet spaces of holomorphic functions satisfying $(\tilde{\Omega})$.

1. Preliminaries.

1.1. *General notation.* We shall use standard notation from the theory of locally convex spaces as presented in the books of Jarchow [11], Pietsch [20] and Schaefer [21]. All locally convex (l.c.) vector spaces E are assumed to be complex vector spaces and Hausdorff.

For a Fréchet space E we always assume that its l.c. structure is generated by an increasing system $(\|\cdot\|_n)_{n\in N}$ of semi-norms. Then we denote by E_n the canonically normed space $E/\|\cdot\|_n^{-1}(0)$ and by $\hat{E}_n$ its completion. $\pi_n\colon E \to \hat{E}_n$ resp. $\pi_{n,m}\colon \hat{E}_m \to \hat{E}_n$ $(m \geq n)$ denote the corresponding canonical maps and U_n denotes the set $\{x \in E \mid \|x\|_n < 1\}$. Sometimes it is convenient to assume that $(U_n)_{n\in N}$ is already a neighbourhood basis of zero.

If M is an absolutely convex subset of E, we define $\|\cdot\|_M^*\colon E' \to [0, \infty]$ by $\|y\|_M^* := \sup_{x\in M} |y(x)|$, where E' denotes the topological dual of E. Obviously $\|\cdot\|_M^*$ is the gauge functional of the polar of M. Instead of $\|\cdot\|_{U_n}^*$ we write $\|\cdot\|_n^*$.

We remark that the adjoint π_n^t of π_n gives an isometry between $(\hat{E}_n)_b'$ and $(E'_{U_n^\circ}, \|\cdot\|_n^*)$.

If M is absolutely convex and bounded in E, then E_M denotes the linear hull of M which becomes a normed space under the gauge functional of M. For l.c. spaces E and F we denote by $L(E, F)$ the space of all continuous linear mappings, while $LB(E, F)$ denotes the set of all $A \in L(E, F)$ for which there exists a zero neighbourhood U in E for which $A(U)$ is bounded.

1.2. *Sequence spaces.* Let $A = (a_{j,k})_{(j,k)\in N^2}$ be a matrix which satisfies

(1) $0 \leq a_{j,k} \leq a_{j,k+1}$ for all $j, k \in N$,

(2) For each $j \in N$ there exists $k \in N$ with $a_{j,k} > 0$.

Then we define the sequence spaces $\lambda^s(A)$ by

$$\lambda^s(A) := \{x \in C^N \mid \|x\|_k := (\sum_{j=1}^{\infty} (|x_j|\, a_{j,k})^s)^{1/s} < \infty \text{ for all } k \in N\}$$

for $1 \leq s < \infty$, and for $s = \infty$ and $s = 0$ by

$$\lambda^\infty(A) := \{x \in C^N \mid \|x\|_k := \sup_{j\in N} |x_j|\, a_{j,k} < \infty \text{ for all } k \in N\},$$
$$\lambda^0(A) := \{x \in \lambda^\infty(A) \mid \lim_{j\to\infty} x_j a_{j,k} = 0 \text{ for all } k \in N\}.$$

Obviously $\lambda^s(A)$ is a Fréchet space under the natural topology induced by the semi-norms $(\|\ \|_k)_{k\in N}$. We write $\lambda(A)$ instead of $\lambda^1(A)$.

We recall that $\lambda^s(A)$ is Schwartz (resp. nuclear) iff for every $k \in N$ there exist $m \in N$ and $\mu \in c_0$ (resp. $\mu \in l^1$) such that $a_{j,k} \leq \mu_j a_{j,m}$ for all $j \in N$.

If α is an increasing unbounded sequence of positive real numbers (called *exponent sequence*) and if $R = 1$ or $R = \infty$ then we define for $1 \leq s \leq \infty$ the *power series space*

$$\Lambda_R^s(\alpha) := \lambda^s(A(R, \alpha)), \quad \text{where} \quad A(R, \alpha) = \{(r_k^{\alpha_j})_{j\in N} \mid k \in N\}, \quad r_k \nearrow R.$$

$\Lambda_R^s(\alpha)$ is called a power series space of *finite* (resp. *infinite*) *type* if $R = 1$ (resp. $R = \infty$). We remark that well-known examples of nuclear power series spaces are $s \simeq C^\infty(S^1) \simeq \Lambda_\infty((\log n + 1)_{n\in N})$, $H(C^k) \simeq \Lambda_\infty((n^{1/k})_{n\in N})$ and $H(D^k) \simeq \Lambda_1((n^{1/k})_{n\in N})$, where D stands for the open unit disk in C and where $H(\Omega)$ denotes the space of all holomorphic functions on Ω endowed with the compact-open topology.

1.3. *Holomorphic functions.* Let E and G be l.c. spaces and let $\Omega \subset E$ be open, $\Omega \neq \emptyset$. $f\colon \Omega \to G$ is called *holomorphic* if f is continuous and if for every $y \in G'$ the function $y \circ f$ is Gâteaux-analytic. By $H(\Omega, G)$ we denote the space of all G-valued holomorphic mappings on Ω; the compact-open topology on $H(\Omega, G)$ is denoted by τ_0. Instead of $H(\Omega, C)$ we write $H(\Omega)$. By $H^\infty(\Omega)$ we denote the bounded holomorphic functions on Ω.

For details concerning holomorphic functions on l.c. spaces we refer to the books of Colombeau [2], Dineen [5], and Noverraz [17].

2. Holomorphic functions of uniformly bounded type. In this section we introduce the class of holomorphic functions of uniformly bounded type in variation of a notion of Nachbin [16]. Then we show that for a Fréchet space E every holomorphic function on E (resp. on some polycylindrical subset of E) is of uniformly bounded type only if E satisfies the linear topological invariant (LB^∞) (resp. $(\bar{\Omega})$) introduced by Vogt [25].

2.1. Definition. Let E be a l.c. space and Ω a p-open subset of E, where p is a continuous semi-norm on E. $f \in H(\Omega)$ is called of *uniformly bounded type* if there exists a continuous semi-norm q on E with $q \geqslant p$ such that f is bounded on each q-bounded subset ω of Ω satisfying $q\text{-dist}(\omega, \Omega^c) > 0$. We put

$$H_{ub}(\Omega) := \{f \in H(\Omega) \mid f \text{ is of uniformly bounded type}\}.$$

The following lemma shows that the identity $H(\Omega) = H_{ub}(\Omega)$ for certain open subsets Ω of a l.c. space E implies linear properties of E.

2.2. Lemma. *Let E be a l.c. k-space and let G be an open subset of C^n. If $H(G \times E) = H_{ub}(G \times E)$ then $L(E, H(G)) = LB(E, H(G))$.*

Proof. Let $A \in L(E, H(G))$ be given. Then it is easy to check that $f\colon G \times E \to C$ defined by

$$f(z, x) := A(x)[z]$$

is a holomorphic function. Hence the hypothesis gives the existence of an absolutely convex zero neighbourhood U in E such that for every compact subset K of G we have

$$\sup_{(z,x) \in K \times U} |f(z, x)| \leqslant M_K < \infty.$$

By the definition of f this implies

$$\sup_{x \in U} \sup_{z \in K} |A(x)[z]| \leqslant M_K.$$

Hence $A(U)$ is bounded in $H(G)$ and consequently $A \in LB(E, H(G))$.

Next we choose G in Lemma 2.2 in such a way that $H(G)$ is isomorphic to a power series space. Then we can use results of Vogt [25] to derive the following two propositions.

2.3. Proposition. *If every entire function on the Fréchet space E is of uniformly bounded type then E satisfies one of the following equivalent conditions:*

(1) *There exists an exponent sequence α with $\sup_{n \in N}(\alpha_{n+1}/\alpha_n) < \infty$ such that $L(E, \Lambda_\infty^\infty(\alpha)) = LB(E, \Lambda_\infty^\infty(\alpha))$.*

(2) *$L(E, \Lambda_\infty^\infty(\alpha)) = LB(E, \Lambda_\infty^\infty(\alpha))$ for all exponent sequences α.*

(3) *E has property (LB^∞):*

For every positive increasing unbounded sequence $(\varrho_N)_{N \in N}$ and every $p \in N$ there exists $q \in N$ such that for all $n_0 \in N$ there are $N_0 \in N$ and $C > 0$ such that for all $y \in E'$ there exists $N \in N$ with $n_0 \leqslant N \leqslant N_0$ with

$$\|y\|_q^{*1+\varrho_N} \leqslant C \|y\|_N^* \|y\|_p^{*\varrho_N}.$$

Proof. Except trivial cases we have $E \simeq C \times E_0$ and hence by hypothesis $H(C \times E_0) = H_{ub}(C \times E_0)$. Since $H(C) \simeq \Lambda_\infty(n)$, we deduce from Lemma 2.2 that E_0 and hence E satisfies (1). Hence the proof is complete, since the equivalence of all the conditions has been shown by Vogt [25], 5.2.

2.4. Corollary. $H_{ub}(\Lambda_\infty^\infty(\alpha)) \subsetneqq H(\Lambda_\infty^\infty(\alpha))$ *for each exponent sequence α.*

Proof. This is an obvious consequence of 2.3 since $\mathrm{id}_{\Lambda_\infty^\infty(\alpha)} \notin LB(\Lambda_\infty^\infty(\alpha), \Lambda_\infty^\infty(\alpha))$.

Remark. (a) For nuclear spaces $\Lambda_\infty(\alpha)$ it is easy to write down $f_\alpha \in H(\Lambda_\infty(\alpha))$ which is not of uniformly bounded type, namely

$$f(z) = \sum_{j=2}^{\infty} z_j \exp(z_1 \alpha_j).$$

(b) Corollary 2.4 extends the example of Nachbin [16]; see also Colombeau [2], 2.72, and Dineen [5], Ex. 2.22.

In order to derive a further consequence from Lemma 2.2, we introduce the following notation:

2.5. Definition. A subset P of a l.c. space E is called *polycylindrical* if there exist $N \in N$, $\varphi_1, \ldots, \varphi_N \in E' \setminus \{0\}$ and $R_1, \ldots, R_N > 0$ such that

$$P = \{x \in E \mid |\varphi_j(x)| < R_j \text{ for } 1 \leqslant j \leqslant N\}.$$

2.6. Proposition. *If the Fréchet space E contains a polycylindrical subset P with $H(P) = H_{ub}(P)$ then E satisfies one of the following equivalent conditions:*

(1) *There exists an exponent sequence α with $\sup_{n \in N}(\alpha_{n+1}/\alpha_n) < \infty$ such that $L(E, \Lambda_1^\infty(\alpha)) = LB(E, \Lambda_1^\infty(\alpha))$.*

(2) *$L(E, \Lambda_1^\infty(\alpha)) = LB(E, \Lambda_1^\infty(\alpha))$ for all exponent sequences α.*

(3) *E has property $(\bar{\Omega})$:*

For each $p \in N$ there exists $q \in N$ such that for each $k \in N$ and each $\varepsilon > 0$ there exists $C > 0$ with $\|\cdot\|_q^{*1+\varepsilon} \leqslant C \|\cdot\|_k^* \|\cdot\|_p^{*\varepsilon}$.

Proof. For P we have N, $\varphi_1, \ldots, \varphi_N$ and $R_1, \ldots, R_N$ according to 2.5. Assume that $\varphi_1, \ldots, \varphi_k$ are linearly independent, while $\varphi_{k+1}, \ldots, \varphi_N$ depend on $\varphi_1, \ldots, \varphi_k$. Then choose $e_1, \ldots, e_k$ in E with $\varphi_j(e_n) = \delta_{jn}$, $1 \leqslant j, n \leqslant k$, and put $E_0 := \bigcap_{j=1}^{k} \ker \varphi_j$. It is easy to check that $\pi\colon E \to E$,

$\pi(x) := \sum_{j=1}^{k} \varphi_j(x) e_j$, is a continuous projector in E with $\ker \pi = E_0$ and $\operatorname{im} \pi = \operatorname{span}\{e_j \mid 1 \le j \le k\} \simeq C^k$. Hence $E \simeq C^k \times E_0$ and it is easy to see that under this identification the set P can be described as $G \times E_0$, where

$$G = \{z \in C^k \mid |z_j| < R_j,\ 1 \le j \le k,\ |\sum_{j=1}^{k} a_{jn} z_j| < R_n,\ k+1 \le n \le N\},$$

for suitable a_{jn}, $1 \le j \le k$, $k+1 \le n < N$. Then it follows from Mityagin and Henkin [15] or (for any absolutely convex and bounded open set G) Petzsche [19], 4.5, and Vogt [24], 7.5, that $H(G) \simeq \Lambda_1(n^{1/k})$. Since $H(P) = H_{ub}(P)$ by hypothesis, we get from Lemma 2.2 that E_0 and hence E satisfies (1). By Vogt [25], 4.2, this completes the proof.

2.7. Examples. (a) From Vogt [25], 5.3, it follows that every quotient space of a power series space of finite type has (LB$^\infty$) and that every Fréchet space E with property $(\bar{\Omega})$ (see 3.5 below) has (LB$^\infty$). By Vogt [25], 5.5, there exist nuclear spaces $\lambda(B)$ with (LB$^\infty$) which do not have $(\bar{\Omega})$. For every exponent sequence α the infinite-dimensional closed linear subspaces of $\Lambda_\infty^\infty(\alpha)$ do not have (LB$^\infty$).

(b) The class of Fréchet spaces with $(\bar{\Omega})$ is fairly small and can be characterized also in a different way (see Vogt [26], 5.2). An example of a nuclear Fréchet space with $(\bar{\Omega})$ and continuous norm is $\lambda(A)$, where $A = (a_{j,k})_{(j,k) \in N^2}$,

$$a_{j,k} := \exp\Big(\sum_{n=1}^{k} j^{1/n}\Big).$$

By Vogt [25], 4.3, it suffices to show that for each $p \in N$, each $k \in N$ and each $\varepsilon > 0$ there exists $C > 0$ such that for all $j \in N$

$$a_{j,k}\, a_{j,p}^{\varepsilon} \le C a_{j,p+1}^{1+\varepsilon},$$

which is easy to check.

In the next section we give sufficient conditions on E for $H(E) = H_{ub}(E)$ resp. for $H(P) = H_{ub}(P)$ for all polycylindrical subsets of E.

3. An interpolation argument and its consequences. In this section we use an interpolation argument in order to enlarge the domain of definition of a holomorphic function with certain properties. We begin with two technical lemmas which are applied to obtain conditions on E which imply that all holomorphic functions on certain subsets of E are of uniformly bounded type.

3.1. Lemma. *Let E and F be Hilbert spaces and X a Banach space. Assume that the following hypotheses are satisfied:*

(a) *$A \in L(E, F)$ is injective and of type s and $A = v \circ u$, where $u \in L(E, X)$ and $v \in L(X, F)$.*

(b) *For every $d > 0$ there exists $C_d > 0$ such that for all $y \in F'$*

$$\|v^t y\|_X^{*1+d} \le C_d \|A^t y\|_E^* \cdot \|y\|_F^{*d}.$$

(c) *G is an open bounded absolutely convex subset of C^N and $f \in H(G \times E)$. There exist a zero neighbourhood W in $C^N \times F$ and $g \in H^\infty(W)$ with $f|(\mathrm{id}_G \times A)^{-1}(W) = g \circ (\mathrm{id}_G \times A)$.*

Then there exists $h \in H_{ub}(G \times X)$ with $f = h \circ (\mathrm{id}_G \times u)$.

Proof. By the spectral mapping theorem (see Pietsch [20], 8.3) there exist a complete orthonormal system $\{e_j \mid j \in N\}$ in E, an orthonormal system $\{y_j \mid j \in N\}$ in F and a decreasing sequence $\lambda = (\lambda_j)_{j \in N}$ in s such that

$$Ax = \sum_{j=1}^{\infty} \lambda_j (x|e_j)_E\, y_j.$$

Let χ_k denote the functional $x \mapsto (x|y_k)_F$ on F and remark that $\|\chi_k\|_F^* = 1$. Then note that

$$\|A^t \chi_k\|_E^* = \sup_{\|x\| \le 1} |(Ax|y_k)_F| = \sup_{\|x\| \le 1} |\lambda_k (x|e_k)_E| = \lambda_k,$$

and put $\varphi_k := v^t(\chi_k) \in X'$. From hypothesis (b) we get

(1) For each $d > 0$ there exists $C_d > 0$ such that for all $k \in N$

$$\|\varphi_k\|_X^{*1+d} = \|v^t(\chi_k)\|_X^{*1+d} \le C_d \|A^t(\chi_k)\|_E^* \|\chi_k\|_F^{*d} = C_d \lambda_k.$$

Next we choose $0 < \delta < 1$ such that for $\mu := (\delta/j)_{j \in N}$ the set

$$\delta \bar{G} \times \{x \in F \mid x = \sum_{j=1}^{\infty} \xi_j y_j,\ |\xi_j| \le \mu_j \text{ for all } j \in N\}$$

is contained in W. We put

$$M := \{m \in N_0^N \mid m_j \ne 0 \text{ only for finitely many } j \in N\}$$

and, following the idea of Boland and Dineen [1], we define for $m = (m_1, \ldots, m_n, 0, \ldots) \in M$ and $k \in N_0$ the k-homogeneous polynomial $a_{k,m}$ on C^N by

$$a_{k,m}(z) := \Big(\frac{1}{2\pi i}\Big)^{n+1} \int_{|\tau|=1} \int_{|\varrho_1|=\mu_1} \cdots \int_{|\varrho_n|=\mu_n} \frac{g(\tau z, \varrho_1 y_1 + \ldots + \varrho_n y_n)}{\tau^{k+1} \varrho_1^{m_1+1} \ldots \varrho_n^{m_n+1}}\, d\tau\, d\varrho_1 \ldots d\varrho_n.$$

Obviously we have with $M = \sup\{|g(w)| \mid w \in W\}$:

(2) $$\sup_{z \in \delta \bar{G}} |a_{k,m}(z)| \le M/\mu^m \quad \text{for all } (k, m) \in N_0 \times M.$$

Since $Ae_j = \lambda_j y_j$ for all $j \in N$ and since $f|(\mathrm{id}_G \times A)^{-1}(W) = g \circ (\mathrm{id}_G \times A)$, we have the following representation of $a_{k,m}$:

$$(3)\qquad a_{k,m}(z) = \left(\frac{1}{2\pi i}\right)^{n+1} \int\limits_{|\tau|=1} \int\limits_{|\varrho_1|=\mu_1} \cdots \int\limits_{|\varrho_n|=\mu_n} \frac{g\left(\tau z, A\left(\frac{\varrho_1}{\lambda_1} e_1 + \ldots + \frac{\varrho_n}{\lambda_n} e_n\right)\right)}{\tau^{k+1} \varrho_1^{m_1+1} \ldots \varrho_n^{m_n+1}} \, d\tau \, d\varrho_1 \ldots d\varrho_n$$

$$= \frac{1}{\lambda^m} \left(\frac{1}{2\pi i}\right)^{n+1} \int\limits_{|\tau|=1} \int\limits_{|w_1|=r_1} \cdots \int\limits_{|w_n|=r_n} \frac{f(\tau z, w_1 e_1 + \ldots + w_n e_n)}{\tau^{k+1} w_1^{m_1+1} \ldots w_n^{m_n+1}} \, d\tau \, dw_1 \ldots dw_n.$$

In order to obtain a further estimate from this representation we remark that for each $0 < \varrho < 1$ and each $t > 0$ the set

$$B(\varrho, t) := \varrho \bar{G} \times \{x \in E \mid x = \sum_{j=1}^{\infty} \xi_j e_j, \, |\xi_j| \leqslant t\mu_j \text{ for all } j \in N\}$$

is compact. Hence f is bounded on $B(\varrho, t)$. We put

$$N(\varrho, t) := \sup \{|f(w)| \mid w \in B(\varrho, t)\}$$

and get from (3) the following estimate:

$$(4)\qquad \sup_{z \in \varrho \bar{G}} |a_{k,m}(z)| \leqslant \frac{N(\varrho, t)}{\lambda^m \mu^m t^{|m|}} \quad \text{for all } (k, m) \in N_0 \times M.$$

Now let $r > 0$ and $0 < \varrho < 1$ be given and choose $\sigma > \sqrt{\varrho}$. Then find $0 < \gamma < 1$ such that $\sigma^\gamma > \sqrt{\varrho}$ and $\delta^{1-\gamma} > \sqrt{\varrho}$. Next put $\beta := 1 - \gamma$, choose $\gamma < \eta < 1$ and put $\nu := \eta - \gamma$ and $d := (1/\eta) - 1$. Since λ is in s, the sequence $(\lambda_j^\nu / \mu_j)_{j \in N} = (j\lambda_j^\nu / \delta)_{j \in N}$ is in l^1 and hence $R := \sup_{k \in N} \lambda_k^\nu \mu_k^{-1} < \infty$. We put $t := (2Rr)^{1/\gamma}$ and $D := C_d^\eta$. Then we get the following estimate from (1), (2) and (4):

$$(5)\qquad \sum_{m \in M} \sum_{k \in N_0} r^{|m|} \sup_{z \in \varrho \bar{G}} |a_{k,m}(z)| \prod_{j=1}^{\infty} \|\varphi_j\|_X^{*m_j}$$

$$\leqslant D \sum_{m \in M} \sum_{k \in N_0} r^{|m|} (\lambda^m)^\eta \sup_{z \in \varrho \bar{G}} |a_{k,m}(z)|$$

$$\leqslant D \sum_{m \in M} \sum_{k \in N_0} r^{|m|} (\lambda^m \sup_{z \in \varrho \bar{G}} |a_{k,m}(z)|)^\gamma (\lambda^m)^{\eta - \gamma} (\sup_{z \in \varrho \bar{G}} |a_{k,m}(z)|)^{1-\gamma}$$

$$\leqslant D \sum_{m \in M} \sum_{k \in N_0} r^{|m|} \left(\left(\frac{\varrho}{\sigma}\right)^k \frac{N(\sigma, t)}{\mu^m t^{|m|}}\right)^\gamma (\lambda^m)^\nu \left(M \left(\frac{\varrho}{\delta}\right)^k \frac{1}{\mu^m}\right)^\beta$$

$$= DN(\sigma, t)^\gamma M^\beta \sum_{m \in M} \sum_{k \in N_0} \left(\frac{r}{t^\gamma}\right)^{|m|} \frac{(\lambda^m)^\nu}{\mu^m} \left(\frac{\varrho}{\sigma^\gamma \delta^\beta}\right)^k$$

$$= DN(\sigma, t)^\gamma M^\beta \left[\sum_{k=0}^{\infty} \left(\frac{\varrho}{\sigma^\gamma \delta^\beta}\right)^k\right] \sum_{m \in M} \left(\frac{\lambda^\nu}{2R\mu}\right)^m$$

$$= DN(\sigma, t)^\gamma M^\beta \left[\sum_{k=0}^{\infty} \left(\frac{\varrho}{\sigma^\gamma \delta^\beta}\right)^k\right] \prod_{k=1}^{\infty} \left(1 - \frac{\lambda_k^\nu}{2R\mu_k}\right)^{-1} < \infty,$$

since $\sigma^\gamma \delta^\beta > \sqrt{\varrho} \cdot \sqrt{\varrho} = \varrho$.

This implies that the series

$$\sum_{m \in M} \sum_{k \in N_0} a_{k,m}(z) \prod_{j=1}^{\infty} \varphi_j(x)^{m_j}$$

converges normally on all sets

$$\varrho \bar{G} \times \{x \in X \mid \|x\| \leqslant r\}, \qquad 0 < \varrho < 1, \, r > 0.$$

Hence it defines a function $h \in H_{\mathrm{ub}}(G \times X)$.

To complete the proof we now show that $h \circ u = f$.

For $x = \sum_{j=1}^{n} \xi_j e_j \in E$ and each $z \in G$ we have with $\xi_j = 0$ for $j > n$ because of $A = v \circ u$ and $(Ax|y_j)_F = \lambda_j \xi_j$,

$$h(z, u(x)) = \sum_{m \in M} \sum_{k \in N_0} a_{k,m}(z) \prod_{j=1}^{\infty} \left(v(u(x))|y_j\right)_F^{m_j}$$

$$= \sum_{m \in M} \sum_{k \in N_0} a_{k,m}(z) \lambda^m \xi^m.$$

Because of (3) and the fact that every $g \in H(G \times C^n)$ can be represented by the mixed polynomial-monomial Taylor series, this shows $h \circ (\mathrm{id}_G \times u) = f$ on a dense subset of $G \times E$ and consequently on $G \times E$.

We shall also need the following variant of Lemma 3.1:

3.2. Lemma. *Let E and F be Hilbert spaces and X a Banach space. Assume that the following hypotheses are satisfied:*

(a) $A \in L(E, F)$ is injective; $A = v \circ u$, where $u \in L(E, X)$ and $v \in L(X, F)$.

(b) There exists $d > 0$ such that A is of type $1/(2+2d)$ and such that for some $C_d > 0$ and all $y \in F'$

$$\|v^t y\|_X^{*1+d} \leqslant C_d \|A^t y\|_E^* \|y\|_F^{*d}.$$

(c) For $f \in H(E)$ there exist a zero neighbourhood W in F and $g \in H^\infty(W)$ with $f|A^{-1}(W) = g \circ A$.

Then there exists $h \in H_{\mathrm{ub}}(X)$ with $h \circ u = f$.

Proof. This follows from the proof of Lemma 3.1 by the remark that hypothesis (b) of 3.1 was used only in (5) to get $\sigma^\gamma \delta^\beta > \varrho$. If we omit G, then these terms do not appear and then the present hypothesis (b) is sufficient, since the estimate corresponding to (5) can be done with the following choices: $\eta := 1/(1+d)$, $\gamma := \nu := \eta/2$, $\beta := 1-\gamma$.

From Proposition 2.6 and Lemma 3.1 we obtain the following holomorphic characterization of the nuclear Fréchet spaces with property $(\bar{\Omega})$.

3.3. Theorem. *Let E be a nuclear Fréchet space. The following are equivalent*:

(1) *E has $(\bar{\Omega})$.*

(2) *There exists an absolutely convex compact subset B in E for which E_B is dense in E having the following property: For every $N \in \boldsymbol{N}$ and every bounded absolutely convex open subset G of $\boldsymbol{C}^N$ and every $f \in H(G \times E_B)$ which has a holomorphic extension to some zero neighbourhood in $\boldsymbol{C}^N \times E$ there exists $h \in H_{\mathrm{ub}}(G \times E)$ with $h|G \times E_B = f$.*

(3) *For every $N \in \boldsymbol{N}$ and every bounded absolutely convex open subset $G \neq \emptyset$ in $\boldsymbol{C}^N$ every holomorphic function on $G \times E$ is of uniformly bounded type.*

(4) *There exist $N \in \boldsymbol{N}$ and a bounded absolutely convex open subset $G \neq \emptyset$ in $\boldsymbol{C}^N$ such that every holomorphic function on $G \times E$ is of uniformly bounded type.*

Proof. (1)$\Rightarrow$(2). Since E is nuclear, we may choose a semi-norm system $(\|\ \|_n)_{n \in \boldsymbol{N}}$ generating the topology of E in such a way that the corresponding canonical spaces are Hilbert spaces. Moreover, since E has $(\bar{\Omega})$ it follows from Vogt [25], 4.4, that there exists a compact absolutely convex subset B of E such that the canonical space E_B is a Hilbert space and such that the following holds:

(*) For every $p \in \boldsymbol{N}$ there exists $q \in \boldsymbol{N}$ such that for each $d > 0$ there exists $C > 0$ with $\|\ \|_q^{*1+d} \leqslant C \|\ \|_B^* \|\ \|_p^{*d}$.

We remark that (*) implies that E_B is dense in E.

Now let N, G and f be as in (2). Then there exist $p \in \boldsymbol{N}$ and an open polydisk $\boldsymbol{D}_p$ in $\boldsymbol{C}^N$ such that f has a holomorphic extension $\tilde{f}$ to $\boldsymbol{D}_p \times U_p$ which is even bounded there. By a standard argument this implies the existence of $g \in H^\infty(\boldsymbol{D}_p \times \hat{U}_p)$ satisfying $f|\boldsymbol{D}_p \times (E_B \cap U_p) = g \circ (\mathrm{id}_{\boldsymbol{D}_p} \times \pi_p)$. We remark that $E_B \cap \ker \pi_p$ is a closed linear subspace of the Hilbert space E_B. Hence there exists an orthogonal projection π on E_B with range E_0 and $\ker \pi = E_B \cap \ker \pi_p$. Since f is bounded on $\boldsymbol{D}_p \times (E_B \cap U_p)$ it follows from Liouville's theorem and analytic continuation that $f = f \circ (\mathrm{id}_G \times \pi)$. In order to apply Lemma 3.1 we choose for $p \in \boldsymbol{N}$ a natural number q according to (*). Then we put $A := \pi_p|E_0$ and remark that A is injective and of type s, since E is nuclear. Furthermore we put $X = \hat{E}_q$, $u := \pi_q|E_0$ and $v := \pi_{pq}$. Then 3.1(a) holds and 3.1(b) follows from (*) since $(\hat{E}_n)'_b$ is isometric isomorphic to $(E')_{U_n^\circ}$. By the hypotheses in (2) and the preceding considerations also 3.1(c) is satisfied for $f|G \times E_0$. Hence we get from 3.1 the existence of $\tilde{h} \in H_{\mathrm{ub}}(G \times \hat{E}_q)$ satisfying $\tilde{h} \circ (\mathrm{id}_G \times u) = f|E_0$. Then $h := \tilde{h} \circ (\mathrm{id}_G \times \pi_q)$ is in $H_{\mathrm{ub}}(G \times E)$. From the definition of $\tilde{h}$ according to 3.1 we get for all $(z, x) \in G \times E_B$

$$h(z, x) = \tilde{h}(z, \pi_q(x)) = \sum_{k \in N_0} \sum_{m \in M} a_{k,m}(z) \prod_{j=1}^{\infty} (\pi_{pq}\pi_q(x)|y_j)^{m_j}$$

$$= \sum_{k \in N_0} \sum_{m \in M} a_{k,m}(z) \prod_{j=1}^{\infty} (\pi_p(x)|y_j)^{m_j} = h(z, \pi(x)),$$

since $\pi_p(x) = \pi_p(\pi(x))$. Because of $f = f \circ (\mathrm{id}_G \times \pi)$ this implies $h|G \times E_B = f$.

(2)$\Rightarrow$(3). Let B be a compact absolutely convex subset of E having the properties mentioned in (2). If $g \in H(G \times E)$ then it is obvious that $f := g|G \times E_B$ has all the properties required in (2). Hence there exists $h \in H_{\mathrm{ub}}(G \times E)$ with $h|G \times E_B = f = g|G \times E_B$. Since $G \times E_B$ is dense in $G \times E$ this implies $h = g$ and hence $H(G \times E) = H_{\mathrm{ub}}(G \times E)$.

(3)$\Rightarrow$(4) holds trivially.

(4)$\Rightarrow$(1). In the proof of 2.6 we have remarked that $H(G) \simeq \Lambda_1(n^{1/N})$. Hence we get from Lemma 2.2 $L(E, \Lambda_1(n^{1/N})) = LB(E, \Lambda_1(n^{1/N}))$. By Vogt [25], 4.2, this implies that E has $(\bar{\Omega})$.

3.4. Corollary. *For a nuclear Fréchet space E the following are equivalent*:

(1) *$H(P) = H_{\mathrm{ub}}(P)$ for every (some nonempty) polycylindrical subset P of E.*

(2) *E has $(\bar{\Omega})$.*

Proof. (1)$\Rightarrow$(2): Proposition 2.6.

(2)$\Rightarrow$(1). If P is a polycylindrical subset of E then we have seen in the proof of 2.6 that we may identify E with $\boldsymbol{C}^k \times E_0$ in such a way that P corresponds to the set $G \times E_0$, where G is a bounded, open and absolutely convex subset of $\boldsymbol{C}^k$. Since E has $(\bar{\Omega})$, also E_0 has $(\bar{\Omega})$. Hence (1) follows from 3.3.

We have not been able to prove or to disprove the converse of Proposition 2.3. To give a sufficient condition for a nuclear Fréchet space E to satisfy $H(E) = H_{\mathrm{ub}}(E)$ we recall from Vogt [25], Sect. 5, the definition of property $(\tilde{\Omega})$.

3.5. Definition. A Fréchet space E has *property* $(\tilde{\Omega})$ if:

For every $p \in \boldsymbol{N}$ there exist $q \in \boldsymbol{N}$ and $d > 0$ such that for every $k \in \boldsymbol{N}$ there exists $C > 0$ such that for every $y \in E'$

$$\|y\|_q^{*1+d} \leqslant C \|y\|_k^* \|y\|_p^{*d}.$$

Remark. (a) It is easy to check that $(\tilde{\Omega})$ is a linear topological invariant, i.e. that $(\tilde{\Omega})$ depends only on the linear topology of E and not on the seminorm system $(\| \ \|_n)_{n \in \mathbf{N}}$.

(b) By standard arguments it follows that $(\tilde{\Omega})$ is inherited by separated quotient spaces.

A suitable modification of the proof of Vogt [24], 1.4 shows

3.6. Lemma. *A Fréchet–Schwartz space E has $(\tilde{\Omega})$ if and only if there exists an absolutely convex compact subset B of E such that E has the following property $(\tilde{\Omega}_B)$:*

For every $p \in \mathbf{N}$ there exist $q \in \mathbf{N}$, $d > 0$ and $C > 0$ such that for all $y \in E'$

$$\|y\|_q^{*\,1+d} \leqslant C \|y\|_B^* \|y\|_p^{*d}.$$

3.7. Examples. (a) A Köthe space $\lambda^s(A)$, $1 \leqslant s < \infty$ or $s = 0$, has $(\tilde{\Omega})$ iff A has the following property:

For every $p \in \mathbf{N}$ there exist $q \in \mathbf{N}$ and $d > 0$ such that for each $k \in \mathbf{N}$ there exists $C > 0$ such that for all $j \in \mathbf{N}$

$$a_{j,k} a_{j,p}^d \leqslant C a_{j,q}^{1+d}.$$

This follows in a standard way (see Vogt and Wagner [28], 2.3, resp. Dineen, Meise and Vogt [8], 4a)).

(b) Using (a) it is easy to see that every power series space $\Lambda_1^s(\alpha)$, but none of the spaces $\Lambda_\infty^s(\alpha)$, has $(\tilde{\Omega})$.

(c) As Vogt [25], 5.6 shows, there exist nuclear Fréchet spaces $\lambda(B)$ with continuous norm which have $(\tilde{\Omega})$ but are not quotients of power series spaces of finite type.

For further examples of nuclear Fréchet spaces with property $(\tilde{\Omega})$ we refer to Section 4 and to Vogt [27], 3.5.

We recall that by Dineen, Meise and Vogt [8], Th. 9 (cf. [9]) a nuclear Fréchet space E has property $(\tilde{\Omega})$ if and only if E contains a bounded subset which is not uniformly polar in E.

3.8. Proposition. *Let E be a Fréchet space and let B be an absolutely convex closed bounded set in E for which E_B is dense. If E does not have property $(\tilde{\Omega}_B)$ then the following holds:*

(a) *There exists $f \in H_{\mathrm{ub}}(E_B)$ which has a holomorphic extension to some zero neighbourhood of E but which is not continuous on E_B for the topology τ_E induced by E. Hence f cannot be extended holomorphically to E.*

(b) *If in addition E has the bounded approximation property and if B is compact, then there exists $f \in H_{\mathrm{ub}}(E_B)$ which is τ_E continuous but cannot be extended holomorphically to E.*

Proof. (a) In the proof of Th. 7 of Dineen, Meise and Vogt [8], the following has been shown: If E does not have $(\tilde{\Omega}_B)$, then there exist $p \in \mathbf{N}$ and

sequences $(y_k)_{k \in \mathbf{N}}$ in E_p', $(x_k)_{k \in \mathbf{N}}$ in E_B and a zero sequence $(a_k)_{k \in \mathbf{N}}$ of positive real numbers such that

(1) $\|y_k\|_p^* = 1/k$, $\quad y_k(x_k) \geqslant a_k^2$ and $\quad a_k^k \geqslant \|y_k\|_B^*$ for all $k \in \mathbf{N}$,

(2) $y_j(x_k) = 0$ for all $j, k \in \mathbf{N}$ with $j \neq k$,

(3) $\sum_{k=1}^{\infty} x_k$ is absolutely converging in E.

From this we obtain the existence of a strictly increasing sequence $(k_j)_{j \in \mathbf{N}}$ in $\mathbf{N}$ with $k_1 > 1$ for which $m_j := [-\log a_{k_j}^2]$ is strictly increasing. Then we have

$$\lim_{j \to \infty} (a_{k_j}^2)^{1/m_j} = 1/e. \tag{4}$$

From (1) and (4) we get

$$\varlimsup_{j \to \infty} (\|y_{k_j}\|_B^*)^{1/m_j} \leqslant \varlimsup_{j \to \infty} (a_{k_j}^2)^{k_j/2m_j} = 0. \tag{5}$$

By (2) and (5) we have for every $r > 0$ and each $x \in rB$

$$\sum_{j=1}^{\infty} |y_{k_j}(x)(y_1(x))^{m_j}| \leqslant r \sum_{j=1}^{\infty} \|y_{k_j}\|_B^* (r a_1)^{m_j} = C_r < \infty.$$

Hence a function $f\colon E_B \to \mathbf{C}$ can be defined by

$$f(x) := \sum_{j=1}^{\infty} y_{k_j}(x)(y_1(x))^{m_j}$$

and is in $H_{\mathrm{ub}}(E_B)$. From (1) it follows that for all $0 < r < 1$ and all $x \in rU_p$ we have

$$\sum_{j=1}^{\infty} |y_{k_j}(x)(y_1(x))^{m_j}| \leqslant \sum_{j=1}^{\infty} \frac{r}{k_j} r^{m_j} \leqslant \sum_{j=1}^{\infty} r^j.$$

This shows that f has a holomorphic extension to the zero neighbourhood U_p in E. In order to prove that f is not τ_E-continuous we put $a := 2e/y_1(x_1)$ and show that f is unbounded on every τ_E-neighbourhood of ax_1. To see this let U be an arbitrary zero neighbourhood of E. From (3) we get the existence of $q \in \mathbf{N}$ such that

$$z_n := \sum_{j=q+1}^{n} x_{k_j} \in U \cap E_B \quad \text{for all } n > q.$$

Hence we get for $\xi_n := ax_1 + z_n$, $n > q$, because of (2) and (1),

$$f(\xi_n) = \sum_{j=q+1}^{n} y_{k_j}(\xi_n)(y_1(\xi_n))^{m_j} = \sum_{j=q+1}^{n} y_{k_j}(x_{k_j})(a y_1(x_1))^{m_j}$$

$$\geqslant \sum_{j=q+1}^{n} a_{k_j}^2 (2e)^{m_j}.$$

Because of (4) this shows that $(f(\xi_n))_{n>q}$ is unbounded and hence f is not τ_E-continuous.

(b) By Dineen, Meise and Vogt [8], Th. 7, E_B is a polar subset of E. Then it follows from Noverraz [18], Prop. 2, that there exists a pseudoconvex open set U in E with $E_B \subset U \subsetneqq E$. By Schottenloher [22], Cor. 3.4, U is a domain of existence for some $g \in H(U)$. If we put $f = g|E_B$ then $f \in H_{ub}(E_B)$ because of the compactness of B. Clearly f has all the desired properties.

3.9. THEOREM. *For a nuclear Fréchet space E the following assertions are equivalent*:

(i) *E has* $(\tilde{\Omega})$.

(ii) *There exists an absolutely convex compact subset B of E for which E_B is dense in E such that for every $f \in H(E_B)$ which has a holomorphic extension to some zero neighbourhood in E there exists $g \in H_{ub}(E)$ with $g|E_B = f$.*

(iii) *There exists an absolutely convex compact subset B of E for which E_B is dense in E such that for every $f \in H(E_B)$ which has a holomorphic extension to some zero neighbourhood in E there exists $g \in H(E)$ with $g|E_B = f$.*

(iv) *There exists an absolutely convex compact subset B of E for which E_B is dense in E such that*

$$H(E_B, \tau_E) = \{f \in H(E_B) \mid f \text{ has a holomorphic extension to some zero neighbourhood in } E\}.$$

If in addition E has the bounded approximation property, then (i)–(iv) *are also equivalent to*

(v) *There exists an absolutely convex compact subset B of E for which E_B is dense in E such that for every $f \in H(E_B, \tau_E)$ there exists $g \in H(E)$ with $g|E_B = f$.*

Proof. (i)$\Rightarrow$(ii). Using Lemma 3.6 and Lemma 3.2 this is shown in the same way as (1)$\Rightarrow$(2) in the proof of Theorem 3.3.

(ii)$\Rightarrow$(iii) and (iii)$\Rightarrow$(iv) hold trivially.

(iv)$\Rightarrow$(i). This follows by 3.6 from 3.8 (a).

(iii)$\Rightarrow$(v). If $f \in H(E_B, \tau_E)$ then clearly $f \in H(E_B)$. Since f is bounded on a zero neighbourhood of (E_B, τ_E), the Cauchy inequalities show by a standard argument that f has a holomorphic extension to some zero neighbourhood in E. Hence f satisfies the hypotheses of (iii) and consequently there exists $g \in H(E)$ with $g|E_B = f$.

(v)$\Rightarrow$(i). Since E has the bounded approximation property, the implication follows by 3.6 from 3.8 (b).

Remark. The implication (i)$\Rightarrow$(ii) of Theorem 3.9 has already been used in the proof of Th. 10 of Dineen, Meise and Vogt [8].

From 3.9 (ii) one gets immediately the following corollary:

3.10. COROLLARY. *If a nuclear Fréchet space E has property* $(\tilde{\Omega})$ *then* $H(E) = H_{ub}(E)$.

3.11. Remark. To abbreviate let us say that a Fréchet space F has *property* (H_{ub}) if $H(F) = H_{ub}(F)$. It is easy to see that (H_{ub}) is a linear topological invariant which is inherited by quotient spaces (see Meise and Vogt [14], 4.b)). By Proposition 2.3 and Corollary 3.13 we have the following implications:

$$(\tilde{\Omega}) \Rightarrow (H_{ub}) \Rightarrow (\mathrm{LB}^\infty).$$

By Vogt [25], 5.5, the class (LB^∞) is strictly larger than the class $(\tilde{\Omega})$. However, we do not know whether one of the implications above can be reversed or whether (H_{ub}) is actually a new invariant for nuclear Fréchet spaces.

4. Applications and examples. In this section we give some applications of the results of the previous section and examples of nuclear Fréchet spaces of holomorphic functions in infinitely many variables which have property $(\tilde{\Omega})$ but are not quotients of power series spaces of finite type.

4.1. PROPOSITION. *Let E be a Fréchet space with* (H_{ub}). *Then* $(H(E), \tau_0)_{bor}$, *the bornological space associated to* $(H(E), \tau_0)$, *is a regular* (LF)-*space.*

Proof. Let $(U_n)_{n\in N}$ be a decreasing neighbourhood basis of zero in E consisting of absolutely convex sets U_n and put

$$H_n := \{f \in H(E) \mid f \text{ is bounded on } kU_n \text{ for all } k \in N\}.$$

H_n endowed with the semi-norms $\pi_{n,k}\colon f \mapsto \sup_{x\in kU_n} |f(x)|$ $(k \in N)$ becomes a Fréchet space. Since E has (H_{ub}), we have $H(E) = \bigcup_{n\in N} H_n$. Obviously the identity map $I\colon \operatorname{ind}_{n\to} H_n \to (H(E), \tau_0)$ is continuous. Since $\operatorname{ind}_{n\to} H_n$ is bornological, I is even continuous in $(H(E), \tau_0)_{bor}$. Since $(H(E), \tau_0)$ is complete $(H(E), \tau_0)_{bor}$ is ultrabornological. Hence the open mapping theorem shows that I is an isomorphism, i.e. $(H(E), \tau_0)_{bor}$ is an (LF)-space. The regularity of $\operatorname{ind}_{n\to} H_n$ follows from this and the fact that the Banach disks are confinal in the bounded sets of $(H(E), \tau_0)$.

Remark. By Dineen [6], Ex. 21, $(H(\Lambda_1(\alpha)), \tau_0)$ is not bornological for any nuclear space $\Lambda_1(\alpha)$.

We recall that $A \in L((H(E), \tau))$ is called a *convolution operator* if A commutes with translations, i.e. $A(f(\cdot - x))[z] = A(f)[z-x]$ for all z and all x in E and all $f \in H(E)$.

4.2. PROPOSITION. *Let E be a nuclear Fréchet space with* (H_{ub}). *Then every nonzero convolution operator on* $(H(E), \tau_0)_{bor}$ *and on* $(H(E), \tau_0)$ *is surjective.*

Proof. For $(H(E), \tau_0)_{bor}$ this follows from Colombeau and Matos [3], 4.9, since the proof of 4.1 shows that $(H(E), \tau_0)_{bor}$ coincides with their l.c. space $H_{ub}(E)$. If $A \neq 0$ is a convolution operator on $(H(E), \tau_0)$, then A is also continuous for the topology of $(H(E), \tau_0)_{bor}$. Hence A is a convolution operator on $(H(E), \tau_0)_{bor}$, and consequently is surjective.

4.3. Corollary. *If a nuclear Fréchet space E has $(\tilde{\Omega})$ then every nonzero convolution operator on $(H(E), \tau_0)$ is surjective.*

4.4. Proposition. *Let G be a locally convex space and let E be a topological linear subspace of G which is a nuclear Fréchet space with property $(\tilde{\Omega})$. Then the restriction map $\varrho_{G,E}$: $H(G) \to H(E)$, $\varrho_{G,E}(f) := f|E$, is surjective.*

Proof. Because of Corollary 3.10 we have $H(E) = H_{ub}(E)$. Hence the result follows from Hollstein [10], Th. 1, since every nuclear space is an ε-space.

4.5. Remark. In [12] we have shown that every infinite-dimensional nuclear Fréchet space F which is not isomorphic to C^N contains a closed linear subspace E for which the restriction map $\varrho_{F,E}$: $H(F) \to H(E)$ is not surjective. This effect does not rely only on the fact that E is not a complemented subspace of F. To see this let $\Lambda_1(\alpha)$ be stable and nuclear. By Vogt and Wagner [29], 2.4, there exists an exact sequence

$$0 \to \Lambda_1(\alpha) \xrightarrow{j} \Lambda_1(\alpha) \to \Lambda_1(\alpha)^N \to 0.$$

Since $\Lambda_1(\alpha)^N$ does not have a continuous norm, $E := j(\Lambda_1(\alpha))$ is not a complemented subspace of $F := \Lambda_1(\alpha)$. However, $\varrho_{F,E}$ is surjective. This follows from 4.4 or from [14], Th. 6 and Corollary 3.10.

In order to give examples of nuclear Fréchet spaces of holomorphic functions which have property $(\tilde{\Omega})$ and which are not quotient spaces of power series spaces of finite type we recall the following notation. If $\lambda(P)$ is a nuclear Fréchet space, then for $a \in \lambda(P)$, $a \geq 0$, the set

$$D_a := \{x \in \lambda(P)' \mid \sup_{j \in N} |x_j| a_j < 1\}$$

is an open subset of $\lambda(P)'_b$, called an *open polydisc*.

4.6. Proposition. *Let $\lambda(P)$ be a nuclear Fréchet space and let $a \in \lambda(P)$, $a \geq 0$, be given. The following are equivalent:*

(1) *$(H(D_a), \tau_0)$ has property $(\tilde{\Omega})$.*

(2) *$a > 0$ and for every $p \in N$ there exist $q \in N$, $d > 0$ and $J_p \in N$ such that, for all $j \geq J_p$, $p_{j,p}^d \leq a_j p_{j,q}^{d+1}$.*

Proof. (1) $\Rightarrow$ (2). First we remark that $a_j = 0$ for some $j \in N$ implies that $H(C)$ is a complemented subspace of $(H(D_a), \tau_0)$. Since $H(C)$ does not have $(\tilde{\Omega})$ this implies that $(H(D_a), \tau_0)$ does not have $(\tilde{\Omega})$. Hence (1) implies $a_j > 0$ for all $j \in N$.

From [13] we recall that $(H(D_a), \tau_0)$ is a nuclear Fréchet space which is isomorphic to the sequence space $\lambda(M, Q^M)$, where $M = \{m \in N_0^N \mid m_j \neq 0$ only for finitely many $j \in N\}$ and where $Q^M = \{(q_k^m)_{m \in M} \mid k \in N\}$. The definition of $q_k = (q_{j,k})_{j \in N}$ is

$$q_{j,k} = \begin{cases} \varrho_k/a_j & \text{for } 1 \leq j < n_k, \\ p_{j,k} & \text{for } n_k \leq j, \end{cases}$$

where $(n_k)_{k \in N}$ and $(\varrho_k)_{k \in N}$ are defined as in [13], 2.4. (Obviously we can assume that the Köthe matrix P satisfies the conditions in 1.2 as well as the following: For each $k \in N$ there exists $\lambda \in c_0$ such that $p_k \leq \lambda p_{k+1}$.) Because of the characterization of $(\tilde{\Omega})$ given in 3.7 (a) we get from (1):

(*) For every $p \in N$ there exist $q \in N$ and $d > 0$ such that for each $k \in N$ there exists $C > 0$ such that for all $m \in M$

$$q_p^{md} q_k^m \leq C q_q^{m(d+1)}.$$

Choosing $m = s(\delta_{ij})_{i \in N}$, $s \in N$, gives

$$q_{j,p}^{sd} q_{j,k}^s \leq C q_{j,q}^{s(d+1)},$$

which implies

(**) $$q_{j,p}^d q_{j,k} \leq q_{j,q}^{d+1} \quad \text{for all } j \in N.$$

Now we put $J_p := n_q$ and fix $j \geq J_p$. By the definition of the sequences q_l we then get for all k with $n_k > j$

$$p_{j,p}^d \frac{\varrho_k}{a_j} = q_{j,p}^d q_{j,k} \leq q_{j,q}^{d+1} = p_{j,q}^{d+1}.$$

Since $\lim_{k \to \infty} \varrho_k = 1$ this implies (2).

(2) $\Rightarrow$ (1). In order to show that (*) holds, it suffices to show (**) for all $j \in N$. To do this choose $p \in N$ arbitrarily and find q, d and J_p according to (2). The proof of [13], 2.4 shows that we can assume without restriction that $n_p \geq J_p$ for all $p \in N$. From (2) we get

$$\frac{p_{j,p}^d}{p_{j,q}^{d+1}} p_{j,k} \leq a_j p_{j,k} \quad \text{for all } j \geq J_p \text{ with } p_{j,q} > 0.$$

Since $\lim_{j \to \infty} a_j p_{j,k} = 0$ we can assume without restriction that $a_j p_{j,k} \leq 1$ for all $j \geq J_k$. Hence we have for all $j \geq n_k \geq J_k$

$$p_{j,p}^d p_{j,k} \leq p_{j,q}^{d+1}.$$

Since we can assume $p < q$, (**) is trivially satisfied if $k \leq q$. For $k > p$ we distinguish the following 4 cases:

1. $n_k \leq j$. Then the preceding consideration shows that from $j \geq n_k \geq J_k$ and the definition of the sequences q_l we get for all $\delta \geq d$

$$q_{j,p}^\delta q_{j,k} = p_{j,p}^\delta p_{j,k} \leq p_{j,q}^{\delta+1} = q_{j,q}^{\delta+1}.$$

2. $n_q \leq j < n_k$. From (2) we get, for all $\delta \geq d$ and $j \geq n_q \geq n_p \geq J_p$, $p_{j,p}^\delta \leq a_j p_{j,q}^{\delta+1}$. This implies

$$q_{j,p}^\delta q_{j,k} = p_{j,p}^\delta \frac{\varrho_k}{a_j} \leq p_{j,q}^{\delta+1} = q_{j,q}^{\delta+1}.$$

3. $n_p \leqslant j < n_q$. By the choice of $(n_k)_{k\in N}$ we have $\sup_{j\geqslant n_p} p_{j,p} a_j < \varrho_p$. If we choose d_1 such that for all $\delta \geqslant d_1$ we have $(\varrho_p/\varrho_q)^\delta \leqslant \varrho_q$, then we get, for all $j \geqslant n_p$, $(p_{j,p} a_j)^\delta \leqslant \varrho_p^\delta \leqslant \varrho_q^{\delta+1}$. This implies

$$q_{j,p}^\delta q_{j,k} = p_{j,p}^\delta \frac{\varrho_k}{a_j} \leqslant \left(\frac{\varrho_q}{a_q}\right)^{\delta+1} = q_{j,q}^{\delta+1}.$$

4. $j < n_p$. For all $\delta \geqslant d_1$ we have

$$q_{j,p}^\delta q_{j,k} = \left(\frac{\varrho_p}{a_j}\right)^\delta \frac{\varrho_k}{a_j} \leqslant \left(\frac{\varrho_q}{a_j}\right)^{\delta+1} = q_{j,q}^{\delta+1}.$$

Hence we have for $\delta = \max(d, d_1)$

$$q_{j,p}^{\delta+1} q_{j,k} \leqslant q_{j,q}^{\delta+1} \quad \text{for all } j\in N \text{ and all } k\in N.$$

This shows that $\lambda(M, Q^M)$ and hence $(H(D_a), \tau_0)$ has $(\tilde{\Omega})$.

4.7. COROLLARY. *A nuclear Fréchet space $\lambda(P)$ has $(\tilde{\Omega})$ if and only if there exists $a\in\lambda(P)$, $a > 0$, such that $(H(D_a), \tau_0)$ has $(\tilde{\Omega})$.*

Proof. If $\lambda(P)$ has $(\tilde{\Omega})$ then P satisfies the condition mentioned in 3.7 (a). From this we get for each $p\in N$ the existence of $x^{(p)}\in\lambda(P)$, $x^{(p)} > 0$, such that there exist $q\in N$ and $d > 0$ satisfying $p_{j,p}^d \leqslant x_j^{(p)} p_{j,q}^{d+1}$ for all $j\in N$. Since $\lambda(P)$ is a nuclear Fréchet space, we can assume for each $p\in N$ $\lim_{j\to\infty} (x_j^{(p)}/x_j^{(p+1)}) = 0$. Then there exists $a\in\lambda(P)$, $a > 0$, such that for each $p\in N$ there exists $C_p > 0$ such that $\sup_{j\in N}(x_j^{(p)}/a_j) \leqslant C_p$. This implies

$$x_j^{(p)} = \frac{x_j^{(p)}}{x_j^{(p+1)}} x_j^{(p+1)} \leqslant \frac{x_j^{(p)}}{x_j^{(p+1)}} C_{p+1} a_j \quad \text{for all } j\in N.$$

Hence there exists J_p such that for all $j \geqslant J_p$ we have $x_j^{(p)} \leqslant a_j$ and hence $p_{j,p}^d \leqslant a_j p_{j,q}^{d+1}$. Because of 4.6 (2) this shows that $(H(D_a), \tau_0)$ has $(\tilde{\Omega})$.

If for some open subset U of $\lambda(P)'_b$ the space $(H(U), \tau_0)$ has $(\tilde{\Omega})$ then $\lambda(P)$ has $(\tilde{\Omega})$, since $\lambda(P)$ is a complemented subspace of $(H(U), \tau_0)$.

4.8. COROLLARY. *Let $\Lambda_1(\alpha)$ be nuclear.*

(a) *For $a\in\Lambda_1(\alpha)$, $a > 0$, $(H(D_a), \tau_0)$ has $(\tilde{\Omega})$ if and only if*

$$\liminf_{j\to\infty} a_j^{1/\alpha_j} > 0.$$

(b) *For $0 < r < 1$ put $a := (r^{\alpha_j})_{j\in N}$. Then $(H(D_a), \tau_0)$ has $(\tilde{\Omega})$ but is not a quotient of a power series space of finite type.*

Proof. (a) If $(H(D_a), \tau_0)$ has $(\tilde{\Omega})$ then we deduce from 4.6 that for each $0 < r < 1$ there exist $r < s < 1$, $d > 0$ and J such that for all $j \geqslant J$

$$r^{d\alpha_j} \leqslant a_j s^{(d+1)\alpha_j}.$$

This implies $a_j^{1/\alpha_j} \geqslant r^d/s^{d+1} > 0$ and hence $\liminf_{j\to\infty} a_j^{1/\alpha_j} > 0$. On the other hand $\liminf_{j\to\infty} a_j^{1/\alpha_j} > 0$ implies that for some $t > 0$ and some $J\in N$ we have, for all $j \geqslant J$, $a_j \geqslant t^{\alpha_j}$. Now let $0 < r < 1$ be arbitrary. Choose $r < s < 1$ and find $d > 0$ such that $(r/s)^d \leqslant st$. Then we get for all $j \geqslant J$

$$r^{d\alpha_j} \leqslant t^{\alpha_j} s^{(d+1)\alpha_j} \leqslant a_j s^{(d+1)\alpha_j}.$$

This shows that a satisfies 4.6 (2) and hence $(H(D_a), \tau_0)$ has $(\tilde{\Omega})$.

(b) By [13], Th. 3.3, $(H(D_a), \tau_0)$ is a quotient of a power series space of finite type iff $\lim_{j\to\infty} a_j^{1/\alpha_j} = 1$. Hence (b) follows from this and part (a).

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Fredholm spectrum and Grunsky inequalities in general domains

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Abstract. We discuss the Fredholm spectrum for general domains and study its applications to conformal and quasi-conformal mappings. In particular, we establish an improvement of the Grunsky inequalities which is valid for general domains. This improvement constitutes an extension of a recent result of Schiffer concerning the sharpening of Grunsky inequalities for the unit disk by a factor smaller than 1, and which is the reciprocal of the least Fredholm eigenvalue of the smooth simply connected image domain.

§ 1. Introduction. Let φ be a univalent holomorphic function of the unit disk Δ onto a simply connected domain $\Omega^* = \varphi(\Delta)$ whose boundary $\partial\Omega^*$ is of class $C^{2,\varepsilon}$ $(0 < \varepsilon < 1)$ and consider the Grunsky coefficients (c_{mn}) of φ, defined by

$$\log\frac{\varphi(z)-\varphi(\zeta)}{z-\zeta} = \sum_{m,n=0}^{\infty} c_{mn} z^m \zeta^n, \qquad z, \zeta \in \Delta.$$

In a recent paper [13], Schiffer has established the following improved Grunsky inequality:

$$\Big|\sum_{m,n=1}^{\infty} \sqrt{mn}\, c_{mn}\alpha_m\alpha_n\Big| \leqslant (\lambda_1^*)^{-1} \sum_{n=1}^{\infty} |\alpha_n|^2 \tag{1.1}$$

where $\lambda_1^* = \lambda_1(\Omega^*)$ is the least Fredholm eigenvalue of Ω^* and $\{\alpha_n\}$ is an arbitrary sequence of complex numbers. The customary Grunsky inequality is inequality (1.1) with $(\lambda_1^*)^{-1}$ replaced by 1, and as $\lambda_1^* > 1$ because of the smoothness assumptions on $\partial\Omega^*$, the present inequality (1.1) constitutes an improvement on it.

The symmetric matrix (g_{mn}) with $g_{mn} = \sqrt{mn}\, c_{mn}$ is known as the *Grunsky operator* $\mathscr{G}_\varphi$. Its domain of definition is l_2, the space of complex sequences $\alpha = (\alpha_1, \alpha_2, \ldots)$ with the norm

$$\|\alpha\|_2 = \Big\{\sum_{n=1}^{\infty} |\alpha_n|^2\Big\}^{1/2} < \infty.$$