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Periodic points of linked twist mappings

by

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Abstract. Consider a connected surface which is the union of two finite families of annuli, $\{P_i\}$ and $\{Q_j\}$, such that P_i's (analogously Q_j's) are pairwise disjoint and P_i's intersect Q_j's transversally. A linked twist mapping (l.t.m.) is a composition of a few families of twists on $\bigcup P_i$ alternately with families of twists on $\bigcup Q_j$, such that on each of P_i and Q_j all the twists go in the same direction. Contrary to the usual definition we do not assume that Lebesgue measures on the intersecting annuli coincide.

It is proved that, for any l.t.m. H composed of sufficiently strong and at least double twists, periodic saddles and homoclinic points are dense in the domain of H. H also turns out to be topologically transitive.

The problems of estimating the number of periodic points and of the existence of Smale horseshoes are also considered.

§ 1. Introduction. In the paper we prove the following

THEOREM. *If a linked twist mapping* (*l.t.m.*) *H is composed of sufficiently strong* (*a certain Condition $\mathscr{E}$ is satisfied*) *and at least double twists, then the periodic saddles and homoclinic points are dense in the domain X of H. H also turns out to be topologically transitive.*

I shall explain precisely all the terms used here (except Condition $\mathscr{E}$) in Section 2. The dynamics properties following from Condition $\mathscr{E}$ needed in this paper will be described in Section 3. Precise definitions of Condition $\mathscr{E}$ and the related Condition $\mathscr{H}$ will be recalled (from [P1]) in the Appendix (as they involve a great deal of notation not needed in the rest of the paper). Let me now give an informal explanation:

The domain of an l.t.m. is a connected surface which is the union of two families of annuli $\{P_i\}_{i=1}^{p}$, $\{Q_j\}_{j=1}^{q}$ such that P_i's (analogously Q_j's) are pairwise disjoint (we call each P_i a "horizontal" strip and Q_j a "vertical" strip as if they were in the torus T^2; P_i and Q_j intersect each other transversally (i.e. the "horizontal" circles of P_i intersect the "vertical" circles of Q_j transversally). An l.t.m. H is a composition of a few families of twists on $\bigcup P_i$ alternately with families of twists on $\bigcup Q_j$, such that on each of P_i and Q_j all the twists in the composition go in the same direction, each of P_i and Q_j being twisted at least once: $H = G_N \circ F_N \circ \ldots \circ G_1 \circ F_1$. Although the twists preserve the Lebesgue measures on P_i, Q_j, respectively, *we do not assume* *anything about H-invariant measures on X.* In particular, we do not assume that the Lebesgue measures on P_i, Q_j coincide on $P_i \cap Q_j$.

In [P1] I studied l.t.m.'s with "good" invariant measures (satisfying the (K–S) condition, see the definition below, and equivalent to the Lebesgue measures on P_i, Q_j; see [P2] for the toral case). I proved there, under the assumptions of the Theorem from the present paper, the ergodicity of H. In fact, I proved topological transitivity purely geometrically, and only then I used a "good" invariant measure to deduce ergodicity from the abstract Pesin theory (in the version with singularities, see [K–S]). (In fact, I proved in [P1] even topological mixing and the Bernoulli property, see the comments at the ends of § 4 and of 5.2 of the present paper.)

The density of periodic points follows, due to this "good" measure, from the Pesin theory and the Katok theory [K]. However, by analogy with topological transitivity, I expected in [P1] that this should be a geometrical fact. Let me discuss the whole problem with more precision.

Let an H-invariant measure μ satisfy the following Katok–Strelcyn [K–S] condition:

(K–S) *There exists $\alpha > 0$ such that for every $\varepsilon > 0$*

$$\mu(B(\operatorname{Sing} H, \varepsilon)) < \varepsilon^\alpha$$

($\operatorname{Sing} H$ is the set where H is not differentiable and $B(\cdot,\cdot)$ is a ball in a metric equivalent to Euclidean metrics on P_i, Q_j).

Assume that μ is hyperbolic, i.e. the Lyapunov characteristic exponents are nonzero μ-almost everywhere. Then Katok's theorem [K] states that the set of periodic saddles is dense in $\operatorname{supp} \mu$.

Recall that for l.t.m.'s considered in [P1] I assumed the existence of a "good" invariant measure, hence satisfying the above Katok assumptions, with $\operatorname{supp} \mu = X$. Here we do not know a priori whether a (K–S) hyperbolic measure μ with $\operatorname{supp} \mu = X$ exists. Of course, this follows a posteriori from the Theorem (one can take the limit of a sequence of suitable measures supported on a finite number of periodic saddles). In particular, I am not able to answer the following

QUESTION 1. Does the Bowen–Ruelle measure exist and satisfy (K–S) if the twists are strong enough?

One can consider, as a candidate for the Bowen–Ruelle measure, any measure $\mu_{\text{B-R}}$ which is the weak limit of a subsequence of the sequence of measures $n^{-1} \sum_{k=0}^{n-1} H_*^k(\nu)$, where ν is the length measure on a local unstable manifold.

If $X = \bigcup P_i = \bigcup Q_j$ (H is uniformly hyperbolic) the answer to Question 1 is: yes. $\mu_{\text{B-R}}$ turns out to be the Bowen–Ruelle measure. The proof is the same as for the Lozi attractor (see for example [Y]). In fact, to use Katok's theorem about density mentioned above one needs only to know that the

trajectory of μ-almost every point approaches Sing more slowly than the shrinking rate in the stable direction. This assumption can be proved (with the use of the Borel–Cantelli lemma) for Lebesgue-almost every point x ($x \in X_{\text{Leb}}$, $\mu_{\text{Leb}}(X \setminus X_{\text{Leb}}) = 0$) if the twists are strong enough (in particular, the stable and unstable manifolds exist for Lebesgue-almost every point). So we arrive at the following

QUESTION 2. Is it true that $\mu_{\text{B-R}}(X \setminus X_{\text{Leb}}) = 0$?

The question of density of periodic points for an l.t.m. was considered by Devaney [D1] for the simple example of a toral l.t.m. Take for example at least double linear twists

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, \quad \alpha\beta > 0.$$

Devaney gave purely geometric arguments for the density: First observe the density of global (un)stable manifolds $W^u(p)$, $W^s(p)$ of a fixed point p. Next observe that since they go, roughly speaking, in different directions and have no turn-back points, they must intersect each other. The points of intersection are dense in X.

In [P1], p. 48, I asked about a geometric proof of the density in the case where turn-back points exist, for example when $\alpha\beta < -4$. See in Fig. 1 how the consecutive images of an unstable segment $\gamma = \gamma^u$ can look like.

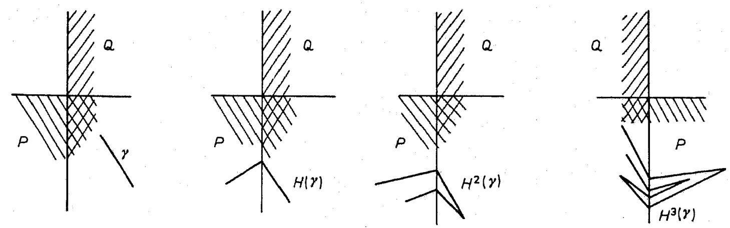

Fig. 1

Although I proved in [P1], under Condition $\mathscr{E}$, that the $H^n(\gamma)$ contain long line segments for large n and I proved the density of global (un)stable manifolds, I did not know how to exclude the possibility shown in Fig. 2. The point was how to deduce the existence of an intersection (homoclinic) point in $W^u(p) \cap W^s(p)$ close to an arbitrary nonperiodic q from the existence of a cycle of length 2 of heteroclinic intersections. If we knew that, the argument for the existence of a periodic point (but not a horseshoe!) near the homoclinic one would be standard, cf. the proof of Proposition 5.3.

From the existence of an invariant measure with support X one could infer (Poincaré Recurrence Theorem) $X = \Omega(H)$. But even then it is not possible to repeat directly the standard proof of the density of periodic saddles for Anosov diffeomorphisms. The trouble is the shadowing argument.

In this paper we cope with these difficulties. Namely we find pieces of trajectories, with beginnings and ends close to an arbitrarily chosen point,

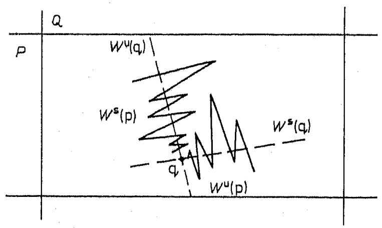

Fig. 2

which do not approach Sing H too closely. Then to shadow these pieces of trajectories by periodic ones we adjust the geometric part of Katok's arguments [K]. We do all this in §§ 3,4. In [D2] Devaney showed the existence of large horsehoes; we do this locally.

Remark. In the proof of the density part of the Theorem in §§ 3,4, I introduce Condition $\mathscr{E}$ and the double twisting assumption since I use topological transitivity. The question arises whether these assumptions are not unnecessarily strong. When a (K–S) hyperbolic measure μ exists, one infers the density of saddles in supp μ even without assumptions about the strength of twists yielding the ergodicity of μ.

The reader will see that in the proof of the Theorem in § 4 the periodic orbit found there need not shadow the original trajectory $z, H(z), \ldots, H^{n_0}(z)$. Thus I cannot cope with the following

Question 3. Is it true that

$$h_{\mathrm{Per}}(H) = \limsup_{n\to\infty} (n^{-1} \log \mathrm{Card}(\mathrm{Fix}\, H^n)) \geqslant h_{\mathrm{top}}(H)?$$

For an l.t.m. preserving a (K–S) measure μ we know at least that $h_{\mathrm{Per}}(H) \geqslant h_\mu(H)$, by Katok's theory [K]. So a positive answer to Question 3 would follow from a positive answer to

Question 4. Is it true that

$$h_{\mathrm{top}}(H) = \sup\{h_\mu(H):\ \mu \text{ is a (K–S) measure}\}?$$

Question 3 has a positive answer in the special case where global (un)stable manifolds do not have turn-back points; e.g. if X is orientable, F_n

are positively oriented and G_n negatively oriented Dehn twists (see Proposition 5.5).

Part of § 5 is devoted to estimating $h_{\mathrm{top}}(H)$, $h_{\mathrm{Per}}(H)$ by computable numbers detecting global expansiveness. I define and use some matrices connected with the horseshoe-like intersections of $F_n(P_i \cap Q_j)$ with $P_i \cap Q_j$, or $G_n(P_i \cap Q_j)$ with $P_i \cap Q_j$. This is a development of Devaney's idea (see [D2]), and of the idea of "graph of linkage" (see [P1]).

The assumption that the twists are at least double in Theorem 1 can be weakened to the assumption of the connectedness of the respective matrix (graph).

§ 2. **Definitions.** Recall some basic definitions from [P1].

2.1. Let $x' < x''$, $y' < y''$ be real numbers and

$$P = \{(x, y) \in \mathbf{R}^2/\mathbf{Z} \times \{0\}:\ y' \leqslant y \leqslant y''\},$$
$$Q = \{(x, y) \in \mathbf{R}^2/\{0\} \times \mathbf{Z}:\ x' \leqslant x \leqslant x''\}.$$

Let $e_i: P \to e_i(P) = P_i \subset X$, $i = 1, \ldots, p$, and $E_j: Q \to E_j(Q) = Q_j \subset X$, $j = 1, \ldots, q$, be continuous mappings into a compact topological space X with the following properties: each $\mathring{e}_i = e_i|_{\mathrm{int}\, P}$ and each $\mathring{E}_j = E_j|_{\mathrm{int}\, Q}$ is a homeomorphism onto its image, $\mathring{e}_i(P) \cap \mathring{e}_j(P) = \emptyset$ and $\mathring{E}_i(Q) \cap \mathring{E}_j(Q) = \emptyset$ for $i \neq j$, $\bigcup_{i=1}^{p} e_i(P) \cup \bigcup_{j=1}^{q} E_j(Q) = X$, $\bigcup_{i=1}^{p} \mathring{e}_i(P) \cup \bigcup_{j=1}^{q} \mathring{E}_j(Q)$ is connected, each of the compositions $\mathring{e}_i^{-1} \circ \mathring{E}_j$, $\mathring{E}_j^{-1} \circ \mathring{e}_i$ is C^2 with first and second derivatives upper-bounded. We shall consider metrics on $\bigcup \mathring{e}_i(P) \cup \bigcup \mathring{E}_j(Q)$ equivalent (i.e. with bounded ratio) to the Euclidean metrics on P_i, Q_j. There will be no need to specify a metric.

Denote the components of $\mathring{e}_i(P) \cap \mathring{E}_j(Q)$ by $\mathscr{C}_{ijs}$ (s labels the components) and denote $\mathscr{C} = \bigcup \mathscr{C}_{ijs}$. Assume that the circles $e_i(\{y = \mathrm{const}\})$ for $y' < y < y''$ intersect the circles $E_j(\{x = \mathrm{const}\})$, $x' < x < x''$, transversally. More exactly, on each $\mathscr{C}_{ijs}$ introduce the coordinates

$$\Phi = \Phi_{ijs}(y \circ e_i^{-1}, x \circ E_j^{-1})$$

and assume that $\Phi \circ \mathring{e}_i$, $\Phi \circ \mathring{E}_j$ and the inverses have upper-bounded C^2-norms.

We call the object described a *pair of transversal families of annuli.*

2.2. Let $N \geqslant 1$, and let $I(n) \subset \{1, \ldots, p\}$, $J(n) \subset \{1, \ldots, q\}$ be some sets chosen for every $n = 1, \ldots, N$ so that

$$\bigcup_{n=1}^{N} I(n) = \{1, \ldots, p\}, \qquad \bigcup_{n=1}^{N} J(n) = \{1, \ldots, q\}.$$

For every n and every $i \in I(n)$, $j \in J(n)$, let $f_{i,n}: \langle y', y''\rangle \to \mathbf{R}$, $g_{j,n}: \langle x', x''\rangle \to \mathbf{R}$ be some C^2-functions with $f_{i,n}(y') = g_{j,n}(x') = 0$, $f_{i,n}(y'') = k_{i,n}$, $g_{j,n}(x'') = l_{j,n}$.

Here $k_{i,n}$, $l_{j,n}$ are nonzero integers and $df_{i,n}/dy$, $dg_{j,n}/dx$ are nowhere zero and for each fixed i (j) have the same sign for all n.

Define the twists $F_{i,n}\colon P\to P$, $G_{j,n}\colon Q\to Q$ by

$$F_{i,n}(x, y) = (x+f_{i,n}(y), y), \qquad G_{j,n}(x, y) = (x, y+g_{j,n}(x)).$$

Define

$$F_n = \begin{cases} e_i\circ F_{i,n}\circ e_i^{-1} & \text{on } P_i,\\ \mathrm{id} & \text{on } X\setminus \bigcup_{i=1}^{p} P_i,\end{cases}$$

$$G_n = \begin{cases} E_j\circ G_{j,n}\circ E_j^{-1} & \text{on } Q_j,\\ \mathrm{id} & \text{on } X\setminus \bigcup_{j=1}^{q} Q_j.\end{cases}$$

Finally define the *linked twist mapping*:

$$H = G_N\circ F_N\circ\ldots\circ G_1\circ F_1.$$

2.3. We call $\alpha_{i,n} = \inf(df_{i,n}/dy)$ for $k_{i,n} > 0$ ($|\sup\ldots|$ if $k_{i,n} < 0$) the *slope* of $F_{i,n}$. Analogously we define the slopes $\beta_{j,n}$ of $G_{j,n}$. By *strong twists* we mean twists with slopes large enough, by *at least double twists* we mean twists with $|k_{i,n}|, |l_{j,n}| \geqslant 2$.

Remark. All the above assumptions about smoothness of functions are only to make correct the discussion involving the Pesin and Katok theories in § 1. In fact, in the rest of the paper, in particular for the Theorem, it is enough to assume that the mappings $\Phi\circ\mathring{e}_i$, $\Phi\circ\mathring{E}_j$ and their inverses are Lipschitz continuous, and that $f_{i,n}$, $g_{j,n}$ are just continuous. In the definition of slopes one then has to consider, instead of derivatives, ratios like $(f(y_2)-f(y_1))/(y_2-y_1)$, of course.

Let me finish the section of definitions by recalling that usually *topological transitivity* means that for all open nonempty sets U, V, $H^n(U)\cap V\neq\emptyset$ for some integer n. Here by topological transitivity I mean that for every pair of smooth curves $\gamma^u, \gamma^s\subset\mathscr{C}$ with tangent directions in unstable and stable cones respectively (see the beginning of § 3) there exists an integer n such that $H^n(\gamma^u)\cap\gamma^s\neq\emptyset$. For an l.t.m. topological transitivity defined in the standard way obviously follows from the above one. For (uniformly) hyperbolic differentiable dynamics the two definitions are equivalent. I do not know whether this is true for an l.t.m. The trouble is how to deduce $\bigcup_n H^n(\gamma^u)\cap\gamma^s\neq\emptyset$ from the density of $\bigcup_n H^n(\gamma^u)$ in X (see Fig. 2 in § 1).

Topological mixing usually means $H^n(U)\cap V\neq\emptyset$ for every n with $|n|$ sufficiently large. Another definition would be $H^n(\gamma^u)\cap\gamma^s\neq\emptyset$ when $|n|$ is

large (the discussion how both definitions are connected is similar to that for topological transitivity).

§ 3. Dynamics properties following from Conditions $\mathscr{H}$ and $\mathscr{E}$. In [P1] I considered two conditions about the strength of twists: Condition $\mathscr{H}$ and a stronger Condition $\mathscr{E}$.

Condition $\mathscr{H}$ ensures the existence of invariant cones. We recall from [P1] what we mean by that:

To simplify notation assume that $N = 1$, i.e. that we compose two families of twists only. Consider the induced (first return) mapping $H_{\mathscr{C}}\colon\mathscr{C}\to\mathscr{C}$. Clearly $H_{\mathscr{C}} = G_{\mathscr{C}}\circ F_{\mathscr{C}}$. Let C_z^u and C_z^s be cones containing the y- and x-axis respectively, considered in the coordinates Φ. We assume that $C_z^u\cap C_z^s = \{0\}$ for every $z\in\mathscr{C}$ and $DF_{\mathscr{C}}(\bigcup_z C_z^u)\subset\bigcup_z C_z^s$, $DG_{\mathscr{C}}(\bigcup_z C_z^s)\subset\bigcup_z C_z^u$. We require also that DH should expand every vector from $\bigcup C_z^u$ (and contract every vector from $\bigcup C_z^s$) by a factor greater than a constant $\lambda > 1$ (see Fig. 3).

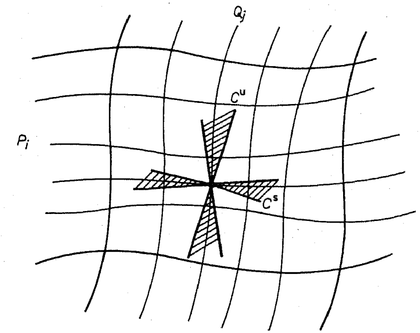

Fig. 3

In particular, in connection with the Remark in § 1, observe that this implies that every H-invariant measure μ such that $\mu(\mathrm{Par}) = 0$ is hyperbolic (Par denotes the set of all H-periodic orbits which undergo exclusively the horizontal twisting F_n or exclusively the vertical twisting G_n; of course Par is finite).

Condition $\mathscr{E}$ is adjusted to proving the topological transitivity of H (which implies ergodicity for a "good" measure).

The dynamics property proved in [P1] and implied by Condition $\mathscr{E}$ is recalled in the following

LEMMA 3.1. *There exist numbers* $\varepsilon_i > 0$, $\varepsilon^j > 0$ *for* $i = 1, \ldots, p$, $j = 1, \ldots, q$ *such that* $\varepsilon_i\varepsilon^j > 1$ *if* $\mathring{e}_i(P)\cap\mathring{E}_j(Q)\neq\emptyset$ *and for which the following* *is satisfied*: *For every smooth curve* $\gamma \subset \mathscr{C}_{ijs}$ *such that every tangent vector* $\dot{\gamma}(t)$ *belongs to* $C^{u}_{\gamma(t)}$, $F_{\mathscr{C}}(\gamma)$ *contains a* smooth *subcurve* γ' *such that either*

(3.1) $$\mathrm{length}_h(\gamma') \geqslant \varepsilon_i \cdot \mathrm{length}_v(\gamma),$$

or

(3.2) $$\gamma' \text{ joins two points from } \partial Q = \bigcup_{j=1}^{q} \operatorname{Fr} Q_j$$

($\mathrm{length}_{h(v)}$ *denotes the length in* Φ *coordinates of the projection onto the* x- (y-) *axis*).

Analogously, if $\dot{\gamma}(t) \in C^{s}_{\gamma(t)}$, *there exists a* $\gamma' \subset G_{\mathscr{C}}(\gamma)$ *such that either* $\mathrm{length}_v(\gamma') \geqslant \varepsilon^j \cdot \mathrm{length}_h(\gamma)$ *or* γ' *joins points from* $\partial P = \bigcup_{i=1}^{p} \operatorname{Fr} P_i$.

To prove the Theorem the following will be found to be important:

COMPLEMENT TO LEMMA 3.1. *There exists a* $\zeta > 0$ *such that for every* γ *with* $\dot{\gamma} \in C^{u}$ *one can choose a* γ' *with the above properties and additionally such that if we denote* $(F_{\mathscr{C}})^{-1}(\gamma') = \delta$ *and denote by* $n(\delta)$ *the integer such that* $F^{n(\delta)}|_{\delta} = F_{\mathscr{C}}|_{\delta}$ *then*

(3.3) $$\operatorname{dist}(F^n(\delta), \partial Q) > \zeta \cdot \mathrm{length}_v(\gamma)$$

for every n *with* $0 < n \leqslant n(\delta)$ *in case* (3.1) *and for* $0 < n < n(\delta)$ *in case* (3.2).

An analogous property holds for γ *with* $\dot{\gamma} \in C^{s}$.

Proof. Choose any system of numbers $\tilde{\varepsilon}_i$, $\tilde{\varepsilon}^j$ satisfying $\tilde{\varepsilon}_i > 0$, $\tilde{\varepsilon}^j > 0$, $\tilde{\varepsilon}_i \tilde{\varepsilon}^j > 1$ if $\mathring{E}_i(P) \cap \mathring{E}_j(Q) \neq \emptyset$ and $\tilde{\varepsilon}_i < \varepsilon_i$, $\tilde{\varepsilon}^j < \varepsilon^j$, where ε_i, ε^j are the numbers from Lemma 3.1. To prove (3.3) one can refer to Lemma 3.1 with each $\mathscr{C}_{ijs}$ thickened in P_i to $\mathscr{C}'_{ijs} = B(\mathscr{C}_{ijs}, r)$ where $r = C \cdot \frac{1}{2}(\varepsilon_i - \tilde{\varepsilon}_i) \cdot \mathrm{length}_v(\gamma)$. ($C$ is a constant connected with the fact that we do not specify what metric from what chart is considered).

The crucial fact is that Condition $\mathscr{E}$ concerns only the number of the components $\mathscr{C}_{ijs}$ in P_j and not the size or position of $\mathscr{C}_{ijs}$. The $\mathscr{C}_{ijs}$'s could even overlap. Condition $\mathscr{E}$ depends on changes of coordinates on $\mathscr{C}_{ijs}$ but the corresponding functions do not change much for small thickenings. The system ε_i, ε^j can be chosen the same for all small thickenings.

Since $F^n(\delta) \cap \bigcup \mathscr{C}_{ijs} = \emptyset$, we have $\operatorname{dist}(F^n(\delta), \bigcup \mathscr{C}_{ijs}) \geqslant r$ for $0 < n < n(\delta)$.

Let $F^{n(\delta)}(\delta) \subset \mathscr{C}_{ij_0s_0}$. Denote

$$\delta' = F^{-n(\delta)}\big(F^{n(\delta)}(\delta) \setminus B(P_i \setminus \mathscr{C}_{ij_0s_0}, r)\big).$$

Then

$$\operatorname{dist}(F^{n(\delta)}(\delta'), \partial Q) \geqslant r$$

and in case (3.1)

$$\mathrm{length}_h(F^{n(\delta)}(\delta')) \geqslant \tilde{\varepsilon}_i \cdot \mathrm{length}_v(\gamma).$$

§ 4. **Density of saddles.** We retain $N = 1$ to simplify notation. Denote

$$\partial = \partial P \cup \partial Q = \bigcup_{i=1}^{p} \operatorname{Fr} P_i \cup \bigcup_{j=1}^{q} \operatorname{Fr} Q_j.$$

We shall use the notation f^n for $(G\circ) F \circ \ldots \circ G \circ F (\circ G)$ (n times), understanding that f denotes F or G alternately. Brackets at the beginning (and end) of the composition mean that it can start (end) from either G or F. Denote analogously $(G_{\mathscr{C}}\circ) F_{\mathscr{C}} \circ \ldots \circ F_{\mathscr{C}} (\circ G_{\mathscr{C}})$ (n times) by $f^n_{\mathscr{C}}$.

LEMMA 4.1. *Under Condition* $\mathscr{E}$ *there exists an* $\eta > 0$ *such that for every* $z \in \mathscr{C}_{ijs}$ *and* r *with* $0 < r \leqslant \frac{1}{2}\operatorname{dist}(z, \partial)$ *there exists an* $x \in B(z, r)$ *such that* $H^{n_0}(x) \in B(z, r)$ *for an integer* $n_0 > 0$ *and* $\operatorname{dist}(f^n(x), \partial) \geqslant \eta r$ *for every* $n = 0, 1, \ldots, 2n_0$.

Proof. Take in $B(z, r)$ arbitrary smooth curves γ^u, $\gamma^s \ni z$ of lengths about r such that $\dot{\gamma}^u \in C^u$, $\dot{\gamma}^s \in C^s$. For $\gamma = \gamma^u$ find $\gamma_1 = \gamma' \subset F_{\mathscr{C}}(\gamma) = F^{n_1}(\gamma)$ from the Complement to Lemma 3.1, next find $\gamma_2 = \gamma'_1 \subset G_{\mathscr{C}}(\gamma_1) = G^{n_2}(\gamma_1)$ and so on. We end with γ_{m_0} joining two points from ∂Q or ∂P. Denote $f_{\mathscr{C}}^{-m_0}(\gamma_{m_0})$ by δ^u. By induction with the use of (3.3) we easily obtain

(4.1) $$\operatorname{dist}(f^n(\delta^u), \partial) \geqslant \mathrm{Const} \cdot \zeta r$$

for every $n = 0, 1, \ldots, \left(\sum_{i=1}^{m_0} n_i\right) - 1$.

Iterate f another few times with the last f equal to G and consider the images $f^m(\gamma_{m_0})$. The set $f(\gamma_{m_0})$ contains a family of arcs in $\mathscr{C}$ with end-points

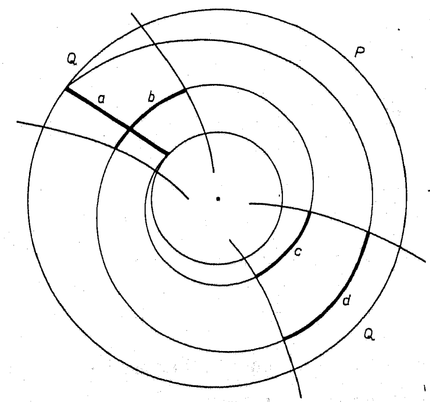

Fig. 4

in ∂P (or ∂Q). Denote the family of arcs by $\{\gamma^t_{m_0+1}\}_t$. Then $f(\bigcup_t \gamma^t_{m_0+1})$ contains a family $\{\gamma^t_{m_0+2}\}$ of analogous arcs and so on. It is easy to prove, due to at least double twisting (see [P1]), that after a time m_1, independent of γ, every $\mathscr{C}_{ijs}$ contains an arc from the family $\{\gamma^t_{m_0+m_1}\}_t$ with ends in ∂P.

Observe that (4.1) also implies $\operatorname{dist}(\gamma_{m_0}, \partial Q) \geqslant \mathrm{Const}\cdot\zeta r$ if the m_0-th $f_{\mathscr{C}}$ is $G_{\mathscr{C}}$ (or $\operatorname{dist}(\gamma_{m_0}, \partial P) \geqslant \mathrm{Const}\cdot\zeta r$ if it is $F_{\mathscr{C}}$). Each $\gamma^t_{m_0+m+1}$ also has this property with the right-hand side quantity $L^{m+1}\cdot\mathrm{Const}\cdot\zeta r$ for a constant L, since $\{\gamma^t_{m_0+m+1}\}$ consists only of components of $f(\gamma^t_{m_0+m})\cap\mathscr{C}$ not containing the end-points of $f(\gamma^t_{m_0+m})$. Call this property (*); the reason it holds is illustrated in Fig. 4 (where $a = \gamma^t_{m_0+m}$, $b = \gamma^{t_1}_{m_0+m+1}$, $c = \gamma^{t_2}_{m_0+m+1}$, $d = \gamma^{t_3}_{m_0+m+1}$).

Concluding, there exists $\tilde{\gamma}^{(u)} = \gamma^{t_0}_{m_0+m_1}$ intersecting an analogous arc $\tilde{\gamma}^{(s)}$ constructed for backward iterations on γ^s. Due to (*) we can shorten $\tilde{\gamma}^{(u)}$, $\tilde{\gamma}^{(s)}$ to $\gamma^{(u)}$, $\gamma^{(s)}$, which still intersect each other. Define $\tilde{\delta}^u = f_{\mathscr{C}}^{-(m_0+m_1)}(\gamma^{(u)})$, and analogously define $\tilde{\delta}^s$. Of course $\tilde{\delta}^{u(s)} \subset \delta^{u(s)}$. Thus we have $\tilde{\delta}^u \subset \gamma^u \subset B(z, r)$, $\tilde{\delta}^s \subset \gamma^s \subset B(z, r)$ such that

$$H^m(\tilde{\delta}^u)\cap H^{-n}(\tilde{\delta}^s) \neq \emptyset \quad \text{for some } m, n > 0,$$
$$\operatorname{dist}(f^i(\tilde{\delta}^u), \partial) > L^{m_1}\cdot\mathrm{Const}\cdot\zeta r \quad \text{for } 0 \leqslant i \leqslant 2n,$$
$$\operatorname{dist}(f^{-j}(\tilde{\delta}^s), \partial) > L^{m_1}\cdot\mathrm{Const}\cdot\zeta r \quad \text{for } 0 \leqslant j \leqslant 2n.$$

Let y be the unique point in $H^m(\tilde{\delta}^u)\cap H^{-n}(\tilde{\delta}^s)$. Define $x = H^{-m}(y)$.

Proof of the Theorem. Take an arbitrary point $z \in \mathscr{C}_{ijs} \setminus \bigcup_{k=-\infty}^{\infty} f^k(\partial)$. Let $k_1(z), \ldots$ $(l_1(z), \ldots)$ be consecutive times for z of return to $\mathscr{C}$ under forward (backward) iterations by H. Let $\lambda > 1$ be, as in § 3, a hyperbolic constant for $H_{\mathscr{C}}$. Choose a large constant $M > 0$. Let $t > 0$ be an integer such that

$$\lambda^t > CM\eta^{-1}$$

for η from Lemma 4.1. What is C will be explained later.

Fix an arbitrary r such that for $-2l_t \leqslant n \leqslant 2k_t$

$$f^n\big(B(z, (M+1)r)\big)\cap\partial = \emptyset. \tag{4.2}$$

Take x found for these z and r by Lemma 4.1. Now we prove that we can approximate (shadow) the trajectory $x, \ldots, H^{n_0}(x)$ by a periodic one, repeating more or less the proof of Katok's main lemma [K].

Denote $y = H^{n_0}(x)$. Let $\gamma^u, \gamma^s \subset \mathscr{C}$ be smooth curves such that $\dot{\gamma}^u \in C^u$, $\dot{\gamma}^s \in C^s$, $\gamma^u \ni x$, $\gamma^s \ni y$. Let

$$\delta^u(x) = H^{-n_0}(H^{n_0}(\gamma^u) \mathbin{\tilde{\cap}} B(y, Mr)), \qquad \delta^s(y) = H^{n_0}(H^{-n_0}(\gamma^s) \mathbin{\tilde{\cap}} B(x, Mr)).$$

By $\tilde{\cap}$ we mean intersection with a ball and the choice of the component containing the origin of the ball.

By construction, for every $k = 0, \ldots, 2n_0$,

$$f^k(\delta^u(x))\cap\partial = \emptyset, \qquad f^{-k}(\delta^s(y))\cap\partial = \emptyset. \tag{4.3}$$

Indeed, in the case of $\delta^u(x)$ for example, (4.3) follows for $k = 2n_0, 2n_0-1, \ldots, 2n_0-2l_t$ from (4.2). For k smaller it follows from Lemma 4.1. To see that exactly one can use induction. I present its beginning and the first step.

Denote

$$S = \Big(\bigcup_{k=0}^{2n_0} f^{2n_0-k}(\partial)\Big)\cap H^{n_0}(\delta^u(x));$$

here each f^{2n_0-k} completes f^k from (4.3) to

$$f^{2n_0-k}\circ f^k = f^{2n_0} = H^{n_0}.$$

(S is just the set of all points of nondifferentiability of $H^{n_0}(\delta^u(x))$. Our aim is to prove $S = \emptyset$).

Let $\bar{\gamma} \subset H^{n_0}(\delta^u(x))$ be a smooth curve joining two points of $S \cup \{$end-points of $H^{n_0}(\delta^u(x))\}$ which are consecutive in $H^{n_0}(\delta^u(x))$ and such that $y \in \bar{\gamma}$. We shall prove

$$f^k(f^{-2n_0}(\bar{\gamma}))\cap\partial = \emptyset. \tag{4.3$'$}$$

For $k \geqslant 2n_0 - 2l_t(z) = 2n_0 - 2l_t(y)$, (4.3$'$) holds. Also

$$\operatorname{length}_v(H^{l_t(y)}(\bar{\gamma})) \leqslant \lambda^{-t} C_1 2Mr < C^{-1}M^{-1}\eta C_1 2Mr = C^{-1}C_1 2\eta r. \tag{4.4}$$

Here C_1 is a constant,

$$C_1 = \sup\{\operatorname{length}(\alpha\colon\langle 0, 1\rangle \to \mathscr{C})/\operatorname{dist}(\alpha(0), \alpha(1))\colon \alpha \text{ is differentiable, } \dot{\alpha} \in C^u\}.$$

Consider the sequence

$$\{\operatorname{length} G^{-\tau}(H^{-l_t(y)}(\bar{\gamma}))\}$$

for $\tau = 0, 1, \ldots, \tau_1 = \inf\{\tau_1(w)\colon w \in H^{-l_t(y)}(\bar{\gamma})\}$ where $\tau_1(w) = \inf\{\tau \in \boldsymbol{R}^+\colon G^{-\tau}(w) \in \mathscr{C}\}$. This sequence is decreasing if we consider the Euclidean metrics on $\{Q_j\}$. Thus, due to Lemma 4.1, if C is sufficiently large we obtain (4.3$'$) for

$k \geq 2n_0 - 2l_t - 2\tau_1$, and $\tau_1(w)$ turns out to be constant for $w \in H^{-l_t(y)}(\bar{\gamma})$. Now we proceed analogously with $F^{-\tau}$ and end with (4.3′) true for $k \geq 2n_0 - 2l_{t+1}$ and with (4.4) true for $t+1$ in place of t. (Choosing C one needs to take into account that the metric in Lemma 4.1 is not specified. At the step from $k = 2n_0 - 2l_t - 2\tau_1$ to $2n_0 - 2l_t - 2\tau_1 - 1$ we pass from the Euclidean metrics on $\bigcup Q_j$ to the Euclidean metrics on $\bigcup P_i$.)

(4.3) implies that in fact $\bar{\gamma} = H^{n_0}(\delta^{\mathrm{u}}(x))$. The end-points of $H^{n_0}(\delta^{\mathrm{u}}(x))$ obviously lie in the sphere $\operatorname{Fr} B(y, Mr)$, assuming that we always consider γ^{u} long (say joining points from ∂) and r is sufficiently small.

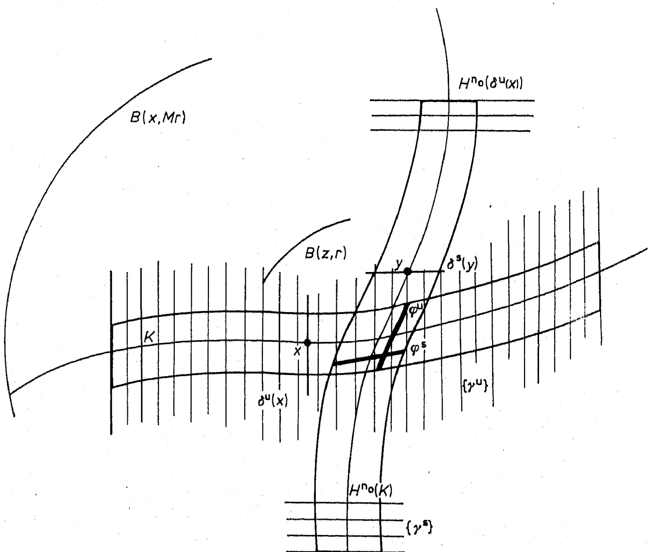

Fig. 5

For every $w \in H^{-n_0}(\delta^{\mathrm{s}}(y))$ choose $\gamma^{\mathrm{u}}(w) \ni w$, $\dot{\gamma}^{\mathrm{u}} \in C^{\mathrm{u}}$ so that $\{\gamma^{\mathrm{u}}\}$ is a foliation by arcs and $\gamma^{\mathrm{u}}(x) \supset \delta^{\mathrm{u}}(x)$. Analogously choose a foliation $\{\gamma^{\mathrm{s}}\}$ of a neighbourhood of $H^{n_0}(\delta^{\mathrm{u}})$. The foliations $\{H^{-n_0}(\gamma^{\mathrm{s}}) \cap B(x, Mr)\}$ and γ^{u} form a coordinate system on a rectangle K (rectangle in these coordinates) in a common domain (see Fig. 5) and $f^k(K) \cap \partial = \emptyset$ for every $k = 0, \ldots, 2n_0$.

All this holds by arguments similar to those which we used to prove (4.3). Now the proof that there exists an H^{n_0}-fixed point in $K \cap H^{n_0}(K)$ is standard; we only sketch it.

Consider a space Γ of graphs in K of Lipschitz continuous functions from $\delta^{\mathrm{u}}(x) \cap K$ to $H^{-n_0}(\delta^{\mathrm{s}}(y)) \cap K$ (these two arcs play the role of axes in our coordinate system on K), with Lipschitz constants uniformly bounded. The operator $\Gamma \ni \varphi \mapsto H^{n_0}(\varphi) \cap K$ is a contraction in C^0-topology. So there exists a fixed point (graph) φ^{u}. Analogously one constructs φ^{s}. The intersection $\varphi^{\mathrm{u}} \cap \varphi^{\mathrm{s}}$ contains exactly one point. This point must be fixed under H^{n_0}.

The topological transitivity of H follows directly from Lemma 3.1 and the assumption $|k_{i,n}|, |l_{j,n}| \geq 2$. The density of homoclinic points of every saddle follows from the density of the set of saddles.

R e m a r k. Even topological mixing of H follows from Lemma 3.1 and $|k_{i,n}|, |l_{j,n}| \geq 2$. So if a "good" measure is preserved, H turns out to be Bernoulli.

§ 5. Strong connectedness of graphs, estimations of $h_{\mathrm{Per}}(H)$, $h_{\mathrm{top}}(H)$.

5.1. Let us begin with the definition of the *graph* $\Gamma(H)$ of an l.t.m. H. Call an h-*curve* (a v-*curve*) a curve $\gamma \subset \mathscr{C}_{ijs}$ such that $\dot{\gamma} \in C^{\mathrm{s(u)}}$ and γ joins the left and right (lower and upper) sides of $\mathscr{C}_{ijs}$.

Repeat each $\mathscr{C}_{ijs}$ as a vertex $\mathscr{C}_{ijst}$ of $\Gamma(H)$, for $t \in I(i)$. Repeat it another $J(j)$ times and denote $\mathscr{C}^t_{ijs}$, $t \in J(j)$.

Consider two vertices, $\mathscr{C}_{ijst}$ and $\mathscr{C}^{t'}_{i'j's'}$. Let $i = i'$ and let $m > 0$ be an odd integer such that $m = 2(t'-t)+1 \pmod{2N}$. Suppose there exists in $\mathscr{C}_{ijs}$ a family of k strips, S_ν, $\nu = 1, \ldots, k$, each with left and right sides in $\operatorname{Fr} Q_j$, with lower and upper sides being h-curves and with the following property: for each v-curve $\gamma \subset \mathscr{C}_{ijs}$ and each ν, $f^m(\gamma \cap S_\nu) \subset \mathscr{C}_{ij's'}$ is an h-curve and

$$f^{\xi}\Big(\bigcup_{\nu=1}^{k} S_\nu\Big) \cap \operatorname{supp} f_{\xi+1} = \emptyset$$

for every odd ξ with $1 \leq \xi \leq m-2$. Here in the composition $f^m = f_m \circ \ldots \circ f_1$ the first $f = f_1$ is F_t, the mth one is $F_{t'}$. (Recall that as in § 4 we denote any composition $(G_r \circ) F_r \circ \ldots \circ F_1 \circ G_N \circ F_N \circ \ldots \circ G_s (\circ F_s)$ (n times) by f^n. To understand what each f in f^n means it is enough to specify the first f.)

Then supply the graph with k edges directed from $\mathscr{C}_{ijst}$ to $\mathscr{C}^{t'}_{i'j's'}$, each of length m (the path of edges, alternating with new vertices, of length m). Analogously supply the graph with edges directed from $\mathscr{C}^t_{ijs}$ to $\mathscr{C}_{i'j's't'}$ for $j = j'$.

Intuitively, a k-edge connection between vertices corresponds to a horseshoe-like intersection with k bends.

In [P1] I considered a slightly different definition; the changes are: omit the fragment about the existence of S_ν, then change the beginning of the sentence which follows that fragment to: for each v-curve $\gamma \subset \mathscr{C}_{ijs}$ there exists a family of subcurves $\gamma_\nu \subset \gamma$ such that $f^m(\gamma_\nu) \subset \mathscr{C}_{i'j's'}$. Call the defined graph a *weak graph* $\Gamma^{(w)}(H)$. We are interested in the strong connectedness of graphs and all assumptions sufficient for the strong connectedness of the weak graph considered in [P1] also imply the strong connectedness of the graph of H. (The weak graph corresponds to the derived graph of the "graph of linkage" from [P1], for $N=1$, where the vertices were annuli. Edges of the "graph of linkage" correspond to vertices of the "weak graph".)

Denote by $M(H)$ $(M^{(w)}(H))$ the matrix corresponding to the (weak) graph.

5.2. A directed graph is called *strongly connected* if for any two vertices there exists a directed path joining them. The Markov chain related to the matrix corresponding to a strongly connected graph is called *irreducible* (and so is the matrix itself). Recall from [P1] some examples of the assumptions under which $\Gamma(H)$ is strongly connected:

1. For $N=1$, each intersection $\mathring{e}_i(P) \cap \mathring{E}_j(Q)$ has at least two components and one intersection at least 3 components. (This is an assumption about the *topology* of X – the pair of transversal families of annuli.)

2. $|k_{i,n}|$, $|l_{j,n}| \geqslant 2$, N arbitrary (the assumption about the *topology of twisting*).

3. Let $N=1$. Let $\{A_i\}_{i=1}^{p}$ be a family of circles in a surface, pairwise disjoint. Let $\{B_j\}_{j=1}^{q}$ be another such family. Assume $\bigcup A_i \cup \bigcup B_j$ to be connected and each pair A_i, B_j in general position. More exactly, each A_i intersects each B_j transversally and for at least one circle A_i (or B_j) if A_i (resp. B_j) intersects $\bigcup_{j=1}^{q} B_j$ (resp. $\bigcup_{i=1}^{p} A_i$) at exactly two points, then these two points are not antipodal in A_i (B_j). Thicken these circles to annuli. Finally assume the thickness of the annuli to be sufficiently small. (This assumption is about the *geometry of twisting* caused by the *geometry of the domain* X. Narrow annuli carry twists which must have large slopes.)

Now we can formulate a *sufficient assumption for the Theorem* as follows:

H satisfies Condition $\mathscr{E}$ and its weak graph $\Gamma^{(w)}(H)$ is strongly connected.

To have topological mixing of H it is sufficient to assume, in addition to the irreducibility of $M^{(w)}(H)$, the aperiodicity of at least one of the matrices $M^{(w)}(H)_t$, $M^{(w)}(H)^t$, defined similarly to $M(H)_t$, $M(H)^t$ at the beginning of 5.4. (An irreducible matrix is called *aperiodic* if all the entries of every sufficiently large power of the matrix are positive.)

The assumption $|k_{i,n}|$, $|l_{j,n}| \geqslant 2$ implies the aperiodicity of all the matrices $M(H)_t$, $M(H)^t$. The proof is in [P1].

5.3. In fact, the following holds:

PROPOSITION 5.3. *Condition $\mathscr{E}$ and the strong connectedness of $\Gamma^{(w)}(H)$ imply the existence of an invariant hyperbolic set on which H is conjugate to a subshift of finite type. In particular, $h_{\text{top}}(H) > 0$ and $h_{\text{Per}}(H) > 0$.*

Proof. Any periodic saddle x has at least two different homoclinic trajectories. This is visible since we can consider iterations $f^n(W^u_{x,\text{loc}})$ ($W^u_{x,\text{loc}}$ denotes the local unstable manifold) until these sets intersect for the first time, say n_0, the boundary set $\partial = \bigcup e_i(\operatorname{Fr} P) \cup \bigcup E_j(\operatorname{Fr} Q)$. The orbits of two short arcs in $f^{n_0}(W^u_{x,\text{loc}})$ on both sides of ∂ are clearly disjoint, and further forward iterations of H on each of these arcs lead to long smooth arcs, hence to homoclinic intersections with a long stable arc. With two homoclinic trajectories one obtains a picture as in Fig. 6.

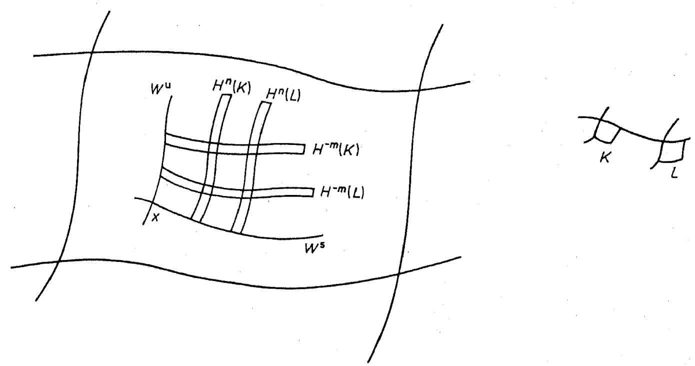

Fig. 6

5.4. Denote by $r(H)$ the spectral radius of the matrix $M(H)$. For every $t=1, \ldots, N$ denote by $M(H)_t$ the submatrix of $(M(H))^{2N}$ consisting exactly of rows and columns indexed by $\mathscr{C}_{ijst}$. Analogously $M(H)^t$ consists of rows and columns $\mathscr{C}^t_{ijs}$. Since these matrices need not be $\{0,1\}$-matrices, consider instead the $\{0,1\}$-matrices $(M(H)_t)^d$, $(M(H)^t)^d$ of graphs derived from the graphs of $M(H)_t$, $M(H)^t$ (one obtains the graph derived from Γ by treating the edges of Γ as vertices and the paths of length 2 as edges).

Condition $\mathscr{E}$ in Proposition 5.3 can almost be omitted. Namely we have the following

PROPOSITION 5.4. *If for an l.t.m. H Condition $\mathscr{H}$ holds and $\log r(H) > 0$, then for every t there exists an H-invariant hyperbolic set X_t (and X^t) on which H is conjugate to the subshift given by the matrix $(M(H)_t)^d$ (or $(M(H)^t)^d$).*

We have

$$h_{\mathrm{Per}}(H),\ h_{\mathrm{top}}(H) \geqslant \max_{t=1,\ldots,N} (h_{\mathrm{Per(top)}} H|X_t) \geqslant 2N \log r(H) > 0.$$

The same holds for X^t.

Proof. The matrix $M(H)$ has been defined in such a way as to ensure the existence of appropriate horseshoes.

It is easy to prove that for each of the examples 1–3, except the toral l.t.m. with $k = -l = 2$, the assumptions implying the transitivity of $\Gamma(H)$ also imply $r(H) > 0$. Proposition 5.4 is better than Proposition 5.3 because it gives a concrete estimate of $h_{\mathrm{top(Per)}}(H)$.

Other important numbers which describe global expansiveness are: $\gamma(H)$, which denotes the exponential growth rate of images of (free) elements of the fundamental group under iterations of H_*, and $r_1(H)$, which is the growth in the first homology group. How is $r(H)$ related to $\gamma(H)$ or $r_1(H)$?

By Manning's theorem (see the version in [F–S], $h_{\mathrm{top}}(H) \geqslant \gamma(H)$. By the Lefschetz formula, $h_{\mathrm{Per}}(H) \geqslant r_1(H)$. In fact, it is even true that $h_{\mathrm{Per}}(H) \geqslant \gamma(H)$. (When H is isotopic to Thurston's pseudo-Anosov homeomorphism on X, this follows from the Nielsen–Thurston Theorem [Th]. In the general case this follows from Thurston's classification of homeomorphisms of surfaces and from the homotopy invariance of the so-called Nielsen numbers; see for example [I] for the definition and references. I owe the Nielsen numbers argument to N. V. Ivanov.)

5.5. Let us go back to Question 3 from § 1. I can prove the following special case:

PROPOSITION 5.5. *If the domain X of an l.t.m. H is orientable, if each twist $e_i \circ F_{i,n} \circ e_i^{-1}$ on any horizontal annulus P_i is positively oriented (i.e. for any curve γ joining two boundary circles of P_i, γ and $F_n(\gamma)$ have a positive index of intersection), if each twist $E_j \circ G_{j,n} \circ E_j^{-1}$ on Q_j is negatively oriented (i.e. the inverse is positively oriented), if the twists are strong enough to imply the existence of invariant families of (un)stable cones, if, finally, the graph $\Gamma(H)$ is strongly connected, then*

(1) $\mathrm{Per}\, H$ *is dense in* X,

(2) $h_{\mathrm{Per}}(H) \geqslant h_{\mathrm{top}}(H)$.

Proof. This is virtually a case where Devaney's proof [D1] of density (see discussion in § 1) works. The curves $H^n(\gamma^u)$ can have points of nondifferentiability but no turn-back points.

To prove (2) observe that singularities are only apparent, and Katok's idea from [K] can be applied; owing to invariant measures one has the return property and keeps track of the return time. Then the geometric part of the proof (shadowing by a periodic orbit) is based on the contraction in the space of Lipschitz continuous graphs of functions as in § 4. (Incidentally, one obtains the existence of Lipschitz continuous local (un)stable manifolds, *not necessarily differentiable*.) One considers of course the first return map to $\mathscr{C}$ (in the case $N = 1$). One can assume that the trajectories to be shadowed have the first and the last points in $\mathscr{C}$ and far away from the boundary $\mathrm{Fr}\,\mathscr{C}$ due to the following

PROPOSITION 5.6. *For every l.t.m. H such that invariant families of cones exist, and for every probabilistic H-invariant measure μ, ergodic and of positive entropy, we have $\mu(\partial) = 0$. (Recall $\partial = \bigcup e_i(\mathrm{Fr}\,P) \cup \bigcup E_j(\mathrm{Fr}\,Q)$.)*

Proof. ∂ intersects its H^n-images in a countable set Y. Since $H^n(\partial \setminus Y) \cap (\partial \setminus Y) = \emptyset$ for every $n \neq 0$, we have $\mu(\partial \setminus Y) = 0$. Also $\mu(\bigcup_{n=-\infty}^{\infty} H^n(Y)) = 0$; if this were equal to 1, we would have $h_\mu(H) = h(H|_{\bigcup H^n(Y)}) = 0$.

Appendix. Conditions $\mathscr{H}$ and $\mathscr{E}$. We begin with supplementing the notation from § 2. Denote $\Phi_{ijs}(\mathscr{C}_{ijs}) = \mathscr{R}_{ijs}$. Consider the disjoint union $\mathscr{R} = \bigcup_{ijs} \mathscr{R}_{ijs}$. So $\Phi\colon \mathscr{C} \to \mathscr{R}$. Denote $\mathscr{C}_i = \mathscr{C} \cap P_i$, $\mathscr{C}^j = \mathscr{C} \cap Q^j$, $\Phi(\mathscr{C}_i) = \mathscr{R}_i$, $\Phi(\mathscr{C}^j) = \mathscr{R}^j$, $e_i^{-1}(\mathscr{C}_i) = \mathscr{P}_i$, $E_j^{-1}(\mathscr{C}^j) = \mathscr{Q}^j$.

Define functions φ_i, ψ_j on the sets $\mathscr{P}_i$, $\mathscr{Q}^j$ respectively by the formulas:

$$\Phi \circ e_i(x, y) = (\varphi_i(x, y), y), \qquad \Phi \circ E_j(x, y) = (x, \psi_j(x, y)).$$

Define functions φ_i', ψ_j' on the sets $\mathscr{R}_i$, $\mathscr{R}^j$ respectively by

$$(\Phi \circ e_i)^{-1}(x, y) = (\varphi_1'(x, y), y), \qquad (\Phi \circ E_j)^{-1}(x, y) = (x, \psi_j'(x, y)).$$

Condition $\mathscr{H}$:

$$(\mathscr{H}1) \qquad \bar{\alpha}_{i,m} = \left(\alpha_{i,m} - \left|\frac{d\varphi_i'}{dy}\right|\right) \cdot \left|\frac{d\varphi_i'}{dx}\right|^{-1} - \left|\frac{d\varphi_i}{dy}\right| > 0,$$

$$(\mathscr{H}2) \qquad \bar{\beta}_{j,n} = \left(\beta_{j,n} - \left|\frac{d\psi_j'}{dx}\right|\right) \cdot \left|\frac{d\psi_j'}{dy}\right|^{-1} - \left|\frac{d\psi_j}{dx}\right| > 0,$$

$$(\mathscr{H}3) \qquad \bar{\alpha}_{i,m} \cdot \bar{\beta}_{j,n} > (1+\mu_i)(1+\mu^j) \quad \text{where}$$

$$\mu_i = \left|\frac{d\varphi_i}{dx}\right| \cdot \left|\frac{d\varphi_i'}{dx}\right|, \qquad \mu^j = \left|\frac{d\psi_j}{dy}\right| \cdot \left|\frac{d\psi_j'}{dy}\right|$$

for all m, n, $i \in I(m)$, $j \in J(n)$ such that $\mathring{e}_i(P) \cap \mathring{E}_j(Q) \neq \emptyset$. By $|\xi|$ we denote the supremum of ξ over its domain, for any positive function ξ. (Geometrically, $\bar{\alpha}_{i,m}$, $\bar{\beta}_{j,n}$ bound from below the slopes of the induced mappings $(F_{i,m})_{\mathscr{P}_i}$, $(G_{j,n})_{\mathscr{Q}_j}$ in the coordinates $\Phi \circ e_i$, $\Phi \circ E_j$.)

Condition $\mathscr{E}$:

(𝓔1) $$\bar{\alpha}_{i,m} = \left(\alpha_{i,m} - (2q(i)+3X(i))\left|\frac{d\varphi_i'}{dy}\right|\right)\cdot\left|\frac{d\varphi_i'}{dx}\right|^{-1} > 0,$$

(𝓔2) $$\bar{\beta}_{j,n} = \left(\beta_{j,n} - (2p(j)+3Y(j))\left|\frac{d\psi_j'}{dx}\right|\right)\cdot\left|\frac{d\psi_j'}{dy}\right|^{-1} > 0,$$

(𝓔3) $$\bar{\alpha}_{i,m}\cdot\bar{\beta}_{j,n} > \max(X(i),\ 1+\mu_i)\max(Y(j),\ 1+\mu^j)$$

for all m, n, $i \in I(m)$, $j \in J(n)$ such that $\mathring{e}_i(P) \cap \mathring{E}_j(Q) \neq \emptyset$. Here $X(i)$ and $Y(j)$ are, respectively, the largest solutions of the equations

$$1 = \frac{2q(i)}{X} + \frac{q(i)}{X-3q(i)} + \frac{2}{X-\mu_i},$$

$$1 = \frac{2p(j)}{Y} + \frac{p(j)}{Y-3p(j)} + \frac{2}{Y-\mu^j},$$

where $q(i)$ is the number of components of $\mathscr{C}_i$, and $p(j)$ is the number of components of $\mathscr{C}^j$.

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Nilpotent groups with T_1 primitive ideal spaces

by

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Abstract. We prove that second countable locally compact nilpotent groups containing a compactly generated normal open subgroup have T_1 primitive ideal spaces.

§ 1. Introduction. We say that a locally compact group G has T_1 *primitive ideal space* if the group C^*-algebra, $C^*(G)$, has the property that every primitive ideal (i.e. kernel of an irreducible representation) is closed in the hull-kernel topology on the space of primitive ideals of $C^*(G)$, denoted Prim G. Long ago Dixmier proved [5] that every connected nilpotent Lie group has T_1 primitive ideal space (in fact, such groups, being type I, are therefore CCR). More recently Poguntke showed [11] that discrete nilpotent groups have T_1 primitive ideal space. This then suggests the obvious conjecture that all locally compact nilpotent groups have T_1 primitive ideal space.

This note proves the following result in that direction.

THEOREM. *If G is a second countable locally compact nilpotent group with a compactly generated open normal subgroup then G has T_1 primitive ideal space.*

The notation of this paper is the same as that of [2], to which we also refer the reader for a more leisurely account of some of the techniques and ideas used here.

Acknowledgement. We would like to thank Mike Keane for the reference to the structure theorem for nilpotent groups cited in Section 2. The first-named author thanks the Pure Mathematics Department at Adelaide University for its hospitality while this work was carried out.

§ 2. Preliminary arguments. We note firstly the following structure theorem for compactly generated nilpotent locally compact groups ([8], p. 104). Namely if G' is such a group then there exists a maximal compact normal subgroup K consisting of all elements whose powers form a relatively compact set such that the quotient $\bar{G}' = G'/K$ is a Lie group. By this last statement we mean that the connected component of the identity $\bar{G}_0$ of $\bar{G}'$ is a (connected) Lie group and $\bar{G}'/\bar{G}_0$ is discrete (possibly infinite). If G is as in