[27] —, *On complex interpolation of Banach spaces of measurable functions*, Vestnik Leningrad. Univ. 1974, no. 19, 64–68 (in Russian).

[28] —, *Transformations of Banach lattices and the interpolation of linear operators*, Bull. Acad. Polon. Sci. 29 (1981), 569–577 (in Russian).

[29] E. M. Stein and G. Weiss, *Interpolation of operators with change of measures*, Trans. Amer. Math. Soc. 87 (1958), 159–172.

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On the splitting of twisted sums, and the three space problem for local convexity

by

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Abstract. We characterize all pairs of topological vector spaces (tvs) (Y, Z) such that Y is semimetrizable (resp. locally bounded) and for every relatively open and continuous map q with $\ker q \simeq Y$ and $\operatorname{im} q \simeq Z$ there is a section (resp. a homogeneous section) continuous at zero (i.e., a map s with $q \circ s = \mathrm{id}$).

A twisted sum of two tvs Y and Z is a tvs X with a subspace $Y_0 \simeq Y$ such that $X/Y_0 \simeq Z$. All twisted sums of an arbitrary pair of tvs are described.

A tvs Z belongs to $S(Y)$ (resp. to the class of TSC-spaces) iff every twisted sum of tvs Y and Z is a direct sum (resp. is locally convex whenever so is Y). We examine hereditary properties of the classes $S(Y)$ and TSC-spaces. As an application we get: (1) all locally convex spaces (lcs) with a weak topology belong to $S(Y)$ for the one-dimensional Y (i.e., they are $\mathcal{K}$-spaces [15]); (2) all nuclear lcs and all metrizable locally convex $\mathcal{K}$-spaces are TSC-spaces. The classes of TSC-spaces and locally pseudoconvex spaces are closed under twisted sums.

We remove the assumption of local boundedness from the following results: (1) Kalton's [14] description of all twisted sums of lcs; (2) p- and q-convexity of Y and Z resp. $(0 < p \neq q \leqslant 1)$ implies $\min(p, q)$-convexity of their twisted sum.

0. Introduction. A *twisted sum* of two topological vector spaces (tvs) Y and Z is a diagram of tvs and linear relatively open continuous mappings:

$$(*) \qquad 0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0,$$

such that $(*)$ is a short exact sequence, i.e., $j(Y) = \ker q$. Sometimes we will simply say that X itself is a twisted sum of Y and Z.

There are two main problems concerning twisted sums.

The first one is the so-called three space problem. We say that a property (P) is a *three space property* if every twisted sum of Y and Z has (P) whenever Y and Z have (P). Now, the question is: what properties are three space properties.

We say that the twisted sum $(*)$ *splits* if there is a continuous linear mapping $T\colon Z \to X$ for which $q \circ T = \mathrm{id}_Z$. The second main question concerning twisted sums is the problem what pairs (Y, Z) of tvs have only splitting twisted sums. There are numerous papers related to these problems: [5], [6], [8]–[14], [16], [21], [25]–[32], [34], [37], [40] (the three space problem) and [1], [2], [10], [13]–[15], [17]–[21], [24], [28], [29], [35],

[36], [38], [39], [41] (the problem of splitting). Our paper is devoted to both of them.

In Section 1 we investigate the three space problem. First, we show that local pseudoconvexity is a three space property. The three space problem seems to be the most interesting for (P) = local convexity. Although it was answered negatively in [13], [28], [29] (independently), it was shown by Kalton in [13], [14] that every twisted sum of two normable spaces must fulfil some convexity conditions. In Section 1 we prove a generalization of this. Namely, it turns out that if Y is locally p-convex and Z is locally q-convex, then their twisted sum is locally $\min(p, q)$-convex whenever $0 < p \neq q \leqslant 1$, and it is locally logconvex whenever $p = q = 1$ (for the definition of local logconvexity see Section 1).

The basic tool in the next part of the paper is the so-called quasilinear technique. It was created and developed in [13], [16], [28] for locally bounded F-spaces, in [14] for any pair of the one-dimensional space and an arbitrary F-space and in [18] for nuclear spaces. The first purpose of the paper is to describe it in a much more general form. We get the appropriate extensions of the main results concerning this technique. For this description, however, we need some facts (proved in Section 2) concerning sections continuous at zero. A map $s\colon Z \to X$ is called a *section* for the twisted sum $(*)$ if $q \circ s = \mathrm{id}_Z$ and $s(0) = 0$. Obviously $(*)$ splits iff it has a linear continuous section.

In Section 2 we characterize precisely those pairs (Y, Z) of a semimetrizable (resp. locally bounded) tvs Y and an arbitrary tvs Z for which every twisted sum has a section (resp. a homogeneous section) continuous at zero. Two counterexamples are given which show that this theorem cannot be improved for an arbitrary Y. Incidentally, we give necessary conditions for the pair (Y, Z) of tvs, where Y is semimetrizable, to have only splitting twisted sums. These conditions will turn out to be sufficient for suitable classes of tvs Y, Z (comp. Proposition 4.3).

The results of Section 2 are widely used in Section 3, which is devoted to the developing of the quasilinear technique. We obtain all twisted sums using quasilinear maps. The main theorem of this Section is Theorem 3.1. We close Section 3 with results on "extensions" and "restrictions" of twisted sums.

Let Y be a tvs. Then $S(Y)$ is the class of all tvs Z such that there are only splitting twisted sums of Y and Z. A tvs Z is a *$\mathscr{K}$-space* [15] iff $Z \in S(Y)$ for the one-dimensional tvs Y.

The second purpose of the paper is to examine the classes $S(Y)$ (Section 4). In particular, we investigate hereditary properties of such classes. The best result in this direction is the following: if Y is a locally bounded F-space, then $S(Y)$ is closed under arbitrary products and reduced projective limits. We show that, for every Y, $S(Y)$ is closed under complemented subspaces and finite products. The key fact of Section 4 is Theorem 4.3.

The paper [13] suggests that a class which may be very interesting is the class of all locally convex spaces (lcs) Z satisfying the condition that every twisted sum of Y and Z is a lcs whenever so is Y. We call such spaces *TSC-spaces* (Twisted Sum-Convex). It was pointed out by Kalton [13] that TSC-spaces (which were not explicitly defined by him) are closely related to $\mathscr{K}$-spaces.

The third main purpose of the paper is to investigate the class of all TSC-spaces (Section 5). In particular, we establish the relation between $\mathscr{K}$-spaces and TSC-spaces. The key result of Section 5 is Theorem 5.1 and its strengthened version, Theorem 5.2. It implies that a metrizable lcs is a $\mathscr{K}$-space iff it is a TSC-space. Moreover, every reduced projective limit of metrizable locally convex $\mathscr{K}$-spaces is a TSC-space. In particular, all nuclear lcs (and lcs with a weak topology) are TSC-spaces. Also, we study hereditary properties of TSC-spaces and $\mathscr{K}$-spaces; for example, "to be a TSC-space" is a three space property. The lists of known TSC-spaces and $\mathscr{K}$-spaces are given. It is rather surprising that we know of no locally convex $\mathscr{K}$-space which is not a TSC-space.

Finally, we identify some elements of $S(Y)$ for an injective Banach space Y.

We close the introduction with some auxiliary notions and facts.

We will consider only vector topologies (not necessarily Hausdorff). Let τ be a topology on a vector space X. By $\mathscr{U}_\tau$ will be denoted the family of all balanced 0-neighbourhoods in (X, τ). $\ker \tau$ is the closure of $\{0\}$ in (X, τ). $(X/\ker \tau, \tau/\ker \tau)$ is called the *Hausdorff associated space* of (X, τ). Obviously $(X, \tau) \simeq (\ker \tau, \tau \cap \ker \tau) \oplus (X/\ker \tau, \tau/\ker \tau)$. By $(\tilde{X}, \tilde{\tau})$ we will denote the direct sum of $(\ker \tau, \tau \cap \ker \tau)$ and the completion of $(X/\ker \tau, \tau/\ker \tau)$. If $j\colon X \to Y$, where X, Y are tvs, then $\tilde{j}\colon \tilde{X} \to \tilde{Y}$ is the natural extension. If Y is a subspace of X, then $\tau \cap Y$ and τ/Y are the induced topology on Y and the quotient topology on X/Y, resp. If $T\colon (X, \tau) \to (Y, \lambda)$ is a linear surjection, then $T(\tau)$ (resp. $T^{-1}(\lambda)$) is the finest topology on Y (resp. the coarsest topology on X) such that T is continuous. $X \simeq Y$ means that X and Y are isomorphic. A continuous and relatively open linear mapping is called a *homomorphism*. $Y \subset Y_1$ means, in fact, that there is a topological linear embedding $i\colon Y \to Y_1$. Then for every map $L\colon Z \to Y$ (where Z is a tvs) we will assume that $L\colon Z \to Y_1$ (in fact, $i \circ L\colon Z \to Y_1$).

A tvs X is called an *F-space* iff it is a metrizable complete tvs. A map $\|\cdot\|\colon X \to \boldsymbol{R}_+$ is called a *quasinorm* if it satisfies the following conditions:

(1) $\|x\| = 0$ iff $x = 0$;

(2) $\|tx\| = |t|\,\|x\|$ whenever t is a scalar and $x \in X$;

(3) $\|x+y\| \leqslant C(\|x\|+\|y\|)$ for all $x, y \in X$ and some constant $C > 0$.

The one-dimensional tvs and the scalar field (of real or complex numbers) are denoted by K. For a tvs X we denote by X' and X^* the topological and the algebraic dual, resp. By $\sigma(X, Y)$ we denote the weak topology on X with respect to the dual pair $\langle X, Y\rangle$.

Two twisted sums

$$0 \to Y \overset{j_1}{\to} X_1 \overset{q_1}{\to} Z \to 0$$

and

$$0 \to Y \overset{j_2}{\to} X_2 \overset{q_2}{\to} Z \to 0$$

are *equivalent* if there is an isomorphism $T\colon X_1 \to X_2$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \overset{j_1}{\longrightarrow} & X_1 & \overset{q_1}{\longrightarrow} & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & Y & \underset{j_2}{\longrightarrow} & X_2 & \underset{q_2}{\longrightarrow} & Z & \longrightarrow & 0 \end{array}$$

In fact, the twisted sum $(*)$ splits iff it is equivalent to the direct sum

$$0 \to Y \overset{j_1}{\to} Y \oplus Z \overset{q_1}{\to} Z \to 0,$$

where $j_1(y) = (y, 0)$, $q_1(y, z) = z$ for every $y \in Y$, $z \in Z$.

The *Hausdorff associated twisted sum* of the twisted sum

$$(**) \qquad 0 \to (Y, \tau) \overset{j}{\to} (X, \lambda) \overset{q}{\to} (Z, \gamma) \to 0$$

is the following twisted sum:

$$0 \to (Y/\ker \tau, \tau/\ker \tau) \overset{j_1}{\to} (X/\ker \lambda, \lambda/\ker \lambda) \overset{q_1}{\to} (Z/q(\ker \lambda), \gamma/q(\ker \lambda)) \to 0,$$

where $q_X \circ j = j_1 \circ q_Y$, $q_Z \circ q = q_1 \circ q_X$, and $q_Y\colon Y \to Y/\ker \tau$, $q_X\colon X \to X/\ker \lambda$, $q_Z\colon Z \to Z/q(\ker \lambda)$ are the natural quotient maps. In general, the second summand of the Hausdorff associated twisted sum may be non-Hausdorff! The twisted sum $(**)$ splits iff its Hausdorff associated twisted sum splits.

The procedure of obtaining Hausdorff associated twisted sums may be divided into two parts. As a result of the first part we get the twisted sum

$$0 \to (Y, \tau) \overset{j_1}{\to} (X/X_1, \lambda/X_1) \overset{q_1}{\to} (Z/q(X_1), \gamma/q(X_1)) \to 0,$$

where $X_1 \cap j(Y) = \{0\}$, $q(X_1) = q(\ker \lambda)$, $j_1 = q_X \circ j$, $q_1 \circ q_X = q_Z \circ q$ for the natural quotient maps $q_X\colon X \to X/X_1$, $q_Z\colon Z \to Z/q(X_1)$. Of course, $\ker(\lambda/X_1) \subset j_1(Y)$.

As a result of the second part of the procedure used for $(**)$ we get the twisted sum

$$0 \to (Y/\ker \tau, \tau/\ker \tau) \overset{j_2}{\to} (X/j(\ker \tau), \lambda/j(\ker \tau)) \overset{q_2}{\to} (Z, \gamma) \to 0,$$

where $j_2 \circ q_Y = q_X \circ j$, $q_2 \circ q_X = q$ for the natural quotient maps $q_Y\colon Y \to Y/\ker \tau$, $q_X\colon X \to X/j(\ker \tau)$.

For other notions and notations see [33].

The following very useful fact is due to W. Roelcke (comp. [30] or [9], Lemma 2.1).

LEMMA A. *Let λ, τ be two vector topologies on a vector space X such that $\lambda \leqslant \tau$. If there is a subspace Y of X for which $\lambda \cap Y = \tau \cap Y$ and $\lambda/Y = \tau/Y$, then $\lambda = \tau$.*

For the sake of completeness we give the proof.

Proof. Let

$$U, U_1 \in \mathscr{U}_\tau, \qquad U_1 + U_1 \subset U$$

and let

$$V \in \mathscr{U}_\lambda, \qquad (V - V) \cap Y \subset U_1 \cap Y.$$

There is $W \in \mathscr{U}_\lambda$ satisfying $W \subset V$ and $W \subset (U_1 \cap V) + Y$. Thus we have immediately

$$W \subset (U_1 \cap V) + \big(Y \cap (W - (U_1 \cap V))\big) \subset (U_1 \cap V) + \big(Y \cap (V - V)\big)$$
$$\subset U_1 + U_1 \subset U.$$

Hence $U \in \mathscr{U}_\lambda$ and this completes the proof.

The following lemma is well known, comp. [7].

LEMMA B. *Let (X, λ) be an arbitrary tvs. If Y is a linear subspace of X, then:*

(a) *for every topology $\tau \leqslant \lambda \cap Y$ on Y there is a unique topology $\lambda_1 \leqslant \lambda$ on X for which $\lambda_1 \cap Y = \tau$ and $\lambda_1/Y = \lambda/Y$;*

(b) *for every topology $\gamma \geqslant \lambda/Y$ on X/Y there is a unique topology $\lambda_2 \geqslant \lambda$ on X for which $\lambda_2 \cap Y = \lambda \cap Y$ and $\lambda_2/Y = \gamma$.*

Proof. The uniqueness is a consequence of Roelcke's Lemma.

Now the families $\mathscr{U}_1 = \{U + V\colon U \in \mathscr{U}_\lambda, V \in \mathscr{U}_\tau\}$ and $\mathscr{U}_2 = \{U \cap q^{-1}(V)\colon U \in \mathscr{U}_\lambda, V \in \mathscr{U}_\gamma\}$ are the neighbourhood bases for λ_1, λ_2, resp., where $q\colon X \to X/Y$ is the natural quotient map.

1. Twisted sums of locally convex spaces. As has been pointed out, a twisted sum of two locally convex spaces need not be locally convex. In this section we will show that it must be "nearly" convex.

Let X be a tvs. A set $U \subset X$ is *pseudoconvex* if there is a constant $C > 0$ such that $U + U \subset DU$ for every $D \geqslant C$. A set $U \subset X$ is *p-convex* for some $0 < p \leqslant 1$ if, for all $a, b > 0$ with $a^p + b^p = 1$, $aU + bU \subset U$.

A balanced set $U \subset X$ is *logconvex* if there is a constant $C > 0$ such that, for every $n \in N$, $a_1, \ldots, a_n > 0$,

$$\sum_{i=1}^{n} a_i U \subset C\Big(\sum_{i=1}^{n} a_i \Big(1 + \ln\Big(\sum_{j=1}^{n} a_j/a_i\Big)\Big)\Big) U.$$

A tvs X is *locally pseudoconvex* (*p-convex*, *logconvex*) iff it has a 0-neighbourhood base containing only pseudoconvex (resp. p-convex, logconvex) sets.

THEOREM 1.1. *Local pseudoconvexity is a three space property. In particular, a twisted sum of two lcs is locally pseudoconvex.*

Proof. Let (X, λ) be a tvs containing a locally pseudoconvex subspace $(Y, \lambda \cap Y)$ with the locally pseudoconvex quotient $(X/Y, \lambda/Y)$. The map q is the quotient map.

Let U be a 0-neighbourhood in (X, λ). Of course, there exists a balanced pseudoconvex $U_1 \in \mathscr{U}_{\lambda \cap Y}$ satisfying $U_1 + V \subset U$ for a fixed $V \in \mathscr{U}_\lambda$. Now, we can define a topology $\lambda_1 \leqslant \lambda$ with a 0-neighbourhood base $\mathscr{U}_1 = \{V + n^{-1} U_1 \colon n \in N,\ V \in \mathscr{U}_\lambda\}$. Thus U is a 0-neighbourhood in λ_1; $\lambda_1/Y = \lambda/Y$; and $\lambda_1 \cap Y$ is a locally bounded topology. We may choose a balanced set $W \in \mathscr{U}_1$ with the following properties:

$$W + W \subset U; \tag{1.1}$$

$$q(W) \text{ is pseudoconvex in } X/Y, \tag{1.2}$$

i.e., $q(W) + q(W) \subset Aq(W)$ for some $A \geqslant 1$;

$$(W + W + W + W + W) \cap Y \text{ is bounded in } (Y, \lambda_1 \cap Y), \tag{1.3}$$

i.e., $(W + W + W + W + W) \cap Y \subset B(W \cap Y)$ for some $B \geqslant 1$.

We will prove that $W + W$ is a pseudoconvex set. Let $x \in W + W + W + W$; obviously, by (1.2) we have

$$q(x) \in q(W) + q(W) + q(W) + q(W) \subset A^2 q(W).$$

Thus there is $y \in Y$ with $x + y \in A^2 W$. Hence

$$y \in A^2 W + W + W + W + W \subset A^2 (W + W + W + W + W).$$

But $y \in Y$ and using (1.3), we have

$$y \in A^2 \big((W + W + W + W + W) \cap Y\big) \subset A^2 B (W \cap Y).$$

Finally,

$$x \in A^2 W + A^2 B (W \cap Y) \subset A^2 B (W + W).$$

We get the desired property $W + W + W + W \subset A^2 B (W + W)$.

The following proposition is obvious.

PROPOSITION 1.1. *For every tvs (X, τ) containing a subspace $(Y, \tau \cap Y)$ and $\ker \tau = Z$, the space $(X/Z, \tau/Z)$ is the twisted sum of $(\bar{Y}/Z, (\tau \cap \bar{Y})/Z)$ and $((X/Y)/\ker(\tau/Y), (\tau/Y)/\ker(\tau/Y))$. In particular, if a tvs (X, τ) is a twisted sum of locally p- and q-convex spaces, then its Hausdorff associated tvs is a twisted sum of Hausdorff locally p- and q-convex spaces.*

THEOREM 1.2. (a) *For every $0 < p, q \leqslant 1$, $p \neq q$, a locally bounded twisted sum of a Hausdorff locally p-convex space Y and a Hausdorff locally q-convex space Z is a locally* $\min(p, q)$*-convex space.*

(b) *A locally bounded Hausdorff tvs is locally logconvex iff it is a Hausdorff quotient of a subspace of a twisted sum of two normable spaces.*

Proof. These facts are due to Kalton: for (a) see [13], Theorem 4.1, [14], Theorems 6.2, 6.4, 6.5; for (b) see [14], Theorem 7.2.

Now, we generalize Theorem 1.2.

THEOREM 1.3. *For every $0 < p, q \leqslant 1$, $p \neq q$, a twisted sum of a locally p-convex space Y and a locally q-convex space Z is a locally* $\min(p, q)$*-convex space. If $p = q = 1$, then it is locally logconvex.*

COROLLARY 1.1. *Every twisted sum of two locally convex spaces is locally p-convex for every $0 < p < 1$.*

Proof of Theorem 1.3. Let (X, λ) be a tvs containing a locally p-convex subspace $(Y, \lambda \cap Y)$ with the locally q-convex quotient $(X/Y, \lambda/Y)$. We will denote by q the natural quotient map. Now, for an arbitrary 0-neighbourhood U in (X, λ), there are balanced 0-neighbourhoods: p-convex V in $(Y, \lambda \cap Y)$, pseudoconvex U_1 in (X, λ) and q-convex V_1 in $(X/Y, \lambda/Y)$, which satisfy the following properties:

$$V + V \subset U \cap Y,$$

$$U_1 \cap Y \subset V, \quad U_1 + V \subset U,$$

$$V_1 \subset q(U_1).$$

This may be established by Theorem 1.1 and the fact that every p-convex set is pseudoconvex. The family

$$\mathscr{U} = \{n^{-1}\big((U_1 \cap q^{-1}(V_1)) + V\big) \colon n \in N\}$$

contains only pseudoconvex balanced sets. Thus $\mathscr{U}$ is a 0-neighbourhood base for a locally bounded vector topology λ_1 (non-Hausdorff in general) with $\lambda_1 \leqslant \lambda$, and U is a 0-neighbourhood in (X, λ_1). $\lambda_1 \cap Y$ and λ_1/Y are p- and q-convex, resp., because

$$q\big((U_1 \cap q^{-1}(V_1)) + V\big) = V_1,$$

$$V \subset \big((U_1 \cap q^{-1}(V_1)) + V\big) \cap Y \subset V + V \subset 2^p V.$$

By Proposition 1.1 and Theorem 1.2 (a), λ_1 is $\min(p, q)$-convex. Similarly, if $p = q = 1$, then, by Proposition 1.1 and Theorem 1.2 (b), λ_1 is locally logconvex. Thus U contains a $\min(p, q)$-convex (resp. logconvex) 0-neighbourhood in (X, λ), and this completes the proof.

THEOREM 1.4. *A tvs is locally logconvex iff it is a quotient of a subspace of a twisted sum of two locally convex spaces.*

Proof. Necessity. A locally logconvex tvs Z may be embedded into a product $\prod_{i\in I} Z_i^1$ of locally bounded Hausdorff locally logconvex tvs. By Proposition 1.2 (b),

$$Z_i^1 \subset X_i/Y_i^1 \quad \text{for } i\in I,$$

where X_i is a twisted sum of two normable tvs Y_i and Z_i; Y_i^1 is a closed subspace of X_i. Thus

$$Z \subset \prod_{i\in I} (X_i/Y_i^1) \simeq \prod_{i\in I} X_i/\prod_{i\in I} Y_i^1.$$

It is enough to prove that $\prod_{i\in I} X_i$ is a twisted sum of two locally convex tvs. But

$$\prod_{i\in I} X_i/\prod_{i\in I} Y_i \simeq \prod_{i\in I}(X_i/Y_i) \simeq \prod_{i\in I} Z_i$$

and $\prod_{i\in I} Y_i$, $\prod_{i\in I} Z_i$ are locally convex.

Sufficiency is an immediate consequence of Theorem 1.3 because every quotient and every subspace of a locally logconvex space are also locally logconvex.

2. Sections for twisted sums. Of course, a twisted sum splits iff it has a linear section continuous at zero (actually continuous). This section is devoted to the study of sections continuous at zero (not necessarily linear).

It is easily seen that the existence of a section continuous at zero is preserved by the equivalence relation of twisted sums. In fact, there is a stronger result:

LEMMA 2.1. *Let*

$$0 \to (Y, \lambda) \xrightarrow{j} (X, \tau_1) \xrightarrow{q} (Z, \gamma_1) \to 0 \tag{2.1}$$

and

$$0 \to (Y, \lambda) \xrightarrow{j} (X, \tau_2) \xrightarrow{q} (Z, \gamma_2) \to 0 \tag{2.2}$$

be twisted sums, where $\tau_1 \geqslant \tau_2$.

Let $s\colon (Z, \gamma_2) \to (X, \tau_2)$ *be a section continuous at zero for the twisted sum* (2.2). *Then* s *is such a section for* (2.1) *too.*

Proof. The family of sets $U \cap q^{-1}(V)$, where U, V are 0-neighbourhoods in τ_2 and γ_1, resp., forms a 0-neighbourhood base in (X, τ_1) (comp. Lemma B).

Let $q\colon X \to Z$ be a continuous map; then we say that a 0-neighbourhood U in X has the *line property* with respect to q iff, for every $z_0 \in Z$, $z_0 \neq 0$, such that $\operatorname{lin}\{z_0\} \subset q(U)$, there is $x_0 \in X$, $x_0 \in q^{-1}(z_0)$, for which $\operatorname{lin}\{x_0\} \subset U$.

Now, we will give a necessary condition for a twisted sum to have a (homogeneous) section continuous at zero.

PROPOSITION 2.1. *Let the following diagram be a twisted sum*:

$$0 \to (Y, \lambda) \xrightarrow{j} (X, \tau) \xrightarrow{q} (Z, \gamma) \to 0. \tag{2.3}$$

(a) *If* (2.3) *has a section continuous at zero, then*

$$q(\ker \tau) = \ker \gamma.$$

(b) *If* (2.3) *has a homogeneous section continuous at zero, then there is a* 0-*neighbourhood base for* τ *whose each element has the line property with respect to* q.

Proof. (a): Let us assume that $s\colon (Z, \gamma) \to (X, \tau)$ is a section continuous at zero. Of course, $q(\ker \tau) \subset \ker \gamma$. Now, let $z \notin q(\ker \tau)$; thus there exists a 0-neighbourhood U in (X, τ) such that $s(z) \notin U$. Hence z is not contained in the 0-neighbourhood $s^{-1}(U)$ in (Z, γ), i.e., $z \notin \ker \gamma$.

(b): Let us assume that $s\colon (Z, \gamma) \to (X, \tau)$ is a homogeneous section continuous at zero. Let U be an arbitrary 0-neighbourhood in (X, τ). The 0-neighbourhood $V = U \cap q^{-1}(s^{-1}(U))$ in (X, τ) has the line property with respect to q, because $q(V) = s^{-1}(U)$ and, for every $z_0 \neq 0$, $z_0 \in Z$, $\operatorname{lin}\{z_0\} \subset s^{-1}(U)$, we have $\operatorname{lin}\{s(z_0)\} \subset V$. Of course, V is a 0-neighbourhood in (X, τ) contained in U. This completes the proof.

The following lemma is a partial converse of Proposition 2.1 (for part (b) comp. [14], proof of Theorem 10.1).

LEMMA 2.2. (a) *If in* (2.3) Y *is semimetrizable and there is a weaker semimetrizable topology* τ_1 *on* X *such that*

$$j^{-1}(\tau_1 \cap j(Y)) = \lambda, \quad q(\tau_1) = \gamma_1, \quad q(\ker \tau_1) = \ker \gamma_1,$$

then (2.3) *has a section continuous at zero.*

(b) *If in* (2.3) Y *is locally bounded and there is a* 0-*neighbourhood base of the topology* τ *containing only sets with the line property with respect to* q, *then* (2.3) *has a homogeneous section continuous at zero.*

Proof. (a): By Lemma 2.1 it is enough to construct a section continuous at zero for the twisted sum

$$0 \to (Y, \lambda) \xrightarrow{j} (X, \tau_1) \xrightarrow{q} (Z, \gamma_1) \to 0. \tag{2.4}$$

Let $\{U_n\}$ be a 0-neighbourhood base in (X, τ_1), $U_\infty = \ker \tau_1$, $U_0 = X$, and let $n(z) = \sup\{n \in N \cup \{\infty\}\colon z \in q(U_n)\}$ for every $z \in Z$. Thus, by the assumption, for every $z \in Z$ we can choose

$$s(z) \in U_{n(z)} \cap q^{-1}(z)$$

with $z \in q(U_{n(z)})$. Hence $s\colon Z \to X$ is the section continuous at zero for (2.4).

(b): Let $\mathscr{U}_1$ be a 0-neighbourhood base in (X, τ) (consisting only of balanced sets) such that, for every $U\in\mathcal{U}_1$, U has the line property with respect to q and $(U+U)\cap Y$ is bounded.

Now, let $U\in\mathcal{U}_1$, $V_U\in\mathcal{U}_Z$, $V_U+V_U\subset q(U)$, and let Z_1 be an arbitrary one-dimensional subspace $Z_1\subset Z$. If $Z_1\subset V_U\subset q(U)$, then there is x for which $0\neq q(x)=z\in Z_1$ and $\operatorname{lin}\{x\}\subset U$. If $Z_1\setminus V_U\neq\emptyset$, then we can choose $z\in\big(q(U)\setminus V_U\big)\cap Z_1$ and $x\in q^{-1}(z)\cap U$. We may define

$$s_U(z)=x,$$

and homogeneously extending s onto the whole space Z we get

$$s_U(V_U)\subset U.$$

Let $U_1, U_2\in\mathcal{U}_1$, $U_1+U_1\subset U_2\subset U$, and let $a\in K$, $|a|\leqslant 1$, for which $a\big((U+U)\cap Y\big)\subset U_1$. We have

$$s_U\big(a(V_{U_1}\cap V_U)\big)\subset(s_U-s_{U_1})\big(a(V_{U_1}\cap V_U)\big)+s_{U_1}(V_{U_1})\subset U_1+U_1\subset U_2.$$

Thus s_U is continuous at zero. This completes the proof.

It may be easily seen that if Y is complete, then in (2.3) $\overline{j(Y)}=j(Y)+\ker\tau=q^{-1}(\ker\gamma)$. Thus in this case the assumption of part (a) of the preceding lemma is satisfied. A similar fact is true for part (b).

LEMMA 2.3. *Let X, Z be tvs and $q\colon X\to Z$ be a surjective homomorphism with a locally bounded and complete kernel. If U is a closed balanced 0-neighbourhood in X which satisfies the condition*

(2.5) $$(U+U)\cap\ker q \quad \text{is a bounded set,}$$

then U has the line property with respect to q.

Proof. Let $0\neq z_0\in Z$ and let $\operatorname{lin}\{z_0\}$ be contained in $q(U)$. Then $\big(q^{-1}(z_0)\cap(n^{-1}U)\big)_{n\in N}$ is a decreasing sequence of nonempty subsets in $q^{-1}(z_0)$. By (2.5), the family $\big(n^{-1}(U+U)\cap\ker q\big)_{n\in N}$ is a 0-neighbourhood base in $\ker q$ and, on the other hand, if $z_1, z_2\in q^{-1}(z_0)\cap(n^{-1}U)$, then $z_1-z_2\in n^{-1}(U+U)$. Thus $\big(q^{-1}(z_0)\cap(n^{-1}U)\big)_{n\in N}$ is a closed Cauchy filter base in the complete uniform space $q^{-1}(z_0)$. Therefore this filter converges, i.e., there exists x_0 which belongs to $\bigcap_{n\in N}\big(q^{-1}(z_0)\cap n^{-1}U\big)$. Obviously $\operatorname{lin}\{x_0\}\subset U$.

Now, we will give some auxiliary facts.

PROPOSITION 2.2 (V. P. Palamodov). *Let X be a complete tvs containing a metrizable subspace Y. Then X/Y is also complete.*

The proof of Proposition 2.2 is contained in [9], Corollary 2.6 or in [32], Theorem 11.18.

COROLLARY 2.1. *Let Y be a semimetrizable tvs and let Z be a Hausdorff tvs. If*

$$0\to Y\xrightarrow{j}X\xrightarrow{q}Z\to 0$$

is a twisted sum, then so is

$$0\to\tilde{Y}\xrightarrow{\tilde{j}}\tilde{X}\xrightarrow{\tilde{q}}\tilde{Z}\to 0.$$

Proof. By passing to the Hausdorff associated twisted sum, we can assume that Y is a Hausdorff tvs. Let us consider the following commutative diagram:

(2.6) $$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\;j\;} & X & \xrightarrow{\;q\;} & Z & \longrightarrow & 0\\ & & \downarrow{\scriptstyle i_Y} & & \downarrow{\scriptstyle i_X} & & & & \\ 0 & \longrightarrow & \tilde{Y} & \xrightarrow[\;\tilde{j}\;]{} & \tilde{X} & \xrightarrow[\;q_1\;]{} & \tilde{X}/\tilde{Y} & \longrightarrow & 0 \end{array}$$

where i_Y, i_X are the respective topological embeddings. Of course, the rows are twisted sums, and $\ker q=\ker q_1\circ i_X$. Now, we can define a linear injection $i_Z\colon Z\to\tilde{X}/\tilde{Y}$ such that (2.6) with the additional map i_Z is commutative. It can be easily verified that i_Z is a homomorphism and $i_Z(Z)$ is a dense subspace in $\tilde{X}/\tilde{Y}$. By Proposition 2.2, $\tilde{X}/\tilde{Y}$ is complete; it may be identified with $\tilde{Z}$, and then q_1 may be identified with $\tilde{q}$. This completes the proof.

We will denote by $\mathfrak{m}(Z)$ the cardinal number $\inf\dim(Z_1/Z_2)$, where inf is taken over all pairs of subspaces $Z_2\subset Z_1\subset Z$ such that there is no weaker semimetrizable topology ξ on Z_1 such that $\ker\xi\subset Z_2$. Similarly, we will denote by $\mathfrak{l}(Z)$ the cardinal number $\inf\dim(Z_1/Z_2)$, where inf is taken over all pairs of subspaces $Z_2\subset Z_1\subset Z$ such that for every 0-neighbourhood U in Z there is a one-dimensional subspace $Z_3\subset Z_1\cap U$ and $Z_3\not\subset Z_2$.

If there are no such Z_1, Z_2 we will assume (in both cases) that $\mathfrak{m}(Z)$ (or $\mathfrak{l}(Z)$) is equal to ∞; this last value is assumed to be greater than all cardinal numbers.

PROPOSITION 2.3. *Let Z be a tvs.*

(a) *$\mathfrak{m}(Z)=\infty$ iff Z has a weaker metrizable topology.*

(b) *$\mathfrak{l}(Z)=\infty$ iff Z has a 0-neighbourhood which contains no line.*

(c) *If Z is not Hausdorff, then $\mathfrak{l}(Z)=\mathfrak{m}(Z)=1$.*

Proof. (a): If Z has no weaker metrizable topology, then the pair of subspaces $Z_1=Z$, $Z_2=\{0\}$ has the required property. On the other hand, if Z has a weaker metrizable topology γ, then, for every $Z_2\subset Z_1\subset Z$, $\ker\gamma\cap Z_1=\{0\}\subset Z_2$.

(b): The proof is quite similar.

(c): If $Z_1\subset Z$ has the trivial topology and $Z_2=\{0\}$, $\dim Z_1=1$, then the pair Z_1, Z_2 has the required property from the definition of $\mathfrak{l}(Z)$ and $\mathfrak{m}(Z)$.

The main theorem of the section is the following:

Theorem 2.1. *Let Z be a tvs, and let Y be a semimetrizable tvs.*

(a) *Every twisted sum of Y and Z has a section continuous at zero iff*

$$\operatorname{codim}_{\tilde{Y}} Y < \mathfrak{m}(Z). \tag{2.7}$$

(b) *Let Y be locally bounded and Z be locally pseudoconvex. Then every twisted sum of Y and Z has a homogeneous section continuous at zero iff*

$$\operatorname{codim}_{\tilde{Y}} Y < \mathfrak{l}(Z). \tag{2.8}$$

(c) *If Y is locally bounded and complete, then every twisted sum of Y and Z has a homogeneous section continuous at zero.*

(d) *If Y is locally bounded and the pair (Y, Z) of tvs does not satisfy the condition* (2.8), *then there is a twisted sum of Y and Z without any homogeneous section continuous at zero.*

Remarks. Theorem 2.1 may be considered as a theorem on the splitting of twisted sums. Of course, for appropriate tvs Y, Z, if $Z \in S(Y)$, then they satisfy (2.7) or (2.8).

As is shown by the following counterexample, it seems that this theorem cannot be strengthened.

E. Michael (see [4], Theorem II.7.1, [3], [22], I, Prop. 7.2) showed that, in fact, every twisted sum of a locally convex F-space and an arbitrary F-space has a continuous section. The first counterexample shows that this section is not homogeneous in general.

Counterexamples. Let us consider the linear spaces

$$X_1 = \{(x_{ij}) \in \prod_{i=1}^{\infty} l_1 \colon \lim_{i\to\infty} \sup_{j\in N} |x_{ij}| = 0\}, \qquad X_2 = \bigoplus_{i=1}^{\infty} l_1 .$$

Let us define for every $n \in N$, $\varepsilon > 0$, $(\varepsilon_i) \in \prod_{i=1}^{\infty} (0, \infty)$,

$$U_1(n, \varepsilon) \subset X_1, \qquad U_2(n, (\varepsilon_i)) \subset X_2,$$

$$U_1(n, \varepsilon) = \{(x_{ij}) \in X_1 \colon |x_{ij}| < \varepsilon \text{ for } i > n, j \in N, \sum_{j=1}^{\infty} |x_{ij}| < \varepsilon \text{ for } i \leqslant n\},$$

$$U_2(n, (\varepsilon_i)) = \{(x_{ij}) \in X_2 \colon |x_{ij}| < \varepsilon_i \text{ for } i > n, j \in N, \sum_{j=1}^{\infty} |x_{ij}| < \varepsilon_i \text{ for } i \leqslant n\}.$$

Now, we will denote by τ_1, τ_2 the topologies generated by the 0-neighbourhood bases

$$\mathscr{U}_1 = \{U_1(n, \varepsilon) \colon n \in N, \varepsilon > 0\},$$

$$\mathscr{U}_2 = \{U_2(n, (\varepsilon_i)) \colon n \in N, \ (\varepsilon_i) \in \prod_{i=1}^{\infty} (0, \infty)\}$$

in X_1, X_2, resp. It is easily seen that (X_1, τ_1), (X_2, τ_2) are complete tvs. The maps

$$T_k \colon (X_k, \tau_k) \to K^N, \qquad T_k((x_{ij})_{i,j\in N}) = (\sum_{j=1}^{\infty} x_{ij})_{i\in N},$$

$k = 1, 2$, are homomorphisms (their kernels are obviously closed).

τ_1 is a metrizable topology and thus, by Theorem 2.1, T_1 has a section continuous at zero. But it has no homogeneous section continuous at zero. This is implied by the fact that $T_1(X_1) = K^N$ and every 0-neighbourhood in K^N contains a line but $U_1(n, \varepsilon)$ contains no line for any $n \in N$, $\varepsilon > 0$.

Similarly, we will prove that T_2 has no section continuous at zero. Let us assume that $s \colon T_2(X_2) = K^{(N)} \to X_2$ is such a section, where $K^{(N)} \subset K^N$ is the space of all finitely nonzero sequences. If $e_1 = (1, 0, 0, \ldots)$, $e_2 = (0, 1, 0, 0, \ldots)$, $\ldots$, then $e_n \to 0$ and $s(e_n) \to 0$ as $n \to \infty$. If $s(e_n) = (x_{ij}^n)$ and $\varepsilon_i = \frac{1}{2} \sup_{j\in N} (|x_{ij}^i|)$, then $\varepsilon_i > 0$ and $s(e_i) \notin U_2(1, (\varepsilon_i))$ for $i \in N$; a contradiction.

Proof of Theorem 2.1. (a), the "only if" part and (d): Let us assume that $\dim (Z_1/Z_2) \leqslant \operatorname{codim}_{\tilde{Y}} Y$ for a pair of subspaces Z_1, Z_2 of Z with the required property. Let $\operatorname{codim}_{Y_1} Y = \dim(Z_1/Z_2)$ for $Y \subset Y_1 \subset \tilde{Y}$. Let Y_0 be a vector space for which $\dim Y_0 = \operatorname{codim}_Z Z_1 + \dim Z_2$ and let Y_0 have the trivial topology. Of course, there is an algebraic isomorphism

$$h \colon Y_1/Y \oplus Y_0 \to Z$$

such that $h^{-1}(Z_2) \cap (Y_1/Y) = \{0\}$. Hence we have the following short exact sequence:

$$0 \to Y \xrightarrow{j} Y_1 \oplus Y_0 \xrightarrow{q} Z \to 0, \tag{2.10}$$

for which $j(y) = (y, 0)$ is an injective homomorphism and $q(y_1, y_2) = h(q_1(y_1), y_2)$ is a surjective homomorphism whenever Z has the trivial topology; $q_1 \colon Y_1 \to Y_1/Y$ is the natural quotient map. Now, using the proof of Lemma B, we can define on $Y_1 \oplus Y_0$ a topology λ with a 0-neighbourhood base consisting of all sets of the form $U \cap q^{-1}(V)$, where U, V are 0-neighbourhoods in the original topologies on $Y_1 \oplus Y_0$ and Z, resp. Then (2.10) is a twisted sum of Y and Z which in either case (a) or case (d) has no respective section. Indeed, by the definition of h, it is easily seen that

$$q^{-1}(Z_2) = Y \oplus Y_2 \subset Y_1 \oplus Y_0$$

for some Y_2 with $Y_2 \cap Y_1 = \{0\}$. Now, if s were a section for (2.10) with the required property, then $s|_{Z_1}\colon Z_1 \to Y_1 \oplus Y_2$ would have the same property whenever Y_1 has the original topology and Y_2 has the trivial topology. The kernel of the topology on $Y_1 \oplus Y_2$ is contained in $Y \oplus Y_2$ and $(s|_{Z_1})^{-1}(Y \oplus Y_2) \subset q(Y \oplus Y_2) = Z_2$.

Thus in case (a) the vector topology ξ induced on Z_1 from $Y_1 \oplus Y_2$ by $s|_{Z_1}$ has the kernel $Z_3 \subset Z_2$. Of course, it is weaker than the original topology and semimetrizable because so is $Y_1 \oplus Y_2$.

In case (d) there is a bounded 0-neighbourhood U in $Y_1 \oplus Y_2$ (Y_2 still with the trivial topology). If a one-dimensional subspace $Z_3 \subset Z$ is contained in $(s|_{Z_1})^{-1}(U)$, then it is contained in $(s|_{Z_1})^{-1}(Y_3)$, where Y_3 is the kernel of the locally bounded topology on $Y_1 \oplus Y_2$. Finally, $Z_3 \subset Z_2$; a contradiction.

(b), the "only if" part: This is an immediate consequence of part (d).

(c): This is an immediate consequence of Lemma 2.2 (b) and Lemma 2.3.

(a) and (b), the "if" parts: Let

$$0 \to (Y, \lambda) \xrightarrow{j} (X, \tau) \xrightarrow{q} (Z, \gamma) \to 0$$

be a twisted sum. We can assume that $\ker \tau \subset j(Y)$. If (Z, γ) were not Hausdorff, then $\operatorname{codim}_{\tilde{Y}} Y < 1$ (see Proposition 2.3 (c)) and Y would be complete. Thus (Z, γ) is in fact Hausdorff and by Corollary 2.1 we get the following twisted sum:

$$0 \to (\tilde{Y}, \tilde{\lambda}) \xrightarrow{\tilde{j}} (X_1, \tau_1) \xrightarrow{q_1} (Z, \gamma) \to 0,$$

for which $\tau_1 = \tilde{\tau} \cap X_1$, $X_1 = \tilde{q}^{-1}(Z)$, $q_1 = \tilde{q}|_{X_1}$.

Of course, for every semimetrizable (resp. locally bounded) topology $\tau_2 \leqslant \tau_1$ for which $\tilde{j}^{-1}(\tau_2 \cap \tilde{j}(\tilde{Y})) = \tilde{\lambda}$ we have

$$q_1(\ker \tau_2) = \ker \gamma_2$$

where $\gamma_2 = q_1(\tau_2)$. Now, we have $\dim \big(\ker \gamma_2 / q(\ker \tau_2 \cap X)\big) \leqslant \operatorname{codim}_{X_1} X = \operatorname{codim}_{\tilde{Y}} Y$. By the assumption, there is a semimetrizable (resp. locally bounded) topology $\gamma_3 \leqslant \gamma$ on Z such that $\ker(\gamma_3 \cap \ker \gamma_2) \subset q(\ker \tau_2 \cap X)$. Using the proof of Lemma B, we can define a topology τ_3 on X_1 such that the following conditions hold:

$$\tilde{j}^{-1}(\tau_3 \cap \tilde{j}(\tilde{Y})) = \tilde{\lambda}, \quad \tau_2 \leqslant \tau_3 \leqslant \tau_1, \quad q_1(\tau_3) = \sup(\gamma_2, \gamma_3) = \gamma_4 \leqslant \gamma.$$

It is easily seen by the proof of Lemma B that the family of all sets of the form $U \cap q_1^{-1}(V)$, where $U \in \mathscr{U}_{\tau_2}$, $V \in \mathscr{U}_{\gamma_4}$, is a 0-neighbourhood base of τ_3. Now, $q(\ker \tau_3 \cap X) \subset \ker \gamma_4$. Let $z \in \ker \gamma_4$; but $\ker \gamma_4 = \ker \gamma_2 \cap \ker \gamma_3 \subset q(\ker \tau_2 \cap X)$ and thus there is $x \in X$ such that $q(x) = z$ and, for every $U \in \mathscr{U}_{\tau_2}$, $x \in U$. Obviously $x \in q_1^{-1}(V)$ for every $V \in \mathscr{U}_{\gamma_4}$. Hence $x \in \ker \tau_3 \cap X$ and we get

$$q(\ker \tau_3 \cap X) = \ker \gamma_4. \tag{2.11}$$

In case (a) it is obvious that we can choose a semimetrizable topology $\tau_2 \leqslant \tau_1$ with the required properties. Thus the preceding construction can be made. By Lemma 2.2 (a) this completes the proof.

In case (b), by Theorem 1.1, τ_1 is locally pseudoconvex. Thus for every 0-neighbourhood U in (X, τ) there is a locally bounded topology $\tau_2 \leqslant \tau_1$ such that U contains a bounded 0-neighbourhood in $\tau_2 \cap X$. Finally, U contains a bounded 0-neighbourhood W in $\tau_3 \cap X$. Let us observe that if V is an arbitrary bounded 0-neighbourhood, then a one-dimensional subspace $Z_3 \subset V$ iff Z_3 is contained in the kernel of the topology. By (2.11), W has the line property with respect to q. This completes the proof by Lemma 2.2 (b).

It may be interesting to find $\mathfrak{m}(Z)$ or $\mathfrak{l}(Z)$ for some tvs Z.

THEOREM 2.2. *Let Z be an infinite-dimensional tvs.*

(a) *For a semimetrizable tvs Z, $\mathfrak{l}(Z) = 1$ iff Z has arbitrarily short lines (i.e., every 0-neighbourhood contains a line).*

(b) *If, for every 0-neighbourhood U in Z, the linear span of all subspaces contained in U is equal to Z, then $\mathfrak{l}(Z) = 1$. For example, $\mathfrak{l}(L_0(0, 1)) = 1$.*

(c) $\mathfrak{l}(Z, \sigma(Z, Z')) = 1$ *(iff $Z' \neq Z^*$) or $= \aleph_0$ (iff $Z' = Z^*$).*

Remark. It can be proved that if $\dim Z \leqslant \dim K^{\aleph_0}$, then $\mathfrak{m}(Z, \sigma(Z, Z^*)) = \infty$. On the other hand, if $\dim Z > \dim K^{\aleph_0}$, then $\mathfrak{m}(Z, \sigma(Z, Z^*))$ is the cardinal number next to $\dim K^{\aleph_0}$.

Proof. (a): The "only if" part is a consequence of Proposition 2.3 (b). If Z is not Hausdorff our fact is a consequence of Proposition 2.3 (c). Let us assume that Z is Hausdorff and has arbitrarily short lines. Let $U_1 \supset U_2 \supset \ldots$ be a 0-neighbourhood base in Z. We can choose a sequence $x_1, x_2, \ldots$ of nonzero vectors in Z such that $\operatorname{lin}\{x_n\} \subset U_n$ for every $n \in N$. If $\dim \operatorname{lin}\{x_n\colon n \in N\} < \aleph_0$, then there is a 0-neighbourhood U in Z such that $U \cap \operatorname{lin}\{x_n\colon n \in N\}$ contains no line (Z is a Hausdorff tvs). But $U \supset U_n$ for some $n \in N$; a contradiction. Thus $\dim \operatorname{lin}\{x_n\colon n \in N\} = \aleph_0$. We can choose a subsequence $n_1 < n_2 < \ldots$ such that $x_{n_1}, x_{n_2}, \ldots$ are linearly independent. Of course, there is a linear functional $f\colon Z \to K$ such that $f(x_{n_i}) = 1$ for every $i \in N$. The pair of subspaces $Z_1 = Z$, $Z_2 = \ker f$ has the desired property. In fact, if U is a 0-neighbourhood in Z, then $U \supset U_{n_i} \supset \operatorname{lin}\{x_{n_i}\}$ for some $i \in N$. But $\operatorname{lin}\{x_{n_i}\} \not\subset \ker f$.

(b): This is obvious: $Z_1 = Z$, Z_2 is an arbitrary hyperplane.

(c): If $Z' \neq Z^*$, then there is a dense hyperplane $Z_2 \subset Z = Z_1$. If U were a 0-neighbourhood in $(Z, \sigma(Z, Z'))$ such that all lines contained in U were contained in Z_2, then Z_2 would contain a closed subspace (in the tvs Z) of finite codimension; a contradiction, because Z_2 would also be closed in Z.

If $Z' = Z^*$, then every subspace Z_2 of finite (nonzero) codimension in $Z_1 \subset Z$ is of the form $\ker f_1 \cap \ker f_2 \cap \ldots \cap \ker f_n \cap Z_1$ for some continuous linear functionals $f_1, \ldots, f_n$ on Z and $n \in N$. Let us denote $U = \{z \in Z\colon |f_i(z)| < 1, i = 1, \ldots, n\}$; thus for every one-dimensional subspace $Z_3 \subset Z_1 \cap U$ we have $Z_3 \subset Z_2$. We have proved that $\mathrm{I}(Z) \geqslant \aleph_0$. Of course, if $\dim Z_1 = \aleph_0$, then every 0-neighbourhood contains a line. Hence $\mathrm{I}(Z) \leqslant \aleph_0$.

3. The quasilinear technique. Let us define for a given mapping $F\colon Z \to Y$, where Y, Z are tvs, two other mappings $A_F\colon Z \times Z \to Y$, $J_F\colon K \times Z \to Y$ such that

$$A_F(x, y) = F(x+y) - F(x) - F(y), \qquad J_F(a, x) = F(ax) - aF(x),$$

for all $x, y \in Z$, $a \in K$. We call F *quasilinear* (comp. [13] or [28] where the definition is slightly different) if $F(0) = 0$, and A_F, J_F are continuous at zero (i.e., at the point (0, 0) in $Z \times Z$ and $K \times Z$, resp.). We denote by $Q(Z, Y)$ the linear space of all quasilinear maps from Z to Y and by $Q(Z, Y, Y_1)$, where $Y \subset Y_1$, the linear subspace of $Q(Z, Y_1)$ containing only maps F which satisfy the following condition: for every $U \in \mathcal{U}_{Y_1}$ there is $W \in \mathcal{U}_Z$ satisfying

$$F^{-1}(Y+U) \supset W. \tag{3.1}$$

Of course, if $Y_1 \subset Y_2$ are tvs, then we may assume that $Q(Z, Y_1) \subset Q(Z, Y_2)$. Similarly, for tvs $Y \subset Y_1 \subset Y_2$ we have $Q(Z, Y, Y_2) \subset Q(Z, Y_1, Y_2)$ and $Q(Z, Y, Y_1) \subset Q(Z, Y, Y_2)$. Obviously $Q(Z, Y, \tilde{Y}) = Q(Z, Y)$.

Let us introduce useful notations:

$$W_F^Y(U, V) = \{(y, z) \in Y \times Z\colon z \in V,\ y - F(z) \in U\},$$

where $V \subset Z$, $U \subset Y_1$, and

$$\mathcal{U}_F^Y = \{W_F^Y(U, V)\colon V \in \mathcal{U}_Z,\ U \in \mathcal{U}_{Y_1}\},$$

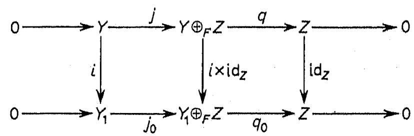
where $F \in Q(Z, Y, Y_1)$, $Y \subset Y_1$. If $Y = Y_1$, then we will omit the superscript Y.

It can be easily seen that $W_F^Y(U, V) = Y \times Z \cap W_F(U, V)$ for $U \subset Y_1$, $V \subset Z$.

PROPOSITION 3.1. *For every $F \in Q(Z, Y, Y_1)$, $\mathcal{U}_F^Y$ is a 0-neighbourhood base for a unique linear topology τ_F^Y on the space $Y \times Z$ (we denote $(Y \times Z, \tau_F^Y)$ by $Y \oplus_F Z$). The following diagram is a twisted sum:*

$$0 \to Y \xrightarrow{j} Y \oplus_F Z \xrightarrow{q} Z \to 0, \tag{3.2}$$

where $j(y) = (y, 0)$, $q(y, z) = z$ for $y \in Y$, $z \in Z$. If $i\colon Y \to Y_1$ is the canonical embedding, then the following diagram commutes:



where $j_0(y_1) = (y_1, 0)$, $q_0(y_1, z) = z$, $i \times \mathrm{id}_Z(y, z) = (i(y), z)$ for all $y \in Y$, $y_1 \in Y_1$, $z \in Z$.

The following short lemma will be useful.

LEMMA 3.1. *$F(az) \to 0$ as $a \to 0$ in K for all $F \in Q(Z, Y)$, $z \in Z$.*

Proof. For every $U \in \mathcal{U}_Y$ there is $c > 0$ such that, for $|a/c|$ sufficiently small, $F(az) - (a/c)F(cz) \in U$. Of course, $(a/c)F(cz) \to 0$ as $a \to 0$, and thus for $|a|$ small we have

$$F(az) \in U + U.$$

Proof of Proposition 3.1. It is easily seen that the unique translation-invariant topology τ_F^Y generated on $Y \times Z$ by $\mathcal{U}_F^Y$ is equal to $\tau_F \cap Y \times Z$, where τ_F is the translation-invariant topology on $Y_1 \times Z$ generated by $\mathcal{U}_F$. This proves the last part of the proposition and allows us to prove the linearity of the topology τ_F only.

By Lemma 3.1, the family $\mathcal{U}_F$ contains only radial sets. We can prove very easily that for every $U \in \mathcal{U}_F$ there are $b > 0$ and $V \in \mathcal{U}_F$ for which

$$V + V \subset U$$

and, for every $a \in K$, $|a| < b$,

$$aV \subset U.$$

So $\mathcal{U}_F$ is a neighbourhood base at zero for a vector topology.

Now, by (3.1),

$$V \supset q(W_F^Y(U, V)) \supset W, \qquad j^{-1}(W_F^Y(U, V) \cap j(Y)) = U \cap Y$$

for every $U \in \mathcal{U}_{Y_1}$, $V \in \mathcal{U}_Z$ and $W \in \mathcal{U}_Z$ satisfying (3.1). This shows that (3.2) is a twisted sum.

Remark. As has been shown, the condition (3.1) is equivalent to the fact that $q\colon Y \oplus_F Z \to Z$ is open for the given $F \in Q(Z, Y_1)$.

It turns out that there exists a very useful characterization of splitting twisted sums of the form $Y \oplus_F Z$.

PROPOSITION 3.2. *The twisted sums $Y \oplus_F Z$, $Y \oplus_G Z$ for $F \in Q(Z, Y, Y_1)$, $G \in Q(Z, Y, Y_2)$, where $Y \subset Y_1$ and $Y \subset Y_2$, are equivalent iff there exists a linear mapping $L\colon Z \to Y$ such that for every $U_2 \in \mathcal{U}_{Y_2}$ there are $V_1 \in \mathcal{U}_Z$ and $U_1 \in \mathcal{U}_{Y_1}$ satisfying*

$$U_2 + G(z) \supset (U_1 + L(z) + F(z)) \cap Y \quad \textit{for every } z \in V_1. \tag{3.3}$$

Of course, $0 \in Q(Z, Y, Y_1)$ and it generates the twisted sum $Y \oplus Z$.

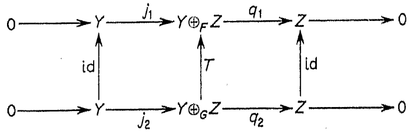
COROLLARY 3.1. *Let $Y \subset Y_1$, Z be tvs and $F \in Q(Z, Y, Y_1)$. Then $Y \oplus_F Z$*

splits iff there is a linear mapping $L\colon Z\to Y$ *such that*

$$F-L\colon Z\to Y_1$$

is continuous at zero.

Proof of Proposition 3.2. Equivalence of the twisted sums $Y\oplus_F Z$, $Y\oplus_G Z$ means (by definition) the commutativity of the following diagram:



for a suitable topological isomorphism T. The mappings j_1, j_2, q_1, q_2 are defined as in Proposition 3.1. By Lemma A, T is a topological isomorphism iff it is a bijective continuous linear map. The above diagram is commutative iff there is a linear map $L\colon Z\to Y$ such that

$$T(y,z)=\big(y+L(z),z\big)$$

for all $y\in Y$, $z\in Z$. But it is easily seen that the continuity of such a map T is equivalent to (3.3).

The following lemma establishes precisely the relation between the results of Sections 2 and 3.

LEMMA 3.2. *The following conditions are equivalent:*

(a) *There exists a section* (*resp. a homogeneous section*) *continuous at zero for the twisted sum* $(*)$.

(b) *There exists a mapping* $F\in Q(Z,Y)$ (*resp. a homogeneous mapping* $F\in Q(Z,Y)$) *such that the twisted sums* $Y\oplus_F Z$ *and* $(*)$ *are equivalent.*

Proof. (b)$\Rightarrow$(a): This is obvious by Lemma 2.1, because the mapping $s\colon Z\to Y\oplus_F Z$,

$$s(z)=\big(F(z),z\big)$$

for every $z\in Z$ and $F\in Q(Z,Y)$, is a section for $Y\oplus_F Z$, continuous at zero.

(a)$\Rightarrow$(b): Let us examine the following diagram:

$$0\to Y\underset{j}{\overset{p}{\leftrightarrows}}X\underset{q}{\overset{s}{\leftrightarrows}}Z\to 0,$$

where s is a given section continuous at zero and the linear map p satisfies the condition $p\circ j=\mathrm{id}_Y$. The desired map is $F=p\circ s$. Of course, F is homogeneous iff so is s.

At first we will show that $F\in Q(Z,Y)$. Let $z_1,z_2\in Z$ and $a\in K$; let us notice that

$$s(z_1+z_2)-s(z_1)-s(z_2)\in j(Y),\qquad s(az_1)-as(z_1)\in j(Y)$$

because

$$q\big(s(z_1+z_2)-s(z_1)-s(z_2)\big)=0=q\big(s(az_1)-as(z_1)\big).$$

Hence

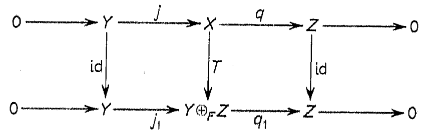
$$A_F(z_1,z_2)=j^{-1}\big(s(z_1+z_2)-s(z_1)-s(z_2)\big)$$

and

$$J_F(a,z_1)=j^{-1}\big(s(az_1)-as(z_1)\big).$$

Of course, $j^{-1}\colon j(Y)\to Y$ is continuous and therefore A_F and J_F are continuous at zero.

It is easily seen that the following diagram commutes:



where $j_1(y)=(y,0)$, $q_1(y,z)=z$, $T(x)=\big(p(x),q(x)\big)$ for all $y\in Y$, $z\in Z$, $x\in X$. Let us notice that

$$\begin{aligned}T^{-1}(y,z)&=T^{-1}\big(y-F(z),0\big)+T^{-1}\big(F(z),z\big)\\&=T^{-1}\big(y-F(z),0\big)+T^{-1}\big(p\circ s(z),z\big)\\&=j\big(y-F(z)\big)+s(z).\end{aligned}$$

Now if $V\in\mathscr{U}_X$, $V_1\in\mathscr{U}_Y$, $V_2\in\mathscr{U}_Z$, $j(V_1)\subset V$, $s(V_2)\subset V$, then obviously $T^{-1}\big(W_F(V_1,V_2)\big)\subset V+V$. Hence T^{-1} is continuous and, by Lemma A, T is a topological isomorphism.

Let (Y,τ) be a tvs and let $(\tau_i,\leqslant)_{i\in I}$ be an arbitrary directed family of semimetrizable topologies on Y such that $\sup_{i\in I}\tau_i=\tau$. Then the space $\prod_{i\in I}(Y,\tau_i)^{\sim}$ is called a *standard product extension* (SPE) of (Y,τ). We call the space $\prod_{i\in I}(Y/\ker\tau_i,\tau_i/\ker\tau_i)^{\sim}$ a *reduced standard product extension* (RSPE) of (Y,τ). It is well known that if Y_1 is a SPE of Y, then $Y\subset Y_1$; the same fact is true if Y is Hausdorff and Y_1 is a RSPE of Y. Actually, Y is isomorphic to the diagonal of the product $\prod_{i\in I}(Y,\tau_i)\subset\prod_{i\in I}(Y,\tau_i)^{\sim}$ (or $\prod_{i\in I}(Y/\ker\tau_i,\tau_i/\ker\tau_i)$, resp.). Of course, if Y is semimetrizable, then $\tilde{Y}$ is a SPE of Y and the associated Hausdorff space Y_1 of $\tilde{Y}$ is a RSPE of Y.

Now, we can prove our main theorem (for particular cases see [16], Theorem 2.4; [14], Theorem 10.1; [18], Theorem 2.5).

THEOREM 3.1. *For every tvs Y, Z and every SPE Y_1 of Y the twisted sum* $(*)$ *is equivalent to $Y\oplus_F Z$ for some $F\in Q(Z, Y, Y_1)$.*

If Y is locally pseudoconvex, then F may be chosen homogeneous (for SPE which are products of locally bounded tvs).

If the pair (Y, Z) satisfies the appropriate conditions from Theorem 2.1, then F may be chosen from $Q(Z, Y)$.

Remarks. 1. As is shown by the counterexample in Section 2 and Lemma 3.2, it seems that this theorem cannot be improved.

2. It is shown (implicitly) in [18], Theorem 2.5, that if Y and Z are nuclear Fréchet spaces and Z has a basis, then F may be chosen linear (for SPE which are products of Banach spaces).

3. In fact, Theorem 3.1 shows that every twisted sum of two tvs Y and Z may be "extended" in the first summand to the twisted sum of an arbitrary SPE Y_1 of Y and Z. Moreover, this extended twisted sum has a section continuous at zero.

4. Changing the proof, we can show that if Y is Hausdorff then "SPE" in the text of the theorem may be replaced by "RSPE".

Proof. The last part of our theorem is an obvious consequence of Theorem 2.1 and Lemma 3.2.

Now, we will prove our theorem for Y semimetrizable and $Y_1=\tilde{Y}$. We need only consider the twisted sums

$$0\to(Y,\lambda)\xrightarrow{j}(X,\tau)\xrightarrow{q}(Z,\gamma)\to 0,$$

where λ is semimetrizable and $\ker\tau\subset j(Y)$. The following diagram is also a twisted sum:

$$0\to(\tilde{Y},\lambda_1)\xrightarrow{j_1}(X,\tau)\xrightarrow{q_1}(Z/\ker\gamma,\gamma/\ker\gamma)\to 0,$$

where $(\tilde{Y},\lambda_1)\simeq\overline{j(Y)}\subset(X,\tau)$. The map j_1 is the corresponding embedding and $q_1=q_0\circ q$, where $q_0\colon Z\to Z/\ker\gamma$ is the natural quotient map. By Corollary 2.1, we get the twisted sum

$$0\to(\tilde{\tilde{Y}},\tilde{\lambda}_1)\xrightarrow{\tilde{j}_1}(X_1,\tau_1)\xrightarrow{q_2}(Z/\ker\gamma,\gamma/\ker\gamma)\to 0,$$

where $X_1=\tilde{q}_1^{-1}(Z/\ker\gamma)$ and $q_2=\tilde{q}_1|_{X_1}$. But $(\tilde{\tilde{Y}},\tilde{\lambda}_1)\simeq(\tilde{Y},\tilde{\lambda})$, $\tilde{j}_1=\tilde{j}$ and $(Z/\ker\gamma,\gamma/\ker\gamma)\simeq(Z_1,\gamma\cap Z_1)$ for every algebraic complement Z_1 of $\ker\gamma$. We can define a linear continuous projection $p_0\colon Z\to Z_1$ with the kernel equal to $\ker\gamma$. Thus we have the next twisted sum:

(3.4) $$0\to(\tilde{Y},\tilde{\lambda})\xrightarrow{\tilde{j}}(X_1,\tau_1)\xrightarrow{q_2}(Z_1,\gamma_1\cap Z_1)\to 0.$$

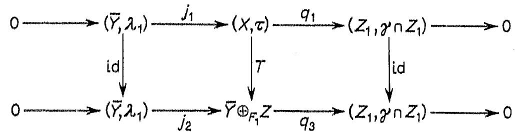
By Theorem 2.1 and Lemma 3.2, there is $F_1\in Q(Z_1,\tilde{Y})$ such that (3.4) is equivalent to

$$0\to(\tilde{Y},\tilde{\lambda})\xrightarrow{j_F}\tilde{Y}\oplus_{F_1}Z_1\xrightarrow{q_F}(Z_1,\gamma_1\cap Z_1)\to 0,$$

where $j_F(y)=(y,0)$, $q_F(y,z)=z$ for all $y\in Y$, $z\in Z_1$. This implies that there exists an isomorphism

$$T\colon X\to\tilde{Y}\oplus_{F_1}Z_1$$

which makes the following diagram commutative:



where $j_2=j_F|_Y$, $q_3=q_F|_{Y\oplus_{F_1}Z}$. Now, let $p_1\colon Z_2\to\tilde{Y}$ be a linear section for the quotient map $q\circ j_1\colon\tilde{Y}\to Z_2\subset Z$. Thus the map $p=(\mathrm{id}_Y-p_1\circ q\circ j_1)\colon\tilde{Y}\to Y$ is a projection onto Y.

We define an algebraic isomorphism (we can assume that $p_1\colon Z\to\tilde{Y}$, $p_1(Z_1)=\{0\}$)

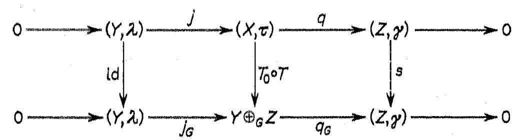
$$T_0\colon\tilde{Y}\oplus_{F_1}Z_1\to Y\oplus_G Z,\qquad T_0(y,z)=\big(p(y),\,q\circ j_1(y)+z\big),$$

where $G\in Q(Z,\tilde{Y})$, $G(z)=F_1\circ p_0(z)-p_1(z)$. For all $U\in U_{\tilde{Y}}$, $V\in\mathcal{U}_Z$ we have:

$$\begin{aligned}W^{\tilde{Y}}_{F_1}&(U,V\cap Z_1)\\ &=\{(y,z)\in\tilde{Y}\times Z_1\colon\ y-F_1\circ p_0(z)\in U,\ z\in V\}\\ &=\{(y,z)\in\tilde{Y}\times Z_1\colon\ p(y)+p_1\circ q\circ j_1(y)-F_1\circ p_0(z)\in U,\ z\in V\}\\ &=\{(y,z)\in\tilde{Y}\times Z_1\colon\ p(y)+(F_1\circ p_0-p_1)(q\circ j_1(y)+z)\in U,\ z\in V\}\\ &=\{(y,z)\in\tilde{Y}\times Z_1\colon\ T_0(y,z)\in W_G(U,V)\}=T_0^{-1}\big(W_G(U,V)\big).\end{aligned}$$

Thus T_0 is a topological isomorphism.

Obviously $j_G\circ\mathrm{id}=T_0\circ T\circ j$, because $p|_Y=\mathrm{id}_Y$. Thus the following diagram commutes:



where $T_0\circ T$ is an isomorphism and s is an appropriately chosen isomorphism. Hence $Y\oplus_F Z$ with $F=G\circ s\in Q(Z,\tilde{Y})$ is a twisted sum equivalent to the given one. Of course, if F_1 is homogeneous, then so is $G\circ s$.

Hence for Y locally bounded F may be chosen homogeneous. This completes the proof of the semimetrizable case.

Now, we will prove our theorem in the general case. Let $(\tau_i, \leqslant)_{i\in I}$ be an arbitrary directed family of semimetrizable topologies on Y such that $\sup_{i\in I} \tau_i = \tau$. By the first part of the proof, we can construct a family of quasilinear maps $\{F(i)\in Q(Z, (Y, \tau_i)^{\sim})\colon i\in I\}$ such that the topology γ_i generated by $\mathscr{U}^{Y}_{F(i)}$ is equal to the unique topology (see Lemma B) on X which satisfies $\gamma_i\cap j(Y) = j(\tau_i)$, $q(\gamma_i) = \lambda$, $\gamma_i \leqslant \gamma$. It can be easily seen that $(\gamma_i, \leqslant)_{i\in I}$ is a directed family of topologies. Thus

$$\sup_{i\in I}\gamma_i \cap j(Y) = \sup_{i\in I} j(\tau_i) = j(\tau), \qquad q(\sup_{i\in I}\gamma_i) = \lambda. \tag{3.5}$$

Of course, $\sup_{i\in I}\gamma_i \leqslant \gamma$ and, by Lemma A, $\sup_{i\in I}\gamma_i = \gamma$. Let us define the map

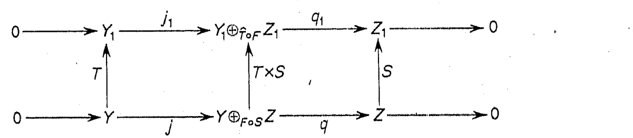
$$F\colon Z\to \prod_{i\in I}(Y, \tau_i)^{\sim} = Y_1, \qquad F(z) = \big(F(i)(z)\big)_{i\in I}.$$

Obviously $F\in Q(Z, Y_1)$ and the topology of $Y\oplus_F Z$ is the supremum of the topologies of $(Y, \tau_i)\oplus_{F(i)} Z$. This fact shows that $Y\oplus_F Z$ and $(*)$ are equivalent twisted sums.

By the remark after the proof of Proposition 3.1 and by (3.5), $F\in Q(Z, Y, Y_1)$. If Y is locally pseudoconvex, then τ_i may be chosen locally bounded and, by the first part of the proof, $F(i)$ are homogeneous. Finally, F is also homogeneous. This completes the proof of our Theorem.

The following theorem will be very useful in Section 4.

THEOREM 3.2. *Let $\hat{Y}_1 \supset Y_1$, $\hat{Y} \supset Y$, Z_1, Z be tvs, $F\in Q(Z_1, Y, \hat{Y})$. Then for all continuous linear mappings $\hat{T}\colon \hat{Y}\to \hat{Y}_1$, $S\colon Z\to Z_1$, $\hat{T}(Y)\subset Y_1$, the following commutative diagram contains only continuous mappings:*



where $T = \hat{T}|_Y$, $(T\times S)(y, z) = (Ty, Sz)$, $y\in Y$, $z\in Z$. If T, S are relatively open, then so is $T\times S$.

Proof. For all $U\in \mathscr{U}_{\hat{Y}_1}$, $V\in \mathscr{U}_{Z_1}$ the following assertion holds:

$$\begin{aligned} W^{Y}_{F\circ S}&(\hat{T}^{-1}(U), S^{-1}(V)) \\ &= \{(y, z)\in Y\times Z\colon y - F\circ S(z)\in \hat{T}^{-1}(U),\ z\in S^{-1}(V)\} \\ &\subset (T\times S)^{-1}\big(\{(y_1, z_1)\in Y_1\times Z_1\colon y_1 - \hat{T}\circ F(z_1)\in U,\ z_1\in V\}\big) \\ &= (T\times S)^{-1}\big(W^{Y_1}_{\hat{T}\circ F}(U, V)\big). \end{aligned}$$

If T, S are relatively open, then for all $U\in \mathscr{U}_{\hat{Y}}$, $V\in \mathscr{U}_Z$ there exist $U_1\in \mathscr{U}_{\hat{Y}_1}$, $V_1\in \mathscr{U}_{Z_1}$ satisfying $\hat{T}(U)\supset \hat{T}(Y)\cap U_1$, $S(V)\supset S(Z)\cap V_1$. Hence

$$\begin{aligned} (T\times S)&\big(W^{Y}_{F\circ S}(U, V)\big) \\ &= (T\times S)\big(\{(y, z)\in Y\times Z\colon y - F\circ S(z)\in U,\ z\in V\}\big) \\ &\supset \{(y_1, z_1)\in Y_1\times Z_1\colon y_1 - \hat{T}\circ F(z_1)\in U_1,\ z_1\in V_1\} \cap \big(T(Y)\times S(Z)\big) \\ &= \big(T(Y)\times S(Z)\big)\cap W^{Y_1}_{\hat{T}\circ F}(U_1, V_1). \end{aligned}$$

This completes the proof.

Remark. For the proof of the openness of $T\times S$ the extended version of Lemma A may be used.

LEMMA 3.3. *For all tvs (Y, λ), (Z, γ) and every continuous linear map $T\colon (Y, \lambda)\to (Z, \gamma)$ there is a pair of tvs $\hat{Y}$, $\hat{Z}$ which are SPE of Y, Z, resp., such that there is a continuous linear extension $\hat{T}\colon \hat{Y}\to \hat{Z}$ of T.*

Remark. If Y and Z are Hausdorff, then "SPE" may be replaced by "RSPE".

Proof. Let $(\gamma_i, \leqslant)_{i\in I}$ be a directed family of semimetrizable topologies on Z such that $\sup_{i\in I} \gamma_i = \gamma$. Of course, $T^{-1}(\gamma_i) = \lambda_i$ is semimetrizable and $(\lambda_i, \leqslant)_{i\in I}$ is a directed family. This family may be extended to a directed family of semimetrizable topologies $(\lambda_j, \leqslant)_{j\in J}$ such that for every $j\in J\setminus I$ there is $i_j\in I$ such that $\lambda_{i_j}\leqslant \lambda_j$ and $\sup_{j\in J}\lambda_j = \lambda$. Let us define $i_j = j$ for $j\in I$. Obviously $T\colon (Y, \lambda_j)\to (Z, \gamma_{i_j})$ is continuous and $\sup_{j\in J}\gamma_{i_j} = \gamma$. Finally, we have the following continuous extension of T to the corresponding SPE:

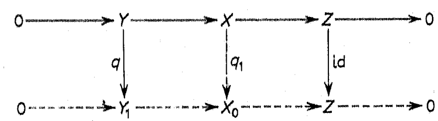
$$\hat{T}\colon \prod_{j\in J}(Y, \lambda_j)^{\sim} \to \prod_{j\in J}(Z, \gamma_{i_j})^{\sim}, \qquad \hat{T}\big((y_j)_{j\in J}\big) = \big(T_j(y_j)\big),$$

where $T_j = \tilde{T}\colon (Y, \lambda_j)^{\sim}\to (Z, \gamma_{i_j})^{\sim}$, $j\in J$.

We can easily show the following results, using Theorem 3.1 and Theorem 3.2 for suitably chosen SPE (Lemma 3.3) and maps T, S.

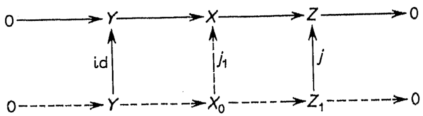
COROLLARY 3.2. *Suppose that in the following diagrams $\to$ denotes given continuous linear mappings and all columns and rows form twisted sums. Then, in each of the cases* (a)–(d), *we can define continuous linear mappings denoted by $- - \to$ and a tvs X_0 such that the diagrams commute. Then new columns and rows are also twisted sums. By* id *we denote identity.*

(a)



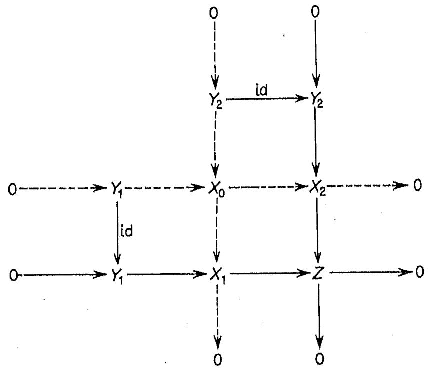
where q, q_1 are surjective homomorphisms.

(b)

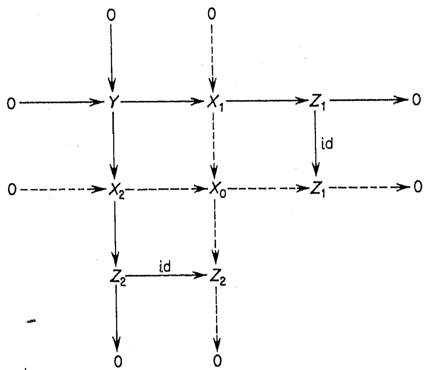


where j, j_1 are injective homomorphisms.

(c)



(d)



4. Splitting twisted sums. A tvs Z belongs to the class $S(Y)$ iff every twisted sum of the tvs Y and Z splits. If $Z \in S(K)$, then Z is called a $\mathscr{K}$-space (comp. Kalton and Peck [15]). In this section we will study the classes $S(Y)$ for suitable tvs Y. By $\hat{Y}$ we will denote an arbitrary SPE of a tvs Y.

The first obvious fact is

PROPOSITION 4.1. *For all tvs Y, Z we have $(Z, \tau) \in S(Y)$, where τ is the finest linear topology on Z.*

It turns out that every class $S(Y)$ has some interesting hereditary properties.

LEMMA 4.1. *Let $Z \in S(Y)$ for given tvs Y, Z. If $Z_1 \subset Z$ is a subspace satisfying the property that every continuous linear operator $T\colon Z_1 \to Y$ may be extended onto the whole space Z, then also $Z/Z_1 \in S(Y)$.*

Remark. A similar method of proof was used independently in [17] for a particular case of Lemma 4.1.

Proof. By Theorem 3.1, we can consider only twisted sums of the form $Y \oplus_F (Z/Z_1)$ for $F \in Q(Z/Z_1, Y, \hat{Y})$. It can be easily seen that $F \circ q = Q(Z, Y, \hat{Y})$; $q\colon Z \to Z/Z_1$ denotes the natural quotient map.

Now, by Corollary 3.1, there exists a linear map $L\colon Z \to Y$ such that for every $U \in \mathscr{U}_{\hat{Y}}$ there is $V \in \mathscr{U}_Z$ satisfying

$$(F \circ q - L)(V) \subset U.$$

But $F \circ q|_{Z_1} \equiv 0$, and thus $L|_{Z_1}\colon Z_1 \to Y$ is continuous and may be extended to a linear continuous operator $L_1\colon Z \to Y$. Hence there is a linear map $L_2\colon Z/Z_1 \to Y$ for which $L = L_1 + L_2 \circ q$. Let U be an arbitrary 0-neighbourhood in $\hat{Y}$. Then for a suitably chosen $V \in \mathscr{U}_Z$ we have

$$(F \circ q - L_2 \circ q)(V) = \big((F \circ q - L) + L_1\big)(V) \subset U.$$

Thus $(F - L_2)(q(V)) \subset U$; this completes the proof by Corollary 3.1.

THEOREM 4.1. *If Y is an arbitrary tvs, then $S(Y)$ is closed under*:

(a) *complemented subspaces*,

(b) *finite products*.

Proof. (a): This is an immediate consequence of Lemma 4.1.

(b): Let tvs $Z_1, Z_2 \in S(Y)$; we will show that also $Z = Z_1 \oplus Z_2 \in S(Y)$. We denote by $P\colon Z \to Z_1$ the projection for which $\ker P = Z_2$. By Theorem 3.1, we need only prove our theorem for twisted sums $Y \oplus_F Z$ with $F \in Q(Z, Y, \hat{Y})$. Now, by Corollary 3.1, there are linear mappings $L_1\colon Z_1 \to Y$, $L_2\colon Z_2 \to Y$ such that $F|_{Z_1} - L_1\colon Z_1 \to \hat{Y}$, $F|_{Z_2} - L_2\colon Z_2 \to \hat{Y}$ are continuous at zero. Hence for $z \in Z$

$$\begin{aligned}(F - L_1 \circ P - L_2 \circ (I - P))(z) \\ = A_F\big(P(z), (I-P)(z)\big) + (F - L_1) \circ P(z) + (F - L_2) \circ (I - P)(z),\end{aligned}$$

and thus

$$F - L_1 \circ P - L_2 \circ (I - P)\colon Z \to \hat{Y}$$

is continuous at zero. This completes the proof by Corollary 3.1.

LEMMA 4.2. *A nontrivial vector space Z with the trivial topology belongs to $S(Y)$ iff Y is complete.*

Proof. The completion $\tilde{Y}$ of Y is a twisted sum of Y and a tvs with the trivial topology.

PROPOSITION 4.2. *Let Y be a complete tvs.*

(a) $Z\in S(Y)$ *iff the Hausdorff tvs associated with Z belongs to* $S(Y)$.

(b) *A dense subspace* Z_1 *of Z belongs to* $S(Y)$ *iff* $Z\in S(Y)$, *whenever Y is semimetrizable.*

Proof. This is an immediate consequence of Lemma 4.2 and Theorem 4.1 (part (a)), and the definition of the class $S(Y)$ (part (b)). The semimetrizability of Y is used only for the "if" part, comp. Corollary 2.1.

We get some more interesting hereditary properties of the class $S(Y)$ when we restrict our considerations to locally bounded F-spaces Y.

THEOREM 4.2. *Let Y be a locally bounded tvs and let Y, Z be a pair of tvs having only twisted sums with a homogeneous section continuous at zero (for example, let Y be a locally bounded F-space and let Z be an arbitrary tvs — comp. Theorem 2.1 (c)). If a tvs* (Z, τ) *satisfies the following condition: for every* $U\in\mathscr{U}_\tau$ *there is a vector topology* $\gamma(U)\le\tau$ *for which* $U\in\mathscr{U}_{\gamma(U)}$ *and*

$$(Z/\ker\gamma(U),\ \gamma(U)/\ker\gamma(U))\in S(Y),$$

then $(Z,\tau)\in S(Y)$.

Remarks. Comp. [14], Proof of Theorem 10.2. A very similar fact is true for semimetrizable spaces Y. By Lemma 4.1, if $(Z,\gamma(U))\in S(Y)$, then also

$$(Z/\ker\gamma(U),\ \gamma(U)/\ker\gamma(U))\in S(Y).$$

Proof. By Lemma 3.2, we may consider only twisted sums of the form $Y\oplus_F Z$ for a homogeneous $F\in Q(Z, Y)$. Let $\|\cdot\|$ be an arbitrary quasinorm generating the given topology on Y. Then there is $U\in\mathscr{U}_\tau$ satisfying

$$\|F(x+y)-F(x)-F(y)\|<1 \quad \text{for every } x, y\in U. \tag{4.1}$$

Let $\gamma=\gamma(U)$ and let $p\colon Z\to\ker\gamma(U)$ be a projection. Then $F|_{\ker p}$ may be considered an element of $Q((Z/\ker\gamma,\ \gamma/\ker\gamma), Y)$ because $(\ker p,\ \gamma\cap\ker p)$ is naturally isomorphic to $(Z/\ker\gamma,\ \gamma/\ker\gamma)$. By Corollary 3.1, there is a linear mapping $f\colon \ker p\to Y$ for which $F|_{\ker p}-f$ is γ-continuous at zero. Thus

$$(F-F\circ(\mathrm{id}-p)-F\circ p)+(F-f)\circ(\mathrm{id}-p)=F-f\circ(\mathrm{id}-p)-F\circ p$$

is γ-continuous at zero and hence τ-continuous at zero.

On the other hand, (4.1) implies the linearity of $F|_{\ker\gamma}\colon \ker\gamma\to Y$. Of course, $f\circ(\mathrm{id}-p)+F\circ p\colon Z\to Y$ is also linear. Using again Corollary 3.1, we conclude that $Y\oplus_F(Z,\tau)$ splits.

Let $(A,\le)$ be a directed family. The *projective limit* of a family of tvs $(X_a)_{a\in A}$ with respect to a family of continuous mappings $(g_{ab})_{a\le b}$, $g_{ab}\colon X_b$ $\to X_a$ is defined as the space

$$\varprojlim g_{ab}X_b=\{(x_a)\in\prod_{a\in A}X_a:\ g_{ab}(x_b)=x_a \text{ for every } a\le b\in A\}.$$

The projective limit is *reduced* iff

$$\overline{p_a(\varprojlim g_{ab}X_b)}=X_a,$$

where $p_a\colon \prod_{a\in A}X_a\to X_a$ is the natural projection.

THEOREM 4.3. *For every locally bounded F-space Y the class* $S(Y)$ *is closed under:*

(a) *reduced projective limits,*

(b) *arbitrary products.*

Remark. This theorem cannot be proved for every F-space Y because, by the Counterexample (Section 2), there are an F-space Y and a twisted sum of Y and $K^{\aleph_0}$ with no homogeneous section continuous at zero.

Proof. (a) is an immediate consequence of Theorem 4.2 and Proposition 4.2 (b).

(b): Every product is the reduced projective limit of its finite subproducts (comp. [33], Ch. II, p.53); thus our fact is a consequence of Theorem 4.3 (a) and Theorem 4.1 (b).

A tvs X *admits* L_0-*structure* iff for every $U\in\mathscr{U}_X$ there is a finite topological decomposition $X=X_1\oplus\ldots\oplus X_n$ for which $X_i\subset U$ for $i=1,\ldots,n$. For example, $L_0(0,1)$ admits L_0-structure.

PROPOSITION 4.3. *For every locally bounded tvs Y the following spaces belong to* $S(Y)$ *iff Y is complete:*

(a) $(Z,\sigma(Z,Z'))$, $Z'\ne Z^*$,

(b) $Z=K^{\mathfrak{m}}$ *for every cardinal number* $\mathfrak{m}\ge\aleph_0$,

(c) *tvs Z admitting* L_0-*structure.*

Similarly:

(d) $(Z,\sigma(Z,Z^*))\in S(Y)$, *where Y is locally bounded, iff* $\operatorname{codim}_{\tilde{Y}}Y<\aleph_0$.

Remark. The "if" part of (c) was proved, in another way, in the form restricted to F-spaces Z by Kalton and Peck [15], Theorem 3.6.

Proof. The "only if" part is an immediate consequence of Theorem 2.1 (d) and Theorem 2.2 (c) and (b) (for part (c) of our theorem). Thus it is enough to prove the "if" part.

(a): A tvs with a weak topology is dense in some reduced projective limit of finite-dimensional tvs; apply Theorem 4.3 and Proposition 4.2.

(b): The topology on $K^{\mathfrak{m}}$ is a weak topology.

(c): By passing to the Hausdorff associated twisted sum, we can assume that Y is Hausdorff. By Theorem 3.1, every twisted sum of Y and Z is of the form $Y \oplus_F Z$ for some homogeneous $F \in Q(Z, Y)$. There are $U \in \mathscr{U}_Z$ and a bounded set $V \in \mathscr{U}_Y$ for which

$$A_F(U \times U) \subset V.$$

Let $Z_1 \oplus \ldots \oplus Z_n = Z$ and $Z_i \subset U$ for $i = 1, \ldots, n$. It is easily seen that $A_F(Z_i \times Z_i) = \{0\}$, and hence $F|_{Z_i}$: $Z_i \to Y$ is linear for $i = 1, \ldots, n$. Thus we can construct a linear mapping L: $Z \to Y$ with $F - L|_{Z_i} \equiv 0$ for $i = 1, \ldots, n$. Finally,

$$(F-L)(x) = \sum_{i=2}^{n} A_F\Big(\sum_{j=1}^{i-1} P_j(x), P_i(x)\Big),$$

where P_i is a continuous projection on Z_i, is continuous at zero. By Corollary 3.1, $Y \oplus_F Z$ splits.

(d): By Theorem 2.1 (b) and 2.2 (c) the pair of tvs Y, Z has only twisted sums with a homogeneous section continuous at zero. Of course, every 0-neighbourhood U in Z contains a convex 0-neighbourhood of the form $V = \{z \in Z: |f_i(z)| < 1, i = 1, \ldots, n\}$, where f_i are continuous functionals. The set V generates the topology $\gamma(U)$ which satisfies the assumptions of Theorem 4.2. By Theorem 4.2, part (d) is proved.

Besides these positive results Kalton and Peck [16] showed the following.

THEOREM 4.4. *Let Y be a locally bounded F-space with an unconditional basis not containing a subsequence equivalent to the natural Schauder basis in c_0. Then $Y \notin S(Y)$.*

Remark. Kalton and Roberts [17] showed that $c_0 \in S(c_0)$ and $l_\infty \in S(l_\infty)$ (comp. Theorem 5.5 (b), Proposition 5.4 (a)).

There are some results on splitting and nonsplitting twisted sums of nuclear Fréchet spaces with bases Y and Z ([39], [18]). In particular, [18] gives some sufficient conditions for $Z \in S(Y)$.

Now, we should describe the class $S(K)$ (i.e. $\mathscr{K}$-spaces); but this class is closely related to the class of TSC-spaces and to the three space problem for local convexity. In fact, a locally convex space $X \in S(K)$ iff every twisted sum of K and X is locally convex. Hence we will study locally convex twisted sums.

5. Locally convex twisted sums and $\mathscr{K}$-spaces. It is a known fact (comp. [13], [28], [29]) that there exist nonlocally convex twisted sums of two locally convex spaces (for example, of K and l_1). A tvs Z will be called a *TSC-space* iff for every locally convex tvs Y every twisted sum of Y and Z is also locally convex.

Remark. Tvs Y with the trivial topology are the only tvs which have only locally convex twisted sums with an arbitrary locally convex tvs Z (in fact, for locally convex $Y = Y_1 \oplus K$ and nonlocally convex $X \supset K$ such that $X/K \simeq l_1$, $Y_1 \oplus X$ is a nonlocally convex twisted sum of Y and l_1).

Of course, every TSC-space belongs to $S(K)$. It is rather surprising that the converse is also true for metrizable tvs. I do not know if it is true in general.

THEOREM 5.1. *Let (Z, τ) be a lcs which has, for every $U \in \mathscr{U}_\tau$, a semimetrizable vector topology $\gamma(U) \leqslant \tau$, $U \in \mathscr{U}_{\gamma(U)}$, such that $(Z, \gamma(U)) \in S(K)$. Then Z is a TSC-space.*

COROLLARY 5.1. *A metrizable locally convex space is a $\mathscr{K}$-space iff it is a TSC-space.*

In the proof we will use the following proposition due to Kalton [13], Theorem 4.10 (comp. Proposition 4.2):

PROPOSITION 5.1. *A semimetrizable lcs Z is a $\mathscr{K}$-space iff for every semimetrizable lcs Y every twisted sum of Y and Z is locally convex.*

Proof of Theorem 5.1. Let

$$0 \to (Y, \eta) \xrightarrow{j} (X, \lambda) \xrightarrow{q} (Z, \tau) \to 0$$

be a twisted sum. Let U be an arbitrary 0-neighbourhood in (X, λ). First, let us define on X a semimetrizable vector topology $\lambda_1 \leqslant \lambda$ such that $U \in \mathscr{U}_{\lambda_1}$ and $\lambda_1 \cap j(Y)$, $q(\lambda_1)$ are locally convex. A 0-neighbourhood base of λ_1 will be $\{U_n\}$ defined inductively as follows:

$$U_0 = U;$$

$$U_{2n+1} \in \mathscr{U}_\lambda, \quad U_{2n+1} + U_{2n+1} \subset U_{2n}, \quad U_{2n+1} \cap j(Y) \text{ is convex};$$

$$U_{2n+2} \in \mathscr{U}_\lambda, \quad U_{2n+2} + U_{2n+2} \subset U_{2n+1}, \quad q(U_{2n+2}) \text{ is convex}.$$

For example, $U_{2n+2} = W \cap q^{-1}(W_1)$, where $W \in \mathscr{U}_\lambda$, $W + W \subset U_{2n+1}$ and $W_1 \in \mathscr{U}_\tau$, $W_1 \subset q(W)$, W_1 is convex.

Now, we can define on X another semimetrizable topology λ_2 satisfying the assumptions of Theorem 4.2 such that $\lambda_1 \leqslant \lambda_2 \leqslant \lambda$. Let us define inductively a sequence $(A_n)_{n \in N}$ of families of 0-neighbourhoods in (X, λ), $A_0 = \{U_n\}$. For every $B \in A_n$ there is $V(B) \in \mathscr{U}_\tau$ such that $V(B) \subset q(B)$ and $V(B)$ is absolutely convex. Let $N(B)$ be a countable 0-neighbourhood base of $\gamma(V(B))$. Then

$$A_{n+1} = \Big\{\bigcap_{i=1}^{k} V_i\colon V_i \in A_n \cup \{q^{-1}(V)\colon V \in N(B),\ B \in A_n\} \text{ for } i, k \in N\Big\}.$$

The family $A = \bigcup_{n \in N} A_n$ is a countable 0-neighbourhood base for the vector topology λ_2. Of course, for $(Z, q(\lambda_2))$ we can use Theorem 4.2 and $(Z, q(\lambda_2))$ is a $\mathscr{K}$-space. Thus (X, λ_2) is a semimetrizable twisted sum of semimetrizable spaces $(Y, j^{-1}(j(Y) \cap \lambda_2))$ and $(Z, q(\lambda_2))$. By Proposition 5.1, λ_2 is locally convex. Hence U contains a convex 0-neighbourhood in (X, λ). This completes the proof because U has been chosen arbitrarily in $\mathscr{U}_\lambda$.

The last proposition may be strengthened in the following manner.

THEOREM 5.2. *Let (Z, τ) be a tvs which, for every semimetrizable topology $\lambda \leqslant \tau$, has a topology $\gamma(\lambda)$ with $\lambda \leqslant \gamma(\lambda) \leqslant \tau$ such that $(Z, \gamma(\lambda))$ is a TSC-space. Then (Z, τ) is also a TSC-space.*

Proof. Let (X, τ_1) be a tvs containing a locally convex subspace Y for which $X/Y \simeq (Z, \tau)$, and let $q\colon X \to Z$ be the quotient map. In a very similar way as in the proof of Theorem 5.1 for every $U \in \mathscr{U}_{\tau_1}$ we can define on X a semimetrizable vector topology $\tau_2 \leqslant \tau_1$ for which $U \in \mathscr{U}_{\tau_2}$ and $\tau_2 \cap Y$, $q(\tau_2)$ are locally convex. Of course, if

$$\sup\left(\tau_2, q^{-1}\left(\gamma\left(q(\tau_2)\right)\right)\right) = \tau_3,$$

then (X, τ_3) is a twisted sum of $(Y, \tau_2 \cap Y)$ and $\left(Z, \gamma\left(q(\tau_2)\right)\right)$. Hence τ_3 is locally convex and $U \in \mathscr{U}_{\tau_3}$. This completes the proof because U has been chosen arbitrarily in $\mathscr{U}_{\tau_1}$ and $\tau_3 \leqslant \tau_1$.

Now, we will consider hereditary properties of the class of all TSC-spaces.

THEOREM 5.3. *The class of all TSC-spaces is closed under:*

(a) *twisted sums,*

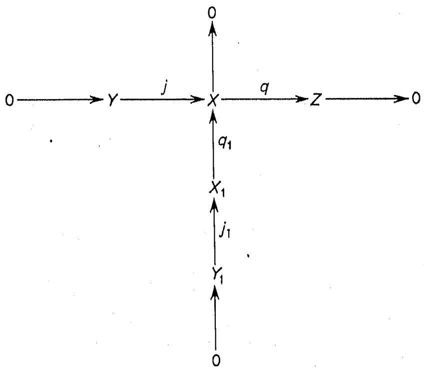
(b) *finite products,*

(c) *quotients,*

(d) *complemented subspaces.*

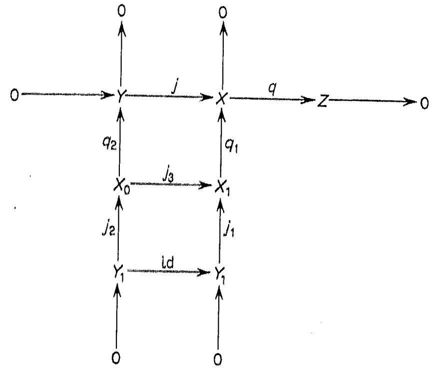
Remark. I do not know whether part (b) can be strengthened to the countable case. If we knew this, we could prove our theorem for arbitrary products (by Theorem 5.2) ([1]).

Proof. (a): Let Y, Z be TSC-spaces. Let us consider the following diagram of tvs, in which the column and the row are twisted sums and Y_1 is a lcs:



([1]) This problem is solved in the affirmative by the author; see P. Domański, *Local convexity of twisted sums*, Suppl. Rend. Circ. Mat. Palermo, Serie II, 5 (1984), Proc. XII Winter School on Abstract Analysis, Srni 1984, 13–31.

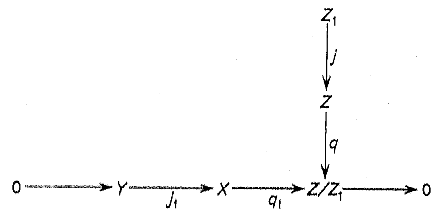
By Corollary 3.2 (b), we can produce a new diagram in which the new column is also a twisted sum and j_3 is an injective homomorphism:



Let $q_3\colon X_1 \to X_1/j_3(X_0)$; then $\ker(q \circ q_1) = \ker q_3$ and $X_1/j_3(X_0) \simeq Z$. Thus X_1 is a twisted sum of the space X_0 and the TSC-space Z. But X_0 is a lcs because it is a twisted sum of the lcs Y_1 and the TSC-space Y. Hence X_1 is also locally convex.

(b) is an immediate consequence of (a).

(c): Let us consider the following diagram:



where Z is a TSC-space, Y is a lcs, the row is a twisted sum and q (resp. j) is the natural quotient map (resp. embedding). By part (c) of Corollary 3.2, X is isomorphic to a quotient of a space X_0 which is a twisted sum of Y and Z. But then X_0 is a lcs, and so is X.

(d) is an obvious consequence of (c).

By Theorem 5.1 and Corollary 5.1, we obviously get the following fact.

THEOREM 5.4. *Every reduced projective limit and every product of metrizable TSC-spaces is a TSC-space.*

Now, let us give a short review of known TSC-spaces.

THEOREM 5.5. *The following tvs are TSC-spaces*:

(a) *B-convex normed spaces, in particular uniformly convex Banach spaces (for example,* l_p, $L_p(0, 1)$ *for* $1 < p < \infty$);

(b) $\mathscr{L}_\infty$-*Banach spaces (for example,* l_∞, c_0, $C(X)$ *for every Hausdorff compact* X);

(c) *nuclear spaces, in particular tvs with a weak topology (for example,* $K^{\mathfrak{m}}$ *for every cardinal number* $\mathfrak{m}$).

Remark. For the definition of B-convexity and $\mathscr{L}_\infty$-spaces see [13], [21], resp.

Proof. (a) and (b) are immediate consequences of [13], Theorem 4.3 (iii) and [17], Theorem 6.2, resp. (comp. Corollary 5.1).

(c): Every nuclear tvs has the topology generated by a family of prehilbertian seminorms (comp. [33], III. 7.3, Corollary 2). Thus (c) is a consequence of Theorem 5.4, since by Theorem 5.5 and Proposition 1.1 every tvs with prehilbertian seminorm is a $\mathscr{K}$-space.

On the other hand, no space which contains uniformly complemented copies of l_1^n for $n = 1, 2, \ldots$ is a $\mathscr{K}$-space (or a TSC-space). This fact is due to Kalton [13], Theorem 4.7. In particular, no $\mathscr{L}_1$-spaces, e.g. l_1 (comp. [13], Theorem 4.6; [28], Theorem 1; [29]), are $\mathscr{K}$-spaces or TSC-spaces. A subspace of a TSC-space (of a locally convex $\mathscr{K}$-space) need not be a TSC-space (a $\mathscr{K}$-space): take e.g. $l_1 \subset C(0, 1)$.

PROBLEM. Is the class of all TSC-spaces equal to the class of all locally convex $\mathscr{K}$-spaces?

Of course, for $\mathscr{K}$-spaces Theorems 4.1, 4.3 and Proposition 4.3 remain true.

A tvs $Y \subset X$ has HBEP (*Hahn–Banach Extension Property*) in a tvs X iff every continuous functional on Y can be extended onto the whole space X. By Lemma 4.1, we obviously get:

PROPOSITION 5.2. *The quotient* X/Y *is a* $\mathscr{K}$-*space if* X *is a* $\mathscr{K}$-*space and* Y *has HBEP in* X. *In particular, every quotient of a locally convex* $\mathscr{K}$-*space is a* $\mathscr{K}$-*space.*

Remark. This result is due to Kalton and Peck [15], where the converse theorem is also proved.

PROPOSITION 5.3. *Every twisted sum of a* $\mathscr{K}$-*space and a TSC-space (in particular, a metrizable locally convex* $\mathscr{K}$-*space) is a* $\mathscr{K}$-*space.*

This can be proved in the same way as Theorem 5.3 (a). We use the fact that metrizable locally convex $\mathscr{K}$-spaces are TSC-spaces.

As was pointed out above, the list of known locally convex $\mathscr{K}$-spaces is given in Theorem 5.5. Other $\mathscr{K}$-spaces are listed in Proposition 4.3. The following result is contained in [13], Theorem 3.6 (even in a more general form for Orlicz spaces).

THEOREM 5.6. l_p *and* $L_p(0, 1)$ *for* $0 < p < 1$ *are* $\mathscr{K}$-*spaces.*

Remark. There are other interesting results for Orlicz spaces [13], Theorem 3.6, [14], Theorem 9.3; and Köthe spaces [14], Theorem 10.3.

We close Section 5 with some additional results on splitting twisted sums. A Banach space X is called *injective* if, for every normed space Y containing an isomorphic copy of X, this copy is complemented. It is well known that l_∞ is injective and c_0 is "injective" for separable Banach spaces.

PROPOSITION 5.4. (a) *For every injective* Y, *every reduced projective limit* Z *of normed* $\mathscr{K}$-*spaces belongs to* $S(Y)$. *In particular,* $l_\infty \in S(l_\infty)$ *(comp. Theorem 4.4 and Theorem 5.5). The same holds for nuclear spaces* Z *and for* Z *with a weak topology.*

(b) *Every reduced projective limit of separable normed* $\mathscr{K}$-*spaces belongs to* $S(c_0)$.

Proof. The proofs are very similar and we will prove only part (a). A twisted sum of Y and a normed $\mathscr{K}$-space is a locally convex locally bounded space (see [13], Theorem 1.1). Thus it splits. This completes the proof by Theorem 4.3. For nuclear spaces we should use Theorem 4.2 instead of Theorem 4.3.

Added in proof. The author proved in [42] a stronger version of Proposition 4.3; namely, every twisted sum of an arbitrary Banach space and a nuclear lcs splits.

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References

[1] H. Ahonen, *On nuclear Köthe spaces defined by Dragilev functions*, Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes 38 (1981), 1–57.

[2] H. Apiola, *Characterization of subspaces and quotients of nuclear* $L_f(\alpha, \infty)$-*spaces*, Compositio Math. 50 (1983), 65–81.

[3] R. G. Bartle and L. M. Graves, *Mappings between function spaces*, Trans. Amer. Math. Soc. 72 (1952), 400–413.

[4] Cz. Bessaga and A. Pełczyński, *Selected Topics in Infinite-Dimensional Topology*, Monografie Mat. 58, PWN–Polish Scientific Publishers, Warszawa 1975.

[5] H. H. Corson, *The weak topology of a Banach space*, Trans. Amer. Math. Soc. 101 (1961), 1–15.

[6] S. Dierolf, *Über Vererbbarkeitseigenschaften in topologischen Vektorräumen*, Dissertation, Ludwig-Maximilians-Universität, München 1973.

[7] —, *A note on the lifting of linear and locally convex topologies on a quotient space*, Collect. Math. 31 (1980), 193–198.
[8] P. Domański, *On the separable topological vector spaces*, Funct. et Approx. 14 (1984), 117–122.
[9] L. Drewnowski, *On minimal topological linear spaces and strictly singular operators*, Comment. Math. Tomus Spec. II (1979), 89–106.
[10] P. Enflo, J. Lindenstrauss and G. Pisier, *On the three space problem*, Math. Scand. 36 (1975), 199–210.
[11] B. V. Godun and S. A. Rakov, *Banach–Saks property and "the three space problem"* (in Russian), Mat. Zametki 31 (1982), 61–74.
[12] S. Heinrich, *Ultraproducts in Banach space theory*, J. Reine Angew. Math. 313 (1980), 72–104.
[13] N. J. Kalton, *The three space problem for locally bounded F-spaces*, Compositio Math. 37 (1978), 234–276.
[14] —, *Convexity, type and the three space problem*, Studia Math. 69 (1981), 247–287.
[15] N. J. Kalton and N. T. Peck, *Quotients of $L_p(0, 1)$ for $0 \leqslant p < 1$*, ibid. 64 (1979), 65–75.
[16] —, —, *Twisted sums of sequence spaces and the three space problem*, Trans. Amer. Math. Soc. 255 (1979), 1–30.
[17] N. J. Kalton and J. W. Roberts, *Uniformly exhaustive submeasures and nearly additive set functions*, ibid. 278 (1983), 803–816.
[18] T. Ketonen and K. Nyberg, *Twisted sums of nuclear Fréchet spaces*, Ann. Acad. Sci. Fenn. Ser. A I Math. 7 (1982), 323–335.
[19] G. Köthe, *Das Reziprokentheorem für zeilenabsolute Matrizen*, Monatshefte Math. Phys. 47 (1939), 224–233.
[20] —, *Hebbare lokalkonvexe Räume*, Math. Ann. 165 (1966), 181–195.
[21] J. Lindenstrauss and H. P. Rosenthal, *The $\mathscr{L}_p$-spaces*, Israel J. Math. 7 (1969), 325–349.
[22] E. Michael, *Continuous selections I, II, III*, Ann. of Math. 63 (1956), 361–382; 64 (1956), 562–580; 65 (1957), 375–390.
[23] —, *A short proof of the Arens–Eells embedding theorem*, Proc. Amer. Math. Soc. 15 (1964), 415–416.
[24] R. S. Phillips, *On linear transformations*, Trans. Amer. Math. Soc. 48 (1940), 516–541.
[25] R. Pol, *The Lindelöf property and its convex analogue in function spaces with the weak topology*, in: Colloquia Math. Soc. János Bolyai 23, Topology, Budapest 1978, North-Holland, 965–969.
[26] S. A. Rakov, *C-convexity and "the three space problem"* (in Russian), Dokl. Akad. Nauk SSSR 228 (1976), 303–305.
[27] —, *Ultraproducts and "the three space problem"* (in Russian), Funktsional. Anal. i Prilozh. 11 (3) (1977), 88–89.
[28] M. Ribe, *Examples for nonlocally convex three space problem*, Proc. Amer. Math. Soc. 73 (1979), 351–355.
[29] J. W. Roberts, *A non-locally convex F-space with the Hahn–Banach approximation property*, in: Proc. Kent State Conf., Lecture Notes in Math. 604, Springer, Berlin 1977.
[30] W. Roelcke, *Einige Permanenzeigenschaften bei topologischen Gruppen und topologischen Vektorräumen*, Vortrag auf der Funktionalanalysistagung in Oberwolfach 1972.
[31] W. Roelcke and S. Dierolf, *On the three space problem for topological vector spaces*, Collect. Math. 32 (1981), 13–35.
[32] —, —, *Uniform Structures on Topological Groups and Their Quotients*, McGraw-Hill, New York 1981.
[33] H. H. Schaeffer, *Topological Vector Spaces*, Springer, New York–Heidelberg–Berlin 1971.

[34] J. J. Schäffer, *Girth, super-reflexivity and isomorphic classification of normed spaces and subspaces*, Bull. Acad. Polon. Sci Sér. Sci. Math. 28 (1980), 573–584.
[35] A. Sobczyk, *Projections of the space (m) on its subspace (c_0)*, Bull. Amer. Math. Soc. 47 (1941), 938–947.
[36] W. A. Veech, *A short proof of Sobczyk's theorem*, Proc. Amer. Math. Soc. 28 (1971), 627–628.
[37] N. Ya. Vilenkin, *On the theory of weakly separable groups* (in Russian), Mat. Sb. 22 (64) (1948), 135–177.
[38] D. Vogt, *Charakterisierung der Unterräume von s*, Math. Z. 155 (1977), 109–117.
[39] D. Vogt and M. J. Wagner, *Charakterisierung der Unterräume und Quotientenräume der nuklearen stabilen Potenzreihenräume von unendlichem Typ*, Studia Math. 70 (1981), 63–80.
[40] K. W. Yang, *The reflexive dimension of an R-space*, Acta Math. Acad. Sci. Hung. 35 (1980), 249–255.
[41] M. Zippin, *The separable extension problem*, Israel J. Math. 26 (1977), 372–387.

Added in proof:

[42] P. Domański, *Twisted sums of Banach and nuclear spaces*, Proc. Amer. Math. Soc., to appear.

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