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Interpolation of Banach lattices

by

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Abstract. For couples of Banach lattices we describe the interpolation spaces generated by the $\pm$ method and by Ovchinnikov's upper method in terms of the Calderón–Lozanovskii spaces.

0. Introduction. In this paper we study the effect of certain interpolation methods on (quasi-) Banach lattices. More specifically, we consider the "$\pm$-method" $\langle \bar{X}, \varphi \rangle$ of Gustavsson–Peetre [10], Ovchinnikov's upper method $\langle \bar{X} \rangle^\varphi$ (see [20]), as well as a variant $\langle \bar{X} \rangle_\varphi$ of Ovchinnikov's lower method. Some results are also obtained for the complex method $[\bar{X}]_\theta$ of Calderón [6]. In fact, we wish to put these interpolation methods in the case of a couple $\bar{X}$ of quasi-Banach lattices in relation to the Calderón–Lozanovskii constructions $\varphi(\bar{X})$.

Not all of our results are new: closely related results may be found in Ovchinnikov [21], [22] as well as in Berezhnoi [2]. However, in contrast to [21], [22], the methods used here are elementary and are similar to those of Gustavsson–Peetre [10].

The plan of the paper is as follows. Section 1 contains definitions and a technical Lemma. In Section 2 we study in the case of a couple $\bar{X}$ of quasi-Banach lattices the connection between $\varphi(\bar{X})$, $\langle \bar{X} \rangle_\varphi$ and $\langle \bar{X}, \varphi \rangle$. As an application we obtain a new proof and an extension of the following theorem of Pisier [25]: a Banach lattice X is p-convex and p'-concave, $1 < p < 2$, $1/p + 1/p' = 1$, if and only if there exists a Banach lattice X_0 such that $X = [X_0, L^2]_\theta$, $\theta = 2/p$. In Section 3 we then extend our considerations to include $\langle \bar{X} \rangle^\varphi$, in the Banach case only. Section 4 is on the Gagliardo closure of $[\bar{X}]_\theta$. Section 5 is concerned with various applications of the previous results.

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1. Preliminaries. Let us recall some notation from interpolation theory. Let A_0 and A_1 be two quasi-Banach spaces. We say that $\bar{A} = (A_0, A_1)$ is a *quasi-Banach couple* if both A_0 and A_1 are continuously imbedded in some Hausdorff topological vector space. We let

$$\Delta(\bar{A}) = A_0 \cap A_1 \quad \text{and} \quad \Sigma(\bar{A}) = A_0 + A_1.$$

Further for $a \in \Sigma(\bar{A})$ and for $t > 0$ we define

$$K(t, a: A) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1}:\ a = a_0 + a_1\}$$

(*K-functional*).

A quasi-Banach space A is called an *intermediate space* with respect to $\bar{A}$ if and only if $\Delta(\bar{A}) \subseteq A \subseteq \Sigma(\bar{A})$ with continuous imbeddings. If this is the case, we denote by A^0 the closure of $\Delta(\bar{A})$ in A. The *Gagliardo closure* of A with respect to $\Sigma(\bar{A})$, written $A^\sim$, is defined as follows: $a \in A^\sim$ if and only if there exists a sequence $(a_i)_0^\infty \in A$ such that $a_i \to a$ as $i \to \infty$ in $\Sigma(\bar{A})$ and $\|a_i\|_A \leqslant \lambda$ for some $\lambda < \infty$. Put $\|a\|_{A^\sim} = \inf \lambda$. If $\bar{A}$ is a quasi-Banach couple we write $\bar{A}^0 = (A_0^0, A_1^0)$. If $\bar{A} = \bar{A}^0$, $\bar{A}$ is called a *regular couple*. For details and any other unexplained notation concerning interpolation see [4].

Let $\mathscr{P}$ denote the set of all positive functions φ on $\boldsymbol{R}_+$ such that both $\varphi(t)$ and $t\varphi(1/t)$ are nondecreasing. We let $\mathscr{P}_0$ denote the subset of $\mathscr{P}$ consisting of all φ with $\min(1, 1/t)\varphi(t) \to 0$ as $t \to 0, \infty$. On $\mathscr{P}$ we define an involution by $\varphi^*(t) = 1/\varphi(1/t)$ and we put $\mathscr{P}^* = \mathscr{P}_0 \cap (\mathscr{P}_0)^*$. Sometimes we will regard φ as a function on $\boldsymbol{R}_+ \times \boldsymbol{R}_+$ by putting $\varphi(s, t) = s\varphi(t/s)$.

We now define the interpolation methods whose study this paper is devoted to.

Definition 1.1. Let $\bar{A}$ be a quasi-Banach couple and let $\varphi \in \mathscr{P}^*$.

(i) Then $a \in \langle \bar{A} \rangle_\varphi$ if there is a sequence $(a_i)_{-\infty}^\infty \subseteq \Delta(\bar{A})$ such that $a = \sum_{i=-\infty}^\infty a_i$ (with convergence in $\Sigma(\bar{A})$) and for any bounded sequence $(\varepsilon_i)_{-\infty}^\infty$ of complex numbers $\sum_{i=-\infty}^\infty \varepsilon_i 2^{ij} a_i/\varphi(2^i)$ is convergent in A_j, $j = 0, 1$. We further require that

$$\Big\|\sum_{i=-\infty}^\infty \varepsilon_i 2^{ij} a_i/\varphi(2^i)\Big\|_{A_j} \leqslant C \sup_{i \in \boldsymbol{Z}} |\varepsilon_i|$$

for some constant C and for $j = 0, 1$. As a quasi-norm we use $\|a\|_{\langle \bar{A} \rangle_\varphi} = \inf C$.

(ii) (the $\pm$ method) Then $a \in \langle \bar{A}, \varphi \rangle$ if there is a sequence $(a_i)_{-\infty}^\infty \in \Delta(\bar{A})$ such that $a = \sum_{i=-\infty}^\infty a_i$ (convergence in $\Sigma(\bar{A})$) and such that for any finite subset F of $\boldsymbol{Z}$ and any bounded sequence $(\varepsilon_i)_{-\infty}^\infty \in \boldsymbol{C}$,

$$\Big\|\sum_{i \in F} \varepsilon_i 2^{ij} a_i/\varphi(2^i)\Big\|_{A_j} \leqslant C \sup_{i \in \boldsymbol{Z}} |\varepsilon_i| \tag{1.1}$$

for $j = 0, 1$, with C independent of F. Put $\|a\|_{\langle \bar{A}, \varphi \rangle} = \inf C$.

Remark 1.2. One easily sees that condition (1.1) is equivalent to demanding that for some C'

$$\Big\|\sum_{i=-\infty}^\infty \varepsilon_i 2^{ij} a_i/\varphi(2^i)\Big\|_{A_j} \leqslant C' \tag{1.2}$$

whenever $\varepsilon = (\varepsilon_i)_{-\infty}^\infty \in c_0$ and $\|\varepsilon\|_{c_0} \leqslant 1$.

Remark 1.3. These interpolation methods can be traced back to the work of Gagliardo. The $\langle \cdot \rangle_\varphi$-method was introduced in this form by Peetre in [23]. The first paper on the $\pm$ method is Gustavsson–Peetre [10]. See also [2], [9], [20], [21], [22], [28].

Let $\omega = (\omega_i)_{-\infty}^\infty$ be a sequence of positive real numbers and let $1 \leqslant p \leqslant \infty$. We denote by $l_p(\omega) = l_p(\omega_i)$ the space of all sequences $\alpha = (\alpha_i)_{-\infty}^\infty$ such that the norm $\|\alpha\|_{l_p(\omega)} = \big(\sum_{i=-\infty}^\infty |\alpha_i \omega_i|^p\big)^{1/p}$ is finite. Let $\bar{l}_p$ denote the couple $(l_p, l_p(2^i))$.

We now give the definition of Ovchinnikov's upper method (see [20]).

Definition 1.4. Let $\bar{A}$ be a couple of Banach spaces and let $\varphi \in \mathscr{P}$. The space $\langle \bar{A} \rangle^\varphi$ consists of all $a \in \Sigma(\bar{A})$ rendering the norm

$$\|a\|_{\langle \bar{A} \rangle^\varphi} = \sup\{\|Ta\|_{l_1(\varphi^*(2^{-i}))}:\ \|T\|_{\bar{A}, \bar{l}_1} = 1\}$$

finite.

If $\varphi(t) = t^\theta$, $0 < \theta < 1$, we simply write $\langle \bar{A} \rangle_\theta$, $\langle \bar{A}, \theta \rangle$, respectively $\langle \bar{A} \rangle^\theta$.

In the sequel we will need the following result, implicit in Janson [11], Theorem 8, as well as in Ovchinnikov [22].

Lemma 1.5. *Let $\bar{A}$ be a couple of p-normed quasi-Banach spaces and let $\varphi \in \mathscr{P}^*$. If $a \in \Delta(\bar{A})$, then there exists an integer N and $a_i \in \Delta(\bar{A})$, $|i| \leqslant N$, such that $a = \sum_{i=-N}^N a_i$ and*

$$\Big\|\sum_{i=-N}^N \varepsilon_i 2^{ij} a_i/\varphi(2^i)\Big\|_{A_j} \leqslant 2^{3(1-p)} 4 \sup_{|i| \leqslant N} |\varepsilon_i|\, \|a\|_{\langle \bar{A}, \varphi \rangle} \tag{1.3}$$

for $j = 0, 1$.

Proof. We may assume that $\|a\|_{\langle \bar{A},\varphi\rangle}=1$. Take $\delta>0$ and choose $b_i\in\Delta(\bar{A})$, $i\in \mathbf{Z}$, such that $a=\sum_{i=-\infty}^{\infty} b_i$ and if $\varepsilon=(\varepsilon_i)_{-\infty}^{\infty}\in c_0$ then

$$(1.4)\qquad \Big\|\sum_{i=-\infty}^{\infty}\varepsilon_i 2^{ij} b_i/\varphi(2^i)\Big\|_{A_j}\leqslant (1+\delta)\|\varepsilon\|_{c_0},\quad j=0,1.$$

Fix a positive integer N and put

$$a^+=\sum_{i=N}^{\infty} b_i\quad\text{and}\quad a^-=\sum_{i=-\infty}^{-N} b_i.$$

Since $\varphi(2^i)/2^i\to 0$ as $i\to\infty$, (1.4) implies that

$$\|a^+\|_{A_1}=\Big\|\sum_{i=N}^{\infty}\frac{\varphi(2^i)}{2^i}\cdot\frac{2^i}{\varphi(2^i)}b_i\Big\|_{A_1}\leqslant (1+\delta)\varphi(2^N)/2^N.$$

Further,

$$\begin{aligned}\|a^+\|_{A_0}&\leqslant 2^{1-p}\Big(\|a\|_{A_0}+\Big\|\sum_{i=-\infty}^{N-1} b_i\Big\|_{A_0}\Big)\\ &\leqslant 2^{1-p}\Big(\|a\|_{A_0}+\Big\|\sum_{i=-\infty}^{N-1}\varphi(2^i)b_i/\varphi(2^i)\Big\|_{A_0}\Big)\\ &\leqslant 2^{1-p}\big(\|a\|_{A_0}+(1+\delta)\varphi(2^N)\big)\end{aligned}$$

since $\varphi(2^i)\to 0$ as $i\to-\infty$. Similarly

$$\|a^-\|_{A_0}\leqslant(1+\delta)\varphi(2^{-N}),\qquad \|a^-\|_{A_1}\leqslant 2^{1-p}\big(\|a\|_{A_1}+(1+\delta)\varphi(2^{-N})2^N\big).$$

If we put $a_{-N}=a^-$, $a_N=a^+$ and $a_i=b_i$ if $-N<i<N$, then these estimates imply that if $|\varepsilon_i|\leqslant 1$, $|i|\leqslant N$, then

$$\Big\|\sum_{i=-N}^{N}\varepsilon_i 2^{ij}a_i/\varphi(2^i)\Big\|_{A_j}\leqslant 2^{2(1-p)}\Big(\Big\|\sum_{i=-N+1}^{N-1}\varepsilon_i 2^{ij}b_i/\varphi(2^i)\Big\|_{A_j}+$$
$$+2^{Nj}|\varepsilon_{-N}|/\varphi(2^{-N})\|a^-\|_{A_j}+2^{Nj}|\varepsilon_N|/\varphi(2^N)\|a^+\|_{A_j}\Big),$$

$j=0,1$. By using (1.4) and the above estimates this expression is in turn dominated by

$$2^{3(1-p)}(1+\delta)\big(3+\max(1/\varphi(2^N),\,2^{-N}/\varphi(2^{-N}))\|a\|_{\Delta(\bar{X})}\big).$$

Since $\varphi\in\mathscr{P}^*$, the lemma will follow if N is chosen sufficiently large.

Let $\bar{A}$ be a couple of quasi-Banach spaces and let $\varphi\in\mathscr{P}^*$. From Definition 1.1 it is evident that $\langle\bar{A}\rangle_\varphi\subseteq\langle\bar{A},\varphi\rangle$. On the other hand, Lemma 1.5 shows that in fact

$$(1.5)\qquad \langle\bar{A}\rangle_\varphi=\langle\bar{A},\varphi\rangle^0.$$

See also [11], Theorem 8.

Let now $\bar{A}$ be a Banach couple and let $\theta\in(0,1)$. Denote by $[\bar{A}]_\theta$ and $[\bar{A}]^\theta$ Calderón's two complex methods of interpolation (see [6] and [4], Chapter 4). One then has the following chain of inclusions:

$$\langle\bar{A}\rangle_\theta\subseteq[\bar{A}]_\theta\subseteq[\bar{A}]^\theta\subseteq[\bar{A}]_\theta^{\sim}\subseteq\langle\bar{A}\rangle^\theta$$

and

$$\langle\bar{A},\theta\rangle\subseteq[\bar{A}]^\theta.$$

More generally, if $\varphi\in\mathscr{P}^*$ one can show that

$$(1.6)\qquad \langle\bar{A}\rangle_\varphi\subseteq\langle\bar{A},\varphi\rangle\subseteq\langle\bar{A}\rangle_\varphi^{\sim}\subseteq\langle\bar{A}\rangle^\varphi.$$

Proofs may be found in Janson [11] and in Ovchinnikov [20], [22]. Let us remark that the inclusion $\langle\bar{A}\rangle_\varphi\subseteq\langle\bar{A}\rangle^\varphi$ (or rather a slightly sharper version of it) was originally proved by Ovchinnikov by using Grothendieck's inequality. Subsequently an elementary proof was found by Janson [11]. We also refer to Peetre [24].

Concerning duality one has the following results.

Lemma 1.6. *Let $\bar{A}$ be a regular Banach couple and let $\varphi\in\mathscr{P}^*$. Denote by $\bar{A}^*=(A_0^*,A_1^*)$ the dual couple. Then*

$$(1.7)\qquad \langle\bar{A}\rangle_\varphi^*=\langle\bar{A}^*\rangle^{\varphi^*},$$

$$(1.8)\qquad \big((\langle\bar{A}\rangle^\varphi)^0\big)^*=\langle\bar{A}^*,\varphi^*\rangle.$$

Proofs may be found in [11] or [22]. See also [19], Section 4.3 for a different approach to (1.8).

Let (Ω,Σ,μ) be a complete σ-finite measure space and denote by $L^0=L^0(\Omega,\Sigma,\mu)$ the space of all equivalence classes of μ-measurable real-valued functions, equipped with the topology of convergence in measure. We will say that a quasi-Banach space X is a *quasi-Banach lattice* (*on* (Ω,Σ,μ)) if X is a quasi-Banach subspace of L^0 with the property that if $f\in X$ and $g\in L^0$ are such that $|g|\leqslant|f|$ μ-a.e. then $g\in X$ and $\|g\|_X\leqslant\|f\|_X$. Note that if X_0 and X_1 are any two quasi-Banach lattices (on (Ω,Σ,μ)) then $\bar{X}=(X_0,X_1)$ forms a couple of quasi-Banach spaces.

Let $\bar{X}$ be a couple of quasi-Banach lattices and let $\varphi\in\mathscr{P}^*$. We denote by $\varphi(\bar{X})=\varphi(X_0,X_1)$ the space of all measurable functions x such that for some $x_i\in X_i$, $\|x_i\|_{X_i}\leqslant 1$, $i=0,1$, and for some $\lambda<\infty$ we have $|x|\leqslant\lambda\varphi(|x_0|,|x_1|)$ a.e. We put $\|x\|_{\varphi(\bar{X})}=\inf\lambda$.

We note that $\varphi(\bar{X})$ is a quasi-Banach lattice as well as an intermediate space with respect to $\bar{X}$. In particular, if we take $\varphi(t)=t^\theta$, $0<\theta<1$, we obtain in this way the spaces $X_0^{1-\theta}X_1^\theta$ introduced by Calderón [6]. The properties of $\varphi(\bar{X})$ have been studied in depth by Lozanovskii (see [15] and the references given there).

Let X be a quasi-Banach lattice and let $1 \leqslant p \leqslant \infty$. The *p-convexification* of X, denoted by $X^{(p)}$, is defined as follows: $x \in X^{(p)}$ if and only if $|x|^p \in X$. We put $\|x\|_{X^{(p)}} = \||x|^p\|_X^{1/p}$. Observe that if $1-\theta = 1/p$ then $X^{(p)} = X^{1-\theta}(L^\infty)^\theta$. Similarly we define the *p-concavification* of X, written $X_{(p)}$, by demanding that $x \in X_{(p)}$ if and only if $|x|^{1/p} \in X$. As a quasi-norm we use $\|x\|_{X_{(p)}} = \||x|^{1/p}\|_X^p$.

Let $0 < p \leqslant \infty$. Recall that a quasi-Banach lattice X is called *p-convex*, respectively *p-concave*, if there exists a constant $M < \infty$ such that if $x_1, x_2, \ldots, x_n \in X$ then

$$\|(\sum_{i=1}^n |x_i|^p)^{1/p}\|_X \leqslant M(\sum_{i=1}^n \|x_i\|_X^p)^{1/p},$$

respectively

$$(\sum_{i=1}^n \|x_i\|_X^p)^{1/p} \leqslant M\|(\sum_{i=1}^n |x_i|^p)^{1/p}\|_X.$$

The smallest possible value of M is denoted by $M^{(p)}(X)$, respectively $M_{(p)}(X)$.

We will also need the following

DEFINITION 1.7. A quasi-Banach lattice X is said to be of *type* $\mathscr{C}$ if there exists an equivalent lattice quasi-norm on X such that, for some $p \geqslant 1$, $X^{(p)}$ is a Banach lattice in this norm.

2. On the interpolation properties of $\varphi(\bar{X})$. Let $\bar{X}$ be a couple of Banach lattices and let $\varphi \in \mathscr{P}^*$. The interest of the Banach lattice $\varphi(\bar{X})$ mainly stems from the fact that it is often computable. For examples we refer to Section 5 as well as to [22]. Hence one is interested in describing the interpolation properties of $\varphi(\bar{X})$. The first general result in this area is due to Shestakov, who in [26] showed that if $\varphi(t) = t^\theta$, $0 < \theta < 1$, then $\varphi(\cdot)^0$ and $\varphi(\cdot)^\sim$ are interpolation methods on the category of couples of Banach lattices. Later he supplemented these results by showing that in fact $(X_0^{1-\theta} X_1^\theta)^0 = [\bar{X}]_\theta$ (see [27] and [3]). If X_0 and X_1 have semicontinuous norms then a theorem of Lozanovskii [16] asserts that $(X_0^{1-\theta} X_1^\theta)^\sim = [\bar{X}]^\theta$. In this section we wish to extend these results to the case of a general $\varphi \in \mathscr{P}^*$ as well as to couples of quasi-Banach lattices of type $\mathscr{C}$.

THEOREM 2.1. *Let X_0 and X_1 be two quasi-Banach lattices of type $\mathscr{C}$ and let $\varphi \in \mathscr{P}^*$. Then*

$$\langle \bar{X} \rangle_\varphi = \varphi(\bar{X})^0, \tag{2.1}$$

$$\varphi(\bar{X}) \subseteq \langle \bar{X}, \varphi \rangle \subseteq \varphi(\bar{X})^\sim. \tag{2.2}$$

The proof of Theorem 2.1 is based on the following lemma, generalizing Carlson's inequality (cf. [10], p. 38, for the case $\varphi \in \mathscr{P}^{+-}$).

LEMMA 2.2. *Let $\varphi \in \mathscr{P}^*$. Then there exists a sequence of disjoint sets $I_h \subset \mathbf{Z}$, $h \in \mathbf{Z}$, two sequences $(\varepsilon_i^j)_{-\infty}^\infty$, $j = 0, 1$, of real numbers with $|\varepsilon_i^j| \leqslant 1$, $i \in \mathbf{Z}$, $j = 0, 1$, such that*

$$|\sum_i x_i \varphi(2^i)| \leqslant C\varphi\big((\sum_h |\sum_{i \in I_h} \varepsilon_i^0 x_i|^2)^{1/2}, (\sum_h |\sum_{i \in I_h} \varepsilon_i^1 2^i x_i|^2)^{1/2}\big)$$

for any finite sequence $(x_i) \in \mathbf{R}$.

Proof. Put $a_\varphi = (\varphi(2^i))_{-\infty}^\infty$ and choose $\eta \in \Sigma(\bar{l}_2)$ such that

$$c_1 \varphi(t) \leqslant K(t, \eta: \bar{l}_2) \leqslant c_2 K(t, a_\varphi: \bar{l}_\infty) \leqslant c_3 \varphi(t)$$

for some constants c_1, c_2 and c_3 (see [4], p. 117). By Theorem 2 in Cwikel [8] we may find $T: \bar{l}_2 \to \bar{l}_\infty$ such that $T(\eta) = a_\varphi$ and $\|T\|_{\bar{l}_2, \bar{l}_\infty} \leqslant C$. An analysis of the proof of Theorem 2 in [8] in this special case shows that we may write $T = \sum_{|k| \leqslant \beta} T_k$ where

$$T_k(\cdot) = \sum_{h=-\infty}^\infty a_\varphi \chi_{I_h} \langle \cdot, \chi_{G_{h+k}} l_h \rangle \delta_h, \quad |k| \leqslant \beta. \tag{2.3}$$

Here $(I_h)_{-\infty}^\infty$ and $(G_h)_{-\infty}^\infty$ are two sequences of pairwise disjoint subsets of $\mathbf{Z}$ whose union is $\mathbf{Z}$ in both cases, δ_h, $h \in \mathbf{Z}$, is a sequence of positive real numbers and l_h, $h \in \mathbf{Z}$, are linear functionals on $\Sigma(\bar{l}_2)$ such that one has the estimates

$$\|l_h\|_{l_2} \delta_h \sup_{j \in I_h} \varphi(2^j) \leqslant C, \tag{2.4}$$

$$\|l_h\|_{l_2(2^i)} \delta_h \sup_{j \in I_h} \varphi(2^j)/2^j \leqslant C \tag{2.5}$$

for some constant C.

For $f \in \Sigma(\bar{l}_2)^*$ we now have

$$|\langle \eta, f \rangle| \leqslant \|f\|_{l_2} K(\|f\|_{l_2(2^i)}/\|f\|_{l_2}, \eta: \bar{l}_2) \leqslant c_3 \varphi(\|f\|_{l_2}, \|f\|_{l_2(2^i)}).$$

(see [4], p. 32, 54). Hence, for any finite sequence $(x_i) \in \mathbf{R}$,

$$|\sum_i x_i \varphi(2^i)| = |\langle x, a_\varphi \rangle| = |\langle x, T(\eta) \rangle| = |\langle T^*(x), \eta \rangle| \tag{2.6}$$
$$\leqslant \sum_{|k| \leqslant \beta} |\langle T_k^*(x), \eta \rangle| \leqslant c_3 \sum_{|k| \leqslant \beta} \varphi(\|T_k^*(x)\|_{l_2}, \|T_k^*(x)\|_{l_2(2^i)}).$$

For $j \in I_h$ put $\varepsilon_j^0 = \varphi(2^j)/\sup_{i \in I_h} \varphi(2^i)$ and note that $0 \leqslant \varepsilon_j^0 \leqslant 1$. Then (2.3) implies that

$$\|T_k^*(x)\|_{l_2}^2 = \sum_{h=-\infty}^\infty (\delta_h |\langle x, \chi_{I_h} a_\varphi \rangle| \, \|\chi_{G_{h+k}} l_h\|_{l_2})^2$$
$$\leqslant \sum_{h=-\infty}^\infty (\sup_{i \in I_h} \varphi(2^i) \delta_h \|l_h\|_{l_2})^2 |\sum_{i \in I_h} \varepsilon_i^0 x_i|^2$$

and by (2.4) this expression is in turn dominated by

$$C\sum_h \Big|\sum_{i\in I_h} \varepsilon_i^0 x_i\Big|^2.$$

Similarly with $\varepsilon_j^1 = \varphi(2^j)2^{-j}/\sup_{i\in I_h}\varphi(2^i)2^{-i}$, $j\in I_h$, one infers from (2.5) that

$$\|T_k^*(x)\|_{l_2(2^i)}^2 \leqslant C\Big(\sum_h \Big|\sum_{i\in I_h} \varepsilon_i^1 2^i x_i\Big|^2\Big).$$

If one inserts these estimates in (2.6), the conclusion of the lemma follows.

Remark 2.3. If $\varphi(t) = t^\theta$, $0<\theta<1$, one may take $I_h = \{h\}$. Hence in this case one has the estimate

$$\Big|\sum_i x_i 2^{i\theta}\Big| \leqslant C\Big(\sum_i |x_i|^2\Big)^{1-\theta/2}\Big(\sum_i |2^i x_i|^2\Big)^{\theta/2},$$

i.e. the classical Carlson inequality. Further one can easily prove analogues to Lemma 2.2 with 2 replaced by any other power $p\in[1,\infty]$ (cf. [10], p. 38).

The existence of the operator T in the proof of Lemma 2.2 is well-known and goes back to the work of Peetre. The main point here is that we have an explicit expression for T which permits us to make the appropriate estimates. The construction of T is based ultimately on the K-divisibility techniques of Brudnyi–Kruglyak (see [5], [8]).

We now give the

Proof of Theorem 2.1. We claim that it suffices to show that (i) $\varphi(\bar X)\subseteq\langle\bar X,\varphi\rangle$ and (ii) if $x\in\Delta(\bar X)$ then $\|x\|_{\varphi(\bar X)}\leqslant C\|x\|_{\langle\bar X\rangle_\varphi}$. Indeed, since $\Delta(\bar X)$ is dense in $\langle\bar X\rangle_\varphi$, (ii) will imply that $\langle\bar X\rangle_\varphi\subseteq\varphi(\bar X)^0$ and from (i) it follows that $\varphi(\bar X)^0\subseteq\langle\bar X,\varphi\rangle^0$. Since $\langle\bar X,\varphi\rangle^0 = \langle\bar X\rangle_\varphi$ (see (1.5)), (2.1) will follow. On the other hand, we have $\langle\bar X,\varphi\rangle\subseteq\langle\bar X\rangle_\varphi^\sim$ and hence (ii) implies that $\langle\bar X,\varphi\rangle\subseteq\varphi(\bar X)^\sim$, i.e. (2.2).

Let us begin with the more complicated case (ii). If $x\in\Delta(\bar X)$ we may choose, in view of Lemma 1.5, an integer N, $x_i\in\Delta(\bar X)$, $-N\leqslant i\leqslant N$, such that $x=\sum_{i=-N}^{N} x_i$ and (1.3) holds. Choose sets I_h, $h\in\boldsymbol{Z}$, and sequences $(\varepsilon_i^j)_{-\infty}^{\infty}$, $j=0, 1$, as in Lemma 2.2. Take $p_i\geqslant 1$, $i=0, 1$, such that $X_i^{(p_i)}$ are Banach lattices. Put $q_i = 1/p_i$. Then Khintchine's inequality implies that for any finite sequence $(y_i)\in X_i$ we have a.e.

$$C_{q_i}\Big(\sum_i |y_i|^2\Big)^{1/2} \leqslant \Big(\int_0^1\Big|\sum_i r_i(t)y_i\Big|^{q_i}dt\Big)^{1/q_i}.$$

Here r_i, $i\in\boldsymbol{Z}_+$, denote the Rademacher functions. Using the fact that $X_i^{(p_i)}$ is a Banach lattice, we obtain

$$(2.7)\qquad C_{q_i}\Big\|\Big(\sum_i |y_i|^2\Big)^{1/2}\Big\|_{X_i} \leqslant \Big\|\Big(\int_0^1\Big|\sum_i r_i(t)y_i\Big|^{q_i}dt\Big)^{1/q_i}\Big\|_{X_i}$$
$$= \Big\|\int_0^1\Big|\sum_i r_i(t)y_i\Big|^{q_i}dt\Big\|_{X_i^{(p_i)}}^{p_i}$$
$$\leqslant \Big(\int_0^1\Big\|\sum_i r_i(t)y_i\Big\|_{X_i}^{q_i}dt\Big)^{1/q_i}$$
$$\leqslant \sup_{0<t<1}\Big\|\sum_i r_i(t)y_i\Big\|_{X_i}.$$

Now put $y_h^j = \sum_{i\in I_h}\varepsilon_i^j 2^{ij}x_i/\varphi(2^i)$. If n is chosen sufficiently large then (2.7) implies the estimate

$$C_{q_j}\Big\|\Big(\sum_h |y_h^j|^2\Big)^{1/2}\Big\|_{X_j} \leqslant \sup_{0<t<1}\Big\|\sum_h r_{h+n}(t)\sum_{i\in I_h}\varepsilon_i^j 2^{ij}x_i/\varphi(2^i)\Big\|_{X_j},$$

and this expression is in turn dominated by $c\|x\|_{\langle\bar X\rangle_\varphi}$ (see (1.3)). On the other hand, Lemma 2.2 implies that

$$|x| = \Big|\sum_i x_i\Big| \leqslant C\varphi\Big(\Big(\sum_h |y_h^0|^2\Big)^{1/2}, \Big(\sum_h |y_h^1|^2\Big)^{1/2}\Big),$$

and hence $x\in\varphi(\bar X)$.

The proof of (i) is essentially the same as the proof of Lemma 8.2.1 in [22]. Fix $\varepsilon>0$ and take $x\in\varphi(\bar X)$. Write $|x| = \eta\varphi(|x_0|, |x_1|)$ where $\|x_i\|_{X_i}=1$, $i=0, 1$, and $|\eta|\leqslant(1+\varepsilon)\|x\|_{\varphi(\bar X)}$. Without loss of generality we may assume that x is nonvanishing a.e., and hence so are x_0 and x_1. If we interpolate the identity map

$$I\colon\ (L^\infty(1/|x_0|), L^\infty(1/|x_1|))\to(X_0, X_1),$$

we find that

$$I\colon\ \langle L^\infty(1/|x_0|), L^\infty(1/|x_1|), \varphi\rangle\to\langle\bar X,\varphi\rangle.$$

Since

$$L^\infty(1/\varphi(|x_0|, |x_1|)) = \langle L^\infty(1/|x_0|), L^\infty(1/|x_1|), \varphi\rangle$$

(see [9], Theorem 2.3) and $x\in L^\infty(1/\varphi(|x_0|, |x_1|))$, it follows that $x\in\langle\bar X,\varphi\rangle$. Hence $\varphi(\bar X)\subseteq\langle\bar X,\varphi\rangle$ and the proof of Theorem 2.1 is complete.

It is evident from the proof of Theorem 2.1 and Remark 2.3 that the assumption that X_j, $j=0, 1$, are of type $\mathscr{C}$ may be replaced with the following assumption: there exists a constant $M<\infty$ such that if

$x_1, x_2, \ldots, x_n \in X_j$ then

$$(*) \qquad \|\max_{1\le i\le n} |x_i|\|_{X_j} \leqslant M \sup_{0<t<1} \|\sum_{i=1}^{n} r_i(t)\, x_i\|_{X_j}.$$

Since any Banach lattice is of type $\mathscr{C}$, Theorem 2.1 applies to any couple of Banach lattices and we recover a theorem of Ovchinnikov [22] (see also Berezhnoi [2] as well as Shestakov [28]). In [21] Ovchinnikov proved the inclusions in (2.2) for a certain class of quasi-Banach lattices, using techniques different from ours. In particular, Ovchinnikov's class contains any quasi-Banach lattice of type $\mathscr{C}$ but its relation to condition $(*)$ is not clear. Let us remark that we do not know of any quasi-Banach lattice which is not of type $\mathscr{C}$.

We now give as an application of Theorem 2.1 a characterization of q-concave Köthe function spaces. Recall that given a Köthe function space X, X' denotes its Köthe dual (see [13]). By $M(X, L^q)$ we denote the Banach lattice of all pointwise multipliers from X to L^q endowed with the operator norm.

THEOREM 2.4. *Let X be a Köthe function space such that $X'' = X$ and let $0 < \theta < 1$. If X is q-concave, $1 < q < \infty$, then there exists a quasi-Banach lattice X_0 such that, with $q_0 = \theta q$,*

$$X = \langle X_0, L^{q_0}, \theta\rangle$$

up to the equivalence of norms. If in addition X is p-convex where $1/p = 1-\theta+\theta/q_0$, then X_0 can be chosen to be a Banach lattice.

In order to prove this result we need the following factorization result due to Maurey [18].

LEMMA 2.5. *Let X be a p-convex Banach lattice with $M^{(p)}(X) = 1$. If $h \in X'$ then there exist $f \in L^{p'}$, $1/p+1/p' = 1$, and $g \in M(X, L^p)$ such that $h = fg$. Further,*

$$\|h\|_{X'} = \inf\{\|f\|_{L^{p'}}\|g\|_{M(X,L^p)}\colon h = fg\}.$$

Proof. If $h \in X'$ then $h \in M(X, L^1)$. Since X is p-convex, Theorem 8 in [18] implies that there exist $g \in L^{p'}$ and a linear operator $V\colon X \to L^p$ such that for any $x \in X$ we have $gV(x) = hx$. Hence V may be chosen as a pointwise multiplication operator and the lemma is proved.

Proof of Theorem 2.4. We may assume that $M_{(q)}(X) = 1$. We claim that as X_0 we may take $M(L^q, X)_{(\tilde p)}$, $1/\tilde p = 1-\theta$. In order to see this, observe that X' is q'-convex with $1/q+1/q' = 1$. Hence Lemma 2.5 implies that

$$X'' = M(X', L^{q'})\, L^q.$$

By duality it follows that

$$X = X'' = M(L^q, X)\, L^q.$$

By taking appropriate powers this expression is in turn equal to

$$X = (M(L^q, X)_{(\tilde p)})^{1-\theta} (L^{q_0})^\theta = X_0^{1-\theta} (L^{q_0})^\theta.$$

Since the quasi-Banach lattice X_0 is of type $\mathscr{C}$, we may apply (2.2) and infer that

$$X = \langle X_0, L^{q_0}, \theta\rangle.$$

If in addition X is p-convex we may assume that $M^{(p)}(X)\, M_{(q)}(X) = 1$. It is easy to check that $M(L^q, X)$ is then $\tilde p$-convex with constant equal to one and hence X_0 is a Banach lattice. The proof is complete.

If $M^{(p)}(X)\, M_{(q)}(X) = 1$, the above proof and Shestakov's theorem [27] imply that the equalities

$$X = X_0^{1-\theta}(L^{q_0})^\theta = [X_0, L^{q_0}]_\theta$$

hold isometrically and we obtain Pisier's theorem [25]. It is not clear if the assumption that $X'' = X$ is necessary. Observe that if X is a sequence space with a monotone unconditional basis this assumption may be dropped. This is a consequence of Corollary 3.2 in [12] which implies that one has the equality

$$X = M(l_q, X)\, l_q$$

whenever X is q-concave.

3. Connections between $\langle \bar X\rangle_\varphi$, $\langle \bar X, \varphi\rangle$ and $\langle \bar X\rangle^\varphi$. In this section we obtain a description of $\langle \bar X\rangle^\varphi$ whenever $\varphi \in \mathscr{P}^*$ and $\bar X$ is a regular couple of Banach lattices.

THEOREM 3.1. *Let $\bar X$ be a regular couple of Banach lattices and let $\varphi \in \mathscr{P}^*$. Then*

$$(3.1) \qquad \langle \bar X\rangle_\varphi = \varphi(\bar X)^0 = (\langle \bar X\rangle^\varphi)^0$$

and

$$(3.2) \qquad \langle \bar X\rangle_\varphi^\sim = \varphi(\bar X)^\sim = \langle \bar X\rangle^\varphi.$$

As a corollary we have

COROLLARY 3.2. *Let $\bar X^*$ be a dual couple of Banach lattices and let $\varphi \in \mathscr{P}^*$. Then*

$$\langle \bar X^*, \varphi\rangle = \varphi(\bar X^*) = \langle \bar X^*\rangle^\varphi.$$

Proof. Since $\varphi(\bar X^*)^\sim = \varphi(\bar X^*)$, this is a consequence of (3.2) applied to $\bar X$ with φ^*, Lemma 1.6, (1.7) and Theorem 2.1, (2.2).

The proof of Theorem 3.1 is based on the following two lemmas. The first one is due to Lozanovskii [15].

LEMMA 3.3. *Let $\bar{X}$ be a regular couple of Banach lattices and let $\varphi \in \mathscr{P}^*$. Then*

$$(\varphi(\bar{X})^0)^* = \varphi^*(\bar{X}^*).$$

Proof. See [15], Theorem 1.

We further require the following result due to Aronszajn–Gagliardo [1].

LEMMA 3.4. *Let A be a dense Banach subspace of the Banach space E. Then, with the usual canonical identifications, we have*

$$A^\sim = E \cap ((A^*)^0)^*.$$

Proof. See [1], Section 10.V.

We now have the

Proof of Theorem 3.1. From (1.6) we infer that we have the dense inclusions $\langle \bar{X} \rangle_\varphi \subseteq (\langle \bar{X} \rangle^\varphi)^0$. Consequently to establish (3.1) it is sufficient in view of Hahn–Banach to show that the two Banach spaces $\langle \bar{X} \rangle_\varphi$ and $(\langle \bar{X} \rangle^\varphi)^0$ have the same dual. By Lemma 1.6 we have $((\langle \bar{X} \rangle^\varphi)^0)^* = \langle \bar{X}^*, \varphi^* \rangle$. On the other hand, Theorem 2.1 implies that $\langle \bar{X} \rangle_\varphi = \varphi(\bar{X})^0$ and hence by Lemma 3.3 we have $\langle \bar{X} \rangle_\varphi^* = \varphi^*(\bar{X}^*)$. Since $\varphi^*(\bar{X}^*)^\sim = \varphi^*(\bar{X}^*)$, Theorem 2.1 asserts that $\langle \bar{X}^*, \varphi^* \rangle = \varphi^*(\bar{X}^*)$, implying now that $\langle \bar{X} \rangle_\varphi^* = ((\langle \bar{X} \rangle^\varphi)^0)^*$. Hence (3.1) is proved.

Let us now prove (3.2). Take $x_0 \in \langle \bar{X} \rangle^\varphi$ with $\|x_0\|_{\langle \bar{X} \rangle^\varphi} = 1$. We wish to show that $x_0 \in \langle \bar{X} \rangle_\varphi^\sim$. In order to do this we consider finite rank operators $T\colon \bar{X} \to \bar{p}_1$ of the form

$$T(\cdot) = \sum_{i=-N}^{N} \langle \cdot, x_i^* \rangle e_{-i}.$$

Here $x_i^* \in \Delta(\bar{X}^*)$ and e_i, $i \in \mathbf{Z}$, denote the canonical unit vectors in l_1. We observe that T has norms less than one if and only if

$$\sum_{i=-N}^{N} 2^{ij} |\langle x, x_i^* \rangle| \leqslant \|x\|_{X_j}$$

for $j = 0, 1$. Hence, whenever $|\varepsilon_i| \leqslant 1$, $-N \leqslant i \leqslant N$, we have

$$\left|\left\langle x, \sum_{i=-N}^{N} \varepsilon_i 2^{ij} x_i^* \right\rangle\right| \leqslant \|x\|_{X_j},$$

implying that

$$\left\| \sum_{i=-N}^{N} \varepsilon_i 2^{ij} x_i^* \right\|_{X_j^*} \leqslant 1. \tag{3.3}$$

Since $T(x_0) \in l_1(1/\varphi(2^i))$, we similarly have

$$\left|\left\langle x_0, \sum_{i=-N}^{N} \varepsilon_i x_i^*/\varphi(2^i) \right\rangle\right| \leqslant 1 \tag{3.4}$$

whenever $|\varepsilon_i| \leqslant 1$, $|i| \leqslant N$. Now fix a sequence $(\varepsilon_i^0)_{-N}^N$ of numbers with modulus less than one and put

$$x^* = \sum_{i=-N}^{N} \varepsilon_i^0 x_i^*/\varphi(2^{-i}).$$

Then (3.3) asserts that $x^* \in \langle \bar{X}^* \rangle_{\varphi^*}$ and, in fact, $\|x^*\|_{\langle \bar{X}^* \rangle_{\varphi^*}} < 1$. Further, (3.4) implies that $|\langle x_0, x^* \rangle| \leqslant 1$ for this particular x^*. But the argument may be reversed. Indeed, if $x^* \in \Delta(\bar{X}^*)$ we use Lemma 1.5 to find N, $x_i^* \in \Delta(\bar{X}^*)$, $|i| \leqslant N$, such that

$$x^* = \sum_{i=-N}^{N} x_i^*/\varphi(2^{-i}),$$

$$\left\| \sum_{i=-N}^{N} \varepsilon_i 2^{ij} x_i^* \right\|_{X_j} \leqslant 4 \sup_{|i| \leqslant N} |\varepsilon_i| \, \|x^*\|_{\langle \bar{X}^* \rangle_{\varphi^*}}.$$

Hence the associated operator T maps $\bar{X} \to \bar{l}_1$ with norms less than $4\|x^*\|_{\langle \bar{X}^* \rangle_{\varphi^*}}$. By the above argument we may now conclude that

$$|\langle x_0, x^* \rangle| \leqslant 4\|x^*\|_{\langle \bar{X}^* \rangle_{\varphi^*}} \tag{3.5}$$

for every $x^* \in \Delta(\bar{X}^*)$.

Next we apply Lemma 3.4 with $A = \langle \bar{X} \rangle_\varphi$ and $E = \Sigma(\bar{X})$. Since Lemma 3.3 and (1.7) implies that $(\langle \bar{X} \rangle_\varphi^*)^0 = \varphi^*(\bar{X}^*)^0 = \langle \bar{X}^* \rangle_{\varphi^*}$, we infer that

$$\langle \bar{X} \rangle_\varphi^\sim = \Sigma(\bar{X}) \cap \langle \bar{X}^* \rangle_{\varphi^*}^*.$$

Now (3.5) implies that $x_0 \in \langle \bar{X}^* \rangle_{\varphi^*}^*$ and since trivially $x_0 \in \Sigma(\bar{X})$, it follows that $x_0 \in \langle \bar{X} \rangle_\varphi^\sim$. Hence the inclusions $\langle \bar{X} \rangle^\varphi \subseteq \langle \bar{X} \rangle_\varphi^\sim \subseteq \varphi(\bar{X})^\sim$ are proved. To prove the converse note that trivially $(\langle \bar{X} \rangle^\varphi)^\sim = \langle \bar{X} \rangle^\varphi$. Hence (1.6) implies that $\langle \bar{X} \rangle_\varphi^\sim \subseteq \varphi(\bar{X})^\sim \subseteq \langle \bar{X} \rangle^\varphi$. This completes the proof of Theorem 3.1.

In [20] Ovchinnikov showed that the equality $\varphi(\bar{X}) = \langle \bar{X} \rangle^\varphi$ holds whenever both X_0 and X_1 have the Fatou property. He further coined the term "weakly tame" for a Banach couple satisfying $\langle \bar{A} \rangle_\varphi = \langle \bar{A} \rangle^\varphi$ for every $\varphi \in \mathscr{P}^*$ (see also [22]). Hence Theorem 3.1 and Corollary 3.2 state that large classes of couples of Banach lattices are weakly tame.

When $\varphi(t) = t^\theta$, $0 < \theta < 1$, it is known that $\varphi(\bar{X})^0 = [\bar{X}]_\theta$ for any couple of Banach lattices (see [27] or [3]). Since trivially $\langle \bar{X} \rangle_\theta = \langle \bar{X}^0 \rangle_\theta$, Theorem 3.1 implies that we also have $\langle \bar{X} \rangle_\theta = [\bar{X}]_\theta$ for any couple of Banach lattices. Now Theorem 7 of [11] and Theorem 1.C.4 of [13] imply that if either X_0 or X_1 is weakly sequentially complete then

$$\langle \bar{X} \rangle_\theta = \langle \bar{X}, \theta \rangle = [\bar{X}]_\theta = X_0^{1-\theta} X_1^\theta.$$

It is not known if the restriction in Theorem 3.1 to regular couples is necessary. Note, however, that trivially $\langle \bar{X} \rangle_\varphi^\sim = \langle \bar{X}^0 \rangle^\varphi$ and $(\langle \bar{X}^0 \rangle^\varphi)^0 = \langle \bar{X} \rangle_\varphi$ since we have $\langle \bar{X} \rangle_\varphi = \langle \bar{X}^0 \rangle_\varphi$.

Theorem 3.1 also implies reiteration theorems for our interpolation methods when restricted to regular couples of Banach lattices. For instance let us prove

THEOREM 3.5. *Let $\bar{X}$ be a regular couple of Banach lattices and take $\varphi_0, \varphi_1, \eta \in \mathscr{P}^*$. Put $\varphi = \eta(\varphi_0, \varphi_1)$. Then*

$$\langle \langle \bar{X} \rangle^{\varphi_0}, \langle \bar{X} \rangle^{\varphi_1} \rangle^\eta = \langle \bar{X} \rangle^\varphi.$$

Proof. It is known (see [11], Theorem 13) that we have

$$X = \langle \langle \bar{X} \rangle^{\varphi_0}, \langle \bar{X} \rangle^{\varphi_1} \rangle^\eta \subseteq \langle \bar{X} \rangle^\varphi,$$

$$X \supseteq \langle \langle \bar{X} \rangle^{\varphi_0}, \langle \bar{X} \rangle^{\varphi_1} \rangle^\eta \supseteq \langle \bar{X} \rangle_\eta.$$

Since Theorem 3.1 asserts that $\langle \bar{X} \rangle_\varphi^\sim = \langle \bar{X} \rangle^\varphi$, it follows that $X^\sim = \langle \bar{X} \rangle^\varphi$. As trivially $X^\sim = X$, the theorem is proved.

4. On the Gagliardo closure of $[\bar{X}]_\theta$. Denote by $\mathscr{F}L^\infty(\omega)$ the space of all sequences $\gamma = (\gamma_j)_{-\infty}^\infty$ such that for some function $h \in L^\infty(0, 2\pi)$ we have

$$\gamma_j \omega_j = (2\pi)^{-1} \int_0^{2\pi} \exp(ij\theta) h(\theta) d\theta, \quad j \in \mathbf{Z}.$$

Put $\|\gamma\|_{\mathscr{F}L^\infty(\omega)} = \|h\|_{L^\infty(0,2\pi)}$. Take $\theta \in (0, 1)$ and put

$$\overline{\mathscr{F}L^\infty} = (\mathscr{F}L^\infty(2^{j\theta}), \mathscr{F}L^\infty(2^{-j(1-\theta)})).$$

Let $\bar{X}$ be a couple of Banach spaces. The interpolation space $[\bar{X}, \theta]$ then consists of all $x \in \Sigma(\bar{X})$ such that the norm

$$\|x\|_{[\bar{X},\theta]} = \sup\{\|T(x)\|_{\mathscr{F}L^\infty}: T: \bar{X} \to \overline{\mathscr{F}L^\infty}, \|T\|_{\bar{X}, \overline{\mathscr{F}L^\infty}} = 1\}$$

is finite. This interpolation method was introduced by Janson in [11], who showed among other things that

$$[\bar{X}]_\theta = ([\bar{X}, \theta])^0.$$

In analogy with Theorem 3.1 we now have the following

THEOREM 4.1. *Let $\bar{X}$ be a regular Banach couple and let $\theta \in (0, 1)$. Then*

$$[\bar{X}]_\theta^\sim = [\bar{X}, \theta].$$

Since the proof is completely analogous to the proof of (3.2), we will leave the details to the reader. Let us only remark that it is mainly based on the discrete definition of $[\bar{X}]_\theta$ due to Cwikel [7]. One also needs an analogue of Lemma 1.5 for the complex method. This may be found in [11] (see the proof of Theorem 23 there). With these tools one analyzes finite rank operators $T: \bar{X} \to \overline{\mathscr{F}L^\infty}$ in a way similar to that followed in the proof of Theorem 3.1.

5. Applications. The main interest of the quasi-Banach lattice $\varphi(\bar{X})$ is that one is often able to compute it. Let us give some examples of this and also illustrate Theorem 2.1. We begin by treating interpolation of couples of quasi-normed Orlicz spaces. Previous treatments of this topic may be found in [10], [20], [21], [22].

EXAMPLE 5.1. Let Φ be an increasing function on $\mathbf{R}_+$ such that $\lim_{t \to 0} \Phi(t) = \Phi(0)$. The Orlicz space L_Φ^* is defined to be the subspace of $L^0(\mu)$ consisting of all $f \in L^0(\mu)$ such that

$$\int_\Omega \Phi(|f|/\lambda) d\mu < \infty$$

for some $\lambda < \infty$. Clearly L_Φ^* is a vector space and a topology is obtained by taking as neighborhoods of 0 the sets

$$U_{\alpha,\beta} = \{f: f \in L^0(\mu) \text{ and } \int_\Omega \Phi(|f|/\alpha) d\mu < \beta\}$$

with $0 < \alpha, \beta < \infty$. Let us further assume that this topology is locally bounded, i.e. L_Φ^* is a quasi-Banach space. By a theorem of Matuszewska–Orlicz [17] we may then assume that Φ is of the form $\omega(t^q)$ where $0 < q \leqslant 1$ and ω is a convex function. Hence with $p = 1/q$ we have isometrically $(L_\Phi^*)^{(p)} = L_\omega^*$ and it follows that L_Φ^* is of type $\mathscr{C}$. The theory of Section 2 thus applies to any couple $(L_{\Phi_0}^*, L_{\Phi_1}^*)$ of quasi-normed Orlicz spaces. We now claim that

$$\varphi(L_{\Phi_0}^*, L_{\Phi_1}^*) = L_\Phi^*$$

for $\varphi \in \mathscr{P}$, where $\Phi^{-1}(t) = \varphi(\Phi_0^{-1}(t), \Phi_1^{-1}(t))$. Indeed, if $f \in L_\Phi^*$ and $\int_\Omega \Phi(|f|/\lambda) d\mu \leqslant 1$ then with $g = \Phi(|f|/\lambda)$ we have $|f| = \lambda\varphi(\Phi_0^{-1}(g), \Phi_1^{-1}(g))$. Since $\|\Phi_i^{-1}(g)\|_{L_{\Phi_i}^*} \leqslant 1$, it follows that $f \in \varphi(L_{\Phi_0}^*, L_{\Phi_1}^*)$. Assume conversely that $|f| \leqslant \lambda\varphi(|f_0|, |f_1|)$ where $\|f_i\|_{L_{\Phi_i}^*} \leqslant 1$, $i = 0, 1$. Put $g_i = \Phi(|f_i|)$ and define h by $h = \max(g_0, g_1)$. We note that

$$\Phi(|f|/\lambda) \leqslant \Phi\big(\varphi(\Phi_0^{-1}(g_0), \Phi_1^{-1}(g_1))\big) \leqslant \Phi\big(\varphi(\Phi_0^{-1}(h), \Phi_1^{-1}(h))\big) = h.$$

Hence

$$\int_\Omega \Phi(|f|/\lambda) d\mu \leqslant \int_\Omega h\, d\mu \leqslant 2$$

and we conclude that $f \in L_\Phi^*$.

Since trivially $L_\Phi^{*\sim} = L_\Phi^*$, Theorem 2.1 now implies that

$$L_\Phi^* = \langle L_{\Phi_0}^*, L_{\Phi_1}^*, \Phi \rangle,$$

i.e. L_Φ^* is an interpolation space with respect to $(L_{\Phi_0}^*, L_{\Phi_1}^*)$. Let us remark that this result has previously been obtained by Ovchinnikov [21] (see also [20], [22] as well as [10]).

EXAMPLE 5.2. Other examples of quasi-Banach lattices of type $\mathscr{C}$ are weak L^p spaces, $0 < p < \infty$, which we denote by $L^{p,\infty}$. Let us show that, for instance, $L^{1,\infty}$ is of type $\mathscr{C}$. If $q < 1$ it is known (see [4]) that the quasi-norm

$$\|f\| = \sup_{t>0} t^{q-1} \Big(\int_0^{t^q} f^*(s)^q \, ds\Big)^{1/q}$$

is an equivalent lattice quasi-norm on $L^{1,\infty}$. With this quasi-norm and $p = 1/q$ it follows that $(L^{1,\infty})^{(p)}$ coincides isometrically with the Banach lattice $L^{p,\infty}$. Hence $L^{1,\infty}$ is of type $\mathscr{C}$.

EXAMPLE 5.3 (Interpolation of weighted L^p spaces). If ω is a positive measurable function on a σ-finite measure space (Ω, Σ, μ), we denote by $L_\omega^p = L_\omega^p(\mu)$, $0 < p \leqslant \infty$, the space of all $f \in L^0$ rendering the quasi-norm

$$\Big(\int_\Omega |f\omega|^p \, d\mu\Big)^{1/p}$$

finite. Let $(L_{\omega_0}^{p_0}, L_{\omega_1}^{p_1})$ be a couple of weighted L^p spaces. In the Banach case, i.e. $1 \leqslant p_0, p_1 \leqslant \infty$, Stein–Weiss [29] showed essentially that

$$[L_{\omega_0}^{p_0}, L_{\omega_1}^{p_1}]^\theta = (L_{\omega_0}^{p_0})^{1-\theta} (L_{\omega_1}^{p_1})^\theta = L_\omega^p$$

where $1/p = 1 - \theta/p_0 + \theta/p_1$ and $\omega = \omega_0^{1-\theta} \omega_1^\theta$ (see [4]). In [9] Gustavsson extended this result to the full range $0 < p_0, p_1 \leqslant \infty$ of parameters using the $\pm$ method. Let us here treat the case of general $\varphi \in \mathscr{P}^*$.

Take $\varphi \in \mathscr{P}^*$ and $0 < p_0 < p_1 \leqslant \infty$. We claim that $f \in \langle L_{\omega_0}^{p_0}, L_{\omega_1}^{p_1}, \varphi \rangle$ if and only if

$$\int_\Omega \psi\big((\omega_1^{1/p_0} \omega_0^{-1/p_1})^q |f|/\lambda\big)(\omega_0/\omega_1)^q \, d\mu \leqslant 1 \tag{5.1}$$

for some $\lambda < \infty$. Here $1/q = 1/p_0 - 1/p_1$ and $\psi^{-1}(t) = t^{1/p_0} \varphi(t^{-1/q})$.

In order to show this put $\tau = (\omega_1^{1/p_0} \omega_0^{-1/p_1})^q$, $\sigma = (\omega_0/\tau)^{p_0}$ and $\nu = \sigma\mu$. For $f \in L^0(\mu)$ put $T(f) = \tau f$ and note that

$$\|T(f)\|_{L^{p_i}(\nu)} = \|f\|_{L_{\omega_i}^{p_i}}$$

for $i = 0, 1$. It follows that

$$T(\langle L_{\omega_0}^{p_0}, L_{\omega_1}^{p_1}, \varphi \rangle) = \langle L^{p_0}(\nu), L^{p_1}(\nu), \varphi \rangle.$$

The latter space may be computed by using Example 5.1. In fact, one sees that $F \in \langle L^{p_0}(\nu), L^{p_1}(\nu), \varphi \rangle$ if and only if

$$\int_\Omega \psi(|F|/\lambda) \, d\nu \leqslant 1$$

for some $\lambda < \infty$. Hence, with $F = \tau f$, $f \in L^0$, $d\nu = \sigma d\mu$, this expression equals (5.1) as claimed.

Let us remark that this proof is based on ideas found in Stein–Weiss [29] (see also Lizorkin [14], in particular Besov's comments on p. 231). In the special case $\varphi(t) = t^\theta$, $0 < \theta < 1$, (5.1) simplifies to the assertion that $f \in \langle L_{\omega_0}^{p_0}, L_{\omega_1}^{p_1}, \theta \rangle$ if and only if $f \in L_\omega^p$, with p and ω as above.

One may note that the results in Section 2 and Section 3 do not solely apply to couples of Banach lattices. In fact, if a Banach couple $\bar{X}$ is isomorphic to a complemented subcouple of a couple of Banach lattices one may extract from Theorem 2.1 and Theorem 3.1 information concerning e.g. $\langle \bar{X}, \varphi \rangle$. Let us give an example of this.

EXAMPLE 5.4 (Bernstein–Szasz type inequalities). This example will be concerned with estimating the Fourier transform of functions from appropriate Besov spaces. Pick $\Phi \in \mathscr{S}(\boldsymbol{R}^n)$ such that

$$\operatorname{supp} \Phi = \{\xi \in \boldsymbol{R}^n \colon \tfrac{1}{2} \leqslant |\xi| \leqslant 2\}, \qquad \hat{\Phi}(\xi) > 0 \text{ if } \tfrac{1}{2} < |\xi| < 2$$

and

$$\sum_{i=-\infty}^{\infty} \hat{\Phi}(2^{-i}\xi) = 1 \quad \text{if } \xi \neq 0.$$

Put $\Phi_i(\cdot) = 2^{in} \Phi(2^i \cdot)$, $i \in \boldsymbol{Z}$. Let $s \in \boldsymbol{R}$, $1 \leqslant q \leqslant \infty$. The homogeneous Besov space $\dot{B}_2^{s,q}$ is defined to be the space of all $f \in \mathscr{S}'(\boldsymbol{R}^n)$ such that $\|f\|_{s,q} < \infty$ where

$$\|f\|_{s,q} = \Big(\sum_{i=-\infty}^{\infty} (2^{is} \|\Phi_i * f\|_{L^2})^q\Big)^{1/q}.$$

For integer $k > 0$ and $f \in \mathscr{S}(\boldsymbol{R}^n)$ put

$$\omega_2^k(t, f) = \sup_{|h| \leqslant t} \|\Delta_h^k f\|_{L^2}$$

(kth order modulus of continuity). Let $s > 0$, $1 \leqslant q < \infty$ and pick $k > s$. As is well known, $\dot{B}_2^{s,q}$ coincides with the completion of $\mathscr{S}(\boldsymbol{R}^n)$ in the norm

$$\Big(\int_0^\infty (\omega_2^k(t, f)/t^s)^q \, dt/t\Big)^{1/q}$$

(see [4], p.144). We further recall that $\dot{B}_2^{s,q}$ is isomorphic to a complemented subspace of $l_q^s(L^2)$. Indeed, if for $f \in \mathscr{S}(\boldsymbol{R}^n)$ we put $T(f) = (\Phi_i * f)_{-\infty}^{\infty}$ then

$T: \dot{B}_2^{s,q} \to l_q^s(L^2)$. For $f = (f_i)_{-\infty}^{\infty} \in \mathscr{S}(\boldsymbol{R}^n)$ we similarly define $S: l_q^s(L^2) \to \dot{B}_2^{s,q}$ by

$$S(f) = \sum_i \sum_{|j| \leqslant 1} \Phi_{i+j} * f_i.$$

Then $ST = \mathrm{id}$ and the claim is proved (see also [4], p. 151).

For $f \in \mathscr{S}'(\boldsymbol{R}^n)$ denote by $\mathscr{F}f = \hat{f}$ the Fourier transform of f. Now, by Parseval's formula and Bernstein's theorem,

$$\mathscr{F}: (\dot{B}_2^{0,2}, \dot{B}_2^{n/2,1}) \to (L^2, L^1).$$

Let us interpolate these estimates using the $\pm$ method. We claim that the following holds: there exist two positive constants c_1 and c_2 such that if $f \in \mathscr{S}(\boldsymbol{R}^n)$ and

$$\int_0^\infty \psi(\omega_2^k(t, f)\, t^{n/2}/\lambda)\, dt/t^{n+1} = c_1$$

for some $\lambda < \infty$ then

$$\int_{\boldsymbol{R}^n} \psi(|\hat{f}(\xi)|/c_2 \lambda)\, d\xi \leqslant 1.$$

Here $k > n/2$, $\psi^{-1}(t) = t^{1/2} \varphi(t^{-1/2})$ and $\varphi \in \mathscr{P}^*$.

In order to show this we need to compute $\langle \dot{B}_2^{0,2}, \dot{B}_2^{n/2,1}, \varphi \rangle$. Using the above two operators T and S we see that this space is isomorphic to a complemented subspace of $\langle l_2(L^2), l_1^{n/2}(L^2), \varphi \rangle$, which in turn clearly equals $\langle l_2, l_1^{n/2}, \varphi \rangle (L^2)$. In Example 5.3 we computed $\langle l_2, l_1^{n/2}, \varphi \rangle$ for $\varphi \in \mathscr{P}^*$. If we fix $\mu > 1$ we infer that $(f_i)_{-\infty}^{\infty} \in \langle l_2(L^2), l_1^{n/2}(L^2), \varphi \rangle$ if and only if

$$\sum_{k=-\infty}^{\infty} \sum_{\mu^{k-1} < 2^{in/2} \leqslant \mu^k} \mu^{2i} \psi(\mu^{-i} \|f_i\|_{L^2}/\lambda) \leqslant 1$$

for some constant $\lambda < \infty$. If we take $\mu = 2^{n/2}$ and $f_i = \Phi_i * f$ we infer that $f \in \langle \dot{B}_2^{0,2}, \dot{B}_2^{n/2,1}, \varphi \rangle$ if and only if

$$\sum_{i=-\infty}^{\infty} 2^{in} \psi(2^{-in/2} \|\Phi_i * f\|_{L^2}/\lambda) \leqslant 1 \tag{5.2}$$

for some $\lambda < \infty$. Take $k > n/2$ and observe that $\|\Phi_i * f\|_{L^2} \leqslant C\omega_2^k(2^{-ik}, f)$ (see [4], p. 161). This estimate inserted in (5.2) now implies that if

$$\int_0^\infty \psi(t^{n/2} \omega_2^k(t, f)/\lambda)\, dt/t^{n+1} \leqslant c_1$$

then $f \in \langle \dot{B}_2^{0,2}, \dot{B}_2^{n/2,1}, \varphi \rangle$. On the other hand, if $\hat{f} \in \langle L^2, L^1, \varphi \rangle$ then by Example 5.3 we have

$$\int_{\boldsymbol{R}^n} \psi(|\hat{f}(\xi)|/\lambda)\, d\xi \leqslant 1$$

for some $\lambda < \infty$. This completes the proof of the claim.

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On the splitting of twisted sums, and the three space problem for local convexity

by

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Abstract. We characterize all pairs of topological vector spaces (tvs) (Y, Z) such that Y is semimetrizable (resp. locally bounded) and for every relatively open and continuous map q with $\ker q \simeq Y$ and $\operatorname{im} q \simeq Z$ there is a section (resp. a homogeneous section) continuous at zero (i.e., a map s with $q \circ s = \mathrm{id}$).

A twisted sum of two tvs Y and Z is a tvs X with a subspace $Y_0 \simeq Y$ such that $X/Y_0 \simeq Z$. All twisted sums of an arbitrary pair of tvs are described.

A tvs Z belongs to $S(Y)$ (resp. to the class of TSC-spaces) iff every twisted sum of tvs Y and Z is a direct sum (resp. is locally convex whenever so is Y). We examine hereditary properties of the classes $S(Y)$ and TSC-spaces. As an application we get: (1) all locally convex spaces (lcs) with a weak topology belong to $S(Y)$ for the one-dimensional Y (i.e., they are $\mathscr{K}$-spaces [15]); (2) all nuclear lcs and all metrizable locally convex $\mathscr{K}$-spaces are TSC-spaces. The classes of TSC-spaces and locally pseudoconvex spaces are closed under twisted sums.

We remove the assumption of local boundedness from the following results: (1) Kalton's [14] description of all twisted sums of lcs; (2) p- and q-convexity of Y and Z resp. $(0 < p \neq q \leqslant 1)$ implies $\min(p, q)$-convexity of their twisted sum.

0. Introduction. A *twisted sum* of two topological vector spaces (tvs) Y and Z is a diagram of tvs and linear relatively open continuous mappings:

$$(*) \qquad 0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0,$$

such that $(*)$ is a short exact sequence, i.e., $j(Y) = \ker q$. Sometimes we will simply say that X itself is a twisted sum of Y and Z.

There are two main problems concerning twisted sums.

The first one is the so-called three space problem. We say that a property (P) is a *three space property* if every twisted sum of Y and Z has (P) whenever Y and Z have (P). Now, the question is: what properties are three space properties.

We say that the twisted sum $(*)$ *splits* if there is a continuous linear mapping $T: Z \to X$ for which $q \circ T = \mathrm{id}_Z$. The second main question concerning twisted sums is the problem what pairs (Y, Z) of tvs have only splitting twisted sums. There are numerous papers related to these problems: [5], [6], [8]–[14], [16], [21], [25]–[32], [34], [37], [40] (the three space problem) and [1], [2], [10], [13]–[15], [17]–[21], [24], [28], [29], [35],