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Holomorphic approximation in infinite-dimensional Riemann domains

by

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Abstract. We establish a sharp version of the classical Oka–Weil theorem for pseudoconvex Riemann domains over Fréchet spaces with basis. As an application of this result we show that the compact-ported topology introduced by Nachbin coincides with the compact-open topology on the space of all holomorphic functions on an arbitrary open subset of a Fréchet–Schwartz space with basis.

Introduction. Let Ω be a domain of holomorphy in C^n and let K be a holomorphically convex compact subset of Ω. A classical result of Oka [28], which extends earlier results of Weil [36], [37] and Oka himself [27], states that every function which is holomorphic on a neighborhood of K can be uniformly approximated on K by functions which are holomorphic on the whole of Ω. The Oka–Weil theorem, as this result is currently called, has been generalized in various directions by different authors. Among them, Noverraz [26] and Schottenloher [33] have obtained infinite-dimensional versions of this theorem. Noverraz considered the case of Banach spaces with basis, whereas Schottenloher dealt with the case of Fréchet spaces with basis. Schottenloher considered, moreover, the case of Riemann domains, whereas Noverraz restricted himself to the case of one-sheeted domains.

In this paper we improve the above-mentioned results by giving a sharp version of the Oka–Weil theorem for pseudoconvex Riemann domains over Fréchet spaces with basis. We show the existence of an open set V containing K such that the given function can be uniformly approximated on compact subsets of V by a sequence of functions which are holomorphic on the whole of Ω. This is essentially the content of Theorem 5.1. The fact that we can approximate by sequences, and not just by nets, is the crucial point, since the open set V is no longer hemicompact, as in the finite-dimensional case. Furthermore, if we restrict ourselves to the case of Fréchet–Schwartz spaces with basis, then we can get even uniform approximation on a suitable neighborhood of K. This is the essential content of Theorem 7.5.

As an application of these approximation theorems, we show that the compact-ported topology τ_ω introduced by Nachbin [24] coincides with the compact-open topology τ_0 on the space of all holomorphic functions on Ω

whenever Ω is an arbitrary Riemann domain over a Fréchet–Schwartz space with basis. This is the content of Theorem 8.2, which extends earlier results of Barroso [3], Schottenloher [34], Barroso and Nachbin [4], Boland and Dineen [7], Meise [18] and the author [22], [23]. Actually we use Theorem 7.5 to prove Theorem 8.2 for pseudoconvex Riemann domains, but then the result for arbitrary Riemann domains follows easily by passing to the envelope of holomorphy. Such an approach has already been followed by Schottenloher [34] in his note on domains over C^N. It is clear why it is important to consider arbitrary Riemann domains, even if one is primarily interested in one-sheeted domains.

Now we describe briefly the organization of this paper. In Section 1 we fix some notation and terminology and collect a few results that will often be used throughout the paper. The first important result is Theorem 2.4, a global approximation theorem for Fréchet spaces with basis which will be the key to the proof of Theorem 5.1. Sections 2 and 3 are devoted to the proof of Theorem 2.4. In Section 2 the theorem is proved when the spaces under consideration have a continuous norm, whereas in Section 3 the theorem is proved for the spaces without a continuous norm. A result of Matyszczyk [17] shows that these two cases are essentially different. The main result in Section 4 is Theorem 4.1. This theorem shows the abundance of certain special open sets and is the key link between Theorems 2.4 and 5.1. In Section 5 we establish Theorem 5.1, a sharp theorem of the Oka–Weil type which we have already mentioned. This theorem will play a key role in the proof of Theorem 8.2. In Section 6 we collect a few results about spaces of holomorphic germs which have been established elsewhere and which will be very useful in subsequent sections. In Section 7 we establish Theorem 7.5, an even sharper theorem of Oka–Weil type, which we have also mentioned before. Theorem 7.5 is derived from Theorem 5.1 with the aid of a topological lemma from Section 6. Theorem 7.5 is used to establish the identity of the topologies τ_0 and τ_ω for pseudoconvex Riemann domains over Fréchet–Schwartz spaces with basis. In Theorem 8.2 the hypothesis of pseudoconvexity is removed by passing to the envelope of holomorphy, as we mentioned before. Finally, with the help of a result of Pełczyński [29], the hypothesis of a basis may be replaced by the hypothesis of the bounded approximation property in each of our main results. We end the paper with a few remarks indicating how this can be accomplished.

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1. Notation and preliminaries. Let E denote a locally convex space, here always assumed to be complex and Hausdorff. Let $\mathrm{cs}(E)$ denote the set of all nontrivial continuous seminorms on E. Let $\mathscr{V}(E)$ denote the set of all open, convex, balanced neighborhoods of zero in E.

Let (Ω, φ) be a *Riemann domain* over E, i.e. Ω is a Hausdorff topological space and $\varphi\colon \Omega \to E$ is a local homeomorphism. For short, we often write Ω instead of (Ω, φ). If (Σ, ψ) is another Riemann domain over E, then a *morphism* is a continuous mapping $f\colon \Omega \to \Sigma$ such that $\psi \circ f = \varphi$. A *section* of Ω is a continuous mapping $\sigma\colon A \to \Omega$, with $A \subset E$, such that $\varphi \circ \sigma = \mathrm{id}$ on A. For $X \subset \Omega$ and $V \in \mathscr{V}(E)$ we write $X+V \subset \Omega$ if for each $x \in X$ there is a section $\sigma\colon \varphi(x)+V \to \Omega$ such that $\sigma \circ \varphi(x) = x$. Then we define $x+a = \sigma(\varphi(x)+a)$ for every $x \in X$ and $a \in V$.

A mapping $f\colon \Omega \to F$, where F is a locally convex space, is said to be *holomorphic* if for each $x \in \Omega$ there is a section $\sigma\colon \varphi(x)+V \to \Omega$, with $V \in \mathscr{V}(E)$, such that $f \circ \sigma$ is holomorphic on $\varphi(x)+V$. Let $\mathscr{H}(\Omega, F)$ denote the vector space of all holomorphic mappings $f\colon \Omega \to F$. When $F = C$ we write $\mathscr{H}(\Omega)$ instead of $\mathscr{H}(\Omega, C)$. Let τ_0 denote the compact-open topology on $\mathscr{H}(\Omega, F)$. If (Σ, ψ) is a Riemann domain over F then a mapping $f\colon \Omega \to \Sigma$ is said to be *holomorphic* if $\psi \circ f \in \mathscr{H}(\Omega, F)$.

Let $f \in \mathscr{H}(\Omega)$. Then for each $x \in \Omega$ there are continuous n-homogeneous polynomials $P^n f(x)\colon E \to C$ and $V \in \mathscr{V}(E)$ such that $x+V \in \Omega$ and

$$f(x+a) = \sum_{n=0}^{\infty} P^n f(x)(a)$$

uniformly for $a \in V$. If we define $P_a^n f(x) = P^n f(x)(a)$ for $a \in E$ and $x \in \Omega$ then $P_a^n f \in \mathscr{H}(\Omega)$ for every $n \in N$ and $a \in E$. If $x \in \Omega$ and $V \in \mathscr{V}(E)$ are such that $x+V \subset \Omega$ then we have the Cauchy inequalities

$$|P_a^n f(x)| \leqslant \sup\{|f(x+\lambda a)|\colon \lambda \in C, |\lambda| < 1\}.$$

For any $X \subset \Omega$ we write $\|f\|_X = \sup\{|f(x)|\colon x \in X\}$. For any $n \in N$, $X \subset \Omega$ and $A \subset E$ we write

$$\|P^n f\|_{X,A} = \sup\{|P_a^n f(x)|\colon x \in X,\ a \in A\}.$$

A function $f\colon \Omega \to [-\infty, +\infty)$ is said to be *plurisubharmonic* if for each $x \in \Omega$ there is a section $\sigma\colon \varphi(x)+V \to \Omega$, with $V \in \mathscr{V}(E)$, such that $f \circ \sigma$ is plurisubharmonic on $\varphi(x)+V$. Let $\mathrm{Ps}(\Omega)$ (resp. $\mathrm{Psc}(\Omega)$) denote the collection of all plurisubharmonic (resp. plurisubharmonic and continuous) functions on Ω.

For $X \subset \Omega$ and $\mathscr{F} \subset \mathscr{H}(\Omega)$ we define

$$\hat{X}_{\mathscr{F}} = \{y \in \Omega\colon |f(y)| \leqslant \|f\|_X \text{ for every } f \in \mathscr{F}\}.$$

Likewise, for $X \subset \Omega$ and $\mathscr{F} \subset \mathrm{Ps}(\Omega)$ we define

$$\hat{X}_{\mathscr{F}} = \{y \in \Omega\colon f(y) \leqslant \sup_{x \in X} f(x) \text{ for every } f \in \mathscr{F}\}.$$

Then $X \subset \hat{X}_{\mathrm{Ps}(\Omega)} \subset \hat{X}_{\mathrm{Psc}(\Omega)} \subset \hat{X}_{\mathscr{H}(\Omega)}$ for every $X \subset \Omega$.

Next we define the distance functions d^α_Ω: $\Omega \to [0, +\infty]$, for $\alpha \in \mathrm{cs}(E)$, and δ_Ω: $\Omega \times E \to (0, +\infty]$, as follows:

$$d^\alpha_\Omega(x) = \sup\{r > 0\text{: there is a section } \sigma\colon B^\alpha_E(\varphi(x), r) \to \Omega \text{ with } \sigma \circ \varphi(x) = x\} \cup \{0\},$$

$$\delta_\Omega(x, a) = \sup\{r > 0\text{: there is a section } \sigma\colon D_E(\varphi(x), a, r) \to \Omega \text{ with } \sigma \circ \varphi(x) = x\},$$

where for $\xi, a \in E$ and $r > 0$ we write

$$B^\alpha_E(\xi, r) = \{\xi + b\colon b \in E,\ \alpha(b) < r\},$$

$$D_E(\xi, a, r) = \{\xi + \lambda a\colon \lambda \in \boldsymbol{C},\ |\lambda| < r\}.$$

If $d^\alpha_\Omega(x) > 0$ then for each $r \in (0, d^\alpha_\Omega(x)]$ there is a unique set $B^\alpha_\Omega(x, r) \subset \Omega$ containing x such that φ: $B^\alpha_\Omega(x, r) \to B^\alpha_E(\varphi(x), r)$ is a bijection. Likewise, for each $x \in \Omega$, $a \in E$ and $r \in (0, \delta_\Omega(x, a)]$ there is a unique set $D_\Omega(x, a, r) \subset \Omega$ containing x such that φ: $D_\Omega(x, a, r) \to D_E(\varphi(x), a, r)$ is a bijection. The function d^α_Ω is continuous, the function δ_Ω is lower semicontinuous, and they are related by the formula

$$d^\alpha_\Omega(x) = \inf\{\delta_\Omega(x, a)\colon a \in E,\ \alpha(a) = 1\}.$$

The domain (Ω, φ) is said to be *one-sheeted* if the mapping φ is injective. The domain Ω is said to be *holomorphically separated* if, given $x, y \in \Omega$ with $x \neq y$, one can find $f \in \mathscr{H}(\Omega)$ such that $f(x) \neq f(y)$. The domain Ω is said to be *holomorphically convex* if $\hat{K}_{\mathscr{H}(\Omega)} \subset\subset \Omega$ for every $K \subset\subset \Omega$, where $X \subset\subset Y$ means that X is relatively compact in Y. The domain Ω is said to be *metrically holomorphically convex* if for each $K \subset\subset \Omega$ there exists $\alpha \in \mathrm{cs}(E)$ such that $d^\alpha_\Omega(\hat{K}_{\mathscr{H}(\Omega)}) > 0$, where we define $d^\alpha_\Omega(X) = \inf_{x \in X} d^\alpha_\Omega(x)$ for any set $X \subset \Omega$. Finally, the domain Ω is said to be *pseudoconvex* if the function $-\log \delta_\Omega$ is plurisubharmonic on $\Omega \times E$. The following useful result gives several characterizations of pseudoconvexity; cf. [25, Théorème 2.4.8] or [32, Satz 2.8].

1.1. Proposition. *For a Riemann domain (Ω, φ) over a locally convex space E, the following conditions are equivalent*:

(a) *Ω is pseudoconvex.*

(b) *$d^\alpha_\Omega(\hat{X}_{\mathrm{Ps}(\Omega)}) = d^\alpha_\Omega(X)$ for every $X \subset \Omega$ and $\alpha \in \mathrm{cs}(E)$.*

(c) *For each $K \subset\subset \Omega$ there exists $\alpha \in \mathrm{cs}(E)$ such that $d^\alpha_\Omega(\hat{K}_{\mathrm{Ps}(\Omega)}) > 0$.*

(d) *$\hat{K}_{\mathrm{Ps}(\varphi^{-1}(M))} \subset\subset \varphi^{-1}(M)$ for each finite-dimensional subspace M of E and each $K \subset\subset \varphi^{-1}(M)$.*

(e) *$\varphi^{-1}(M)$ is pseudoconvex for each finite-dimensional subspace M of E.*

The following result will also be useful, and may be proved by adapting the proof of [12, Proposition I.G.2].

1.2. Proposition. *Every connected Riemann domain over a separable Fréchet space is second countable.*

Throughout this paper we shall often use several results from Hörmander's book [13]. We restate these results here in a way more convenient to us.

1.3. Theorem. *Let (Ω, φ) be a pseudoconvex Riemann domain over $\boldsymbol{C}^n$. Then*:

(a) *Ω is holomorphically convex.*

(b) *Ω is holomorphically separated.*

(c) *The equation $\bar{\partial} f = g$ has a solution $f \in C^\infty(\Omega)$ for every $g \in \mathscr{C}^\infty_{(0,1)}(\Omega)$ with $\bar{\partial} g = 0$.*

(d) *Let K be a compact subset of Ω such that $\hat{K}_{\mathscr{H}(\Omega)} = K$. Then each function which is holomorphic on a neighborhood of K can be uniformly approximated on K by functions belonging to $\mathscr{H}(\Omega)$.*

(e) *Let U be an open subset of Ω. Then $\hat{K}_{\mathscr{H}(\Omega)} \subset U$ for every $K \subset\subset U$ if and only if U is pseudoconvex and $\mathscr{H}(\Omega)$ is dense in $(\mathscr{H}(U), \tau_0)$.*

(f) *$\hat{K}_{\mathrm{Ps}(\Omega)} = \hat{K}_{\mathrm{Psc}(\Omega)} = \hat{K}_{\mathscr{H}(\Omega)}$ for every $K \subset\subset \Omega$.*

Proof. (a) follows from [13, Theorem 5.4.6]. Note that Hörmander assumes by definition that Riemann domains are holomorphically separated, but he does not really use this assumption in the proof of [13, Theorem 5.4.6]. (b) follows from [13, Corollary 5.2.12]. (c) follows from [13, Theorem 5.1.6 and Corollary 5.2.6]. (d) follows from [13, Corollary 5.2.9]. To (e) the proof of [13, Theorem 4.3.3] applies. Finally, to prove (f) we adapt the proof of [13, Theorem 4.3.4]. Let U be an open neighborhood of $\hat{K}_{\mathrm{Ps}(\Omega)}$. By [13, Theorem 5.1.6] there is a function $u \in \mathrm{Psc}(\Omega)$ such that $\{z \in \Omega\colon u(z) < c\} \subset\subset \Omega$ for every $c \in \boldsymbol{R}$. Then the proof of [13, Theorem 2.6.11] yields a strictly plurisubharmonic function $v \in \mathscr{C}^\infty(\Omega)$ such that $v < 0$ on K, $v > 0$ on $\Omega \setminus U$ and $\{z \in \Omega\colon v(z) < c\} \subset\subset \Omega$ for every $c \in \boldsymbol{R}$. Let $\Omega_0 = \{z \in \Omega\colon v(z) < 0\}$. Then $\mathscr{H}(\Omega)$ is dense in $(\mathscr{H}(\Omega_0), \tau_0)$ by [13, Theorem 5.2.8]. On the other hand, Ω_0 is obviously pseudoconvex, and thus (e) implies that $\hat{K}_{\mathscr{H}(\Omega)} \subset \Omega_0 \subset U$. The inclusion $\hat{K}_{\mathscr{H}(\Omega)} \subset \hat{K}_{\mathrm{Ps}(\Omega)}$ follows and the proof is complete.

We conclude this preliminary section with a few remarks on Schauder bases. A sequence (e_n) in a Fréchet space E is said to be a *Schauder basis* if every $x \in E$ admits a unique representation as a series $x = \sum_{n=1}^\infty \xi_n(x) e_n$, where the series converges in the ordinary sense for the topology of E. Let E_n denote the subspace generated by $e_1, \ldots, e_n$ and let T_n: $E \to E_n$ denote the canonical projection. Then it follows from the Open Mapping Theorem that the sequence (T_n) is equicontinuous and converges to the identity uniformly on compact sets, and that the space E has a fundamental sequence of continuous seminorms α_j which satisfy the conditions $\alpha_j = \sup_n \alpha_j \circ T_n$. This was essentially known to Banach; see [2, Chapitre VII, § 3].

2. Global approximation in spaces with a continuous norm.

2.1. Definition. Let (Ω, φ) be a Riemann domain over a locally convex space E. An open set $U \subset \Omega$ is said to have the *property* (P) if

$$\hat{K}_{\mathrm{Ps}(\Omega)} \subset U \quad \text{for every } K \subset\subset U.$$

The open set U is said to have the *finite property* (P) if $U \cap \varphi^{-1}(M)$ has the property (P) (with respect to the domain $\varphi^{-1}(M)$) for each finite-dimensional subspace M of E. Clearly the property (P) implies the finite property (P).

2.2. Examples. Let (Ω, φ) be a Riemann domain over a locally convex space E.

(a) If V is a convex open set in E then $U = \varphi^{-1}(V)$ has the property (P).

(b) If $f \in \mathrm{Ps}(\Omega)$ then $U_c = \{x \in \Omega\colon f(x) < c\}$ has the property (P) for every $c \in \boldsymbol{R}$.

(c) More generally, let $f\colon \Omega \to [-\infty, +\infty]$ be an upper semicontinuous function which is the supremum of a family $(f_i) \subset \mathrm{Ps}(\Omega)$. Then $U_c = \{x \in \Omega\colon f(x) < c\}$ has the property (P) for every $c \in \boldsymbol{R}$.

(d) If Ω is pseudoconvex and $\alpha \in \mathrm{cs}(E)$ then it follows from (c) that $U_\varepsilon = \{x \in \Omega\colon d_\Omega^\alpha(x) > \varepsilon\}$ has the property (P) for every $\varepsilon > 0$.

(e) If an open set U is the intersection of a family of open sets each of which has the property (P), then U has the property (P) as well.

2.3. Lemma. *Let (Ω, φ) be a pseudoconvex Riemann domain over a locally convex space E. Let U be an open subset of Ω with the finite property* (P).

(a) *If M is any finite-dimensional subspace of E then*

$$\hat{K}_{\mathscr{H}(\varphi^{-1}(M))} \subset\subset U \cap \varphi^{-1}(M) \quad \textit{for every } K \subset\subset U \cap \varphi^{-1}(M).$$

In particular, $\mathscr{H}(\varphi^{-1}(M))$ is dense in $(\mathscr{H}(U \cap \varphi^{-1}(M)), \tau_0)$.

(b) *U is pseudoconvex.*

(c) *The union of an arbitrary collection of components of U has the finite property* (P) *as well.*

Proof. (a) follows from Theorem 1.3 (a), (f) and (e). Statement (b) is clear. To show (c) let V be the union of a collection of components of U, and let M be a finite-dimensional subspace of E. By (a), $\mathscr{H}(\varphi^{-1}(M))$ is dense in $(\mathscr{H}(U \cap \varphi^{-1}(M)), \tau_0)$. But this obviously implies that $\mathscr{H}(\varphi^{-1}(M))$ is dense in $(\mathscr{H}(V \cap \varphi^{-1}(M)), \tau_0)$. Then Theorem 1.3 (e) implies that

$$\hat{K}_{\mathscr{H}(\varphi^{-1}(M))} \subset V \cap \varphi^{-1}(M) \quad \text{for every } K \subset\subset V \cap \varphi^{-1}(M).$$

It follows that $V \cap \varphi^{-1}(M)$ has the property (P) with respect to $\varphi^{-1}(M)$ and the proof of the lemma is complete.

Now we come to one of the key results in this paper.

2.4. Theorem. *Let (Ω, φ) be a pseudoconvex Riemann domain over a Fréchet space E which has a Schauder basis. Let U be an open subset of Ω with the finite property* (P). *If E has a continuous norm, or if U has finitely many components, then each $g \in \mathscr{H}(U)$ is the limit in $(\mathscr{H}(U), \tau_0)$ of a sequence $(f_n) \subset \mathscr{H}(\Omega)$. Furthermore, there is an increasing sequence of open sets $C_k(U)$ with $U = \bigcup_{k=1}^{\infty} C_k(U)$ such that, whenever $g \in \mathscr{H}(U)$ is bounded on U, the corresponding sequence $(f_n) \subset \mathscr{H}(\Omega)$ can be taken uniformly bounded on each $C_k(U)$.*

Let (ξ_k) denote the sequence of coordinate functionals on $\boldsymbol{C}^N$. Matyszczyk [17, Theorem 2.7] has shown that if U is a polynomially convex open set in $\boldsymbol{C}^N$ with infinitely many components $U_1, U_2, \ldots$ (there are plenty of such open sets), then the function $g \in \mathscr{H}(U)$ defined by $g = \xi_k$ on U_k $(k = 1, 2, \ldots)$ is not the limit in $(\mathscr{H}(U), \tau_0)$ of any sequence $(f_n) \subset \mathscr{H}(\boldsymbol{C}^N)$. This shows that the hypotheses that E has a continuous norm and that U has finitely many components cannot be deleted simultaneously in Theorem 2.4.

The author [23, Theorem 2.1] has already established Theorem 2.4 in the case where the domain Ω is one-sheeted and the space E has a continuous norm. As we shall see, the proof in the case of Riemann domains is considerably more involved. An earlier result of Matyszczyk [17, Theorem 2.9] on polynomially convex domains may be regarded also as a special case of Theorem 2.4; see also [20, Theorem 4.7] for a complement of this result of Matyszczyk.

In this section we shall prove Theorem 2.4 under the assumption that E has a continuous norm, whereas in Section 3 we shall prove the theorem under the assumption that U has finitely many components.

Before proving the theorem for E with a continuous norm, we shall need three auxiliary lemmas. Lemma 2.5 is due to Schottenloher [33, p. 227], whereas Lemmas 2.6 and 2.7 are inspired by the ideas of Schottenloher [33] and Gruman and Kiselman [11], respectively.

2.5. Lemma [33]. *Let (Ω, φ) be a Riemann domain over a Fréchet space E which has a Schauder basis. Set $\Omega_n = \varphi^{-1}(E_n)$ for every n. Then there exist a sequence of open sets $A_n \subset \Omega$ and a sequence of holomorphic mappings $\tau_n\colon A_n \to \Omega_n$ with the following properties:*

(a) $\Omega = \bigcup_{n=1}^{\infty} A_n$, $A_n \subset A_{n+1}$ *and* $\Omega_n \subset A_n$ *for every* n.

(b) $\tau_n = \mathrm{id}$ *on* Ω_n, $\varphi \circ \tau_n = T_n \circ \varphi$ *on* A_n *and* $\tau_n \circ \tau_{n+1} = \tau_{n+1} \circ \tau_n = \tau_n$ *on* A_n *for every* n.

(c) *For each* $K \subset\subset \Omega$ *and* $V \in \mathscr{V}(E)$ *with* $K+V \subset \Omega$ *there exists* $n \in N$ *such that* $K \subset A_n$ *and* $\tau_k(x) \in x+V$ *for every* $x \in K$ *and* $k \geqslant n$.

(d) *If* Ω *is pseudoconvex then each* A_n *has the property* (P).

Proof. If U is any open subset of Ω then we consider the functions $\eta_U^n\colon U \to [0, +\infty]$ defined by

$$\eta_U^n(x) = \inf_{k \geqslant n} \delta_U\big(x,\ T_k \circ \varphi(x) - \varphi(x)\big).$$

These functions were introduced by Schottenloher [33, p. 226], who proved that they are strictly positive and lower semicontinuous on U. Thus the functions $-\log \eta_U^n$ are plurisubharmonic on U whenever U is pseudoconvex. If we define

$$A_n = \{x \in \Omega\colon\ \eta_\Omega^n(x) > 1\},$$

$$\tau_n(x) = (\varphi|D_x)^{-1} \circ T_n \circ \varphi(x) \qquad (x \in A_n),$$

where

$$D_x = D_\Omega\big(x,\ T_n \circ \varphi(x) - \varphi(x),\ \eta_\Omega^n(x)\big),$$

then the assertions in the lemma can be readily verified.

2.6. Lemma. *Let* (Ω, φ) *be a connected Riemann domain over a Fréchet space E which has a Schauder basis and a continuous norm. Let* (A_n) *and* (τ_n) *be two sequences satisfying the conditions in Lemma* 2.5. *Then there are two sequences of open sets* $C_n \subset B_n \subset A_n$ *and a sequence* $(V_n) \subset \mathscr{V}(E)$ *with the following properties*:

(a) $\Omega = \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} C_n$, $B_n \subset B_{n+1}$ *and* $C_n + V_n \subset C_{n+1}$ *for every* n.

(b) $B_n \cap \Omega_k \subset\subset A_n \cap \Omega_k$ *for every* n *and* k.

(c) $\tau_k(C_n) \subset B_n \cap \Omega_k$ *whenever* $k \geqslant n$.

(d) *If* Ω *is pseudoconvex then*

$$(B_n \cap \Omega_k)\hat{}_{\mathscr{H}(\Omega_k)} \subset\subset A_n \cap \Omega_k \quad \textit{for every } n \textit{ and } k.$$

Proof. Let (α_n) be a fundamental sequence of continuous norms on E such that

$$\alpha_{n+1} \geqslant 2\alpha_n \quad \text{and} \quad \alpha_n = \sup_k \alpha_n \circ T_k \quad \text{for every } n.$$

Let us consider the following auxiliary open sets:

$$X_n = \{x \in A_n\colon\ d_{A_n}^{\alpha_n}(x) > 1\}, \qquad Y_n = \{x \in X_n\colon\ \eta_{X_n}^n(x) > 1\}.$$

Certainly $\Omega = \bigcup_{n=1}^{\infty} X_n = \bigcup_{n=1}^{\infty} Y_n$ and the sequences (X_n) and (Y_n) are increasing.

We also see that

$$\tau_k(Y_n) \subset X_n \cap \Omega_k \quad \text{whenever } k \geqslant n,$$

and hence

$$\tau_n(Y_n) \subset X_n \cap \Omega_n = Y_n \cap \Omega_n \quad \text{for every } n.$$

In particular, we see that

$$(1) \qquad D_\Omega(x, a, r) \subset Y_n \quad \text{implies} \quad D_\Omega\big(\tau_n(x), T_n(a), r\big) \subset Y_n \cap \Omega_n$$

for every n. By modifying an idea of Schottenloher [33, p. 227] we shall construct a sequence of continuous functions $\gamma_n\colon Y_n \to [0, +\infty]$ such that

$$(2) \qquad \{x \in Y_n \cap \Omega_n\colon\ \gamma_n(x) < m\} \subset\subset \Omega_n$$

for every n and m. Without loss of generality we may assume that $Y_1 \cap \Omega_1$ is nonvoid. Fix a point $x_1 \in Y_1 \cap \Omega_1$. For each point $x \in Y_n$ let $\Gamma_n(x)$ denote the set of all finite sequences $(x_1, \ldots, x_{s+1})$ in Y_n with $x_{s+1} = x$ such that

$$D_\Omega\big(x_r,\ \varphi(x_{r+1}) - \varphi(x_r),\ 1+\varepsilon_r\big) \subset Y_n$$

for suitable $\varepsilon_r > 0$ and $r = 1, \ldots, s$. It is easy to see that $\Gamma_n(x)$ is nonempty if and only if x belongs to the component of Y_n which contains x_1. We define $\gamma_n\colon Y_n \to [0, +\infty]$ by

$$\gamma_n(x) = \inf\Big\{\sum_{r=1}^{s} \alpha_1\big(\varphi(x_{r+1}) - \varphi(x_r)\big)\colon\ (x_1, \ldots, x_{s+1}) \in \Gamma_n(x)\Big\}$$

if $\Gamma_n(x)$ is nonempty, and $\gamma_n(x) = +\infty$ if $\Gamma_n(x)$ is empty. It is easy to see that each γ_n is continuous. Indeed, if $x \in Y_n$ with $\gamma_n(x) < +\infty$ and $d_{Y_n}^{\alpha}(x) > 1$ for some $\alpha \in \mathrm{cs}(E)$, then

$$|\gamma_n(y) - \gamma_n(x)| \leqslant \alpha_1\big(\varphi(y) - \varphi(x)\big) \quad \text{for every } y \in B_{Y_n}^{\alpha}(x, \tfrac{1}{2}).$$

Let us fix n and inductively define a sequence of sets K_m as follows:

$$K_1 = Y_n \cap B_{\Omega_n}^{\alpha_n}(x_1, \tfrac{1}{2}), \qquad K_{m+1} = Y_n \cap \bigcup\{B_{\Omega_n}^{\alpha_n}(x, \tfrac{1}{2})\colon\ x \in K_m\}.$$

In order to show (2) we shall prove (3) and (4) below for every $m \in N$ and for a suitable constant c, to be defined shortly.

$$(3) \qquad K_m \subset\subset \Omega_n,$$

$$(4) \qquad \{x \in Y_n \cap \Omega_n\colon\ \gamma_n(x) < m/(2c)\} \subset K_m.$$

Let us show (3) by induction on m. Clearly, (3) is true for $m = 1$. If we assume $K_m \subset\subset \Omega_n$ then we can find points $y_1, \ldots, y_s \in K_m$ such that

$$K_m \subset \bigcup_{r=1}^{s} B_{\Omega_n}^{\alpha_n}(y_r, \tfrac{1}{2}),$$

hence

$$K_{m+1} \subset \bigcup_{r=1}^{s} B_{\Omega_n}^{\alpha_n}(y_r, 1) \subset\subset \Omega_n$$

and (3) is proved. To prove (4) let us choose a constant $c \geqslant 1$ such that $\alpha_n \leqslant c\alpha_1$ on E_n. This is possible since α_1 and α_n are norms. Let us prove (4) by induction on m. Clearly (4) is true for $m = 1$. Let us assume that (4) is true for some m and let $x \in Y_n \cap \Omega_n$ with

$$\frac{m}{2c} \leqslant \gamma_n(x) < \frac{m+1}{2c}.$$

Choose $(x_1, \ldots, x_{s+1}) \in \Gamma_n(x)$ such that

$$\sum_{r=1}^{s} \alpha_1(\varphi(x_{r+1}) - \varphi(x_r)) < \frac{m+1}{2c}.$$

In view of (1) we may assume that

$$D_\Omega(x_1, \varphi(x_{r+1}) - \varphi(x_r), 1+\varepsilon_r) \subset Y_n \cap \Omega_n \tag{5}$$

for every $r = 1, \ldots, s$, for otherwise we could replace each x_r by $\tau_n(x_r)$. Let $t \leqslant s$ be the first integer such that

$$\sum_{r=1}^{t} \alpha_1(\varphi(x_{r+1}) - \varphi(x_r)) \geqslant \frac{m}{2c}.$$

Using the Intermediate Value Theorem we can then find a point y in the line segment $[x_t, x_{t+1}]$ such that

$$\sum_{r<t} \alpha_1(\varphi(x_{r+1}) - \varphi(x_r)) + \alpha_1(\varphi(y) - \varphi(x_t)) < \frac{m}{2c},$$

$$\alpha_1(\varphi(x_{t+1}) - \varphi(y)) + \sum_{r>t} \alpha_1(\varphi(x_{r+1}) - \varphi(x_r)) < \frac{1}{2c}.$$

Then $\gamma_n(y) < m/(2c)$, and since (5) implies that $y \in Y_n \cap \Omega_n$, the induction hypothesis implies that $y \in K_m$. Since $\alpha_n(\varphi(x) - \varphi(y)) < \frac{1}{2}$ we conclude that $x \in K_{m+1}$ and (4) is proved. Certainly (2) follows from (3) and (4). If we define

$$B_n = \{x \in Y_n:\ \gamma_n(x) < n\}$$

then (2) implies that $B_n \cap \Omega_n \subset\subset \Omega_n$. But this obviously implies that

$$B_n \cap \Omega_k \subset\subset A_n \cap \Omega_k \quad \text{for every } n \text{ and } k,$$

and (b) is proved. Then (d) follows from Lemmas 2.5 (d) and 2.3 (a). Clearly $\Omega = \bigcup_{n=1}^{\infty} B_n$, and the sequence (B_n) is increasing since $\gamma_{n+1} \leqslant \gamma_n$ on Y_n. Finally, we define

$$Z_n = \{x \in B_n:\ \eta_{B_n}^{n}(x) > 1\}, \qquad C_n = \{x \in Z_n:\ d_{Z_n}^{\alpha_n}(x) > 1\}.$$

Then clearly

$$\tau_k(C_n) \subset \tau_k(Z_n) \subset B_n \cap \Omega_k \quad \text{whenever } k \geqslant n.$$

If we set $V_n = \{a \in E:\ \alpha_{n+1}(a) < 1\}$ then, since $\alpha_{n+1} \geqslant 2\alpha_n$, we see that $C_n + V_n \subset C_{n+1}$. The remaining assertions in the lemma are clear.

2.7. Lemma. *Let (Ω, φ) be a connected pseudoconvex Riemann domain over a Fréchet space E which has a Schauder basis and a continuous norm. Then, with the notation of Lemmas 2.5 and 2.6, for each $f_n \in \mathscr{H}(\Omega_n)$ and for each $\varepsilon > 0$ there exists $f \in \mathscr{H}(\Omega)$ such that*

(a) $f = f_n$ *on* Ω_n;

(b) $\|f - f_n \circ \tau_n\|_{C_n} \leqslant \varepsilon$;

(c) $\|f\|_{C_j} < +\infty$ *for every* j.

Proof. Consider the sets

$$X = \Omega_n \cup (B_n \cap \Omega_{n+1})\hat{}_{\mathscr{H}(\Omega_{n+1})}, \qquad Y = \Omega_{n+1} \setminus A_n.$$

Then X and Y are disjoint closed subsets of Ω_{n+1}. Let $\psi \in \mathscr{C}^\infty(\Omega_{n+1})$ with $\psi \equiv 1$ on a neighborhood of X and $\psi \equiv 0$ on a neighborhood of Y. Let $g \in \mathscr{C}^\infty(\Omega_{n+1})$ be defined by

$$g = \psi(f_n \circ \tau_n) - (\xi_{n+1} \circ \varphi) u$$

where (ξ_j) denotes the sequence of coordinate functionals on E and where $u \in \mathscr{C}^\infty(\Omega_{n+1})$ will be chosen so that $\bar\partial g = 0$ on Ω_{n+1}. The equation $\bar\partial g = 0$ is equivalent to the equation $\bar\partial u = v$, where $v = (f_n \circ \tau_n / \xi_{n+1} \circ \varphi) \bar\partial \psi$. Since $v \in \mathscr{C}^\infty_{(0,1)}(\Omega_{n+1})$ is well defined and verifies $\bar\partial v = 0$, Theorem 1.3 (c) guarantees that the equation $\bar\partial u = v$ has a solution, and thus $g \in \mathscr{H}(\Omega_{n+1})$. Since $\bar\partial u = 0$ on a neighborhood of the compact set $(B_n \cap \Omega_{n+1})\hat{}_{\mathscr{H}(\Omega_{n+1})}$, Theorem 1.3 (d) yields a function $h \in \mathscr{H}(\Omega_{n+1})$ such that

$$\|h - u\|_{B_n \cap \Omega_{n+1}} \leqslant \frac{\varepsilon}{2C},$$

where $C = \|\xi_{n+1} \circ \varphi\|_{B_n \cap \Omega_{n+1}}$. If we define

$$f_{n+1} = g + (\xi_{n+1} \circ \varphi) h = \psi(f_n \circ \tau_n) + (\xi_{n+1} \circ \varphi)(h - u),$$

then clearly $f_{n+1} = f_n$ on Ω_n and

$$\|f_{n+1} - f_n \circ \tau_n\|_{B_n \cap \Omega_{n+1}} \leqslant \varepsilon/2,$$

and it follows that

$$\|f_{n+1} \circ \tau_{n+1} - f_n \circ \tau_n\|_{C_n} \leqslant \varepsilon/2,$$

Proceeding inductively, we can find a sequence $(f_j)_{j=n+1}^{\infty}$ with $f_j \in \mathscr{H}(\Omega_j)$ such that $f_{j+1} = f_j$ on Ω_j and

$$\|f_{j+1} \circ \tau_{j+1} - f_j \circ \tau_j\|_{C_j} \leqslant \varepsilon/2^{j-n+1}$$

for every $j \geqslant n$. It follows easily that

$$\|f_k \circ \tau_k - f_j \circ \tau_j\|_{C_j} \leqslant \varepsilon/2^{j-n}$$

whenever $k \geqslant j \geqslant n$, and thus the sequence $(f_j \circ \tau_j)$ converges uniformly on each C_i to a function $f \in \mathscr{H}(\Omega)$. Clearly, $f = f_j$ on Ω_j and

$$\|f - f_j \circ \tau_j\|_{C_j} \leqslant \varepsilon/2^{j-n}$$

for every $j \geqslant n$. From this estimate we also conclude that f is bounded on each C_j, for

$$\tau_j(C_j) \subset B_j \cap \Omega_j \subset\subset \Omega_j.$$

Proof of Theorem 2.4 when E has a continuous norm. We may assume that Ω is connected. Let $A_n(U)$ be open sets associated with U as in Lemma 2.5 and let $B_n(U)$ and $C_n(U)$ be defined by

$$B_n(U) = \{x \in B_n \cap A_n(U):\ d^{\alpha_n}_{A_n(U)}(x) > 1\},$$

$$C_n(U) = \{x \in C_n \cap B_n(U):\ \eta^n_{B_n(U)}(x) > 1\}.$$

Let $g \in \mathscr{H}(U)$ be given. By Lemmas 2.6 (b) and 2.3 (a),

$$(B_n(U) \cap \Omega_n)^{\wedge}_{\mathscr{H}(\Omega_n)} \subset\subset U \cap \Omega_n$$

and thus Theorem 1.3 (d) yields a sequence (h_n), with $h_n \in \mathscr{H}(\Omega_n)$, such that

$$\|h_n - g\|_{B_n(U) \cap \Omega_n} \leqslant 1/n$$

for every n. It follows that

$$\|h_n \circ \tau_n - g \circ \tau_n\|_{C_n(U)} \leqslant 1/n$$

for every n. An application of Lemma 2.7 yields a sequence (f_n) in $\mathscr{H}(\Omega)$ such that

$$\|f_n - h_n \circ \tau_n\|_{C_n} \leqslant 1/n$$

and therefore

(1) $$\|f_n - g \circ \tau_n\|_{C_n(U)} \leqslant 2/n$$

for every n. We claim that the sequence (f_n) converges to g in $(\mathscr{H}(U), \tau_0)$. Indeed, let $K \subset\subset U$ and $\varepsilon > 0$ be given. By the continuity of g we can easily

find $V \in \mathscr{V}(E)$ such that $K + V \subset U$ and

$$|g(y) - g(x)| \leqslant \varepsilon \quad \text{for every } x \in K \text{ and } y \in x + V.$$

Using Lemma 2.5 (c), we can find $n_0 \geqslant 1/\varepsilon$ such that $K \subset C_{n_0}(U)$ and

(2) $$\|g \circ \tau_n - g\|_K \leqslant \varepsilon$$

for every $n \geqslant n_0$. From (1) and (2) we conclude that $\|f_n - g\|_K \leqslant 3\varepsilon$ for every $n \geqslant n_0$. From (1) we also see that the sequence (f_n) is uniformly bounded on each $C_k(U)$ if g is bounded on U.

3. Global approximation in spaces without a continuous norm. In Section 2 we proved Theorem 2.4 when E is assumed to have a continuous norm. In this section we shall prove the theorem when U is assumed to have finitely many components. Before proving the theorem in this case we shall need two more auxiliary lemmas, the first of which is due to Dineen [9, Example 2.4].

3.1. Lemma [9]. *Let E be a Fréchet space with a Schauder basis. Let α be a continuous seminorm on E satisfying the condition*

$(*)$ $$\alpha(x) = \sup_m \alpha\Big(\sum_{n=1}^{m} \xi_n(x) e_n\Big) \quad \textit{for every } x \in E.$$

If we set

$$Z^\alpha = \{n \in N:\ \alpha(e_n) = 0\}, \qquad E^\alpha = \{x \in E:\ \xi_n(x) = 0 \textit{ for every } n \in Z^\alpha\},$$

then E^α has a Schauder basis and a continuous norm, and E is the topological direct sum of E^α and $\alpha^{-1}(0)$.

3.2. Lemma. *Let (Ω, φ) be a connected pseudoconvex Riemann domain over a Fréchet space E which has a Schauder basis. Let $x_0 \in \Omega$ and let α be a continuous seminorm on E satisfying the condition $(*)$ in Lemma 3.1 and such that $d^\alpha_\Omega(x_0) > 0$. Let π_α: $E \to E^\alpha$ denote the canonical projection and set $\Omega^\alpha = \varphi^{-1}(E^\alpha)$. Then:*

(a) *There is a holomorphic mapping σ_α: $\Omega \to \Omega^\alpha$ such that $\sigma_\alpha = \mathrm{id}$ on Ω^α and $\varphi \circ \sigma_\alpha = \pi_\alpha \circ \varphi$ on Ω.*

(b) *Let U be any connected pseudoconvex open subset of Ω such that $d^\alpha_U(y_0) > 0$ for some $y_0 \in U$. Then $U = \sigma_\alpha^{-1}(U \cap \Omega^\alpha)$ and $f \circ \sigma_\alpha = f$ on U for every $f \in \mathscr{H}(U)$ which is bounded on an α-neighborhood of y_0.*

(c) *For $x, y \in \Omega$ we have $x = y$ if and only if $\varphi(x) = \varphi(y)$ and $\sigma_\alpha(x) = \sigma_\alpha(y)$.*

(d) *For each $a \in E$ and $t \in \Omega^\alpha$ with $\pi_\alpha(a) = \varphi(t)$ there is a unique $x \in \Omega$ such that $\varphi(x) = a$ and $\sigma_\alpha(x) = t$.*

(e) *A net (x_i) in Ω converges in Ω if and only if $(\varphi(x_i))$ converges in E and $(\sigma_\alpha(x_i))$ converges in Ω^α.*

Proof. (a) If $0 < r < d^{\alpha}_{\Omega}(x_0)$ then $\delta_{\Omega}(x, a) = +\infty$ for every $x \in B^{\alpha}_{\Omega}(x_0, r)$ and $a \in \alpha^{-1}(0)$. Since Ω is pseudoconvex and connected we conclude that $\delta_{\Omega}(x, a) = +\infty$ for every $x \in \Omega$ and $a \in \alpha^{-1}(0)$. If we define $\sigma_{\alpha}\colon \Omega \to \Omega^{\alpha}$ by

$$\sigma_{\alpha}(x) = (\varphi|D_x)^{-1} \circ \pi_{\alpha} \circ \varphi(x),$$

where

$$D_x = D_{\Omega}(x, \pi_{\alpha} \circ \varphi(x) - \varphi(x), +\infty),$$

then σ_{α} has the required properties.

(b) As in (a) we see that $\delta_U(x, a) = +\infty$ for every $x \in U$ and $a \in \alpha^{-1}(0)$. This implies that

$$D_{\Omega}(x, \pi_{\alpha} \circ \varphi(x) - \varphi(x), +\infty) \subset U \quad \text{for every } x \in U,$$

and it follows that $x \in U$ if and only if $\sigma_{\alpha}(x) \in U$. Thus $U = \sigma_{\alpha}^{-1}(U \cap \Omega^{\alpha})$. To prove the second assertion let $f \in \mathscr{H}(U)$ be bounded on some α-neighborhood of y_0, and define

$$U_j = \{x \in U\colon |f(x)| < j\} \quad \text{for } j = 1, 2, \ldots$$

Fix $y \in U$ and choose j such that y_0, y and $\sigma_{\alpha}(y)$ all lie in some component V_j of U_j. This is possible since U is pathwise connected and $U = \bigcup_{j=1}^{\infty} U_j$. Since V_j contains an α-neighborhood of y_0, and since V_j is pseudoconvex and connected, we conclude as before that $\delta_{V_j}(x, a) = +\infty$ for every $x \in V_j$ and $a \in \alpha^{-1}(0)$. In particular,

$$D_y = D_{\Omega}(y, \pi_{\alpha} \circ \varphi(y) - \varphi(y), +\infty) \subset V_j.$$

Thus f is bounded on D_y and hence constant there, by Liouville's theorem. In particular, $f \circ \sigma_{\alpha}(y) = f(y)$.

(c) Suppose $\varphi(x) = \varphi(y) = a$. Then

$$\sigma_{\alpha}(x) = (\varphi|D_x)^{-1} \circ \pi_{\alpha}(a), \qquad \sigma_{\alpha}(y) = (\varphi|D_y)^{-1} \circ \pi_{\alpha}(a),$$

where

$$D_x = D_{\Omega}(x, \pi_{\alpha}(a) - a, +\infty), \qquad D_y = D_{\Omega}(y, \pi_{\alpha}(a) - a, +\infty).$$

Since D_x and D_y are either disjoint or identical, we see that $\sigma_{\alpha}(x) = \sigma_{\alpha}(y)$ if and only if $x = y$.

(d) Let $x = (\varphi|D_{\Omega}(t, a - \pi_{\alpha}(a), +\infty))^{-1}(a)$.

(e) Assume $(\varphi(x_i))$ converges to some a in E and $(\sigma_{\alpha}(x_i))$ converges to some t in Ω^{α}. Then $\pi_{\alpha}(a) = \varphi(t)$ and by (d) there is a unique $x \in \Omega$ such that $\varphi(x) = a$ and $\sigma_{\alpha}(x) = t$. We claim that (x_i) converges to x. Let $V \in \mathscr{V}(E)$ be such that $x + V \subset \Omega$, $\sigma_{\alpha}(x) + V \subset \Omega$ and $\pi_{\alpha}(V) \subset V$. Choose i_0 such that

(1) $$\varphi(x_i) \in a + V = \varphi(x) + V,$$

(2) $$\sigma_{\alpha}(x_i) \in t + V = \sigma_{\alpha}(x) + V$$

for every $i \geq i_0$. By (1) for each $i \geq i_0$ there is a unique $y_i \in x + V$ such that $\varphi(y_i) = \varphi(x_i)$. Thus

(3) $$\sigma_{\alpha}(y_i) \in \sigma_{\alpha}(x) + V,$$

(4) $$\varphi \circ \sigma_{\alpha}(y_i) = \pi_{\alpha} \circ \varphi(y_i) = \pi_{\alpha} \circ \varphi(x_i) = \varphi \circ \sigma_{\alpha}(x_i)$$

for every $i \geq i_0$. From (2), (3) and (4) we see that $\sigma_{\alpha}(y_i) = \sigma_{\alpha}(x_i)$ and thus $y_i = x_i$ for every $i \geq i_0$, by (c). Thus $x_i \in x + V$ for every $i \geq i_0$ and the proof is complete.

Proof of Theorem 2.4 when U has finitely many components. Without loss of generality we may assume that the domain Ω is connected. Let $(A_n(U))$ denote a sequence of open sets associated with U as in Lemma 2.5. Given $g \in \mathscr{H}(U)$, it will be sufficient to find a sequence $(f_n) \subset \mathscr{H}(\Omega)$ and an increasing sequence of open sets $C_n(U) \subset A_n(U)$ with $U = \bigcup_{n=1}^{\infty} C_n(U)$ such that

(1) $$\|f_n - g \circ \tau_n\|_{C_n(U)} \leq 2/n \quad \text{for every } n,$$

for then the proof may continue as in the case where E has a continuous norm. Let $U_1, \ldots, U_m$ denote the components of U. Choose points $x_1 \in U_1, \ldots, x_m \in U_m$ and a continuous seminorm α on E satisfying the condition $(*)$ in Lemma 3.1 and such that $d^{\alpha}_U(x_i) > 0$ and g is bounded on an α-neighborhood of x_i, for every $i = 1, \ldots, m$. If $\sigma_{\alpha}\colon \Omega \to \Omega^{\alpha}$ denotes the mapping given by Lemma 3.2 then it follows from Lemma 3.2 (b) that

(2) $$U = \sigma_{\alpha}^{-1}(U \cap \Omega^{\alpha}).$$

There is no guarantee that $A_n(U) = \sigma_{\alpha}^{-1}(A_n(U) \cap \Omega^{\alpha})$, but we can remedy this as follows. Since $U = \bigcup_{n=1}^{\infty} A_n(U)$ we may assume without loss of generality that $x_1, \ldots, x_m \in A_n(U)$ for every n. Then we define $A'_n(U)$ to be the union of those components of $A_n(U)$ which contain some x_i. Then it is clear that the open sets

$$A'_n(U) \subset U \quad \text{and} \quad A'_n(U) \cap \Omega^{\alpha} \subset U \cap \Omega^{\alpha}$$

also have the properties stated in Lemma 2.5. Furthermore, Lemma 3.2 (b) guarantees that

(3) $$A'_n(U) = \sigma_{\alpha}^{-1}(A'_n(U) \cap \Omega^{\alpha}) \quad \text{for every } n.$$

Let

$$C_n(U \cap \Omega^{\alpha}) \subset B_n(U \cap \Omega^{\alpha}) \subset A'_n(U) \cap \Omega^{\alpha}$$

denote open sets associated with $U \cap \Omega^{\alpha}$ as on page 118. Since $U \cap \Omega^{\alpha}$ has the finite property (P) with respect to Ω^{α}, and since E^{α} has a continuous norm, we can find a sequence (f_n^α) in $\mathscr{H}(\Omega^\alpha)$ such that

(4) $$\|f_n^\alpha - g\circ\tau_n\|_{C_n(U\cap\Omega^\alpha)} \leqslant 2/n \quad \text{for every } n.$$

Define $f_n = f_n^\alpha \circ \sigma_\alpha \in \mathscr{H}(\Omega)$ and $C_n(U) = \sigma_\alpha^{-1}(C_n(U\cap\Omega^\alpha))$ for every n. Using (2) we see that $U = \bigcup_{n=1}^{\infty} C_n(U)$, and using (3) we see that $C_n(U) \subset A_n'(U) \subset A_n(U)$ for every n. Finally, from (4) it follows that

$$\|f_n - g\circ\tau_n\circ\sigma_\alpha\|_{C_n(U)} \leqslant 2/n \quad \text{for every } n,$$

and since $g\circ\tau_n\circ\sigma_\alpha = g\circ\tau_n$ on $A_n'(U)$ by Lemma 3.2 (b), (1) follows. The proof of Theorem 2.4 is now complete.

4. Convexity properties.

4.1. THEOREM. *Let (Ω, φ) be a pseudoconvex Riemann domain over a Fréchet space with a Schauder basis. Let K be a compact subset of Ω with $\hat{K}_{\mathrm{Psc}(\Omega)} = K$. Then for each open set U with $K \subset U \subset \Omega$ there is an open set V with $K \subset V \subset U$ such that V has the property* (P) *with respect to Ω.*

The conclusion of Theorem 4.1 is also valid for a pseudoconvex open subset Ω of a quasi-complete locally convex space; see [23, Theorem 1.3].

Before proving Theorem 4.1 we need some preparation. Following Schottenloher [31], we introduce the following definition adapted to our purpose.

4.2. DEFINITION. Let (Ω, φ) be a Riemann domain over a locally convex space E. A sequence $\mathscr{C} = (C_n)$ of open subsets of Ω is said to be an *admissible covering* of Ω if $\Omega = \bigcup_{n=1}^{\infty} C_n$ and there is a sequence $(V_n) \subset \mathscr{V}(E)$ such that $C_n + V_n \subset C_{n+1}$ for every n. The family

$$A_{\mathscr{C}} = \{f \in \mathscr{H}(\Omega)\colon \|f\|_{C_n} < +\infty \text{ for every } n\}$$

is called the *regular class* associated with the admissible covering $\mathscr{C}$. Then $A_{\mathscr{C}}$ is a Fréchet algebra in a natural way.

The following result is essentially due to Schottenloher; cf. [31, Sätze 3.2 und 2.4].

4.3. LEMMA [31]. *Let (Ω, φ) be a Riemann domain over a locally convex space E. Let $\mathscr{C}$ be an admissible covering of Ω. Then*:

(a) *$P_a^n f \in A_{\mathscr{C}}$ for every $f \in A_{\mathscr{C}}$, $n \in N$ and $a \in E$.*

(b) *Let $X, Y \subset \Omega$ and $V \in \mathscr{V}(E)$ be such that $X + V \subset Y$ and $\hat{X}_{A_{\mathscr{C}}} + V \subset \Omega$. Then $\hat{X}_{A_{\mathscr{C}}} + V \subset \hat{Y}_{A_{\mathscr{C}}}$.*

4.4. EXAMPLE. Let (Ω, φ) be a connected Riemann domain over a Fréchet space with a Schauder basis and with a continuous norm. Then the sequence $\mathscr{C} = (C_n)$ constructed in Lemma 2.6 is an admissible covering of Ω.

4.5. EXAMPLE. Let (Ω, φ) be a connected pseudoconvex Riemann domain over a Fréchet space E which has a Schauder basis. Let α be a continuous seminorm on E satisfying the hypotheses in Lemmas 3.1 and 3.2. Then the sequence $\mathscr{C}^\alpha = (C_n^\alpha)$ constructed in Lemma 2.6 is an admissible covering of Ω^α, whereas the sequence $\mathscr{C} = (C_n)$ defined by $C_n = \sigma_\alpha^{-1}(C_n^\alpha)$ is an admissible covering of Ω.

4.6. LEMMA. *Let (Ω, φ) be a connected pseudoconvex Riemann domain over a Fréchet space with a Schauder basis and with a continuous norm. Then, with the notation of Lemmas 2.5 and 2.6 and Example 4.4, we have*:

(a) $(C_n)\hat{}_{A_{\mathscr{C}}} \cap \Omega_k \subset (B_n \cap \Omega_k)\hat{}_{\mathscr{H}(\Omega_k)}$ *whenever* $k \geqslant n$.

(b) $(C_n)\hat{}_{A_{\mathscr{C}}} \cap \Omega_k \subset\subset \Omega_k$ *for every n and k.*

(c) $(C_n)\hat{}_{A_{\mathscr{C}}} + V_{n-1} \subset \Omega$ *for every n.*

(d) $(C_n)\hat{}_{A_{\mathscr{C}}} + V_n \subset (C_{n+1})\hat{}_{A_{\mathscr{C}}}$ *for every n.*

Proof. To prove (a) let $x \in (C_n)\hat{}_{A_{\mathscr{C}}} \cap \Omega_k$ and let $f_k \in \mathscr{H}(\Omega_k)$ with $k \geqslant n$. By Lemma 2.7, given $\varepsilon > 0$ we can find $f \in A_{\mathscr{C}}$ such that $f = f_k$ on Ω_k and $\|f - f_k\circ\tau_n\|_{C_k} \leqslant \varepsilon$. Then

$$|f_k(x)| = |f(x)| \leqslant \|f\|_{C_n} \leqslant \|f_k\circ\tau_k\|_{C_n} + \varepsilon \leqslant \|f_k\|_{B_n\cap\Omega_k} + \varepsilon,$$

and since $\varepsilon > 0$ is arbitrary, (a) follows. From (a) and Lemma 2.6 (d) we see that (b) is true for $k \geqslant n$. But this obviously implies that (b) is true for n and k arbitrary. To prove (c) let $x \in (C_n)\hat{}_{A_{\mathscr{C}}}$. Then a slight modification of the proof of (a) shows that

$$\tau_k x \in (B_n \cap \Omega_k)\hat{}_{\mathscr{H}(\Omega_k)}$$

for every $k \geqslant n$ such that $x \in C_k$. An examination of the proof of Lemma 2.6 shows that the set B_n is contained in the set

$$F_n = \{x \in \Omega\colon d_\Omega^{\alpha_n}(x) \geqslant 1\}$$

and clearly $(F_n)\hat{}_{\mathrm{Ps}(\Omega_k)} = F_n$. It follows that

$$\tau_k x \in (B_n \cap \Omega_k)\hat{}_{\mathscr{H}(\Omega_k)} = (B_n \cap \Omega_k)\hat{}_{\mathrm{Ps}(\Omega_k)} \subset F_n$$

for all sufficiently large k. Letting k tend to infinity, we obtain (c). Finally, since by Lemma 2.6, $C_n + V_n \subset C_{n+1}$, (d) follows from (c) and Lemma 4.3 (b).

4.7. LEMMA. *Let (Ω, φ) be a connected pseudoconvex Riemann domain over a Fréchet space E which has a Schauder basis. Then, with the notation of Example 4.5, the set*

$$L = \varphi^{-1}(K) \cap (C_n)\hat{}_{A_{\mathscr{C}}}$$

is compact for every compact set $K \subset E$ and $n \in N$.

Proof. We have $C_n = \sigma_\alpha^{-1}(C_n^\alpha)$, where $\alpha \in \operatorname{cs}(E)$ satisfies the conditions in Lemmas 3.1 and 3.2, and where $\mathscr{C}^\alpha = (C_n^\alpha)$ is a sequence of open sets associated with Ω^α as in Lemma 2.6. Thus there is a sequence $(V_n^\alpha) \subset \mathscr{V}(E^\alpha)$ such that $C_n^\alpha + V_n^\alpha \subset C_{n+1}^\alpha$ and it follows from Lemma 4.6 (d) that

(1) $$(C_n^\alpha)\hat{}_{A_{\mathscr{C}^\alpha}} + V_n^\alpha \subset (C_{n+1}^\alpha)\hat{}_{A_{\mathscr{C}^\alpha}}.$$

Let (x_i) be a net in L. Then $(\varphi(x_i)) \subset K$ and, after passing to a subnet if necessary, we may assume that $(\varphi(x_i))$ converges to a point $a \in K$. The idea now is to find a point $t \in \Omega^\alpha$ with $\varphi(t) = \pi_\alpha(a)$ such that a subnet of $(\sigma_\alpha(x_i))$ converges to t. Then it will suffice to apply Lemma 3.2 (d) and (e).

Since $(x_i) \subset (C_n)\hat{}_{A_\mathscr{C}}$, it follows that

(2) $$(\sigma_\alpha(x_i)) \subset (C_n^\alpha)\hat{}_{A_{\mathscr{C}^\alpha}}.$$

Since $\varphi(x_i) \to a$, it follows that

(3) $$\varphi \circ \sigma_\alpha(x_i) = \pi_\alpha \circ \varphi(x_i) \to \pi_\alpha(a),$$

and hence there is i_0 such that

(4) $$\varphi \circ \sigma_\alpha(x_i) - \pi_\alpha(a) \in \tfrac{1}{5} V_n^\alpha \quad \text{for every } i \geqslant i_0.$$

Since $\bigcup_{k=1}^{\infty} E_k^\alpha$ is dense in E^α there are $k \in N$ and $b \in E_k^\alpha$ such that

(5) $$b - \pi_\alpha(a) \in \tfrac{1}{5} V_n^\alpha,$$

and by (4) it follows that

$$\varphi \circ \sigma_\alpha(x_i) - b \in \tfrac{2}{5} V_n^\alpha \quad \text{for every } i \geqslant i_0.$$

Thus for each $i \geqslant i_0$ there is a unique point y_i with

(6) $$y_i \in \sigma_\alpha(x_i) + \tfrac{2}{5} V_n^\alpha \quad \text{and} \quad \varphi(y_i) = b.$$

Using (1) and (2), we see that

$$y_i \in (C_{n+1}^\alpha)\hat{}_{A_{\mathscr{C}^\alpha}} \cap \Omega_k^\alpha \quad \text{for every } i \geqslant i_0.$$

By Lemma 4.6 (b) the last set is compact. Hence, after passing to a subnet if necessary, we may assume that (y_i) converges to a point $y \in \Omega_k^\alpha$. From (6), (2) and (1) it follows that

$$y_i + \tfrac{3}{5} V_n^\alpha \subset \Omega_n^\alpha \quad \text{for every } i \geqslant i_0,$$

and hence

(7) $$y + \tfrac{3}{5} V_n^\alpha \subset \Omega_n^\alpha$$

too. It is also clear that $\varphi(y) = b$, and thus by (5) there is a unique point t with

(8) $$t \in y + \tfrac{1}{5} V_n^\alpha \quad \text{and} \quad \varphi(t) = \pi_\alpha(a).$$

We claim that $\sigma_\alpha(x_i) \to t$. Indeed, there is $i_1 \geqslant i_0$ such that

$$y_i \in y + \tfrac{1}{5} V_n^\alpha \quad \text{for every } i \geqslant i_1,$$

and using (6) we see that

(9) $$\sigma_\alpha(x_i) \in y + \tfrac{3}{5} V_n^\alpha \quad \text{for every } i \geqslant i_1.$$

By (3) and (8), $\varphi \circ \sigma_\alpha(x_i) \to \varphi(t)$, and because of (8) and (9) we may conclude that $\sigma_\alpha(x_i) \to t$. The proof of the lemma is complete.

Proof of Theorem 4.1. Let E denote the Fréchet space with basis under consideration. Without loss of generality we may assume that the domain Ω is connected. Let $\mathscr{C} = (C_n)$ denote the admissible covering of Example 4.5. Choose n such that $K \subset C_n$. If $\bar{\Gamma}(\varphi(K))$ denotes the closed convex hull of $\varphi(K)$ in E then by Lemma 4.7 the set

$$L = \varphi^{-1}\big(\bar{\Gamma}(\varphi(K))\big) \cap (C_n)\hat{}_{A_\mathscr{C}}$$

is compact and contains K. Since $\hat{K}_{\operatorname{Psc}(\Omega)} = K$, for each point $x \in L \setminus U$ we can find $f_x \in \operatorname{Psc}(\Omega)$ such that $f_x(x) > 0$ and $\sup_K f_x < 0$. By the compactness of $L \setminus U$ we can find $f_1, \ldots, f_m \in \operatorname{Psc}(\Omega)$ such that

$$\sup_K f_j < 0 \quad \text{for } j = 1, \ldots, m,$$

$$L \setminus U \subset \bigcup_{j=1}^{m} \{x \in \Omega: f_j(x) > 0\}, \quad \text{i.e.} \quad L \cap \bigcap_{j=1}^{m} \{x \in \Omega: f_j(x) \leqslant 0\} \subset U.$$

Set $f = \sup\{f_1, \ldots, f_m\} \in \operatorname{Psc}(\Omega)$. Then

(1) $$\varphi^{-1}\big(\bar{\Gamma}(\varphi(K))\big) \cap (C_n)\hat{}_{A_\mathscr{C}} \cap \{x \in \Omega: f(x) \leqslant 0\} \subset U.$$

We claim there exists $W \in \mathscr{V}(E)$ such that

(2) $$\varphi^{-1}\big(\bar{\Gamma}(\varphi(K)) + W\big) \cap (C_n)\hat{}_{A_\mathscr{C}} \cap \{x \in \Omega: f(x) \leqslant 0\} \subset U.$$

Let (W_k) be a decreasing fundamental sequence in $\mathscr{V}(E)$ such that $C_k + W_k \subset C_{k+1}$ for every k. If there is no W_k satisfying (2) then for each $k \geqslant n$ there is a point x_k such that

$$x_k \in \varphi^{-1}\big(\bar{\Gamma}(\varphi(K)) + W_k\big) \cap (C_n)\hat{}_{A_\mathscr{C}} \cap \{x \in \Omega: f(x) \leqslant 0\} \setminus U.$$

For each $k \geqslant n$ there is a point $b_k \in \bar{\Gamma}(\varphi(K))$ such that $\varphi(x_k) - b_k \in W_k$. Then for each $k \geqslant n$ there is a unique point $y_k \in x_k + W_k$ with $\varphi(y_k) = b_k$. From Lemma 4.3 (b) it follows that

$$y_k \in (C_n)\hat{}_{A_\mathscr{C}} + W_n \subset (C_{n+1})\hat{}_{A_\mathscr{C}}$$

and hence

$$y_k \in \varphi^{-1}\big(\bar{\Gamma}(\varphi(K))\big) \cap (C_{n+1})\hat{}_{A_\mathscr{C}}$$

for every $k \geqslant n$. By Lemma 4.7 the last set is compact. Hence, after passing to a subnet if necessary, we may assume that (y_k) converges to a point y. But then (x_k) also converges to y and it follows that

$$y \in \varphi^{-1}(\bar{\Gamma}(\varphi(K))) \cap (C_n)\hat{}_{A_{\mathscr{C}}} \cap \{x \in \Omega: f(x) \leqslant 0\} \setminus U,$$

contradicting (1). This shows the existence of $W \in \mathscr{V}(E)$ satisfying (2). Then we define

$$V = \varphi^{-1}(\bar{\Gamma}(\varphi(K)) + W) \cap \operatorname{int}(C_n)\hat{}_{A_{\mathscr{C}}} \cap \{x \in \Omega: f(x) < 0\}.$$

We claim that the set $\operatorname{int}(C_n)\hat{}_{A_{\mathscr{C}}}$ has the property (P). Indeed, given a compact set $J \subset \operatorname{int}(C_n)\hat{}_{A_{\mathscr{C}}}$, we choose $W' \in \mathscr{V}(E)$ such that $J + W' \subset (C_n)\hat{}_{A_{\mathscr{C}}}$. By Lemma 3.3 (b)

$$\hat{J}_{A_{\mathscr{C}}} + W' \subset (C_n)\hat{}_{A_{\mathscr{C}}}$$

and therefore $\hat{J}_{\mathrm{Ps}(\Omega)} \subset \hat{J}_{A_{\mathscr{C}}} \subset \operatorname{int}(C_n)\hat{}_{A_{\mathscr{C}}}$. From Examples 2.2 we conclude that V has the property (P) and the proof is complete.

Using Lemma 4.7, we may also rederive a result of Schottenloher [33, Proposition 4.1].

4.8. Theorem [33]. *Let (Ω, φ) be a pseudoconvex Riemann domain over a Fréchet space with a Schauder basis. Then every bounding set in Ω is relatively compact.*

Proof. Let E denote the Fréchet space with basis under consideration. Without loss of generality we may assume that the domain Ω is connected. Let B be a bounding set in Ω. Then $\varphi(B)$ is a bounding set in E and is therefore relatively compact there (see [10, Proposition 4.26]). Thus there is a compact set $K \subset E$ such that $\varphi(B) \subset K$. Now let $\mathscr{C} = (C_n)$ denote the admissible covering of Example 4.5. Then the set $\{f \in A_{\mathscr{C}}: \|f\|_B \leqslant 1\}$ is a barrel in $A_{\mathscr{C}}$, and hence a neighborhood of zero, since $A_{\mathscr{C}}$ is a Fréchet space. Thus there are $n \in N$ and $c > 0$ such that $\|f\|_B \leqslant c\|f\|_{C_n}$ for every $f \in A_{\mathscr{C}}$. Replacing f by f^k, taking the kth root and letting k tend to $+\infty$, we see that $\|f\|_B \leqslant \|f\|_{C_n}$ for every $f \in A_{\mathscr{C}}$. Thus $B \subset \varphi^{-1}(K) \cap (C_n)\hat{}_{A_{\mathscr{C}}}$ and it suffices to apply Lemma 4.7.

4.9. Corollary [33]. *Let (Ω, φ) be a pseudoconvex Riemann domain over a Fréchet space with a Schauder basis. Then Ω is holomorphically convex, i.e.*

$$\hat{K}_{\mathscr{H}(\Omega)} \subset\subset \Omega \quad \textit{for every } K \subset\subset \Omega.$$

5. Local approximation.

5.1. Theorem. *Let (Ω, φ) be a pseudoconvex Riemann domain over a Fréchet space with a Schauder basis. Let K be a compact subset of Ω with $\hat{K}_{\mathrm{Psc}(\Omega)} = K$. Then for each open set U with $K \subset U \subset \Omega$ there is an open set V with $K \subset V \subset \Omega$ such that each $g \in \mathscr{H}(U)$ is the limit in $(\mathscr{H}(V), \tau_0)$ of a*

sequence $(f_n) \subset \mathscr{H}(\Omega)$. Furthermore, whenever $g \in \mathscr{H}(U)$ is bounded on U, the corresponding sequence $(f_n) \subset \mathscr{H}(\Omega)$ can be taken uniformly bounded on V.

Proof. By Theorem 4.1 there is an open set W with $K \subset W \subset U$ such that W has the finite property (P) with respect to Ω. By Lemma 2.3 (c) we may assume without loss of generality that W has only finitely many components. Let $(C_k(W))_{k=1}^{\infty}$ be the corresponding sequence of open subsets given by Theorem 2.4. Choose k_0 so large that $K \subset C_{k_0}(W)$ and let $V = C_{k_0}(W)$. Then all the conclusions follow from Theorem 2.4.

As already mentioned, Theorem 5.1 extends and sharpens the earlier results of Noverraz [26, Theorem 2] and Schottenloher [33, Proposition 4.5]. Theorem 5.1 has already been established by the author [23, Theorem 3.1′] in the case of one-sheeted domains.

5.2. Proposition. *Let (Ω, φ) be a pseudoconvex Riemann domain over a Fréchet space with a Schauder basis. Then*

$$\hat{K}_{\mathrm{Ps}(\Omega)} = \hat{K}_{\mathrm{Psc}(\Omega)} = \hat{K}_{\mathscr{H}(\Omega)} \quad \textit{for every } K \subset\subset \Omega.$$

Proof. In view of Corollary 4.9 and Theorem 5.1, the proof of [21, Theorem 11.1] can be easily adapted.

5.3. Proposition. *Let (Ω, φ) be a pseudoconvex Riemann domain over a Fréchet space with a Schauder basis. Let U be an open subset of Ω. Then the following conditions are equivalent:*

(a) $\hat{K}_{\mathscr{H}(\Omega)} \subset U$ *for every* $K \subset\subset U$.

(b) *U has the property* (P).

(c) *U has the finite property* (P).

(d) *U is pseudoconvex and $\mathscr{H}(\Omega)$ is dense in $(\mathscr{H}(U), \tau_0)$.*

If E has a continuous norm, or if U has finitely many components, then these conditions are also equivalent to:

(e) *U is pseudoconvex and $\mathscr{H}(\Omega)$ is sequentially dense in $(\mathscr{H}(U), \tau_0)$.*

Proof. In view of Corollary 4.9 and Theorems 2.4 and 5.1, the proof of [13, Theorem 4.3.3] can be easily adapted.

6. Spaces of holomorphic germs. Let (Ω, φ) be a Riemann domain over a locally convex space E. For an open set $U \subset \Omega$, let τ_0 and τ_ω denote respectively the compact-open topology and the *compact-ported topology* on $\mathscr{H}(U)$. We recall that a seminorm p on $\mathscr{H}(U)$ is said to be *ported* by a compact set $K \subset U$ if for each open set V with $K \subset V \subset U$ there is a constant $c > 0$ such that $p(f) \leqslant c\|f\|_V$ for every $f \in \mathscr{H}(U)$. The topology τ_ω, introduced by Nachbin [24], is generated by all those seminorms which are ported by compact subsets of U.

For a compact set $K \subset \Omega$ let $\mathscr{H}(K)$ denote the vector space of all germs of holomorphic functions on K. By abuse of notation we also denote by τ_0

and τ_ω the locally convex inductive limit topologies on $\mathscr{H}(K)$ which are defined by

$$(\mathscr{H}(K), \tau_0) = \operatorname*{ind}_{U \supset K} (\mathscr{H}(U), \tau_0)$$

and

$$(\mathscr{H}(K), \tau_\omega) = \operatorname*{ind}_{U \supset K} (\mathscr{H}(U), \tau_\omega).$$

One can readily see (cf. [19, Proposition 2.3]) that $(\mathscr{H}(K), \tau_\omega)$ can also be represented as an inductive limit of Banach spaces, namely

$$(\mathscr{H}(K), \tau_\omega) = \operatorname*{ind}_{U \supset K} \mathscr{H}^\infty(U),$$

where $\mathscr{H}^\infty(U)$ denotes the Banach space of all bounded holomorphic functions on U with the supremum norm.

One can say very little about the spaces $(\mathscr{H}(K), \tau_0)$ and $(\mathscr{H}(K), \tau_\omega)$ when E is an arbitrary locally convex space, but these spaces have very nice properties when E is a Fréchet space; see the survey of Bierstedt and Meise [6], the book of Dineen [10] and the recent article of the author [22].

Fix a compact set $K \subset \Omega$. If E is a Fréchet space then we fix a decreasing fundamental sequence (V_j) in $\mathscr{V}(E)$ with $K + V_j \subset \Omega$ for every j. Consider the sets

$$\mathscr{K}_j = \{f \in \mathscr{H}^\infty(K+V_j) \colon \|f\|_{K+V_j} \leqslant j\}.$$

Then $\mathscr{K}_j$ is equicontinuous and it follows from the Ascoli theorem that $\mathscr{K}_j$ is a compact subset of $(\mathscr{H}(K+V_j), \tau_0)$. Hence $\mathscr{K}_j$ is also a compact subset of $(\mathscr{H}(K), \tau_0)$, and the spaces $(\mathscr{H}(K+V_j), \tau_0)$ and $(\mathscr{H}(K), \tau_0)$ induce the same topology on $\mathscr{K}_j$.

6.1. Theorem [22]. *Let (Ω, φ) be a Riemann domain over a Fréchet space E, and let K be a compact subset of Ω. Then, with the preceding notation, a subset $\mathscr{G}$ of $(\mathscr{H}(K), \tau_0)$ is open if and only if $\mathscr{G} \cap \mathscr{K}_j$ is open in $\mathscr{K}_j$ for the induced topology for every j.*

In other words, the space $(\mathscr{H}(K), \tau_0)$ is the inductive limit of the compact subsets $\mathscr{K}_j$ in the category of all topological spaces. In particular, $(\mathscr{H}(K), \tau_0)$ is a k-space.

6.2. Theorem [5], [22]. *Let (Ω, φ) be a Riemann domain over a Fréchet–Schwartz space. Then $(\mathscr{H}(K), \tau_\omega)$ is a (DFS)-space for each compact subset K of Ω.*

Theorem 6.2 was established by Bierstedt and Meise [5, Theorem 7] in the case of one-sheeted domains and their proof can be adapted to our more general situation. However, for future convenience we have preferred in [22] to derive Theorem 6.2 from the more precise Lemma 6.3 below.

6.3. Lemma [22]. *Let (Ω, φ) be a Riemann domain over a Fréchet–Schwartz space E, and let K be a compact subset of Ω. Let (V_j) be a fundamental sequence in $\mathscr{V}(E)$ such that $V_{j+1} \subset \frac{1}{2} V_j$, the canonical mapping $E_{V_{j+1}} \to E_{V_j}$ is precompact, and $K + V_j \subset \Omega$ for every j. Then:*

(a) *The spaces $(\mathscr{H}(K+V_j), \tau_0)$ and $\mathscr{H}^\infty(K+V_{j+1})$ induce the same topology on $c\mathscr{K}_j$ for every $c > 0$ and every j.*

(b) *The set $\mathscr{K}_j$ is compact in $\mathscr{H}^\infty(K+V_{j+1})$ for every j.*

(c) *The spaces $(\mathscr{H}(K), \tau_0)$ and $(\mathscr{H}(K), \tau_\omega)$ induce the same topology on each $\mathscr{K}_j$.*

From Lemma 6.3 (c) and Theorem 6.1 the following theorem results at once:

6.4. Theorem [22]. *Let (Ω, φ) be a Riemann domain over a Fréchet–Schwartz space. Then the topologies τ_0 and τ_ω coincide on $\mathscr{H}(K)$ for each compact subset K of Ω.*

7. Spaces of holomorphic functions. Let (Ω, φ) be a Riemann domain over a locally convex space E. We would like to obtain information about $\mathscr{H}(\Omega)$ using the information we have on the spaces $\mathscr{H}(K)$. In particular, we would like to use Theorem 6.4 to show that the topologies τ_0 and τ_ω coincide on $\mathscr{H}(\Omega)$ for every Riemann domain Ω over a Fréchet–Schwartz space. Now, there is a canonical algebraic isomorphism

$$\mathscr{H}(\Omega) = \operatorname*{proj}_{K \subset \Omega} \mathscr{H}(K)$$

(cf. Bierstedt and Meise [6, Lemma 32]), and this isomorphism is clearly a homeomorphism when each of the spaces involved is endowed with the topology τ_0, i.e.

$$(\mathscr{H}(\Omega), \tau_0) = \operatorname*{proj}_{K \subset \Omega} (\mathscr{H}(K), \tau_0).$$

However, it is an open problem whether a similar result holds for τ_ω in general. Thus our original problem is reduced to the study of the topological isomorphism

$$(\mathscr{H}(\Omega), \tau_\omega) = \operatorname*{proj}_{K \subset \Omega} (\mathscr{H}(K), \tau_\omega).$$

Following [19], we introduce the following definition.

7.1. Definition. Let (Ω, φ) be a Riemann domain over a locally convex space. For each compact subset K of Ω we consider the following inductive limits:

$$\mathscr{H}^K(\Omega) = \operatorname*{ind}_{U \supset K} (\mathscr{H}(\Omega) \cap \mathscr{H}^\infty(U), \| \ \|_U), \quad \tilde{\mathscr{H}}^K(\Omega) = \operatorname*{ind}_{U \supset K} (\overline{\mathscr{H}(\Omega) \cap \mathscr{H}^\infty(U)}, \| \ \|_U)$$

where the closure is taken in $\mathscr{H}^\infty(U)$.

Then one can easily get the following result (cf. Mujica [19, Lemmas 5.3 and 5.4] or Dineen [10, Proposition 6.12]).

7.2. Proposition. *Let (Ω, φ) be a Riemann domain over a locally convex space. Then we have the topological isomorphisms*

$$(\mathscr{H}(\Omega), \tau_\omega) = \operatorname*{proj}_{K \subset \Omega} \mathscr{H}^K(\Omega) = \operatorname*{proj}_{K \subset \Omega} \tilde{\mathscr{H}}^K(\Omega).$$

Following Chae [8], we introduce the following definition.

7.3. Definition. Let (Ω, φ) be a Riemann domain over a locally convex space.

(a) A compact set $K \subset \Omega$ is said to be *Ω-Runge* if for each open set U with $K \subset U \subset \Omega$ there is an open set V with $K \subset V \subset U$ such that each $g \in \mathscr{H}^\infty(U)$ is the uniform limit on V of a sequence $(f_n) \subset \mathscr{H}(\Omega)$.

(b) The domain Ω is said to have the *Runge property* if each compact subset of Ω is contained in another one which is Ω-Runge.

In view of Proposition 7.2 the following result is clear (cf. Chae [8, Proposition 6.5] or Mujica [19, Theorem 6.1]).

7.4. Proposition. *Let (Ω, φ) be a Riemann domain over a locally convex space.*

(a) *If a compact set K is Ω-Runge then there is a topological isomorphism*

$$\tilde{\mathscr{H}}^K(\Omega) = (\mathscr{H}(K), \tau_\omega).$$

(b) *If the domain Ω has the Runge property then there is a topological isomorphism*

$$(\mathscr{H}(\Omega), \tau) = \operatorname*{proj}_{K \subset \Omega} (\mathscr{H}(K), \tau_\omega).$$

Since a balanced open set $\Omega \subset E$ always has the Runge property, we conclude at once from Proposition 7.4 and Theorem 6.4 that the topologies τ_0 and τ_ω coincide on $\mathscr{H}(\Omega)$ whenever Ω is a balanced open subset of a Fréchet-Schwartz space, a result already established by the author [22, Proposition 5.7]. But if in addition we use Theorem 5.1 then we obtain the following results.

7.5. Theorem. *Let (Ω, φ) be a pseudoconvex Riemann domain over a Fréchet–Schwartz space with a Schauder basis. Then each compact set $K \subset \Omega$ with $\hat{K}_{\mathrm{Psc}(\Omega)} = K$ is Ω-Runge, i.e. for each open set U with $K \subset U \subset \Omega$ there is an open set V with $K \subset V \subset U$ such that each $g \in \mathscr{H}^\infty(U)$ is the uniform limit on V of a sequence $(f_n) \subset \mathscr{H}(\Omega)$.*

Proof. Apply Theorem 5.1 and Lemma 6.3 (a).

7.6. Proposition. *Let (Ω, φ) be a pseudoconvex Riemann domain over a Fréchet–Schwartz space with a Schauder basis. Then:*

(a) *Ω has the Runge property.*

(b) *The topologies τ_0 and τ_ω coincide on $\mathscr{H}(\Omega)$.*

Proof. (a) follows from Theorem 7.5 and Corollary 4.9. Statement (b) follows from (a), Proposition 7.4 and Theorem 6.4.

Proposition 7.6 (b) will be superseded by Theorem 8.2 in the next section.

We remark that we have adopted here the original version of the Runge property, introduced by Chae [8]. In [19] and [22] we adopted a slightly different version. We refer to [19, Section 6] for a discussion on this matter.

8. The envelope of holomorphy. Let (Ω, φ) be a Riemann domain over a quasi-complete locally convex space E. Let Σ denote the spectrum of the topological algebra $(\mathscr{H}(\Omega), \tau_0)$, i.e. Σ is the set of all continuous nonzero algebra homomorphisms h: $(\mathscr{H}(\Omega), \tau_0) \to \boldsymbol{C}$. Let δ: $x \in \Omega \to \hat{x} \in \Sigma$ denote the mapping defined by $\hat{x}(f) = f(x)$ for every $f \in \mathscr{H}(\Omega)$. For each function $f \in \mathscr{H}(\Omega)$ let $\hat{f}$: $\Sigma \to \boldsymbol{C}$ denote the function defined by $\hat{f}(h) = h(f)$ for every $h \in \Sigma$. The Mackey–Arens theorem yields a mapping ψ: $\Sigma \to E$ such that $\mu(\psi(h)) = h(\mu \circ \varphi)$ for every $\mu \in E'$.

8.1. Theorem. *Under the preceding setting there is a Hausdorff topology on Σ with the following properties:*

(a) *(Σ, ψ) is a Riemann domain over E.*

(b) *(Σ, ψ) is holomorphically separated.*

(c) *(Σ, ψ) is metrically holomorphically convex, in particular pseudoconvex.*

(d) *The mapping δ: $\Omega \to \Sigma$ is a morphism which is injective if and only if Ω is holomorphically separated.*

(e) *$\hat{f} \in \mathscr{H}(\Sigma)$ for every $f \in \mathscr{H}(\Omega)$.*

(f) *The mapping δ^*: $g \in \mathscr{H}(\Sigma) \to g \circ \delta \in \mathscr{H}(\Omega)$ is continuous for τ_0 and for τ_ω.*

(g) *The mapping G: $f \in \mathscr{H}(\Omega) \to \hat{f} \in \mathscr{H}(\Sigma)$ is continuous for τ_0 and for τ_ω.*

(h) *$\delta^* \circ G(f) = f$ for every $f \in \mathscr{H}(\Omega)$.*

Theorem 8.1 is essentially due to Alexander [1, Sections 2 and 4] (see also Matos [16] for the statements concerning τ_ω), who adapted to the case of Banach spaces a construction of Rossi [30] in the finite-dimensional case, in an attempt to construct the envelope of holomorphy of Ω as a subset of Σ. Later on a counterexample of Josefson [14] showed that this is not possible in general. Very recently Schottenloher [35] has proved that Σ is indeed the envelope of holomorphy of Ω when Ω is a Riemann domain over a Fréchet space with a Schauder basis. However, we do not need this deep result here. With the help of Theorem 8.1 we can already improve Proposition 7.6 (b) as follows:

8.2. Theorem. *Let (Ω, φ) be a Riemann domain over a Fréchet–Schwartz* *space with a Schauder basis. Then the topologies τ_0 and τ_ω coincide on $\mathscr{H}(\Omega)$.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} (\mathscr{H}(\Sigma),\tau_0) & \xrightarrow{\text{id}} & (\mathscr{H}(\Sigma),\tau_\omega) \\ \uparrow G & & \downarrow \delta^* \\ (\mathscr{H}(\Omega),\tau_0) & \xrightarrow[\text{id}]{} & (\mathscr{H}(\Omega),\tau_\omega) \end{array}$$

The vertical mappings are continuous by Theorem 8.1. The upper horizontal mapping is continuous by Proposition 7.6 (b). Hence the lower horizontal mapping is also continuous, and the proof is complete.

As we mentioned in the Introduction, Theorem 8.2 extends the earlier results of Barroso [3, Theorem 2.2], Schottenloher [34, Proposition], Barroso and Nachbin [4, Proposition 10], Meise [18, Proposition 6] and the author [22, Proposition 5.7], [23, Corollary 4.5].

9. Final remarks. We shall say that a locally convex space E has the *bounded approximation property* if there is an equicontinuous family $\mathscr{T}$ of continuous linear operators of finite rank on E such that for each compact set $K \subset E$ and $V \in \mathscr{V}(E)$ there is $T \in \mathscr{T}$ such that $Tx - x \in V$ for every $x \in K$. By adapting the first part of the proof of [15, Theorem 1.e. 13] one can easily show that a separable Fréchet space has the bounded approximation property if and only if there is a sequence of continuous linear operators of finite rank which converges pointwise to the identity. Pełczyński has shown that every separable Fréchet space with the bounded approximation property is topologically isomorphic to a complemented subspace of a Fréchet space with a Schauder basis; see the announcement in [29] or the detailed proof in [17, Theorem 2.11]. Using Pełczyński's result, one can easily extend Theorems 2.4 and 4.1 and Corollary 4.9 to the case of separable Fréchet spaces with the bounded approximation property. But then it follows at once that Theorems 5.1 and 7.5, Proposition 7.6 and Theorem 8.2 (taken in this order) are also valid for separable Fréchet spaces with the bounded approximation property.

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Interpolation of Banach lattices

by

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Abstract. For couples of Banach lattices we describe the interpolation spaces generated by the $\pm$ method and by Ovchinnikov's upper method in terms of the Calderón–Lozanovskii spaces.

0. Introduction. In this paper we study the effect of certain interpolation methods on (quasi-) Banach lattices. More specifically, we consider the "$\pm$-method" $\langle \bar{X}, \varphi \rangle$ of Gustavsson–Peetre [10], Ovchinnikov's upper method $\langle \bar{X} \rangle^\varphi$ (see [20]), as well as a variant $\langle \bar{X} \rangle_\varphi$ of Ovchinnikov's lower method. Some results are also obtained for the complex method $[\bar{X}]_\theta$ of Calderón [6]. In fact, we wish to put these interpolation methods in the case of a couple $\bar{X}$ of quasi-Banach lattices in relation to the Calderón–Lozanovskii constructions $\varphi(\bar{X})$.

Not all of our results are new: closely related results may be found in Ovchinnikov [21], [22] as well as in Berezhnoi [2]. However, in contrast to [21], [22], the methods used here are elementary and are similar to those of Gustavsson–Peetre [10].

The plan of the paper is as follows. Section 1 contains definitions and a technical Lemma. In Section 2 we study in the case of a couple $\bar{X}$ of quasi-Banach lattices the connection between $\varphi(\bar{X})$, $\langle \bar{X} \rangle_\varphi$ and $\langle \bar{X}, \varphi \rangle$. As an application we obtain a new proof and an extension of the following theorem of Pisier [25]: a Banach lattice X is p-convex and p'-concave, $1 < p < 2$, $1/p + 1/p' = 1$, if and only if there exists a Banach lattice X_0 such that $X = [X_0, L^2]_\theta$, $\theta = 2/p$. In Section 3 we then extend our considerations to include $\langle \bar{X} \rangle^\varphi$, in the Banach case only. Section 4 is on the Gagliardo closure of $[\bar{X}]_\theta$. Section 5 is concerned with various applications of the previous results.

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