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Automatic continuity, local type and causality

by

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Abstract. Using a general gliding hump technique, the authors prove first some abstract results of uniform boundedness type. To demonstrate the power of the abstract theory, three concrete examples are then given of significant simplifications and improvements of earlier results. Next, the abstract theory is applied to obtain some basic automatic continuity principles. These results are formulated in the general context of convex operators, but they yield new information even in the case of linear operators. In the last two sections, the basic principles are used to prove continuity for causal operators and for operators of local type. In particular, it is shown that some important operators from systems theory are automatically continuous. The class of operators of local type includes differential operators, multiplication operators, and certain singular integral operators.

Introduction. The present note arose out of several discussions the two authors held on different occasions and represents a synthesis of the views which they hold on the subject of automatic continuity.

The work extended over several years and proceeded over several stages: each stage represented a significant simplification of the previous results and, we believe, a better understanding of the basic abstract principles which underlie the main automatic continuity results. The abstract result presented in the present paper is the last of a series of attempts to isolate the essential factors which make automatic continuity results possible. It seems to have reached a degree of simplicity which greatly contributed to the decision to write this paper. Another factor which influenced this decision was the fact that the recent investigations of M. Neumann make it possible to formulate the automatic continuity results for convex mappings, not only for linear ones.

The simplest case of an automatic continuity result is the following. If T is a linear mapping of a Hilbert space H into itself which satisfies

$$\langle Tx, y\rangle = \langle x, Ty\rangle$$

for all $x, y \in H$, then T is continuous.

It is easy to see that this result may be obtained as an immediate consequence of the uniform boundedness theorem. It is perhaps less obvious that another important automatic continuity result, namely the continuity of strictly irreducible representations of Banach algebras, admits a formulation under which it becomes a theorem of uniform boundedness type. Indeed, if T

is any representation of a Banach algebra A on a normed space E, we have, for each $x \in E$ with $\|x\| \leqslant 1$, a linear mapping S_x: $A \to E$ given by $S_x(a) := T(a)x$ for all $a \in A$ and the continuity of T: $A \to B(E)$ is then equivalent to the equicontinuity of the family of the mappings S_x. Now, this family is certainly pointwise bounded, since $\|S_x(a)\| \leqslant \|T(a)\|$ for each $a \in A$. Unfortunately, the classical uniform boundedness theorem does not apply, since we do not have the continuity of the S_x at this stage. However, we have a weakened condition of continuity: for a strictly irreducible T, the kernel of each S_x is closed, since it is a maximal modular left ideal.

Starting from these observations, V. Pták proved [20] a theorem of uniform boundedness type, the statement of which was formulated in such a manner that the important theorem of B. E. Johnson could be obtained from it as a consequence. The proof was based on a gliding hump argument, the main emphasis being on establishing a general technical result which would yield automatic continuity results in concrete situations. The gliding hump lemma given in the present paper is much more satisfactory in this respect.

Subsequent investigations, notably by M. Neumann and E. Albrecht, contributed greatly towards clarifying the abstract principle underlying the automatic continuity results. These authors established a number of automatic continuity results for Fréchet spaces and spaces of distributions. On leaving the framework of normed spaces one becomes aware of the dual role played by the unit ball of a normed space: it is a neighbourhood of zero and at the same time a bounded set. Of course, these two notions are essentially different in the general case and a careful analysis of the gliding hump method in Banach spaces is needed to realize in which of the two roles the unit ball appears at the different stages of the argument. We believe that the gliding hump lemma given in this note makes a clear distinction between the assumptions concerning continuity and boundedness and brings out their meaning for the proof. A careful analysis shows that completeness of the spaces is not essential: it suffices to assume σ-convexity for certain sets. E. Albrecht and M. Neumann proved, in [1], automatic continuity results for sublinear operators. Recently M. Neumann [17] succeeded in extending the classical uniform boundedness and closed graph theorems to convex operators; this makes it possible to prove our results for this larger class of mappings as well.

The paper is divided into six sections. In the first section we prove an abstract gliding hump theorem: it has reached now a remarkable degree of simplicity. It assumes the form of a statement about two families of subsets of a vector space. To demonstrate its power, we present, in Section 2, three examples of results where its application leads to quite simple proofs. At the same time they are selected to show in what manner the sets occurring in the statement of the gliding hump theorem are to be chosen in concrete situations. In Section 3, we collect some auxiliary material on ordered

topological vector spaces and convex operators so that the present paper can be read independently of [17]. We then proceed, in Section 4, to the main automatic continuity principles for convex operators. All these results are based on the gliding hump theory from Section 1. For our present purpose, the most important result is Theorem 4.4, which states, roughly speaking, that a convex operator leaving invariant certain families of closed linear subspaces has to be continuous on one of these subspaces. Actually, the precise formulation of this abstract principle is somewhat more general and applies both to causal operators and to operators of local type. These applications will be carried out in the last two sections; in particular, it will be seen that some important operators from the theory of linear systems are automatically continuous.

The list of publications at the end of this paper contains only papers immediately connected with the topics discussed here. The literature on automatic continuity is very extensive and the reader is referred to [8], [16], [21] for a fairly complete list of references up to 1980.

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1. The gliding hump. In this section, we present an abstract refinement of the classical method known as the gliding hump construction. It is not difficult to see that the proofs using the gliding hump method do not use completeness of the underlying space in its full force: it is, indeed, only the convergence of series of a certain type that is all we need. Thus it turns out that a weaker notion of completeness is sufficient for our considerations; it is the notion of σ-convexity — a notion which is more general than completeness and which possesses, at the same time, better permanence properties than (sequential) completeness.

For the convenience of the reader, we first recall the definition and some useful properties of σ-convex sets. Throughout this note, all topological vector spaces are assumed to be Hausdorff, but not necessarily locally convex. A subset K of a topological vector space X is said to be *σ-convex* if every countable convex combination of its elements converges to a point of K; more precisely, if the limit

$$x := \lim_{n \to \infty} \sum_{k=1}^{n} \sigma_k x_k$$

exists in X and belongs to K whenever $x_k \in K$ and $\sigma_k \geqslant 0$ such that $\sigma_1 + \sigma_2 + \ldots = 1$. Thus the σ-convex sets are exactly the CS-compact sets in the sense of G. Jameson [11].

To illustrate the generality as well as the permanence properties of the notion of σ-convexity, lest us mention the following facts: both the closed and the open unit ball of a Banach space are σ-convex; given a continuous linear operator between two Banach spaces, the image of the closed unit ball is σ-convex, but fails to be complete in general except when the range of the operator is closed.

In the following proposition, we collect some examples and results concerning σ-convex sets which are occasionally useful. The σ-convexity of all bounded convex G_δ-sets in a complete locally convex space was obtained in [10]. The proof of the remaining assertions is elementary and therefore omitted; let us also refer the reader to [10] and [11] for more information on σ-convex sets.

1.1. Proposition. (i) *Every σ-convex subset K of a topological vector space X is convex and bounded. Conversely, a bounded convex subset K of X is necessarily σ-convex if one of the following conditions is fulfilled*:

– *X is finite-dimensional.*
– *K is sequentially complete.*
– *K is open, and X is a Fréchet space.*
– *More generally: K is a G_δ-set, X is locally convex and complete.*

(ii) *If K is a σ-convex subset of X, then the absolutely convex hull of K and every continuous affine image of K are σ-convex as well.*

(iii) *If K is both σ-convex and absolutely convex in X, then the linear hull of K in X is a Banach space with respect to the Minkowski functional $\|\cdot\|_K$ corresponding to K. Moreover, the $\|\cdot\|_K$-topology is finer than the original topology on the linear hull of K.*

It is a well-known experience in functional analysis that it may be appropriate to reformulate certain results on linear operators as statements about families of subsets of the underlying spaces. This holds, in particular, whenever the Baire category theorem is involved. The same idea turns out to be useful in automatic continuity theory. Here, we shall present a series of abstract results concerning finite intersections of σ-convex sets, which will be applied later to various concrete situations.

For the remainder of this section, we consider a topological vector space X and an arbitrary nonempty set I. For all $\alpha \in I$ and $m \in N$, let $U(\alpha, m) \subset X$ be a balanced subset of X, and let $V(\alpha) \subset X$ be a σ-convex subset of X. The following *gliding hump conditions* will be used in the sequel:

(GH1) *$U(\alpha, m) + U(\alpha, n) \subset U(\alpha, m+n)$ for all $\alpha \in I$ and $m, n \in N$.*

(GH2) *There exists an $\alpha_0 \in I$ such that*

$$V(\alpha_0) \subset \bigcup_{m=1}^{\infty} \bigcap_{\alpha \in I} U(\alpha, m).$$

(GH3) *There exists an $n \in N$ such that $V(\alpha) \subset U(\alpha, n)$ for all $\alpha \in I$.*

(GH4) *For all $\alpha \in I$ there is an $n(\alpha) \in N$ such that $V(\alpha) \subset U(\alpha, n(\alpha))$.*

(GH5) *Given $\alpha \in I$ and $m, n \in N$, for all $x \in X \backslash U(\alpha, m)$ there exists an $\varepsilon > 0$ such that $U(\alpha, m)$ and $x + \varepsilon U(\alpha, n)$ are disjoint.*

In concrete applications concerning families of operators, we shall see that these conditions are fulfilled because of properties like linearity or convexity, pointwise boundedness, and continuity.

1.2. Proposition. *Assume that* (GH1) *and* (GH2) *are satisfied. Then there exist an $m \in N$ and finitely many $\alpha_1, \ldots, \alpha_r \in I$ such that*

$$\bigcap_{k=1}^{r} V(\alpha_k) \subset \bigcap_{\alpha \in I} [U(\alpha, m) - V(\alpha)].$$

Proof. By condition (GH2), for every $x \in V(\alpha_0)$ there exists a $p(x) \in N$ such that $x \in U(\alpha, p(x))$ for all $\alpha \in I$. We now assume that the assertion is false. Then there exist an $\alpha_1 \in I$ and an $x_1 \in V(\alpha_0)$ such that $x_1 \notin U(\alpha_1, 2) - V(\alpha_1)$. Moreover, for $n \geqslant 2$ we may choose by induction $\alpha_n \in I$ and $x_n \in V(\alpha_0) \cap \ldots \cap V(\alpha_{n-1})$ such that

$$x_n \notin U(\alpha_n, 2^n n + 2^{n-1} p(x_1) + \ldots + 2p(x_{n-1})) - V(\alpha_n).$$

Since $x_k \in V(\alpha_0)$ for all $k \in N$, we may consider

$$x := \sum_{k=1}^{\infty} (\tfrac{1}{2})^k x_k \in V(\alpha_0).$$

Now set $n := p(x)$ and observe that $x_j \in V(\alpha_n)$ if $j > n$. Hence

$$z := \sum_{j>n} (\tfrac{1}{2})^{n-j} x_j \in V(\alpha_n)$$

and clearly

$$x_n = 2^n x - \sum_{k=1}^{n-1} 2^{n-k} x_k - z.$$

Since $x \in U(\alpha_n, n)$ and $x_k \in U(\alpha_n, p(x_k))$ for all k, condition (GH1) yields

$$x_n \in U(\alpha_n, 2^n n + 2^{n-1} p(x_1) + \ldots + 2p(x_{n-1})) - V(\alpha_n).$$

This contradiction to our inductive choice completes the proof.

1.3. Proposition. *Assume that* (GH1), (GH2), (GH3) *are satisfied. Then there exist an $m \in N$ and finitely many $\alpha_1, \ldots, \alpha_r \in I$ such that*

$$\bigcap_{k=1}^{r} V(\alpha_k) \subset \bigcap_{\alpha \in I} U(\alpha, m).$$

Proof. According to Proposition 1.2, there exist a $q \in N$ and finitely many $\alpha_1, \ldots, \alpha_r \in I$ such that $V(\alpha_1) \cap \ldots \cap V(\alpha_r) \subset U(\alpha, q) - V(\alpha)$ for all $\alpha \in I$. By condition (GH3), the set on the right-hand side is contained in

$U(\alpha, q)+U(\alpha, n)$. From (GH1) it follows that $m := q+n$ possesses the required property.

The preceding proposition may be viewed as our main *gliding hump lemma* and will be the basis for most of the subsequent automatic continuity results. Sometimes, however, it may be difficult to construct appropriate sets $V(\alpha)$ and $U(\alpha, m)$ satisfying the strong condition (GH3), whereas the considerably weaker condition (GH4) can be fulfilled easily. We therefore supplement Proposition 1.3 with the following slight variant, where condition (GH3) is replaced by (GH4) and (GH5).

Let us note that condition (GH5) is not at all inconvenient for the applications we have in mind. Indeed, if $(B_m)_m$ is a sequence of closed, bounded, and balanced subsets B_m of some topological vector space Y and if $T\colon X \to Y$ denotes an arbitrary linear mapping, then one can check immediately that condition (GH5) holds for the typical choice $U(\alpha, m) := T^{-1}(B_m) \subset X$.

1.4. Proposition. *Assume that* (GH1), (GH2), (GH4), *and* (GH5) *are satisfied. Then there exist an $m \in N$ and finitely many $\alpha_1, \ldots, \alpha_r \in I$ such that*

$$\bigcap_{k=1}^{r} V(\alpha_k) \subset \bigcap_{\alpha\in I} U(\alpha, m).$$

Proof. Take $m \in N$ and $\alpha_1, \ldots, \alpha_r \in I$ from Proposition 1.2 and define W to be the absolutely convex hull of the set $V := V(\alpha_0) \cap V(\alpha_1) \cap \ldots \cap V(\alpha_r)$. From Proposition 1.1 we deduce that the linear hull Z of W is a Banach space with respect to the Minkowski functional $\|\cdot\|_W$ of W. We note that $W \subset U(\alpha, 2m+2n(\alpha))$ holds for all $\alpha \in I$. Indeed, if $w \in W$ is arbitrarily given, we have $w = sx - ty$ for suitable $x, y \in V$ and real $s, t \geq 0$ satisfying $s+t \leq 1$, and from (GH4) and (GH1) we conclude that

$$\begin{aligned} w = sx - ty &\in sU(\alpha, m) - sV(\alpha) - tU(\alpha, m) + tV(\alpha) \\ &\subset U(\alpha, m) + U(\alpha, n(\alpha)) + U(\alpha, m) + U(\alpha, n(\alpha)) \\ &\subset U(\alpha, 2m+2n(\alpha)) \end{aligned}$$

for all $\alpha \in I$. Next, we claim that for all $\alpha \in I$, $n \in N$ the intersection $Z \cap U(\alpha, n)$ is closed with respect to the norm $\|\cdot\|_W$. To see that, consider an $x \in Z \setminus U(\alpha, n)$. By condition (GH5) there exists an $\varepsilon > 0$ for which

$$U(\alpha, n) \cap [x + \varepsilon U(\alpha, 2m+2n(\alpha))] = \emptyset.$$

Since $W \subset U(\alpha, 2m+2n(\alpha))$ we see that $U(\alpha, n)$ and $x + \varepsilon W$ are disjoint, which proves our claim. It follows that each of the sets

$$Z_n := Z \cap \bigcap_{\alpha\in I} U(\alpha, n) \quad \text{for } n \in N$$

is closed with respect to $\|\cdot\|_W$. On the other hand, from (GH1) and (GH2) one easily deduces that $Z_n \uparrow Z$ as $n \to \infty$. Hence, by the Baire category theorem, there exist $k, n \in N$ and $z \in Z_n$ such that $z + (1/k)W \subset Z_n$. Using condition (GH1) again, we conclude that $W \subset U(\alpha, 2kn)$ for all $\alpha \in I$. The assertion follows.

In many applications, the index set I will be endowed with a canonical order structure, and in this case a slight modification of the preceding results will sometimes be more appropriate.

1.5. Proposition. *Let I be endowed with some transitive relation $<$ such that $V(\beta) \subset V(\gamma)$ holds for all $\beta, \gamma \in I$ satisfying $\beta < \gamma$, and assume that either* (GH1), (GH2), (GH3) *or* (GH1), (GH2), (GH4), (GH5) *are fulfilled. Then there exist an $m \in N$ and a $\beta \in I$ such that*

$$V(\beta) \subset U(\alpha, m) \quad \textit{for all } \alpha \in I \textit{ satisfying } \alpha < \beta.$$

Proof. We suppose again that the assertion is false and define $\beta_1 := \alpha_0$. Proceeding by induction, we then obtain $\beta_k \in I$ such that $\beta_{k+1} < \beta_k$ and $V(\beta_k) \not\subset U(\beta_{k+1}, k)$ for all $k \in N$. Let us consider

$$\tilde{V}(k) := V(\beta_k), \quad \tilde{U}(k, m) := U(\beta_k, m) \quad \text{for all } k, m \in N.$$

By 1.3 or 1.4 there exist an $m \in N$ and finitely many $k_1, \ldots, k_r \in N$ such that $\tilde{V}(k_1) \cap \ldots \cap \tilde{V}(k_r) \subset \tilde{U}(n, m)$ for all $n \in N$. But this implies $V(\beta_k) \subset U(\beta_{k+1}, k)$ whenever $k \geq k_1, \ldots, k_r, m$. This contradiction completes the proof.

2. Some examples. In this section we intend to collect several simple applications of the preceding general theory. Some of these results are already known; the main point here is to exhibit a common principle from which apparently different types of automatic continuity results may be deduced. Moreover, the present arguments should help to understand some more complicated constructions in the following sections. Here, we shall only be concerned with certain linear mappings between Banach spaces. In this context, there is a natural way to construct suitable σ-convex sets so that particularly short and easy proofs are available.

We start with an extension of the classical uniform boundedness principle. The ancestor of this result from [20] proved to be useful for the automatic continuity of mappings into spaces of linear operators, in particular of representations of Banach algebras. A similar result involving more general topological vector spaces and some further applications can be found in [15] and [16].

2.1. Proposition. *Let $\{T_\alpha\colon \alpha \in I\}$ be a pointwise bounded family of linear mappings $T_\alpha\colon X \to Y$ from a Banach space X into a normed linear space Y. Suppose that each operator T_α is continuous on some closed linear subspace X_α of X. Then there exist finitely many $\alpha(1), \ldots, \alpha(r) \in I$ such that $\{T_\alpha\colon \alpha \in I\}$ is equicontinuous on the intersection $X_{\alpha(1)} \cap \ldots \cap X_{\alpha(r)}$.*

Proof. For $\alpha \in I$ and $m \in N$ let

$$U(\alpha, m) := \{x \in X\colon \|T_\alpha(x)\| \leqslant m\}, \qquad V(\alpha) := \{x \in X_\alpha\colon \|x\| \leqslant (1 + \|T_\alpha|X_\alpha\|)^{-1}\}.$$

Then the conditions (GH1), (GH2), and (GH3) from Section 1 correspond to the linearity, the pointwise boundedness, and the continuity property of the mappings T_α, respectively. Hence the assertion follows immediately from Proposition 1.3.

We next turn to the continuity of derivations. A theorem of Singer and Wermer says that there are no nonzero *continuous* derivations on a semisimple commutative Banach algebra. According to a remarkable theorem due to B. E. Johnson [12], every derivation on a semisimple commutative Banach algebra is continuous; it follows that there are no nonzero derivations on such an algebra. The theorem of Johnson follows by a closed graph argument from the following proposition, which is crucial in the proof of his result; see also § 18 of [7].

2.2. Proposition. *Let D be a derivation on a semisimple commutative Banach algebra X. Then the set of those multiplicative linear functionals φ on X for which the composition φD is discontinuous on X is finite.*

Proof. Suppose there is a sequence of distinct multiplicative linear functionals φ_p on X such that $\varphi_p D$ is discontinuous for each $p \in N$. According to a lemma of B. E. Johnson [7; p. 93], there exists a sequence of elements $x_q \in X$ such that $\varphi_p(x_q) = 0$ for $p \leqslant q$ and $\varphi_p(x_q) \neq 0$ for $p > q$. Denote by U the closed unit ball of X. Replacing the x_q by suitable multiples, we may assume that $x_q \in U$ and $D(x_1 \ldots x_q) \in U$ for $q \in N$. Define, for all k, $m \in N$, the sets

$$U(k, m) := \{x \in X\colon |\varphi_k D(x)| \leqslant m\} \quad \text{and} \quad V(k) := x_1 \ldots x_k U.$$

Note that the sets $V(k)$ are σ-convex and satisfy $V(k+1) \subset V(k)$ for all $k \in N$. Using $\varphi_k(x_k) = 0$, we easily establish the inclusion $V(k) \subset U(k, 1)$ for all $k \in N$. According to Proposition 1.3, there exists an m such that $V(m) \subset U(k, m)$ for all $k \in N$ so that $\varphi_k D(x_1 \ldots x_m)$ is continuous on X for every $k \in N$. Now, D being a derivation, we have

$$\varphi_k D(x_1 \ldots x_m x) = \varphi_k(x_1 \ldots x_m)\, \varphi_k D(x) + \varphi_k(x)\, \varphi_k D(x_1 \ldots x_m) \quad \text{for all } x \in X.$$

Since $\varphi_k(x_1 \ldots x_m) = \varphi_k(x_1) \ldots \varphi_k(x_m)$ is different from zero as soon as $k > m$, we conclude that $\varphi_k D$ is continuous on X for all $k > m$. This contradiction completes the proof.

We close with a useful continuity principle, which applies both to causal and to local linear operators. A more sophisticated version of this result will be established later. Given a topological vector space X, let $\mathscr{S}(X)$ stand for the family of all closed linear subspaces of X. On $\mathscr{S}(X)$ we consider the order structure given by inclusion.

2.3. Proposition. *Suppose we are given two Banach spaces X and Y, a linear mapping $T\colon X \to Y$, a set $\mathscr{F}$ endowed with a transitive relation $<$, and a pair of monotone mappings $\mathscr{E}_X\colon \mathscr{F} \to \mathscr{S}(X)$ and $\mathscr{E}_Y\colon \mathscr{F} \to \mathscr{S}(Y)$. Suppose that the following two conditions are satisfied:*

(i) $T\mathscr{E}_X(F) \subset E_Y(F)$ *for all* $F \in \mathscr{F}$.

(ii) $\bigcap_{G<F} \mathscr{E}_Y(G) = \{0\}$ *for all* $F \in \mathscr{F}$.

Then T is continuous on $\mathscr{E}_X(F)$ for some $F \in \mathscr{F}$.

Proof. For $F \in \mathscr{F}$ and $m \in N$ let

$$V(F) := \{x \in \mathscr{E}_X(F)\colon \|x\| \leqslant 1\}, \qquad U(F, m) := \{x \in X\colon \|\pi_F T(x)\| \leqslant m\}$$

where $\pi_F\colon Y \to Y/\mathscr{E}_Y(F)$ denotes the canonical quotient map. Then conditions (GH1) and (GH2) from Section 1 are obviously fulfilled if $I := \mathscr{F}$. It follows from assumption (i) that $\pi_F TV(F) = \{0\}$ for all $F \in \mathscr{F}$, so that condition (GH3) is satisfied as well. Thus condition (i) alone guarantees, by Proposition 1.5, the existence of an $F \in \mathscr{F}$ and an $m \in N$ such that $V(F) \subset U(G, m)$ whenever G satisfies $G < F$. It follows that $\pi_G T|\mathscr{E}_X(F)\colon \mathscr{E}_X(F) \to Y/\mathscr{E}_Y(G)$ is continuous for all $G \in \mathscr{F}$ with $G < F$. From the additional assumption (ii) one easily deduces that the restriction $T|\mathscr{E}_X(F)\colon \mathscr{E}_X(F) \to Y$ is closed and hence continuous by the closed graph theorem.

The preceding result is related to the automatic continuity theory for generalized local operators from [2], but the technicalities are somewhat different. Moreover, the present approach includes the case of certain causal operators. To indicate a typical example, let us consider a linear operator $T\colon L^p(\boldsymbol{R}) \to L^p(\boldsymbol{R})$ for some $1 \leqslant p \leqslant \infty$. Recall that T is said to be *causal* if for all $f \in L^p(\boldsymbol{R})$ and $t \in \boldsymbol{R}$ we have:

$$\operatorname{supp} f \subset [t, \infty[\;\Rightarrow\; \operatorname{supp} Tf \subset [t, \infty[.$$

It follows from a construction in [1; p. 269] that causal operators on $L^p(\boldsymbol{R})$ need not be continuous. However, the following positive result holds:

2.4. Corollary. *For every causal linear operator T on $L^p(\boldsymbol{R})$ there exists a $t \in \boldsymbol{R}$ such that T is continuous on the subspace $\{f \in L^p(\boldsymbol{R})\colon \operatorname{supp} f \subset [t, \infty[\}$.*

Proof. Let $X := Y := L^p(\boldsymbol{R})$, endow the family $\mathscr{F}$ of all intervals $[t, \infty[$ for $t \in \boldsymbol{R}$ with the transitive relation given by inclusion, and for each $t \in \boldsymbol{R}$ define $\mathscr{E}_X([t, \infty[) := \mathscr{E}_Y([t, \infty[) := \{f \in L^p(\boldsymbol{R})\colon \operatorname{supp} f \subset [t, \infty[\}$. The assertion is then an immediate consequence of Proposition 2.3.

Continuity on the whole space may be obtained if causality is replaced by a certain stronger condition which makes sense for an arbitrary open subset G of $\mathbf{R}^n$: A linear mapping $T: L^p(G) \to L^p(G)$ is called *local* if

$$\operatorname{supp} Tf \subset \operatorname{supp} f \quad \text{for all } f \in L^p(G).$$

The following easy consequence of Proposition 2.3 is also contained in a recent somewhat more involved result from [3; p. 351]. Later we shall examine a considerably more general situation.

2.5. Corollary. *Every local linear operator T on $L^p(G)$ is continuous.*

Proof. First, let $\mathscr{F}$ be the family of all open subsets F of G such that $G \setminus F$ is compact, with the order relation given by inclusion. Then Proposition 2.3 applies to $X := Y := L^p(G)$ and $\mathscr{E}_X(F) := \mathscr{E}_Y(F) := \{f \in L^p(G)\colon \operatorname{supp} f \subset F^-\}$ for all $F \in \mathscr{F}$. Hence T is continuous on the linear subspace $X(\infty) := \{f \in X\colon \operatorname{supp} f \cap K = \emptyset\}$ for a suitable compact subset K of G. Next, for every $t \in G$, let $\mathscr{F}_t$ be the family of all compact neighbourhoods of t, ordered again by inclusion. Then another application of Proposition 2.3 yields some $U(t) \in \mathscr{F}_t$ such that T is continuous on the subspace $X(t) := \{f \in X\colon \operatorname{supp} f \subset U(t)\}$ for every $t \in G$. As K is compact, we obtain finitely many points $t_1, \ldots, t_r \in K$ and continuous functions $\varphi_1, \ldots, \varphi_r \in C(G)$ such that $\operatorname{supp} \varphi_j \subset U(t_j)$ for $j = 1, \ldots, r$ and $\varphi_1 + \ldots + \varphi_r = 1$ on some neighbourhood of K. Let $\varphi_0 := 1 - \varphi_1 - \ldots - \varphi_r$, $M_0 := \|T|X(\infty)\|$ and $M_j := \|T|X(t_j)\|$ for $j = 1, \ldots, r$. Then

$$\|Tf\|_p = \Big\|\sum_{j=0}^{r} T(\varphi_j f)\Big\|_p \leqslant \sum_{j=0}^{r} M_j \|\varphi_j\|_\infty \|f\|_p$$

for all $f \in L^p(G)$, so that T is continuous on $L^p(G)$.

3. Preliminaries on convex operators. We start by recalling some notions from the theory of ordered topological vector spaces; see for instance [19] or [23]. Given an ordered vector space Y, a subset A of Y is said to be *full* if for all $u, w \in A$ and $v \in Y$ satisfying $u \leqslant v \leqslant w$ it follows that $v \in A$. We denote by $[A]$ the full hull of an arbitrary subset A of Y; clearly $[A] = \{v \in Y\colon$ there exist $u, w \in A$ such that $u \leqslant v \leqslant w\}$. If Y is endowed with a vector space topology, then the positive cone Y_+ of Y is called *normal* if there exists a neighbourhood-base at 0 consisting of full sets. Many standard spaces from analysis are ordered by a normal cone. For example, the usual cone of nonnegative functions is normal for quite a lot of function spaces, including the spaces $L^p(\mu)$ for an arbitrary positive measure μ and any $0 < p \leqslant \infty$.

Now, let D be a convex subset of a vector space X, and consider an ordered vector space Y. Then an operator $T: D \to Y$ is said to be *convex* if $T(su + (1-s)v) \leqslant sT(u) + (1-s)T(v)$ holds for all $u, v \in D$ and all real $0 \leqslant s \leqslant 1$. It is shown in [17] that some basic principles of linear functional analysis remain valid for convex operators, provided the positive cone of the range space is normal. Here, we shall only need the following three results from [17]. The first assertion generalizes an elementary property of real-valued convex functionals, the second statement is the principle of uniform boundedness for convex operators, and the last result is a version of the closed graph theorem which is well suited for our present purpose.

Let us note that the trivial cone $Y_+ = \{0\}$ is normal for every vector space topology. Hence the following results apply to affine operators and, in particular, to linear operators without any order-theoretic restriction on the range space.

3.1. Theorem. *Let X and Y be topological vector spaces, let D denote an open convex subset of X, and suppose that Y is ordered by a normal cone.*

(i) *If a convex operator $T: D \to Y$ is continuous at some point of D, then T is continuous on D.*

(ii) *Assume that X is ultrabarrelled, resp. barrelled if X and Y are locally convex. Then every pointwise bounded family of continuous convex operators from D into Y is equicontinuous on D.*

(iii) *Assume that X is a Baire space and that Y is countably boundedly generated, boundedly summing, and sequentially complete. Then every convex operator $T: D \to Y$ having a closed graph in $D \times Y$ is continuous on D.*

Recall that a topological vector space Y is said to be *countably boundedly generated* if it is the union of some countable family of bounded subsets, and Y is called *boundedly summing* if for every bounded subset B of Y there exists another bounded set C and a sequence fo real $\sigma_k > 0$ such that $\sigma_1 B + \ldots + \sigma_k B \subset C$ for all $k \in \mathbf{N}$. It is well known and easily seen that the metrizable, the almost convex, and the locally pseudoconvex spaces are all boundedly summing; see [5; p. 76]. To give a concrete example, let us note that the spaces $L^p(\mu)$ for any $0 < p \leqslant \infty$ satisfy all the assumptions of the preceding theorem.

We close with a useful technical characterization concerning the continuity of convex operators between certain topological vector spaces.

3.2. Lemma. *Let X and Z be topological vector spaces, and suppose that X is a Baire space and that Z is countably boundedly generated and ordered by a normal cone. Moreover, choose a sequence of balanced bounded subsets H_m of Z such that*

$$H_m + H_m \subset H_{m+1} \quad \text{for all } m \in \mathbf{N} \quad \text{and} \quad H_m \uparrow Z \quad \text{as } m \to \infty.$$

Finally, let $D \subset X$ be open and convex, and consider a convex operator $S: D \to Z$. Then the following assertions are equivalent:

(i) *S is continuous on D.*

(ii) *For every $x \in D$ there exist an open neighbourhood $U = U(x)$ of x in D and an integer $m = m(x)$ such that $S(U) \subset [H_m^-]$.*

(iii) *There exist a nonempty open subset U of D and an $m\in N$ such that* $S(U)\subset [H_m^-]$.

Proof. (i)$\Rightarrow$(ii). Given an arbitrary $x\in D$, let W be a closed balanced neighbourhood of zero in X such that $x+W+W\subset D$ and define

$$C_k := \{w\in W:\ S(x\pm 2w)\in S(x)+H_k^-\}\quad \text{for all } k\in N.$$

These sets are closed and satisfy

$$X=\bigcup_{j=1}^{\infty}\bigcup_{k=1}^{\infty} jC_k.$$

Since X is a Baire space, there exist a $k\in N$, a $v\in C_k$, and a balanced neighbourhood of zero $U\subset W$ such that $v+U\subset C_k$. Now, given any $u\in U$, we shall show that

$$S(x+u)\subset S(x)+[\tfrac{1}{2}(H_k^-+H_k^-)].$$

Since $u+v$, $v\in C_k$, we have $S(x+2(u+v))=S(x)+h_1$ and $S(x-2v)=S(x)+h_2$ for suitable $h_1, h_2\in H_k^-$. This implies by convexity

$$\begin{aligned}S(x+u) &= S\big(\tfrac{1}{2}(x+2(u+v))+\tfrac{1}{2}(x-2v)\big)\\ &\leqslant \tfrac{1}{2}S\big(x+2(u+v)\big)+\tfrac{1}{2}S(x-2v)=S(x)+\tfrac{1}{2}(h_1+h_2).\end{aligned}$$

Applying the same argument to $-u\in U$, we obtain

$$S(x-u)\leqslant S(x)+\tfrac{1}{2}(h_3+h_4)$$

for suitable $h_3, h_4\in H_k^-$. Since $U\subset W$, we have

$$S(x)=S\big(\tfrac{1}{2}(x+u)+\tfrac{1}{2}(x-u)\big)\leqslant \tfrac{1}{2}S(x+u)+\tfrac{1}{2}S(x-u),$$

whence $S(x)-S(x-u)\leqslant S(x+u)-S(x)$. We conclude that

$$-\tfrac{1}{2}(h_3+h_4)\leqslant S(x+u)-S(x)\leqslant \tfrac{1}{2}(h_1+h_2),$$

which proves our claim. Let $l\geqslant k+1$ such that $S(x)\in H_l$. Then it follows that $S(x+U)\subset [H_{l+1}^-]$, so that $U(x):=x+U$ and $m(x):=l+1$ have the required property.

(ii)$\Rightarrow$(iii) is immediate.

(iii)$\Rightarrow$(i). By assumption, there exist an $x\in D$, an $m\in N$, and a balanced neighbourhood of zero U in X such that $x+U\subset D$ and $S(x+U)\subset [H_m^-]$. By the first assertion in Theorem 3.1, it suffices to prove the continuity of S at the point x. Let V be an arbitrary balanced neighbourhood of zero in Z. Then there exists an $\varepsilon>0$ such that $\varepsilon([H_m^-]+[H_m^-])\subset V$. If $\varepsilon\geqslant 1$ we have $S(x+U)-S(x)\subset V$. Now suppose that $0<\varepsilon<1$. Then

$$S(x)=S\big(\tfrac{1}{2}(x+\varepsilon u)+\tfrac{1}{2}(x-\varepsilon u)\big)\leqslant \tfrac{1}{2}S(x+\varepsilon u)+\tfrac{1}{2}S(x-\varepsilon u),$$

whence $S(x)-S(x-\varepsilon u)\leqslant S(x+\varepsilon u)-S(x)$. Furthermore, we have

$$\begin{aligned}S(x+\varepsilon u)-S(x) &= S\big(\varepsilon(x+u)+(1-\varepsilon)x\big)-S(x)\\ &\leqslant \varepsilon S(x+u)+(1-\varepsilon)S(x)-S(x)\\ &= \varepsilon\big(S(x+u)-S(x)\big).\end{aligned}$$

Replacing u by $-u$, we obtain

$$S(x-\varepsilon u)-S(x)\leqslant \varepsilon\big(S(x-u)-S(x)\big).$$

Combining all these estimates, we arrive at

$$-\varepsilon\big(S(x-u)-S(x)\big)\leqslant S(x+\varepsilon u)-S(x)\leqslant \varepsilon\big(S(x+u)-S(x)\big),$$

so that $S(x+\varepsilon U)-S(x)\subset [V]$. This implies the continuity of S at the point x, since the positive cone of Z is normal. The assertion follows.

Note that the implication (iii)$\Rightarrow$(i) of the preceding lemma does not depend on the assumption that X is a Baire space; it corresponds to the well-known fact that a linear mapping between topological vector spaces is necessarily continuous if it is bounded on some neighbourhood of zero. In the sequel, we shall sometimes use the following simple observation.

3.3. Remark. Let $T\colon D\to Y$ be a mapping from a topological space D into a topological vector space Y, and let $\{\pi_F\colon F\in\mathscr{F}\}$ be a family of continuous linear mappings $\pi_F\colon Y\to Y_F$ from Y into topological vector spaces Y_F. Suppose that this family is separating in the sense that the intersection of the kernels of the π_F is zero. If the composition $\pi_F T\colon D\to Y_F$ is continuous for each $F\in\mathscr{F}$, then the graph of T is closed in $D\times Y$.

Indeed, if $x\in D$ and $y\in Y$ are such that $(x, y)\notin G(T)$, then $y-Tx\neq 0$, so that there exists an $F\in\mathscr{F}$ with $\pi_F y\neq \pi_F Tx$. Since $\pi_F T$ is continuous, there exist a neighbourhood U of x in D and a neighbourhood V of $\pi_F y$ in Y_F such that $(U\times V)\cap G(\pi_F T)=\emptyset$. It follows that $(U\times \pi_F^{-1}V)\cap G(T)=\emptyset$, so that $(x, y)\notin G(T)^-$.

4. Some general principles of automatic continuity. We first extend a fundamental continuity result from [1; p. 254] to the case of convex operators. From the technical point of view, this situation is considerably more involved than the former one; the present proof will be based on a suitable combination of our results 1.3 and 3.2. We regret that Theorem 4.1 does not look very attractive at first glance. But even the more restrictive and complicated version of this principle from [1; p. 254] has already proved to be very useful for various types of automatic continuity problems including, for instance, the case of translation-invariant and causal linear operators between spaces of functions or distributions [1]. And we shall present some consequences of Theorem 4.1, which are still quite useful and which are much easier to understand and to handle. To formulate this theorem, we assume the following *situation*, which generalizes the essential features of the standard situation given in [1; p. 254]:

(a) Let A be a nonempty set endowed with a transitive relation $<$.

(b) Consider a topological vector space X and, for each $\alpha \in A$, continuous linear operators $S_\alpha: X_\alpha \to X$ and $Q_\alpha: W_\alpha \to X$ from (F)-spaces X_α and W_α into X such that $S_\beta(X_\beta) \subset Q_\alpha(W_\alpha)$ holds for all $\alpha, \beta \in A$ satisfying $\alpha < \beta$.

(c) For each $\alpha \in A$, let Y and Y_α be ordered topological vector spaces such that the positive cone of Y_α is normal, and consider a continuous positive linear operator $\pi_\alpha: Y \to Y_\alpha$.

Let us note that, in most applications, $X_\alpha = W_\alpha$ and $S_\alpha = Q_\alpha$ for all $\alpha \in A$. Moreover, the condition on the ranges in (b) is fulfilled whenever the following factorization property holds: for all $\alpha, \beta \in A$ satisfying $\alpha < \beta$ there exists a linear mapping $S_{\alpha\beta}: X_\beta \to W_\alpha$ such that $S_\beta = Q_\alpha S_{\alpha\beta}$.

4.1. THEOREM. *Suppose that Y is countably boundedly generated, and consider an open convex subset D of X such that $0 \in D$. Moreover, let $T: D \to Y$ be a convex operator such that*

$$\pi_\gamma TQ_\beta \quad \text{is continuous on } Q_\beta^{-1}(D)$$

for all $\beta, \gamma \in A$ with $\gamma < \beta$. Then there exists an $\alpha \in A$ such that

$$\pi_\gamma TS_\alpha \quad \text{is continuous on } S_\alpha^{-1}(D)$$

for all $\gamma \in A$ with $\alpha < \gamma$.

Proof. Replacing T by $T - T(0)$ if necessary, we may assume $T(0) = 0$. Since Y is countably boundedly generated, we may choose a sequence of balanced bounded subsets B_m of Y such that $B_m + B_m \subset B_{m+1}$ for all $m \in N$ and $B_m \uparrow Y$ as $m \to \infty$. Now assume that the assertion is false. Then we construct, by induction, a sequence of indices $\alpha(k) \in A$ satisfying $\alpha(k) < \alpha(k+1)$ for all $k \in N$ such that the convex operator

$$\pi_{\alpha(k+1)} TS_{\alpha(k)}: S_{\alpha(k)}^{-1}(D) \to Y_{\alpha(k+1)}$$

is discontinuous for each $k \in N$. For brevity, we write k for $\alpha(k)$. Given an arbitrary $k \in N$, we choose an (F)-norm $|\cdot|_k$ on W_k which generates the topology of the metrizable space W_k. By assumption, we know that the convex operator

$$\pi_k TQ_{k+1}: Q_{k+1}^{-1}(D) \to Y_k$$

is continuous, so that Lemma 3.2 may be applied to this operator and to the sets $H_m := \pi_k(B_m)$ for all $m \in N$. Hence there exists a $\delta_{k+1} > 0$ and an $m(k) \in N$ such that the 0-neighbourhood $N_{k+1} := \{z \in W_{k+1}: |z|_{k+1} \leq \delta_{k+1}\}$ in W_{k+1} satisfies

$$Q_{k+1}(N_{k+1}) \subset D \quad \text{and} \quad \pi_k TQ_{k+1}(N_{k+1}) \subset [(\pi_k B_{m(k)})^{-}].$$

On the other hand, the convex operator $\pi_{k+1} TS_k$ is known to be discontinuous on $S_k^{-1}(D)$. Using Lemma 3.2 again, we conclude that

$$\pi_{k+1} TS_k(U) \not\subset [(\pi_{k+1} B_{km(k+1)})^{-}]$$

for every 0-neighbourhood $U \subset S_k^{-1}(D)$. Now, for $j = 1, \ldots, k-1$ we endow the range $Q_j(W_j)$ with the (F)-space topology coming from W_j so that $Q_j(W_j)$ is isomorphic to the quotient $W_j/\mathrm{Ker}\, Q_j$. Since $S_k: X_k \to X$ is continuous with respect to the original topology of X and satisfies $S_k(X_k) \subset Q_j(W_j)$, it follows that $S_k: X_k \to Q_j(W_j)$ is closed with respect to the (F)-space topology on $Q_j(W_j)$ and hence continuous by the classical closed graph theorem. Thus $S_k^{-1}(Q_j(M))$ is a 0-neighbourhood in X_k for every 0-neighbourhood M in W_j. Hence we obtain an $x_k \in S_k^{-1}(D)$ and $w_{jk} \in W_j$ for $j = 2, \ldots, k-1$ such that the following conditions are fulfilled:

$$\pi_{k+1} TS_k\left(\frac{1}{2^k} x_k\right) \notin [(\pi_{k+1} B_{km(k+1)})^{-}],$$

$$S_k(x_k) = Q_j(w_{jk}), \qquad |w_{jk}|_j \leq \frac{1}{2^k} \delta_j$$

for $j = 2, \ldots, k-1$. Now, let $n \in N$ be arbitrarily given. Then it is easily seen that all the σ-convex combinations exist in the (F)-space W_{n+1} for any sequence of vectors $\pm w_{n+1,k}$ where $k \geq n+2$. Denote the set of all these σ-convex combinations by

$$Z_{n+1} := \sigma\text{-co}\{\pm w_{n+1,k}: k \geq n+2\}.$$

Obviously Z_{n+1} is contained in N_{n+1}. Moreover, it is not hard to check that Z_{n+1} is σ-convex, a fact which can also be deduced from Proposition 2 and Theorem 3 in [11]. Being the continuous affine image of a σ-convex set,

$$V(n) := Q_{n+1}(Z_{n+1}) \subset D$$

turns out to be σ-convex as well. Furthermore, by our construction we have

$$V(n) = \sigma\text{-co}\{\pm S_k(x_k): k \geq n+2\}.$$

Hence $V(n+1) \subset V(n)$, and it is clear that

$$\pi_n T(V(n)) = \pi_n TQ_{n+1}(Z_{n+1}) \subset [(\pi_n B_{m(n)})^{-}]$$

for all $n \in N$. We finally introduce the sets

$$U(n, p) := 2^{p-1}\{x \in X: \pm x \in D, \pi_n T(\pm x) \in [(\pi_n B_{pm(n)})^{-}]\}$$

for all $n, p \in N$ and claim that the conditions (GH1), (GH2), (GH3) from Section 1 are satisfied in this situation. Certainly we have $V(n) \subset U(n, 1)$ for all $n \in N$, so that (GH3) is fulfilled. Since $\pi_n T$ is convex with $\pi_n T(0) = 0$, one easily verifies that the sets $U(n, p)$ are balanced and satisfy $U(n, p) \subset$

$\subset U(n, p+1)$ for all $n, p \in N$. In order to check condition (GH2), we fix an arbitrary $x \in V(1)$. Then there exists a $p \in N$ such that $T(\pm x) \in B_p$, which implies $x \in U(n, p)$ for all $n \in N$. It remains to show that (GH1) is fulfilled. Let $n, p \in N$ and consider $x_1, x_2 \in X$ such that

$$\pm x_j \in D \quad \text{and} \quad \pi_n T(\pm x_j) \in [(\pi_n B_{pm(n)})^-] \quad \text{for } j = 1, 2.$$

Then $x := \frac{1}{2}(x_1 + x_2)$ satisfies $\pm x \in D$ as well as

$$u \leqslant -\tfrac{1}{2}\pi_n T(-x_1) - \tfrac{1}{2}\pi_n T(-x_2) \leqslant -\pi_n T(-x)$$
$$\leqslant \pi_n T(x) \leqslant \tfrac{1}{2}\pi_n T(x_1) + \tfrac{1}{2}\pi_n T(x_2) \leqslant v$$

for a suitable pair of elements

$$u, v \in \big(\pi_n(B_{pm(n)} + B_{pm(n)})\big)^- \subset (\pi_n B_{pm(n)+1})^-.$$

We conclude that

$$\pi_n T(\pm x) \in [(\pi_n B_{(p+1)m(n)})^-].$$

This implies $U(n, p) + U(n, p) \subset U(n, p+1)$ and hence by monotonicity $U(n, p) + U(n, q) \subset U(n, p+q)$ for all $n, p, q \in N$. Now we are in a position to apply the gliding hump lemma 1.3. Using again the monotonicity properties of the families of sets $V(n)$ and $U(n, p)$, we thus arrive at a $p \in N$ such that $V(p) \subset U(n, p)$ holds for every $n \in N$. In particular, it follows that

$$\pi_n TS\left(\frac{1}{2^k} x_h\right) \in [(\pi_n B_{pm(n)})^-]$$

for all $k \geqslant p+2$ and all n. Take an arbitrary $k \geqslant p+2$ and set $n := k+1$. Then we obtain

$$\pi_{k+1} TS_k\left(\frac{1}{2^k} x_k\right) \in [(\pi_{k+1} B_{km(k+1)})^-].$$

This contradiction to the choice of x_k completes the proof.

Let us note that some parts of the preceding proof can be considerably simplified in the case of a linear mapping $T: X \to Y$. For instance, in this case it appears to be quite natural to choose x_k in a suitable neighbourhood of zero so that $\pi_{k+1} TS_k(x_k) \notin (\pi_{k+1} B_k)^-$ and to consider the sets $U(n, p) := \{x \in X: \pi_n T(x) \in (\pi_n B_p)^-\}$ for all $n, p \in N$. Then one easily arrives at the desired contradiction by means of Proposition 1.4. We now state an illuminating special case of the preceding result.

4.2. Theorem. *Consider a sequence of (F)-spaces X_n for $n = 0, 1, 2, \ldots$ and continuous linear operators $S_n: X_n \to X_{n-1}$ for all $n \in N$. Moreover, for $n = 0, 1, 2, \ldots$ let Y_n be ordered topological vector spaces such that Y_0 is countably boundedly generated and the positive cone of Y_n is normal for each $n \in N$, and consider continuous positive linear operators $\pi_n: Y_0 \to Y_n$ for all $n \in N$. Finally, let D be an open convex subset of X_0 with $0 \in D$, and let $T: D \to Y_0$ denote a convex operator. Assume that*

$$\pi_n TS_1 \ldots S_n: (S_1 \ldots S_n)^{-1}(D) \to Y_n$$

is continuous for all $n \in N$. Then there exists an $n \in N$ such that

$$\pi_k TS_1 \ldots S_n: (S_1 \ldots S_n)^{-1}(D) \to Y_k$$

is continuous for all $k \in N$.

Proof. Endow $A := N$ with the usual order relation $\leqslant$ and consider the spaces $\tilde{X}_n := \tilde{W}_n := X_n$ and operators $\tilde{S}_n := \tilde{Q}_n := S_1 \ldots S_n$ for all $n \in N$. Then the assertion follows from Theorem 4.1.

4.3. Remarks. (i) It is interesting to observe that the continuity assumption of Theorem 4.2 means exactly the continuity of the operators $\pi_k TS_1 \ldots S_n$ for all $k, n \in N$ satisfying $k \leqslant n$. And the conclusion states that, for a suitable n, the operators $\pi_k TS_1 \ldots S_n$ are continuous for all $k \geqslant n$.

(ii) The theorem ensures for some n the continuity of the convex operator $TS_1 \ldots S_n$ with respect to the projective topology on Y_0 generated by the mappings π_k for $k \in N$. Under certain additional assumptions, this forces $TS_1 \ldots S_n$ to be continuous with respect to the (finer) original topology on Y_0. Indeed, in the situation of Theorem 4.2 suppose, in addition, that the following two conditions are fulfilled:

1° The intersection of the kernels of the linear operators π_k for $k \in N$ is zero.

2° Y_0 is boundedly summing and sequentially complete, and its positive cone is normal.

Then it follows from 3.3 that $TS_1 \ldots S_n$ is closed and hence continuous for the original topology on Y_0 by the closed graph theorem for convex operators stated in Theorem 3.1. For applications to concrete spaces of analysis, the assumptions on Y_0 are not very restrictive, whereas the construction of suitable mappings π_k turns out to be crucial.

(iii) Theorem 4.2 is the natural convex extension of Satz 1.4 in [1] and can be applied in a similar fashion; see [1] and [16] for a discussion of the topological assumptions and for several applications. If one specializes Theorem 4.2 to the case of linear operators between Banach spaces, then the result can be seen to be equivalent to a basic stability theorem due to K. B. Laursen [13]; see also [1; p. 259], where this stability theorem is generalized to the case of (F)-spaces.

(iv) Theorem 4.2 provides a common generalization of Propositions 2.2 and 2.3 in [22] concerning the continuity of certain linear mappings on Banach algebras. Moreover, we also obtain a positive answer to a conjecture of J. D. Stein stated in [22; p. 196]; it turns out that the continuity ideals considered there cannot be disjoint. The details are very easy and therefore omitted.

(v) By a suitable combination of Theorem 4.2 and part (ii) of Theorem 3.1, one may extend the uniform boundedness principle given in Proposition 2.1 to the case of convex operators mapping an open convex subset of an (F)-space into a topological vector space which has a fundamental sequence of bounded sets and is ordered by a normal cone. We omit the proof which is sufficiently similar to the corresponding argument for linear operators given in [16; p. 274].

We close this section with another useful application of Theorem 4.1, which generalizes our previous result 2.3 and which will be seen to be basic for the continuity properties of causal operators and of operators of local type.

We consider the following situation. Let $\mathscr{F}$ be a nonempty set endowed with a transitive relation $<$, consider an (F)-space X, and let Y denote a topological vector space which is countably boundedly generated, boundedly summing, sequentially complete, and ordered by a normal cone. Moreover, consider a pair of monotone mappings $\mathscr{E}_X\colon \mathscr{F}\to\mathscr{S}(X)$ and $\mathscr{E}_Y\colon \mathscr{F}\to\mathscr{S}(Y)$, and for each $F\in\mathscr{F}$ let $\pi_F\colon Y\to Y_F$ be a continuous positive linear mapping from Y into a topological vector space Y_F which is ordered by a normal cone. Then we have the following *basic principle*:

4.4. THEOREM. *Let D be an open convex subset of X satisfying $0\in D$, and consider a convex operator $T\colon D\to Y$. Assume that the following conditions are fulfilled:*

(BP1) *The restriction $\pi_G T|\mathscr{E}_X(F)\cap D$ is continuous for all $F, G\in\mathscr{F}$ with $F<G$.*

(BP2) *The kernel of π_F is contained in $\mathscr{E}_Y(F)$ for every $F\in\mathscr{F}$.*

(BP3) $\bigcap_{G<F}\mathscr{E}_Y(G)=\{0\}$ *for every $F\in\mathscr{F}$.*

Then the restriction $T|\mathscr{E}_X(F)\cap D$ is continuous for some $F\in\mathscr{F}$.

Proof. We endow $A:=\mathscr{F}$ with the transitive relation inverse to the given relation $<$ and consider the spaces $X_F:=W_F:=\mathscr{E}_X(F)$ for all $F\in\mathscr{F}$. Moreover, for $F\in\mathscr{F}$ we define S_F and Q_F to be the inclusion operator from $\mathscr{E}_X(F)$ into X. By Theorem 4.1 there exists an $F\in\mathscr{F}$ such that the restriction $\pi_G T|\mathscr{E}_X(F)\cap D\colon \mathscr{E}_X(F)\cap D\to Y_G$ is continuous for every $G\in\mathscr{F}$ with $G<F$. According to Remark 3.3, it follows from our assumptions (BP2) and (BP3) that the convex operator $T|\mathscr{E}_X(F)\cap D\colon \mathscr{E}_X(F)\cap D\to Y$ has a closed graph. Hence the version of the closed graph theorem given in Theorem 3.1 finishes the proof.

In the preceding result, there is a natural choice for the spaces Y_F and operators $\pi_F\colon Y\to Y_F$. Indeed, one may take π_F to be the canonical quotient mapping from Y onto the quotient space $Y/\mathscr{E}_Y(F)$ provided that the canonical positive cone $\pi_F(Y_+)$ in $Y/\mathscr{E}_Y(F)$ is normal with respect to the quotient topology. This condition is certainly fulfilled for the case $Y_+=\{0\}$, which is relevant to the theory of affine operators. In general, however, the positive cone of the quotient space need not be normal again: there are elementary examples of closed linear subspaces of Banach lattices such that the corresponding positive cone in the quotient space is closed and pointed, but *fails* to be normal. The authors are grateful to Peter Greim for providing them with a simple example of this type involving sequence spaces. On the other hand, in most cases of practical interest the positive cone of the quotient space will be normal. This is due to the well-known fact that for every ideal of a locally solid Riesz space the corresponding quotient space is again a locally solid Riesz space, so that its positive cone turns out to be normal; see for instance Theorem 5.9 in [6].

5. Causal operators. The automatic continuity problem for causal linear operators on $L^p(\boldsymbol{R})$ was already studied in Section 2. Here, we shall investigate some considerably more general situations.

First, let $(G,\circ)$ be a locally compact group endowed with a left-invariant regular Haar measure λ, let M denote an arbitrary closed subset of G, and consider a commutative subsemigroup A of G. We assume that for each $t\in M$ the following *compatibility conditions* are fulfilled:

$$\alpha\circ t\in M \quad \text{for all } \alpha\in A;\qquad \alpha^{-1}\circ t\notin M \text{ for some } \alpha\in A.$$

Further, given any $0<p\leqslant\infty$, let $X=X^p(M)$ denote the space of all $f\in L^p(G)$ vanishing on the complement $M^c:=G\setminus M$, and for each $\alpha\in A$ let X_α consist of all $f\in L^p(G)$ vanishing on $(\alpha\circ M)^c$. Because of the first compatibility condition on M and A, the spaces X_α are closed linear subspaces of X. For each $\alpha\in A$, let S_α denote the corresponding shift operator on X given by $S_\alpha f(t):=f(\alpha^{-1}\circ t)$ for all $f\in X$ and $t\in G$. We finally consider a convex operator $T\colon D\to X$, where D is an open convex subset of X satisfying $0\in D$. Clearly, T is said to be *causal* with respect to the subsemigroup A if

(C) $$T(X_\alpha\cap D)\subset X_\alpha \quad \text{for all } \alpha\in A.$$

The present situation naturally arises in the general theory of systems; see [1] and the references given there for the relevant background information including a discussion of the compatibility conditions. Note that (C) is fulfilled whenever $S_\alpha^{-1}(D)\subset D$ and $TS_\alpha=S_\alpha T$ on $S_\alpha^{-1}(D)$ for every $\alpha\in A$.

We now exhibit a remarkable continuity property of causal convex operators on $X^p(M)$, thus extending Theorem 4.2 of [16]; see also [1] for several closely related results, but observe that the present approach is somewhat different. It should be obvious that the assertion remains valid for other spaces of functions and distributions including, for instance, the case of certain vector-valued bounded or continuous functions.

5.1. THEOREM. *Let D be an open convex subset of $X = X^p(M)$ satisfying $0 \in D$, consider a causal convex operator $T\colon D \to X$, and assume that the compatibility conditions are fulfilled. Then there exists an $\alpha \in A$ such that the restriction $T|X_\alpha \cap D$ is continuous. Moreover, if T satisfies in addition $TS_\alpha = S_\alpha T$ on $D \cap S_\alpha^{-1}(D)$, then T is necessarily continuous on D.*

Proof. In order to apply Theorem 4.4, we endow $\mathscr{F} := \{\alpha \circ M\colon \alpha \in A\}$ with the order relation given by inclusion and consider the spaces $X := Y := X^p(M)$ and $\mathscr{E}_X(\alpha \circ M) := \mathscr{E}_Y(\alpha \circ M) := X_\alpha$ for all $\alpha \in A$. Then the assumptions (BP1) and (BP2) of Theorem 4.4 are certainly fulfilled if we define $\pi_{\alpha \circ M}$ to be the canonical quotient mapping from Y onto $Y/\mathscr{E}_Y(\alpha \circ M)$ for each $\alpha \in A$. It follows from Theorem 5.9 in [6] and can also be easily verified directly that the natural positive cone of these quotient spaces is normal. Hence it remains to check condition (BP3) of Theorem 4.4. To this end, let $\alpha \in A$ be arbitrarily given and consider an $f \in Y$ satisfying $f \neq 0$. Then there exists a compact subset K of M such that $\lambda(K) > 0$ and $f(t) \neq 0$ λ-almost everywhere on K. Since one may easily deduce from the compatibility conditions (6) that

$$\bigcap_{\beta \in A} \alpha \circ \beta \circ M = \emptyset,$$

we obtain by compactness finitely many $\beta_1, \ldots, \beta_m \in A$ such that

$$K \subset \bigcup_{j=1}^{m} (\alpha \circ \beta_j \circ M)^c.$$

As A is commutative, it follows that $\beta := \beta_1 \circ \ldots \circ \beta_m \in A$ satisfies $K \subset (\alpha \circ \beta \circ M)^c$ and hence $f \notin \mathscr{E}_Y(\alpha \circ \beta \circ M)$. We have shown that

$$\bigcap_{\beta \in A} \mathscr{E}_Y(\alpha \circ \beta \circ M) = \{0\} \quad \text{for all } \alpha \in A,$$

so that condition (BP3) is fulfilled. By Theorem 4.4 there exists an $\alpha \in A$ such that the restriction $T|X_\alpha \cap D$ is continuous. Finally, if T commutes with the shift operator corresponding to this particular α, it is immediate that T is continuous at 0 and hence on D by Theorem 3.1.

We next turn to causal operators in the sense of V. Dolezal [9], which requires the notion of a scale in $\mathbf{R}^n$. The following definition of a scale is slightly more restrictive than the original notion from [9], but it is general enough to include the most interesting examples.

5.2. DEFINITION. A family $\{S_a\colon a \in \mathbf{R}^n\}$ consisting of subsets S_a of $\mathbf{R}^n$ is said to be a *scale* in $\mathbf{R}^n$ if for every $a \in \mathbf{R}^n$ the following four conditions are fulfilled:

(a) $(\operatorname{Int} S_a)^- = S_a \neq \emptyset$.

(b) There is an $x_a \in \mathbf{R}^n$ such that $S_a^{c-} - tx_a \subset S_a^c$ for all $t > 0$.

(c) $b \in S_a$ implies that $S_b \subset S_a$.

(d) $a \in \partial S_a$.

5.3. EXAMPLES. (i) Let K be a closed convex cone in $\mathbf{R}^n$ such that $\mathbf{R}^n \neq K$ and $\mathbf{R}^n = K - K$, and define $S_a := a + K$ for all $a \in \mathbf{R}^n$. Then it is easily seen that $\{S_a\colon a \in \mathbf{R}^n\}$ is a scale in $\mathbf{R}^n$.

(ii) Fix a real $\mu > 0$, and for each $x \in \mathbf{R}^n$ let $\hat{x} \in \mathbf{R}$ be given by $\hat{x} := \|x\|^2 - x_n^2 - \mu x_n$, where $\|\cdot\|$ denotes the Euclidean norm on $\mathbf{R}^n$. Let $S_a := \{x \in \mathbf{R}^n\colon \hat{x} \leqslant \hat{a}\}$ for every $a \in \mathbf{R}^n$. Again, it is not hard to see that the family $\{S_a\colon a \in \mathbf{R}^n\}$ is a scale in $\mathbf{R}^n$.

The following result will be crucial for the continuity of causal operators in the sense of V. Dolezal.

5.4. LEMMA. *Let $\{S_a\colon a \in \mathbf{R}^n\}$ be a scale in $\mathbf{R}^n$. Then $\bigcap_{b \in S_a} S_b = \emptyset$ holds for every $a \in \mathbf{R}^n$.*

Proof. Let S denote the intersection on the left-hand side and suppose that S is nonempty. Given an arbitrary $x \in S$, we have $S_x \subset S_b$ for all $b \in S_a$ by condition (c) and therefore $S_x \subset S$. On the other hand, condition (d) implies that $x \in S_a$ and hence $S \subset S_x$. Thus $S_x = S$ for all $x \in S$. Again by condition (d), we conclude that $x \in \partial S_x = \partial S$ for all $x \in S$ and hence $\operatorname{Int} S = \emptyset$. But $(\operatorname{Int} S)^- = S_x \neq \emptyset$ for all $x \in S$ by condition (a). This contradiction completes the proof.

5.5. Remark. There is only one difference between the definition of a scale in [9] and the present one: instead of condition (d) in 5.2, V. Dolezal requires only $a \in S_a$ for every $a \in \mathbf{R}^n$. In this more general situation, Lemma 5.4 ceases to be true. To give an easy counterexample, let

$$S_a := \{x \in \mathbf{R}^n\colon x_j \geqslant \min\{a_j, 0\} \text{ for } j = 1, \ldots, n\}$$

for all $a \in \mathbf{R}^n$. Then $\{S_a\colon a \in \mathbf{R}^n\}$ is certainly a scale in the sense of [9], but condition (d) is not fulfilled. Moreover, we have $\mathbf{R}^n_+ \subset S_a$ for all $a \in \mathbf{R}^n$, so that Lemma 5.4 does not hold for this situation.

Similarly to [9], we now consider the following *situation*. Let X and Y be sequentially complete locally convex spaces such that X is metrizable and Y is countably boundedly generated, and assume that X and Y are both continuously embedded in $\mathscr{D}'(\mathbf{R}^n)$. For instance, one may take X as the space $\mathscr{S}(\mathbf{R}^n)$ of all rapidly decreasing test functions and Y as the space $\mathscr{S}'(\mathbf{R}^n)$ of all tempered distributions on $\mathbf{R}^n$. Moreover, let $\{S_a\colon a \in \mathbf{R}^n\}$ be a scale in $\mathbf{R}^n$, and for each $a \in \mathbf{R}^n$ endow the space $X_a := \{f \in X\colon \operatorname{supp} f \subset S_a\}$ with the topology inherited from X. Then we have:

5.6. THEOREM. *Let $a \in \mathbf{R}^n$ and consider a linear operator $T\colon X_a \to Y$ which* *is causal in the sense that* $\operatorname{supp} f \subset S_b$ *implies that* $\operatorname{supp} Tf \subset S_b$ *for all* $f \in X_a$ *and* $b \in S_a$. *Then there exists a* $b \in S_a$ *such that the restriction* $T|X_b$ *is continuous.*

Proof. We endow the family $\mathscr{F} := \{S_b : b \in S_a\}$ with the transitive relation given by inclusion and consider the spaces $\mathscr{E}_X(S_b) := X_b$ and similarly $\mathscr{E}_Y(S_b) := \{f \in Y: \operatorname{supp} f \subset S_b\}$ for all $b \in S_a$. Since the inclusion mappings from X and Y into $\mathscr{D}'(\boldsymbol{R}^n)$ are supposed to be continuous, it is clear that these spaces are closed linear subspaces of X_a and of Y, respectively. Now the assertion follows immediately from Theorem 4.4 in combination with Lemma 5.4.

Of course, a similar result holds for causal convex operators whenever the order structure of the range space is reasonably related to its topology. Moreover, if the given causal operator commutes with appropriate translation operators, one may deduce the continuity on the whole domain space. In this connection, let us mention that the main examples of operators causal with respect to a scale are given by certain convolution-type operators [9].

6. Operators of local type. To formulate the main proposition it will be convenient to describe the general situation and introduce the notation to be used in this section.

We shall consider an (F)-space X and a topological vector space Y which is countably boundedly generated, boundedly summing, sequentially complete and ordered by a normal cone.

Furthermore, G will denote a regular Hausdorff topological space and $\mathscr{F}(G)$ stands for the family of all closed subsets of G. We are given two monotone mappings

$$\mathscr{E}_X: \mathscr{F}(G) \to \mathscr{S}(X), \qquad \mathscr{E}_Y: \mathscr{F}(G) \to \mathscr{S}(Y)$$

such that the following conditions are fulfilled:

1° $\mathscr{E}_Y(\emptyset) = \{0\}$ and if $\mathscr{M}$ is a family of closed subsets of Y, then

$$\mathscr{E}_Y\Big(\bigcap_{F \in \mathscr{M}} F\Big) = \bigcap_{F \in \mathscr{M}} \mathscr{E}_Y(F);$$

2° if U and V are two open subsets of G such that $U \cup V = G$, then

$$X = \mathscr{E}_X(U^-) + \mathscr{E}_X(V^-);$$

3° if $t \in G$ is an isolated point of G then $\mathscr{E}_X(\{t\})$ is finite-dimensional; if $t \in G$ is not isolated then $\mathscr{E}_Y(\{t\}) = \{0\}$.

For each $F \in \mathscr{F}(G)$, a continuous positive linear mapping $\pi_F: Y \to Y_F$ from Y into a normally ordered topological vector space Y_F is given such that:

4° $\operatorname{Ker} \pi_F \subset \mathscr{E}_Y(F)$ for all $F \in \mathscr{F}(G)$.

6.1. Proposition. *Let D be an open convex subset of X such that $0 \in D$, and let $T: D \to Y$ be a convex operator which is of* local type *in the sense that the following condition is satisfied:*

(LT) *The operator* $\pi_H T|\mathscr{E}_X(F) \cap D$ *is continuous for all* $F, H \in \mathscr{F}(G)$ *with* $F \subset \operatorname{Int} H$.

Then T is continuous on D.

Proof. (i) We begin by proving the following assertion:

For any $t \in G$ there exists an open neighbourhood $U(t)$ of t such that $T|\mathscr{E}_X(U(t)^-) \cap D$ *is continuous.*

According to 3° it suffices to take $U(t) = \{t\}$ if t is an isolated point of G. If t is not isolated, denote by $\mathscr{F}$ the family of all closed neighbourhoods of t with the relation: $F < H$ if and only if $F \subset \operatorname{Int} H$, and apply Theorem 4.4. Conditions (BP1) and (BP2) are satisfied in view of (LT) and 4°. The space G being Hausdorff, every point is the intersection of all its closed neighbourhoods.

Thus we have, for each $F \in \mathscr{F}$,

$$\bigcap_{H<F} \mathscr{E}_Y(H) = \mathscr{E}_Y\Big(\bigcap_{H<F} H\Big) = \mathscr{E}_Y(\{t\}) = \{0\},$$

so that (BP3) is satisfied. Our assertion then follows immediately from 4.4.

(ii) The second step consists in proving an assertion which is, in a sense, dual to the preceding one.

For any $t \in G$ there exists an open neighbourhood W of t such that $\pi_{G\setminus W} T$ is continuous on D.

Consider a point $t \in G$; according to what we have already proved, there exists an open neighbourhood U of t such that $T|\mathscr{E}_X(U^-)$ is continuous on D. Now let V and W be two open neighbourhoods of t such that $t \in W \subset W^- \subset V \subset V^- \subset U$. The existence of such neighbourhoods follows from the regularity of the space G.

Now $V^c \subset W^{-c}$ and W^{-c} is nothing more than the interior of W^c. Thus

$$\pi_{G\setminus W} T|\mathscr{E}_X(G\setminus V) \cap D$$

is continuous by our assumption (LT). At the same time

$$\pi_{G\setminus W} T|\mathscr{E}_X(U^-) \cap D$$

is continuous as well. Since $G = U \cup V^{-c}$ and $V^{-c-} \subset V^c$ we have $X = \mathscr{E}_X(U^-) + \mathscr{E}_X(V^c)$. An application of the closed graph theorem yields the continuity of $\pi_{G\setminus W} T$ on D: for brevity, let us introduce the following notation

$$\pi_{G\setminus W} T = S, \qquad \mathscr{E}_X(U^-) = X_1, \qquad \mathscr{E}_X(V^c) = X_2.$$

Let $x \in D$ and a balanced 0-neighbourhood V in Y be given. There exist balanced neighbourhoods of zero U_j in X_j such that $S(x+2U_j) \subset Sx+V$ for $j = 1, 2$. The open mapping theorem yields the existence of a neighbourhood of zero U in X such that $U \subset U_1+U_2$. If $u \in U$ we have

$$S(x+u) = S(x+u_1+u_2) = S\left(\tfrac{1}{2}(x+2u_1)+\tfrac{1}{2}(x+2u_2)\right) \leqslant S(x)+\tfrac{1}{2}(v_1+v_2)$$

for suitable $u_1 \in U_1$, $u_2 \in U_2$, v_1, $v_2 \in V$. In a similar manner, we obtain

$$S(x-u) \leqslant S(x)+\tfrac{1}{2}(v_1'+v_2')$$

for suitable v_1', $v_2' \in V$. Since

$$S(x+u) \geqslant 2S(x)-S(x-u) \geqslant S(x)-\tfrac{1}{2}v_1'-\tfrac{1}{2}v_2',$$

we have $S(x+u) \subset S(x)+[V+V]$. The continuity of $\pi_{G\setminus W}T$ on D is thus established.

(iii) To complete the proof, it suffices to use Remark 3.3 to show that the graph of T is closed in $D \times Y$. For each $t \in G$ we have an open neighbourhood $W(t)$ of t such that $\pi_{G\setminus W(t)}T$ is continuous on D. Consider the family $\mathscr{F}$ of all complements $G \setminus W(t)$, $t \in G$; clearly the intersection of this family is empty. The corresponding family of mappings π_F, $F \in \mathscr{F}$ is separating since

$$\bigcap_{F\in\mathscr{F}} \operatorname{Ker} \pi_F \subset \bigcap_{F\in\mathscr{F}} \mathscr{E}_Y(F) = \mathscr{E}_Y\Big(\bigcap_{F\in\mathscr{F}} F\Big) = \{0\}.$$

Now the closed graph theorem 3.1 applies.

In the following application of Proposition 6.1, let μ be a regular Borel measure on some locally compact Hausdorff space G, and for any $0 < p \leqslant \infty$ consider the locally bounded (F)-space $L^p(\mu)$. For each closed subset F of G, the operator on $L^p(\mu)$ given by multiplication with the characteristic function of F will be denoted by p_F. Then the continuity of a convex operator on $L^p(\mu)$ may be characterized as follows.

6.2. THEOREM. *Assume that μ has no point atoms, and consider a convex operator T: $D \to L^p(\mu)$, where D is an open convex subset of $L^p(\mu)$ with $0 \in D$. Then T is continuous on D if and only if $p_K T p_F$ is continuous on $p_F^{-1}(D)$ for all compact $K \subset G$ and all closed $F \subset G$ satisfying $K \cap F = \emptyset$.*

Proof. For the sufficiency of the condition, we begin by proving the following assertion:

For every compact $K \subset G$, the operator $p_K T$ is continuous.

We shall use Proposition 6.1. For that purpose, we set $X = Y = L^p(G)$, $\mathscr{E}_X(F) = \mathscr{E}_Y(F) = \{f \in X\colon \operatorname{supp} f \subset F\}$ for $F \in \mathscr{F}(G)$. Furthermore, we define for each closed $F \subset G$ a mapping π_F as the multiplication by the characteristic function of $G \setminus \operatorname{Int} F$. It is easy to verify conditions 1°, 2° and 4°. Condition 3° follows from the fact that μ has no point atoms. For each compact $K \subset G$ let T_K be the operator $p_K T$. If F, H are two closed sets such that $F \subset \operatorname{Int} H$ then

$$\pi_H T_K | \mathscr{E}(F) \cap D = p_{K\setminus \operatorname{Int} H} T p_F | \mathscr{E}(F) \cap D$$

and $K \setminus \operatorname{Int} H$ is a compact set disjoint from F. Thus (LT) is satisfied, so that T_K is continuous by 6.1.

To complete the proof, we apply the principle of uniform boundedness given in Theorem 3.1 to the family $\{T_K\colon K \subset G \text{ compact}\}$ of continuous convex operators T_K: $D \to L^p(\mu)$ to obtain the equicontinuity of this family at the point 0. We next observe that, for every $f \in L^p(\mu)$, the (F)-norm $|f|_p$ equals the supremum of $|p_K f|_p$, where K ranges over all compact subsets of G. Thus T is continuous at 0. By the first assertion in 3.1, we arrive at the continuity of T on D.

The necessity of the condition being obvious, the theorem is established.

The condition of the preceding result is certainly fulfilled if the operator T is *local* in the sense that $\operatorname{supp} Tf \subset \operatorname{supp} f$ holds for all $f \in D$. Thus the algebraic conditions of being convex and local force an operator on $L^p(\mu)$ to be continuous whenever μ has no point atoms. Without this additional assumption, the result ceases to be true even for linear operators. A counter-example was recently given in Remark 4.7 of [3]. Let us note that typical examples of local linear operators on $L^p(\mu)$ are given by multiplication operators.

If G is a compact Hausdorff space, a linear mapping T: $L^p(\mu) \to L^p(\mu)$ is said to be an *operator of local type* if $p_K T p_F$ is a compact operator on $L^p(\mu)$ for all disjoint closed K, $F \subset G$. This notion naturally arises in the general theory of singular integral operators; see for instance [14]. In fact, the class of these operators contains all singular integral operators of the Calderón–Zygmund type and, of course, all local linear operators on $L^p(\mu)$. Now, Theorem 6.2 confirms that every operator of local type on $L^p(\mu)$ is automatically continuous, provided that μ has no point atoms. This result was also obtained in Theorem 4.6 of [3], but the present approach is somewhat different and actually more general than the former one.

Let us describe an application of Theorem 6.2 in a somewhat more concrete situation. Let $1 < p, q < \infty$ with $1/p+1/q = 1$, and consider a measurable function f: $\boldsymbol{R}^n \times S^n \to \boldsymbol{C}$ satisfying the Calderón–Zygmund condition

$$\sup_{x\in\boldsymbol{R}^n} \int_{S^n} |f(x, v)|^q \, dv < \infty,$$

where S^n denotes the unit sphere in $\boldsymbol{R}^n$. According to a remarkable theorem of Calderón and Zygmund [14; p. 288], in this situation the formula

$$(Tu)(x) := \int_{\boldsymbol{R}^n} \|x-y\|^{-n} f\big(x, (x-y)\|x-y\|^{-1}\big) u(y)\, dy$$

for $u \in L^p(\mathbf{R}^n)$ defines a continuous linear operator T from $L^p(\mathbf{R}^n)$ into itself. Now, the continuity of this mapping can also be deduced from our general theory as soon as Tu is known to belong to $L^p(\mathbf{R}^n)$ for every $u \in L^p(\mathbf{R})$. Indeed, it is easily seen that $p_K T p_F$ is then continuous on $L^p(\mathbf{R}^n)$ whenever $K \subset \mathbf{R}^n$ is compact and $F \subset \mathbf{R}^n$ is closed with $K \cap F = \emptyset$. Hence the continuity of T follows from Theorem 6.2.

We next turn to the natural counterpart of Theorem 6.2 for spaces of continuous functions. Let $C(G)$ consist of all continuous functions on the locally compact Hausdorff space G. Note that we do not assume G to be countable at infinity, so that the compact-open topology of $C(G)$ is not necessarily metrizable. By $C_c(G)$ and $C_*(G)$ we denote the spaces consisting of all those continuous functions on G which are, respectively, zero and constant outside some compact subset of G. On $C_c(G)$ we consider the natural inductive limit topology. For every $\varphi \in C(G)$, the multiplication by φ defines a continuous linear mapping on $C(G)$ and on $C_c(G)$, which will be denoted by M_φ. For operators on spaces of continuous functions, we have the following two characterizations of continuity.

6.3. THEOREM. *Let T: $D \to C(D)$ be a convex operator on some open convex subset D of $C(G)$ with $0 \in D$. Then T is continuous on D if and only if $M_\varphi T M_\psi$ is continuous on $M_\psi^{-1}(D)$ for all $\varphi \in C_c(G)$ and all $\psi \in C_*(G)$ satisfying* $\operatorname{supp}\varphi \cap \operatorname{supp}\psi = \emptyset$.

6.4. THEOREM. *Let T: $D \to C_c(G)$ be a convex operator on some open convex subset D of $C_c(G)$ with $0 \in D$. Then T is continuous on D if and only if $M_\varphi T M_\psi$ is continuous on $M_\psi^{-1}(D)$ for all $\varphi \in C_c(G)$ and all $\psi \in C_*(G)$ satisfying* $\operatorname{supp}\varphi \cap \operatorname{supp}\psi = \emptyset$.

In these results, the necessity of the respective condition is obvious. For the sufficiency, we first prove the subsequent lemma. Note that both Theorem 6.3 and Theorem 6.4 follow immediately from this lemma if the underlying space G is compact. Let us point out that, in Theorem 6.3, it is *not* sufficient to require the continuity of $M_\varphi T M_\psi$ for all φ, $\psi \in C_c(G)$ satisfying $\operatorname{supp}\varphi \cap \operatorname{supp}\psi = \emptyset$, as can easily be inferred from the special case $G = \mathbf{N}$. A similar remark holds for Theorem 6.2.

6.5. LEMMA. *Let G be a locally compact Hausdorff space, and K a compact subset of G. Let X be the subspace of $C(G)$ consisting of those $f \in C(G)$ whose support is contained in K. Let T: $D \to C(K)$ be a convex operator on some open convex subset D of X with $0 \in D$ such that the following condition is satisfied: $M_{\varphi|K} T M_\psi$ is continuous on $M_\psi^{-1}(D)$ for all pairs φ, $\psi \in C_c(G)$ with* $\operatorname{supp}\varphi \cap \operatorname{supp}\psi = \emptyset$.

Then T is continuous on D.

Proof. We shall apply Proposition 6.1 in the following situation. The underlying locally compact space will be K, we shall consider our space X and take $Y = C(K)$, so that both X and Y are Banach spaces. For $F \in \mathcal{F}(K)$ we set

$$\mathcal{E}_X(F) = \{f \in X\colon \operatorname{supp} f \subset F\},$$
$$\mathcal{E}_Y(F) = \{f \in Y\colon \operatorname{supp} f \subset F\},$$
$$Y_F = C\big((K \setminus F)^-\big)$$

and define π_F: $Y \to Y_F$ as the restriction operator to $(K \setminus F)^-$.

The monotonicity of $\mathcal{E}_X$ and $\mathcal{E}_Y$ as well as condition 1° are obvious. To prove 4° consider an $f \in \operatorname{Ker}\pi_F$. Since f is zero on $K \setminus \operatorname{Int} F$, its support is a subset of F, so that $f \in \mathcal{E}_Y(F)$. If $t \in G$ then $\mathcal{E}_X(\{t\})$ is at most one-dimensional; for a nonisolated t we have $\mathcal{E}_Y(\{t\}) = \{0\}$. Thus 3° is satisfied. To verify 2° consider two sets U and V relatively open in K with $U \cup V = K$. There exist sets $\hat{U}$, $\hat{V}$ open in G such that $U = \hat{U} \cap K$ and $V = \hat{V} \cap K$; furthermore, there exist φ, $\psi \in C_c(G)$ with $\operatorname{supp}\varphi \subset \hat{U}$, $\operatorname{supp}\psi \subset \hat{V}$, $\varphi + \psi = 1$ on K. Given any $f \in X$ we have $f = \varphi f + \psi f$ and $\varphi f \in \mathcal{E}_X(U^-)$, $\psi f \in \mathcal{E}_X(V^-)$.

To prove condition (LT) consider two closed subsets F, H of K such that F is contained in the relative interior H_i of H with respect to K. Then F and $K \setminus H_i$ are two disjoint compact sets, so that there exist two functions φ, $\psi \in C_c(G)$ with disjoint supports such that $\varphi = 1$ on $K \setminus H_i$ and $\psi = 1$ on F. Now we have on $M_\psi^{-1}(D)$

$$\pi_H T|\mathcal{E}_X(F) = \pi_H M_{\varphi|K} T M_\psi|\mathcal{E}_X(F),$$

which is continuous by our assumption.

Proof of Theorem 6.3. Without loss of generality, we may assume $T(0) = 0$. In view of the definition of the topology on $C(G)$, it suffices to prove the continuity of $R_K T$: $D \to C(K)$, where K is an arbitrary compact subset of G and R_K stands for the restriction operator from $C(G)$ into $C(K)$. The proof will be divided into three steps.

(i) We first show that, for each compact $K \subset G$, the operator

$$R_K T|\mathcal{E}(K) \cap D\colon \mathcal{E}(K) \cap D \to C(K)$$

is continuous, where $\mathcal{E}(K)$ consists of all $f \in C(G)$ whose support is contained in K. To prove this assertion, we use Lemma 6.5; thus we have to prove the continuity of $M_{\varphi|K} R_K T M_\psi$ on $\mathcal{E}(K) \cap M_\psi^{-1}(D)$ for all pairs φ, $\psi \in C_c(G)$ with disjoint supports. Now we note that $M_{\varphi|K} R_K = R_K M_\varphi$ and that $C_c(G) \subset C_*(G)$, so that the assumptions of Lemma 6.5 are satisfied.

(ii) Given a compact $K \subset G$, we choose a compact L such that $K \subset \operatorname{Int} L$, and a $\varphi \in C_c(G)$ such that $\varphi = 1$ on a neighbourhood of K and $\operatorname{supp}\varphi \subset L$. Then $\psi = 1 - \varphi$ is an element of $C_*(G)$ and $\operatorname{supp}\psi \cap K = \emptyset$. Now take a $\xi \in C_c(G)$ such that $\xi = 1$ on K and $\operatorname{supp}\xi \cap \operatorname{supp}\psi = \emptyset$. Then

$$R_K T M_\psi = R_K M_\xi T M_\psi$$

is continuous on $M_\psi^{-1}(D)$ by our assumption. On the other hand, in view of the inclusion $\operatorname{supp}\varphi \subset L$ assertion (i) yields the continuity of $R_K TM_\varphi$ on $M_\varphi^{-1}(D)$.

(iii) Take an arbitrary convex neighbourhood of zero V in $C(K)$. Since D is open and $R_K TM_\psi$, $R_K TM_\varphi$ are continuous, there exists a balanced neighbourhood of zero U in $C(G)$ such that $U \subset D$ and

$$2M_\varphi(U) \subset D, \quad R_K T2M_\varphi(U) \subset V;$$
$$2M_\psi(U) \subset D, \quad R_K T2M_\psi(U) \subset V.$$

Now suppose $f \in U$. Then

$$R_K Tf \leqslant \tfrac{1}{2} R_K T(2M_\varphi f) + \tfrac{1}{2} R_K T(2M_\psi f) \in V.$$

In a similar manner we obtain

$$R_K Tf \geqslant -R_K T(-f) \in -V,$$

since $T(0) = 0$. This proves the continuity of $R_K T$ at 0 and hence on the whole of D by the first assertion in 3.1.

Proof of Theorem 6.4. We shall prove the sufficiency in three steps. For each compact subset K of G, consider the Banach space $X_K := \{f \in C(G)\colon \operatorname{supp} f \subset K\}$ and the restriction operator $R_K\colon C_c(G) \to C(K)$ given by $R_K f := f|K$ for all $f \in C_c(G)$. Then Lemma 6.5 yields the continuity of the convex operator

$$R_K T|X_K \cap D\colon X_K \cap D \to C(K) \text{ for all compact } K \subset G.$$

Since $C_c(G)$ is endowed with the inductive limit topology of the spaces X_K, it is easy to derive the continuity of the composition

$$R_K T\colon D \to C(K) \quad \text{for all compact } K \subset G.$$

The proof will be completed by means of a uniform boundedness argument. To this end, let M consist of all $\varphi \in C_c(G)$ satisfying $0 \leqslant \varphi \leqslant 1$, and for each $\varphi \in M$ consider the convex operator $T_\varphi\colon D \to C_c(G)$ given by $T_\varphi := M_\varphi T$. It is easily seen that the family $\{T_\varphi\colon \varphi \in M\}$ is pointwise bounded on D. Moreover, each of the operators T_φ is continuous on D. Indeed, given any $\varphi \in M$, we choose a compact subset K of G satisfying $\operatorname{supp}\varphi \subset K$. By the preceding result,

$$R_K M_\varphi T = M_{\varphi|K} R_K T\colon D \to C(K)$$

is continuous, which immediately implies the continuity of $T_\varphi = M_\varphi T$. By the principle of uniform boundedness given in Theorem 3.1, the family $\{T_\varphi\colon \varphi \in M\}$ turns out to be equicontinuous. Hence, given an arbitrary neighbourhood of zero V in $C_c(G)$, there exists a neighbourhood of zero $U \subset D$ in $C_c(G)$ such that

$$M_\varphi T(U) = T_\varphi(U) \subset V \quad \text{for all } \varphi \in M.$$

Now let $f \in U$. Then $Tf \in X_K$ for some compact $K \subset G$ and hence $Tf = T_\varphi f$ for some $\varphi \in M$. We conclude that $T(U) \subset V$, so that T is continuous at 0. By the first assertion in 3.1, we finally arrive at the continuity of T on D.

It should be obvious that similar results can be obtained for further spaces of functions, including for instance the case of locally p-integrable functions. Moreover, it is also possible to derive from our basic principle 4.4 certain continuity results for operators of local type and, in particular, for local operators between spaces of differentiable functions and distributions. In this context, however, one has to face certain difficulties whenever the range of the operator contains nontrivial distributions with finite support. Actually, the best one can prove in this case is the continuity of the given operator of local type outside some discrete subset of the underlying space; see [2], [3], [4] for some counterexamples and for several positive results in this direction. These results can also be subsumed under the present general theory, but we omit the details. The investigation of local operators in the context of distribution theory dates back to J. Peetre [18], who characterized differential operators by the algebraic condition of being local. A similar characterization of ultradifferential operators was given in [4].

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Some combinatorial and probabilistic inequalities and their application to Banach space theory

by

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Abstract. Some combinatorial and probabilistic estimates are proved. As applications they are used to study invariants of Banach spaces, such as the projection constant.

Introduction. We consider here for $x, y \in \mathbf{R}^n$, $1 \leqslant p \leqslant \infty$,

$$\operatorname*{Ave}_{\pi} \|(x_i y_{\pi(i)})_{i=1}^n\|_p$$

and give the order of this expression in terms of the vectors x and y. For special vectors x or y this was already considered by E. D. Gluskin [4] and, independently, in [7].

It seems that the estimates that we obtain are, in a sense, crucial if one wants to compute projection constants of symmetric Banach spaces and related invariants.

We give some examples and applications. We characterize the symmetric sublattices of $l^1(c_0)$ and the symmetric subspaces of l^1. We compute the positive projection constant of a (finite-dimensional) Orlicz space and show that it is, up to a universal constant, the same as the one of the dual space. For symmetric spaces this is in general not true.

The order of the projection constant of the Lorentz space $l_n^{2,1}$, $n \in N$, is estimated. The result seems to be rather peculiar.

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0. Preliminaries. In this paper we are mainly concerned with finite-dimensional Banach spaces that have a 1-symmetric basis, i.e. a basis $\{e_i\}_{i=1}^n$ such that for all $a_i \in \mathbf{R}$, $\varepsilon_i = \pm 1$, $i = 1, \ldots, n$ and all permutations π we have

$$\Big\|\sum_{i=1}^n a_i e_i\Big\| = \Big\|\sum_{i=1}^n \varepsilon_i a_i e_{\pi(i)}\Big\|.$$

The projection constant of a finite-dimensional Banach space E is given by

$$\gamma_\infty(E) = \inf\{\|P\| \mid P \text{ is a projection from } l^\infty \text{ onto } E\}.$$