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An atomic theory of ergodic H^p spaces

by

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Abstract. Let T be an invertible measure-preserving ergodic transformation on a probability space. We define elementary functions associated with T, called "atoms", and we use them to define ergodic Hardy spaces H^p for $p \leqslant 1$. From this atomic definition we obtain maximal function characterizations of H^p. We identify the duals of H^p and of H^1, and finally we obtain interpolation theorems between H^p and L_q, $p \leqslant 1 < q$.

Introduction. In this paper we study the Hardy spaces induced by an invertible, ergodic, measure-preserving transformation on a probability space X.

In [2], Coifman and Weiss studied the space $H^1(X)$, which they defined as the space of functions in $L_1(X)$ whose ergodic Hilbert transform is in $L_1(X)$. Their main results are that, as in the classical case, H^1 can be characterized in terms of maximal operators and that the dual of H^1 can be identified with the space of functions of bounded mean oscillation. (See [4] for the case $H^1(\boldsymbol{R}^n)$).

It was found later that $H^p(\boldsymbol{R}^n)$ can be defined in terms of elementary functions called "atoms" [1], this atomic characterization being very useful in studying interpolation, duality, etc.

Since the methods of [2] do not seem to work for $p < 1$, we use an "atomic" approach. We define $H^{p,q}(X)$ for $1/2 < p \leqslant 1$, $p < q$, as the spaces of functions that can be written in terms of (p, q) atoms. In the first section we show that $H^{p,q}$ can be characterized in terms of maximal operators as in the case $p = 1$. As a corollary we show that $H^{p,q}$ depends only on p, i.e. $H^{p,q} = H^{p,\infty}$, so that we may write simply H^p.

In the second section we use our atoms to study the dual of H^p. One easily sees then that the dual of H^1 is BMO, obtaining another proof of the result in [2]. For $p < 1$ the analogy with the case $H^p(\boldsymbol{R}^n)$ breaks down since the dual of $H^p(X)$ $(p < 1)$ is made only of multiples of the functional induced by the measure on X, while in the classical case H^{p*} is a space of Lipschitz functions. For ergodic H^p spaces, defined by an ergodic action of $\boldsymbol{R}$ in X, this result was obtained by Muhly in [6], but his methods are entirely different and do not seem to be applicable to the discrete case. Our "atomic" proof has also the advantage of giving both H^{1*} and H^{p*} at one stroke, thus explaining why there is such a sharp difference.

Finally, in the third section we extend to the ergodic case the interpolation theorems which, in the real case, are due to Fefferman, Rivière, and Sagher [5]. We obtain interpolation theorems between H^p and $L_{p'}$, for $1/2 < p \leqslant 1 < p' \leqslant \infty$, using once more our atoms and the fact that $H^{p,\infty} = H^{p,q} = H^p$.

The restriction $p > 1/2$ is only technical, and it is only made to simplify the proofs. One can extend the results to $p < 1/2$ by asking for more cancellations in the definition of the atoms, in the same way as in [3].

$H^{p,q}$ **spaces. Definitions.** Let (X, m) be a nonatomic probability space and $T\colon X \to X$ an invertible, measure-preserving, ergodic transformation.

DEFINITION. Let $B \subset X$ be a set of positive measure such that for some $k \geqslant 1$ we have

$$T^i B \cap T^j B = \emptyset, \quad i \neq j, \ 0 \leqslant i, j \leqslant k-1.$$

Then the set $R = \bigcup_{i=0}^{k-1} T^i B$ will be called an *ergodic rectangle* with *base* B and *length* k.

DEFINITION. Let p, q be real numbers, $1/2 < p \leqslant 1 \leqslant q \leqslant \infty$, $p < q$. A real-valued function a is a (p, q) *atom* if either

(1) a is zero outside an ergodic rectangle $R = \bigcup_{i=0}^{k-1} T^i B$ and satisfies

(a) $\sum_{j=0}^{k-1} a(T^j x) = 0$ for $x \in B$,

(b) $k^{-1} \sum_{j=0}^{k-1} |a(T^j x)|^q \leqslant m(R)^{-q/p}$, $x \in B$, $q < \infty$,

$\|a\|_\infty \leqslant m(R)^{-1/p}$ if $q = \infty$,

or

(2) $a \in L_q(X)$, $\|a\|_q \leqslant 1$.

We will say that an atom is *of type* 1 if it verifies (1) and *of type* 2 if it verifies (2).

It is easy to show that (b) implies that

$$\|a\|_q \leqslant m(R)^{(1/q)-(1/p)}.$$

DEFINITION. For p, q as before, $H^{p,q}(X)$ is the set of functions f that can be written as

$$f = \sum_{i=1}^{\infty} c_i a_i$$

where a_i are (p, q) atoms, $\sum_{i=1}^{\infty} |c_i|^p < \infty$, and the convergence is in the L_p-metric.

We introduce a metric in $H^{p,q}(X)$ by

$$d_{p,q}(h_1, h_2) = |h_1 - h_2|_{p,q}$$

where

$$|h|_{p,q} = \{\inf \sum_{i=1}^{\infty} |c_i|^p\colon h = \sum_{i=1}^{\infty} c_i a_i, \ \sum_{i=1}^{\infty} |c_i|^p < \infty\}.$$

It is clear that $H^{p,\infty} \subset H^{p,q} \subset H^{p,r}$ if $r \leqslant q \leqslant \infty$, the inclusion being continuous.

We will show that the converse also holds by using a characterization of the $H^{p,q}$ spaces in terms of a maximal operator.

The maximal operator. Let $\varphi \in L_1(\boldsymbol{Z})$ and $f \in L_1(X)$. The *convolution* of f and φ is defined as $(f * \varphi)(x) = \sum_{n \in \boldsymbol{Z}} f(T^{-n} x)\varphi(n)$. It is obvious that $\|f * \varphi\|_1 \leqslant \|\varphi\|_1 \|f\|_1$.

DEFINITION. Let φ be a nonnegative $C^\infty(\boldsymbol{R})$ function with support in $(-1, 1)$ and $L > 0$. For $f \in L_1(X)$ we define

$$M(L, \varphi) f(x) = \sup_{|i| < n < L} |(f * \varphi_n)(T^i x)|$$

and

$$M(\varphi) f(x) = \lim_{L \to \infty} M(L, \varphi) f(x)$$

where

$$\varphi_n(m) = \frac{1}{n} \varphi\left(\frac{m}{n}\right), \quad m \in \boldsymbol{Z}.$$

We also define

$$M(L) f(x) = \sup M(L, \varphi) f(x) \bigl(A(\varphi)\bigr)^{-1}$$

where the sup is taken over all C^∞ functions with support in $(-1, 1)$ and where $A(\varphi)$ is the normalizing factor

$$A(\varphi) = \|\varphi\|_\infty + \|\varphi'\|_\infty.$$

Finally,

$$Mf(x) = \lim_{L \to \infty} M(L) f(x).$$

It is not difficult to show that this operator is dominated by the ergodic maximal operator, and therefore is of weak type (1, 1) and bounded in any L_q, $1 < q$.

We now show that atoms behave well with respect to the maximal operator. More precisely:

PROPOSITION. *Let a be a (p, q) atom. Then $Ma \in L_p$, and $\|Ma\|_p \leqslant C$, where C is an absolute constant.*

Proof. If a is a type 2 atom then since our space has measure one we have

$$\|Ma\|_p \leqslant \|Ma\|_q \leqslant C\|a\|_q \leqslant C \quad \text{for } q > 1.$$

For $q = 1$ we use Kolmogorov's inequality to obtain

$$\|Ma\|_p \leqslant C\|a\|_1 \leqslant C.$$

If a is a type 1 atom, we use a transference argument.

We say that a function $A: \boldsymbol{Z} \to \boldsymbol{R}$ is a (p, q) *atom in the integers* if its support is contained in an interval $(l, l+1, \ldots, l+k-1)$ and

(a) $\sum_n A(n) = 0$,

(b) $k^{-1} \sum_{i=0}^{k-1} |A(l+i)|^q \leqslant k^{-q/p}$.

We will show that if we consider MA where M is the maximal operator defined above, we have

$$\sum_{n=-\infty}^{\infty} |MA(n)|^p \leqslant C(p, q)$$

where C depends only on p and q. The proof is an adaptation to the integers of the standard argument for the continuous case, and we include it only for completeness.

First of all, since M commutes with translations, we can assume that $l = 0$. Now

$$\begin{aligned}\sum_{m=-4k}^{4k} |MA(m)|^p &\leqslant (8k+1)\Big(\frac{1}{8k+1}\sum_{m=-4k}^{4k} |MA(m)|^q\Big)^{p/q} \\ &\leqslant C(p, q)(8k+1)^{1-(p/q)}\Big(\sum_{m=-\infty}^{\infty} |A(m)|^q\Big)^{p/q} \\ &\leqslant C(p, q)(8k+1)^{1-(p/q)} k^{-(1-(p/q))} \\ &\leqslant C'(p, q).\end{aligned}$$

Let us now fix $|m| > 4k$, and let us fix n, L, φ and $|i| < n < L$. Then

$$\begin{aligned}(A * \varphi_n)(m+i) &= \sum_j A(m+i-j)\varphi_n(j) = \sum_{j=0}^{k-1} A(j)\varphi_n(m+i-j) \\ &= \sum_{j=0}^{k-1} A(j)\Big(\frac{1}{n}\Big(\varphi\Big(\frac{m+i-j}{n}\Big) - \varphi\Big(\frac{m+i}{n}\Big)\Big)\Big).\end{aligned}$$

Now the sum is zero unless $n > |m/4|$, for if $n \leqslant |m/4|$ then $|i| < |m|/4$, $j < k < |m|/4$, and thus $|m+i-j| > |m| - |m/2| > n$. Therefore

$$\begin{aligned}|(A * \varphi_n)(m+i)| &\leqslant \frac{1}{n^2}\sum_{j=0}^{k-1} |A(j)| A(\varphi) j \\ &\leqslant \frac{A(\varphi) C}{m^2} k \sum_{j=0}^{k-1} |A(j)| \leqslant \frac{CA(\varphi)}{m^2} k^{2-(1/p)}.\end{aligned}$$

This means that

$$|MA(m)| \leqslant \frac{C}{m^2} k^{2-(1/p)} \quad \text{if } |m| > 4k$$

and

$$\sum_{|m|>4k} |MA(m)|^p \leqslant Ck^{2p-1} \sum_{|m|>4k} \frac{1}{m^{2p}} \leqslant C' \quad \text{if } p > 1/2.$$

We can now go back to the ergodic case. Let a be a (p, q) atom with support in

$$R = \bigcup_{i=0}^{k-1} T^i B.$$

For each $x \in X$ we consider the function

$$a_x(n) = a(T^n x).$$

Since the orbit of x enters B infinitely many times, let us call y_i the points of the orbit that belong to B. Then for each n, $a_x(n) = a(T^l y_i)$ for some i, $0 \leqslant l \leqslant k$. It is then clear that we can write

$$a_x(n) = \sum_{i \in \boldsymbol{Z}} m(B)^{-1/p} A_{i,x}(n)$$

where $A_{i,x}(n) = m(B)^{1/p} a(T^n x)$ are (p, q) atoms in the integers with support in $(l_i, \ldots, l_i+k-1)$ with $y_i = T^{l_i} x \in B$. Therefore for $N > L > k$

$$\begin{aligned}\int_X |M(L) a(x)|^p\,dx &= \int_X \frac{1}{2N+1}\sum_{n=-N}^{N} |M(L) a_x(n)|^p\,dx \\ &= \int_X \frac{1}{2N+1}\sum_{n=-N}^{N}\sum_i |M(L) m(B)^{-1/p} A_{i,x}(n)|^p\,dx.\end{aligned}$$

Now the atoms $A_{i,x}$ whose support does not cut the interval $(-3N, 3N)$ do not contribute anything to $M(L)$ since it is easy to check that

$$M(L)A_{i,x}(n) = 0 \quad \text{if } n \notin (-3N, 3N);$$

since $k < L < N$, we can thus restrict our attention to those atoms whose support is contained in $(-4N, 4N)$.

Now

$$\begin{aligned}\int_X |M(L)a(x)|^p dx &\leqslant \int_X \frac{1}{2N+1} \sum_{n=-N}^{N} \sum_{\operatorname{supp} A_i \subset (-4N,4N)} |M(L)m(B)^{-1/p} A_{i,x}(n)|^p dx \\ &\leqslant \frac{1}{2N+1} m(B)^{-1} \int_X \sum_{\operatorname{supp} A_i \subset (-4N,4N)} \sum_{n=-\infty}^{\infty} |M(L)A_{i,x}(n)|^p dx \\ &\leqslant \frac{m(R)^{-1}}{2N+1} \int_X k \sum_{\operatorname{supp} A_i \subset (-4N,4N)} C\, dx \\ &= \frac{m(R)^{-1}}{2N+1} \int_X k \{\text{number of } A_i\text{'s with } \operatorname{supp} A_i \subset (-4N, 4N)\} \\ &= \frac{m(R)^{-1}}{2N+1} \int_X \sum_{i=-4N}^{4N} \chi_R(T^i x)\, dx \\ &= C\frac{8N+1}{2N+1} m(R)^{-1} \int_X \chi_R = C\frac{8N+1}{2N+1}.\end{aligned}$$

Letting N and L go to infinity, we are done.

Our proposition obviously implies that if $f \in H^{p,q}$ then $Mf \in L_p$ and

$$\int |Mf|^p \leqslant C \sum |c_i|^p.$$

This means that any $H^{p,q}$ function has a maximal function in L_p. Our next aim is to prove the converse, namely that if f is an L_p function whose maximal function Mf is in L_p then f is in $H^{p,\infty}$. As a corollary we will have $H^{p,q} = H^{p,\infty}$.

First of all we need two technical lemmas.

Lemma 1. *Let $O \subset X$ be a set of positive measure such that the subset of* $\boldsymbol{Z}$ *defined by $O^x = \{n \in \boldsymbol{Z};\ T^n x \in O\}$ does not contain an interval of infinite length. Then O can be written as a disjoint union of ergodic rectangles R_i of length i.*

Proof. One just defines $R_i = \{x \in O; l(O_0^x) = i\}$, i.e. the set of points in O such that the interval of O^x that contains the origin has length i. It is obvious that $O = \bigcup_{i=1}^{\infty} R_i$ and that the R_i are pairwise disjoint. To see that R_i is a rectangle one just observes that $R_i = \bigcup_{j=0}^{i-1} T^j B_i$, where B_i is the set of points x in O such that $T^{-1}x \notin O$, $T^j x \in O$ for $j = 1, 2, \ldots, i-1$, and $T^i x \notin O$.

Lemma 2. *Let I be an interval in* $\boldsymbol{R}$ *of the form $[a, b]$, $a, b \in \boldsymbol{Z}$. Then there exist a finite number of C^∞ functions $\{\Psi_j\}$ such that*

(a) $\sum_j \Psi_j(x) = 1$ *for* $x \in I$; $\operatorname{supp} \Psi_j \subset [a-1/2, b+1/2]$,

(b) $\|\Psi_j'\|_\infty \leqslant C|\operatorname{supp}\Psi_j|^{-1}$,

(c) $d(\operatorname{supp}\Psi_j, \boldsymbol{R}-\bar{I}) \sim |\operatorname{supp}\Psi_j|$; $\bar{I} - [a-1, b+1]$,

(d) $\sum_{n\in\boldsymbol{Z}} \Psi_j(n) \geqslant C|\operatorname{supp}\Psi_j|$.

This is just a version adapted to the integers of a smooth partition of the characteristic function of an interval.

Proof. Let $N = b-a$. If $N \leqslant 3$, we do it with just one function since it is clear that one can always construct a C^∞ function identically 1 on $[a, b]$ with support in $[a-1/2, b+1/2]$ and satisfying (b), (c) and (d).

If $N > 3$, we consider an interval $I_1 = [a_1, b_1]$, $a_1, b_1 \in \boldsymbol{Z}$, $b_1 - a_1 = [N/3]$ and situated in the middle of I. We take a C^∞ function ξ_1 satisfying

$$\xi_1(x) = 1, \quad x \in I_1, \quad 0 \leqslant \xi_1 \leqslant 1,$$
$$\operatorname{supp}\xi_1 \subset [a_1 - [N/6],\ b_1 + [N/6]],$$
$$\|\xi_1'\|_\infty \leqslant 6/N.$$

From now on we consider only what is left on the right-hand side of I_1 (and proceed on the left side in the same way).

Let $J = [b_1, b]$. We cut it in half, consider $[b_1, b_1 + [|J|/2]] = [b_1, b_2]$ and construct ξ_2 such that $0 \leqslant \xi_2 \leqslant 1$, $\xi_2 \in C^\infty$ and

$$\operatorname{supp}\xi_2 \subset [b_1 - |J|/4,\ b_2 + |J|/4],\ \|\xi_2'\| \leqslant 4/|J|,$$
$$\xi_2(x) = 1, \quad x \in [b_1, b_2].$$

One then repeats the process on $[b_2, b]$ until one gets an interval $[b_k, b]$ of length less than 4, in which case we define ξ_{k+1} as above, identically 1 on $[b_k, b]$, with support in $[b_k - 1/2, b+1/2]$, and $\|\xi_{k+1}'\|_\infty \leqslant 2$.

It is clear now that if we define

$$\Psi_j = \frac{\xi_j}{\sum_j \xi_j}$$

we have the family of functions satisfying (a), (b), (c) and (d).

We are ready to prove our main result.

THEOREM 1. *Let f be an L_p function such that Mf is in L_p. Then f can be written as $f(x) = \sum_i c_i a_i(x)$ where the a_i are (p, ∞) atoms and*

$$\sum_i |c_i|^p \leqslant C\,\|Mf\|_p^p.$$

Proof. For each $\lambda > 0$ consider the set

$$O(\lambda) = \{x \in X;\ Mf(x) > \lambda\}.$$

Let $\lambda_0 = \inf\{\lambda > 0;\ m(O_\lambda) < 1\}$. If $\lambda_0 \neq 0$, we consider the sequence $\lambda_k = 2^k \lambda_0$, $k = 0, 1, \ldots$

Let $k \neq 0$. Then $m(O(\lambda_k)) < 1$ and since T is ergodic we can use Lemma 1 and write

$$O(\lambda_k) = \bigcup_i R_i^k$$

where the R_i^k are disjoint rectangles of length i; we now write for k fixed

$$f = \sum_i f\chi_{R_i^k} + f(1-\chi_{O(\lambda_k)}).$$

For each i fixed, we use Lemma 2 on the interval $[0, l(i)]$ ($l(i)+1$ = length of R_i) and we call $\{\Psi_{i,j}^k\}$ the corresponding partition of unity.

Let B_i^k be the base of R_i^k and write

$$m_{i,j}^k(T^m x) = \frac{\sum_n f(T^n x)\,\Psi_{i,j}^k(n)}{\sum_n \Psi_{i,j}^k(n)}, \qquad x \in B_i^k,\ m \in [0, l(i)],$$

$$m_{i,j}^k(T^m x) = 0 \qquad \text{if } T^m x \notin R_i^k.$$

Each $\Psi_{i,j}^k$ can be used to define a function on X, which we will call by the same name, as

$$\Psi_{i,j}^k(T^n x) = \Psi_{i,j}^k(n) \quad \text{for } x \in B_i^k,\ 0 \leqslant n \leqslant l(i),$$

and zero otherwise. It is clear that $\sum_j \Psi_{i,j}^k = \chi_{R_i^k}$. Now for k fixed we may write

$$f = \sum_i \sum_j (f - m_{i,j}^k)\,\Psi_{i,j}^k + \sum_{i,j} m_{i,j}^k\,\Psi_{i,j}^k + f(1-\chi_{O(\lambda_k)}) = b_k + g_k$$

where

$$b_k = \sum_i \sum_j (f - m_{i,j}^k)\,\Psi_{i,j}^k,$$

$$g_k = \sum_{i,j} m_{i,j}^k\,\Psi_{i,j}^k + f(1-\chi_{O(\lambda_k)}).$$

Let $(a-h, a+h)$ be the smallest interval containing the support of $\Psi = \Psi_{i,j}^k$ with $a, h \in \boldsymbol{Z}$. Let N be a fixed number (independent of f, i, j, k) such that $(a-Nh, a+Nh)$ intersects $\boldsymbol{R}-(-1, l(i)+1)$. The let

$$\varphi(s) = Nh\,\Psi(a - sNh).$$

It is clear that

$$\|\varphi\|_\infty \leqslant Nh \quad \text{and} \quad \|\varphi'\|_\infty \leqslant (Nh)^2\,\|\Psi'\|_\infty \leqslant Ch$$

and

$$(f*\varphi_{Nh})(T^a x) = \sum_n f(T^{a-n}x)\,\varphi_{Nh}(n) = \sum_n f(T^{a-n}x)\,\Psi(a-n)$$
$$= \sum_n f(T^n x)\,\Psi(n).$$

Remembering the definition of our maximal operator, we have

$$\Big|\sum_n f(T^n x)\,\Psi(n)\Big| \leqslant C\,Mf(T^{a+l}x)\,h$$

provided $|l| < Nh$. Now because of property (c) in Lemma 2 we choose l such that $T^{a+l}(x) \notin O(\lambda_k)$ and by property (d)

$$m_{i,j}^k \leqslant C\,Mf(x') \quad \text{with } x' \in X - O(\lambda_k).$$

Therefore

$$m_{i,j}^k \leqslant C\lambda_k.$$

On the other hand, if $x \notin O(\lambda_k)$ then

$$f(x) \leqslant Mf(x) < \lambda_k,$$

and we obtain

$$g_k(x) \leqslant C\lambda_k.$$

Since we are assuming $Mf \in L_p$, it is clear that $m(O(\lambda_k)) \to 0$ as $\lambda_k \to \infty$, and therefore, since b_k has support in $O(\lambda_k)$, we have

$$f(x) = \lim_{k\to\infty} g_k,$$

and defining

$$b_0(x) = f(x)$$

we have

$$f(x) = \sum_{k=0}^{\infty} (g_{k+1} - g_k) = \sum_{k=0}^{\infty} (b_k - b_{k+1}).$$

Now we observe that $O(\lambda_{k+1}) \subset O(\lambda_k)$ and therefore each

$$R_j^{k+1} \subset O(\lambda_k) = \bigcup_i R_i^k;$$

so if we write

$$R_j^{k+1} = \bigcup_i (R_j^{k+1} \cap R_i^k)$$

we obtain R_j^{k+1} as a disjoint union of $R_{i,j}^{k+1} = R_j^{k+1} \cap R_i^k$, where each $R_{i,j}^{k+1}$ is a rectangle with base $B_{i,j}^{k+1} = B_j^{k+1} \cap R_i^k$, and length that of R_j^{k+1}.

If we write

$$a_{i,j}^k = (f - m_{i,j}^k)\,\Psi_{i,j}^k$$

then

$$b_k = \sum_i b_{i,k} \quad \text{with } b_{i,k} = \sum_j a_{i,j}^k.$$

Let us fix i and write

$$A_{i,k} = b_{i,k} - \sum_j b_{j,k+1}\,\chi_{R_{i,j}^{k+1}},$$

the sum extended over all j such that $R_{i,j}^{k+1} \neq \emptyset$. From the above observation on the decompositions of $O(\lambda_{k+1})$ and $O(\lambda_k)$ it is clear that

$$b_k - b_{k+1} = \sum_i A_{i,k} = g_{k+1} - g_k.$$

But since the $A_{i,k}$ have disjoint supports for k fixed, it follows that for any x

$$|A_{i,k}(x)| = |g_{k+1}(x) - g_k(x)| \leqslant C\lambda_k.$$

Also from the definition of $m_{i,j}^k$ it follows that

$$\sum_{l=0}^{i-1} a_{i,j}(T^l x) = 0, \qquad x \in B_i^k.$$

These last two observations imply that

$$\tilde{A}_{i,k} = (C\lambda_k\, m(R_i^k)^{1/p})^{-1} A_{i,k}$$

is a (p, ∞) atom for $k = 1, 2, \ldots$

For $k = 0$ we define $A_0 = b_0 - b_1 = g_1 - g_0$, which implies

$$|A_0(x)| \leqslant C\lambda_0,$$

so that

$$\tilde{A}_0 = (C\lambda_0)^{-1} A_0$$

is a (p, ∞) atom of type 2. We may then write

$$f = \sum_{k=0}^{\infty} (b_k - b_{k+1}) = C\lambda_0 \tilde{A}_0 + \sum_{i,k} C\lambda_k \left(m(R_{i,k})\right)^{1/p} \tilde{A}_{i,k}$$

where $\tilde{A}_0$, $\tilde{A}_{i,k}$ are (p, ∞) atoms, while the sum of the pth powers of the coefficients is dominated by

$$C \sum_k \lambda_k^p\, m(O(\lambda_k)) = C \sum_k \lambda_k^p \int_{\{X: Mf > \lambda_k\}} 1\, dx = \int_X \sum_{k=0}^{\lambda_k < Mf(x)} \lambda_k^p\, dx$$
$$\leqslant C \int_X |Mf(x)|^p\, dx.$$

If $\lambda_0 = 0$ we choose $\lambda_k = 2^k$, $k \in \mathbf{Z}$. Then for each k, $m(O(\lambda_k)) < 1$ and we may write as above

$$f = b_k + g_k$$

with $|g_k| < C\lambda_k$, which means that $g_k \to 0$ as $k \to -\infty$. On the other hand, since the support of b_k is contained in $O(\lambda_k)$, it follows that $b_k \to 0$ as $k \to \infty$ and therefore

$$f = \lim_{k \to -\infty} b_k = \lim_{k \to \infty} g_k$$

and

$$f(x) = \sum_{k=-\infty}^{\infty} (g_{k+1} - g_k)(x) = \sum_{k=-\infty}^{\infty} (b_k - b_{k+1})(x).$$

From this equality we proceed as in the case $\lambda_0 \neq 0$.

The theorem implies that the set $\{f \in L_p;\ Mf \in L_p\}$ is contained in $H^{p,\infty} \subset H^{p,q}$ and the metric induced by Mf is equivalent to that of $H^{p,\infty}$. We thus have

$$H^{p,q} = H^{p,\infty} = \{f \in L_p;\ Mf \in L_p\}$$

with equivalent metrics. We may therefore drop the q and write simply H^p.

The characterization of H^p in terms of the maximal function allows us to show that H^p is a complete metric space.

THEOREM 2. *H^p is complete.*

Proof. Let $\{f_n\}$ be a Cauchy sequence, i.e.,

$$\|M(f_n - f_m)\|_p^p \to 0.$$

Then $\{f_n\}$ is a Cauchy sequence in L_p. Let f be its L_p-limit. We will show that $f_n \to f$ in H^p.

First of all, for f, f_n, L, x and ε fixed, there exist $k(x)$, $i = i(x)$ such that

$$M(L)(f - f_n)(x) \leqslant \Big| \sum_{j=-k(x)}^{k(x)} (f - f_n)(T^{i-j} x)\, \Psi_{k(x)}(j) \Big| + \varepsilon,$$

which means that

$$\int_X M(L)(f-f_n)(x)\,dz \leqslant \int_X \Big|\sum_{j=-k(x)}^{k(x)} (f-f_n)(T^{i-j}x)\,\Psi_{k(x)}(j)\Big|^p dx+\varepsilon^p.$$

But since

$$\sum_{j=-k(x)}^{k(x)} (f-f_n)(T^{i-j}x)\,\Psi_{k(x)}(j) \overset{L_p}{=} \lim_{m\to\infty} \sum_{j=-k(x)}^{k(x)} (f_m-f_n)(T^{i-j}x)\,\Psi_{k(x)}(j),$$

we have

$$\begin{aligned}\int_X |M(L)(f-f_n)(x)|^p dx &\leqslant \lim_{m\to\infty} \int_X \Big|\sum_{j=-k(x)}^{k(x)} (f_m-f_n)(T^{i-j}x)\,\Psi_{k(x)}(j)\Big|^p dx+\varepsilon^p\\ &\leqslant \lim_{m\to\infty} \int_X |M(f_m-f_n)(x)|^p dx+\varepsilon^p \leqslant \tilde{\varepsilon}\end{aligned}$$

if n is big enough. This shows that f is in H^p and $f_n\to f$ in H^p.

Dual spaces. In this section we will show that the dual of H^1 is the space of functions of bounded mean oscillation (BMO), while the dual of $H^p\,(p<1)$ is trivial.

For $p=1$ the result was first proved by Coifman and Weiss [2], while for $p<1$, in a different setting, Muhly [6] proved that $(H^p)^*$ is trivial. We present a very simple proof based on the atomic decomposition, which gives both cases and explains the reason for the sharp differences. We recall that for any integrable function f one can define f^* as

$$f^*(x) = \sup_n n^{-1} \sum_{i=0}^{n-1} |f(T^i x)-T_n f(x)|$$

where

$$T_n f(x) = n^{-1} \sum_{i=0}^{n-1} f(T^i x).$$

A function is said to belong to BMO iff f^* is bounded. We norm BMO by

$$\|f\|_{\mathrm{BMO}} = \|f\|_1+\|f^*\|_\infty.$$

We start by showing that any BMO function "is" a linear functional in H^1.

PROPOSITION. *Let $f\in\mathrm{BMO}$. Then for any $h\in H^1$ of the form $h=\sum_{i=1}^N \lambda_i a_i$,*

$$\langle f, h\rangle \equiv \int fh$$

satisfies

$$|\langle f, h\rangle| \leqslant \|f\|_{\mathrm{BMO}} \sum_{i=1}^N |\lambda_i| \leqslant \|f\|_{\mathrm{BMO}}\,\|h\|_{H^1}$$

Therefore f induces a linear functional in H^1 with norm at most $\|f\|_{\mathrm{BMO}}$.

Proof. It is enough to show that if a is a $(1,\infty)$ atom, then

$$\Big|\int fa\Big| \leqslant \|f\|_{\mathrm{BMO}}.$$

If a is a type 2 atom then

$$\Big|\int fa\Big| \leqslant \int |f| \leqslant \|f\|_{\mathrm{BMO}}$$

since $\|a\|_\infty\leqslant 1$. If a is a type 1 atom supported in the rectangle $R=\bigcup_{i=0}^{k-1} T^i B$, then

$$\begin{aligned}\Big|\int_X f(x)\,a(x)\Big| &= \Big|\int_R f(x)\,a(x)\Big| = \Big|\int_B \sum_{i=0}^{k-1} a(T^i x) f(T^i x)\Big|\\ &= \Big|\int_B \sum_{i=0}^{k-1} a(T^i x)\big(f(T^i x)-T_n f(x)\big)\Big|\\ &\leqslant \int_B \sum_{i=0}^{k-1} |a(T^i x)|\,|f(T^i x)-T_n f(x)|\\ &\leqslant k\,m(R)^{-1} \int_B k^{-1} \sum_{i=0}^{k-1} |f(T^i x)-T_n f(x)|\\ &\leqslant km(R)^{-1}\,m(B)\,\|f\|_{\mathrm{BMO}} = \|f\|_{\mathrm{BMO}}.\end{aligned}$$

It follows that $\mathrm{BMO}\subset(H^1)^*$. For $p<1$ it is trivial that any constant produces a continuous linear functional on H^p.

Let now L be an element of $(H^p)^*$. Fix any q, $1<q<\infty$. Then if h is in L_q, we have

$$\|Mh\|_q \leqslant C_q \|h\|_q$$

and therefore

$$\Big(\int |Mh|^p\Big)^{1/p} \leqslant \|Mh\|_q \leqslant C\,\|h\|_q,$$

and this means that L defines a linear functional in L_q, and so it can be represented by a function f in $L_{q'}\subset L_1$. Let now a be a (p,∞) atom of type 1. Then

$$|\langle L, a\rangle| = \Big|\int_R fa\Big| \leqslant \|L\|.$$

Let us now fix an ergodic rectangle $R=\bigcup_{i=0}^{k-1} T^i B$, and let us write, for any $y\in R$,

$$T_k f(y) = k^{-1}\sum_{i=0}^{k-1} f(T^i x) \quad \text{where } x\in B,\ y=T^j x,\ 0\leqslant j\leqslant k-1.$$

We observe that

$$\|f-T_k f\|_{L_1(R)} = \sup_{\|g\chi_R\|_\infty \leqslant 1} \left|\int_R (f-T_k f)g\right|.$$

But

$$\left|\int_R (f-T_k f)g\right| = \left|\int_R (f-T_k f)(g-T_k g)\right|$$

since

$$\int_R (f-T_k f)\, T_k g = \int_B \sum_{i=0}^{k-1} (f-T_k f)(T^i x)\, T_k g(T^i x),$$

but $T_k g(T^i x)$ is independent of i and

$$\sum_{i=0}^{k-1} (f-T_k f)(T^i x) = T_k f(x) - T_k f(x) = 0.$$

The same argument shows that

$$\int_R (T_k f)(g-T_k g) = 0$$

and therefore

$$\left|\int_R (f-T_k f)g\right| = \left|\int_R f(g-T_k g)\right|$$

$$= \left|\int_R f\frac{g-T_k g}{(km(B))^{1/p}}(km(B))^{1/p}\right| \leqslant 2(km(B))^{1/p}\left|\int_R fa\right|$$

with

$$a = \frac{g-T_k g}{2(km(B))^{1/p}}\chi_R,$$

which is a (p, ∞) atom. Therefore

$$\|f-T_k f\|_{L_1(R)} \leqslant 2m(R)^{1/p}\|L\|.$$

This can be written as

$$\frac{1}{m(B)}\int_B k^{-1}\sum_{i=0}^{k-1} |f(T^i x)-T_k f(x)| \leqslant 2m(R)^{(1/p)-1}\|L\|.$$

Let now $A \subset B$, $m(A) > 0$. It is obvious that A is the base of the rectangle $\tilde{R} = \bigcup_{i=0}^{k-1} T^i A$ and so

$$\frac{1}{m(A)}\int_A k^{-1}\sum_{i=0}^{k-1} |f(T^i x)-T_k f| \leqslant 2\|L\|\, m(\tilde{R})^{(1/p)-1} \leqslant 2\|L\|\, m(R)^{(1/p)-1}.$$

Since A is arbitrary, this implies that for almost all x in B we have

$$k^{-1}\sum_{i=0}^{k-1} |f(T^i x)-T_k f| \leqslant 2\|L\|\, m(R)^{(1/p)-1}.$$

If $p = 1$, this means that

$$k^{-1}\sum_{i=0}^{k-1} |f(T^i x)-T_k f| \leqslant 2\|L\| \quad \text{a.e.}$$

If $p < 1$, by choosing small rectangles one gets

$$k^{-1}\sum_{i=0}^{k-1} |f(T^i x)-T_k f| = 0 \quad \text{a.e.},$$

i.e. f is constant on orbits, and since T is ergodic, f is constant.

We have thus shown that for $p < 1$, f must be constant, while for $p = 1$ we have $f \in \mathrm{BMO}$ and

$$\|f^\#\| \leqslant 2\|L\|.$$

Furthermore, since any function g such that $\|g\|_\infty \leqslant 1$ is a $(1, \infty)$ atom, we have $\left|\int fg\right| \leqslant \|L\|$ for any such g. Therefore $\|f\|_1 \leqslant \|L\|$ and finally

$$\|f\|_{\mathrm{BMO}} = \|f\|_1 + \|f^\#\|_\infty \leqslant 3\|L\|.$$

Interpolation. In this section we will show that one can interpolate between H^p, $1/2 < p \leqslant 1$ and L^q, $q > 1$.

Definition. We will say that a sublinear operator T is of *weak type* (H^p, p), $p \leqslant 1$ if

$$m\{x;\ |Tf(x)| > \lambda\} \leqslant (M/\lambda)^p |f|_{p,\infty}.$$

We will prove that if an operator T is of weak type (H^{p_1}, p_1) and (p_2, p_2) with $1/2 < p_1 \leqslant 1 < p_2 \leqslant \infty$, then T is bounded in H^p, $p_1 < p \leqslant 1$, and also in L_p, $1 < p \leqslant p_2$.

We will split the proof into two theorems.

Theorem 1. *Let T be a sublinear operator of weak type (H^{p_1}, p_1) and (p_2, p_2), $1/2 < p_1 \leqslant 1 < p_2$. Then T is bounded from H^p into L_p, $p_1 < p \leqslant 1$.*

Proof. It is enough to show that $\|Ta\|_p < C$ for any (p, ∞) atom.

If a is a type 1 atom with support in $R = \bigcup_{i=0}^{k-1} T^i B$, then it follows that

$$\|a\|_{p_2} \leqslant m(R)^{(1/p_2)-(1/p)}$$

and

$$\Big(k^{-1} \sum_{i=0}^{k-1} |a(T^i x)|^{p_2}\Big)^{1/p_2} \leqslant m(R)^{-1/p},$$

which means that

$$b = m(R)^{(1/p)-(1/p_1)} a$$

is a (p_1, p_2) atom since

$$\Big(k^{-1} \sum_{i=0}^{k-1} |b(T^i x)|^{p_2}\Big)^{1/p_2} \leqslant m(R)^{-1/p_1}.$$

Since $H^{p_1,p_2} = H^{p_1,\infty} = H^{p_1}$ with equivalent "norms" it follows that $a \in H^{p_1}$ and

$$|a|_{p_1} \leqslant m(R)^{((1/p_1)-(1/p))p_1} = m(R)^{1-(p_1/p)}.$$

Therefore we know that

$$m\{x;\ |Ta(x)| > \lambda\} \leqslant (M_1/\lambda)^{p_1} |a|_{p_1} \leqslant (M_1/\lambda)^{p_1} m(R)^{1-(p_1/p)}$$

and

$$m\{x;\ |Ta(x)| > \lambda\} \leqslant (M_2/\lambda)^{p_2} \int |a|^{p_2} \leqslant (M_2/\lambda)^{p_2} m(R)^{1-(p_2/p)}.$$

From these two estimates one obtains a bound for $\|Ta\|_p^p$ in the usual way: we fix a number D and write

$$\int |Ta|^p = p \int_0^{Dm(R)^{-1/p}} \lambda^{p-1} m\{x;\ |Ta(x)| > \lambda\}\, d\lambda + p \int_{Dm(R)^{-1/p}}^{\infty} \lambda^{p-1} m\{x;\ |Ta(x)| > \lambda\}\, d\lambda$$

$$\leqslant p \int_0^{Dm(R)^{-1/p}} \lambda^{p-1-p_1} M_1^{p_1} m(R)^{1-(p_1/p)}\, d\lambda + p \int_{Dm(R)^{-1/p}}^{\infty} \lambda^{p-1-p_2} M_2^{p_2} m(R)^{1-(p_2/p)}\, d\lambda.$$

Since $p_1 < p < p_2$, the last expression is bounded by

$$\frac{p}{p-p_1} M_1^{p_1} m(R)^{1-(p_1/p)} \big(Dm(R)^{-1/p}\big)^{p-p_1} + \frac{p}{p_2-p} M_2^{p_2} m(R)^{1-(p_2/p)} \big(Dm(R)^{-1/p}\big)^{p-p_2}$$

$$= \frac{p}{p-p_1} M_1^{p_1} D^{p-p_1} + \frac{p}{p_2-p} M_2^{p_2} D^{p-p_2}.$$

Taking $D = (M_2^{p_2} M_1^{-p_1})^{1/(p_2-p_1)}$, we obtain

$$\|Ta\|_p \leqslant \left(\frac{p}{p-p_1} + \frac{p}{p_2-p}\right)^{1/p} M_1^t M_2^{1-t}$$

with

$$t = \frac{p_1(p_2-p)}{p(p_2-p_1)}.$$

If a is a type 2 atom, i.e. if $a \in L_\infty$, $\|a\|_\infty \leqslant 1$, then obviously $a \in H^{p_1}$ with $|a|_{p_1} \leqslant 1$ and $a \in L_{p_2}$ with $\|a\|_{p_2} \leqslant 1$, and we may write

$$\int |Ta|^p = p \int_0^D \lambda^{p-1} m\{x:\ |Ta| > \lambda\}\, d\lambda + p \int_D^\infty \lambda^{p-1} m\{x;\ |Ta| > \lambda\}\, d\lambda$$

$$\leqslant p \int_0^D \lambda^{p-1-p_1} M_1^{p_1}\, d\lambda + p \int_D^\infty \lambda^{p-1-p_2} M_2^{p_2}\, d\lambda$$

$$\leqslant \frac{p}{p-p_1} M_1^{p_1} D^{p-p_1} + \frac{p}{p_2-p} M_2^{p_2} D^{p-p_2}.$$

Choosing D as before, we have the same bound for $\|Ta\|_p$. This ends the proof of Theorem 1.

Next, we want to show that an operator of weak type $(H^1, 1)$ and (p_2, p_2), $1 < p_2$, is bounded in $L_p(1 < p < p_2)$. The idea is, as in the Marcinkiewicz interpolation theorem, to split f in L_p into two functions f_1 and f_2, with f_1 in H_1 and f_2 in L_{p_2}. In order to be able to do this we need a technical lemma that will play the role of the Calderón-Zygmund decomposition.

For $p > 1$, let us fix p_0, $1 < p_0 < p$, and let us consider the operator

$$\Lambda_{p_0}(f) = (|f|^{p_0 *})^{1/p_0}$$

where

$$g^*(x) = \sup_k k^{-1} \sum_{i=0}^{k-1} |g(T^i x)|.$$

Then obviously, since $p/p_0 > 1$, we have

$$\int |A_{p_0}(f)|^p = \int (|f|^{p_0 *})^{p/p_0} \leqslant C_{p/p_0} \int |f|^p,$$

which means that A_{p_0} is a bounded operator in L_p, and in particular the set $O(\lambda) = \{x;\ A_{p_0}(f)(x) > \lambda\}$ has measure strictly less than 1 if $\lambda > C_{p/p_0}\|f\|_p$.

LEMMA 3. *Let $f \in L_p$ and $\lambda > C_{p/p_0}\|f\|_p$. Then $O(\lambda) = \{x;\ A_{p_0} f(x) > \lambda\} = \bigcup R_j$ where the R_j are ergodic rectangles, pairwise disjoint, and for each $x \in B_j$ (the base of R_j) we have*

$$\Big(j^{-1} \sum_{i=0}^{j-1} |f(T^i x)|^{p_0}\Big)^{1/p_0} \leqslant 2\lambda.$$

Proof. As in Lemma 1, we just define

$$B_j = \{x \in O(\lambda);\ T^{-1}x \notin O(\lambda),\ x \in O(\lambda), \ldots, T^{j-1}x \in O(\lambda),\ T^j x \notin O(\lambda)\},$$

and it follows that $O(\lambda) = \bigcup R_j$ with $R_j = \bigcup_{i=0}^{j-1} T^i B_j$. Finally

$$j^{-1} \sum_{i=0}^{j-1} |f(T^i x)|^{p_0} \leqslant \frac{2}{j+1} \sum_{i=-1}^{j-1} |f(T^i x)|^{p_0} \leqslant 2 f^{p_0 *}(T^{-1}x) \leqslant 2\lambda^{p_0}$$

since $T^{-1}x \notin O(\lambda)$.

THEOREM 2. *Let T be a sublinear operator of weak type $(H^1, 1)$ and (p_2, p_2), $1 < p_2 < \infty$. Then T is bounded in L_p, $1 < p < p_2$.*

For $p_2 = \infty$, the result holds assuming that T is bounded in L_∞.

Proof. We will prove the theorem only in the case $p_2 < \infty$, since the other case is similar.

Let f be an L_p function, $1 < p < p_2$. We choose p_0, $1 < p_0 < p$, and we consider the operator A_{p_0}. For each $\lambda > C_{p/p_0}\|f\|_p$, we use Lemma 3 to write

$$O(\lambda) = \{x;\ A_{p_0} f(x) > \lambda\} = \bigcup R_j.$$

For each $y \in R_j$, we define

$$(T_j f)(y) = j^{-1} \sum_{i=0}^{j-1} f(T^i x)$$

where $x \in B_j$, $y = T^l x$, $0 \leqslant l \leqslant j-1$. We may then write

$$f = \sum_j (f - T_j f)\chi_{R_j} + \sum_j (T_j f)\chi_{R_j} + f(1 - \chi_{O(\lambda)}) \equiv b_\lambda + g_\lambda$$

where

$$b_\lambda = \sum_j (f - T_j f)\chi_{R_j}.$$

Since

$$|(T_j f)(y)| \leqslant \Big(j^{-1} \sum_{i=0}^{j-1} |f(T^i x)|^{p_0}\Big)^{1/p_0} \leqslant 2\lambda,$$

we have for each j, and for each $x \in B_j$,

$$\Big(j^{-1} \sum_{i=0}^{j-1} |f(T^i x) - T_j f(T^i x)|^{p_0}\Big)^{1/p_0}$$

$$\leqslant \Big(j^{-1} \sum_{i=0}^{j-1} |f(T^i x)|^{p_0}\Big)^{1/p_0} + |T_j f(x)| \leqslant 4\lambda.$$

Therefore the function

$$a_j = \frac{1}{4\lambda m(R_j)} (f - T_j f)\chi_{R_j}$$

is a $(1, p_0)$ atom supported in the rectangle R_j. This means that we can write b_λ as an H^{1,p_0} function, since

$$b_\lambda = \sum_j 4\lambda m(R_j) a_j$$

with norm bounded by $4\lambda \sum m(R_j) = 4\lambda m(O(\lambda))$. Since $H^{1,p_0} = H^1$ with equivalent norms, we have $b_\lambda \in H^1$ and $|b|_{H^1} \leqslant C\lambda m(O(\lambda))$. On the other hand, g_λ is in L_{p_2} since

$$g_\lambda = \sum_j (T_j f)\chi_{R_j} + f(1 - \chi_{O(\lambda)});$$

so if $y \in O(\lambda)$ we have $g_\lambda(y) = T_j f(y)$ for some j, and then $|g_\lambda(y)| \leqslant 2\lambda$, while if $y \notin O(\lambda)$, then

$$|g_\lambda(y)| = |f(y)| \leqslant (A_{p_0} f)(y) \leqslant \lambda.$$

These are the type of estimates one needs to make the argument in the Marcinkiewicz interpolation theorem work.

Let us consider a constant L larger than $C_{p/p_0}\|f\|_p$. Then we have

$$\int |Tf|^p \leqslant p \int_0^L \lambda^{p-1} m\{x;\ |Tf(x)| > \lambda\}\, d\lambda +$$

$$+ p \int_L^\infty \lambda^{p-1} m\{x;\ |Tg_\lambda(x)| > \lambda/2\}\, d\lambda +$$

$$+ p \int_L^\infty \lambda^{p-1} m\{x;\ |Tb_\lambda(x)| > \lambda/2\}\, d\lambda$$

$$= I_1 + I_2 + I_3.$$

In order to estimate I_1, we recall that $f \in L_p \Rightarrow a = f/\|f\|_p$ is a $(1, p)$ atom, and therefore f belongs to H^1 with norm bounded by $C\|f\|_p$. Therefore

$$I_1 \leqslant Cp \int_0^L \lambda^{p-2} \|f\|_p \, d\lambda = C \frac{p}{p-1} \|f\|_p L^{p-1}.$$

For I_3 we use the fact that $b_\lambda \in H^1$ with norm bounded by $C\lambda m(O(\lambda))$ to obtain

$$I_3 \leqslant Cp \int_L^\infty \lambda^{p-2} \lambda m(O(\lambda))\, d\lambda \leqslant Cp \int_0^\infty \lambda^{p-1} m\{x;\ \Lambda_{p_0} f > \lambda\}\, d\lambda$$

$$= C \int (\Lambda_{p_0} f)^p \leqslant C \cdot C^p_{p/p_0} \int |f|^p = C' \|f\|_p^p.$$

Finally, for I_2 we use the fact that T is of weak type (p_2, p_2), and we have

$$I_2 \leqslant Cp \int_L^\infty \lambda^{p-p_2-1} \int |g|^{p_2}\, dx\, d\lambda$$

$$= Cp \int_L^\infty \lambda^{p-p_2-1} \Big(\int_{O(\lambda)} |g|^{p_2}\, dx + \int_{X-O(\lambda)} |g|^{p_2}\, dx \Big) d\lambda$$

$$\leqslant Cp \int_0^\infty \lambda^{p-p_2-1} (2\lambda)^{p_2} m(O(\lambda))\, d\lambda$$

$$+ Cp \int_L^\infty \lambda^{p-p_2-1} \int_{\{\Lambda_{p_0} f \leqslant \lambda\}} |f|^{p_2}\, dx\, d\lambda$$

$$\leqslant C \cdot 2^{p_2} p \int_0^\infty \lambda^{p-1} m\{x;\ \Lambda_{p_0} f > \lambda\}\, d\lambda +$$

$$+ Cp \int |f|^{p_2} \int_{\Lambda_{p_0} f}^\infty \lambda^{p-p_2-1}\, d\lambda\, dx$$

$$\leqslant C \cdot 2^{p_2} \int (\Lambda_{p_0} f)^p\, dx + C \frac{p}{p_2 - p} \int |f|^{p_2} (\Lambda_{p_0} f)^{p-p_2}\, dx$$

$$\leqslant \Big(C \cdot 2^{p_2} + C \frac{p}{p_2 - p} \Big) \int (\Lambda_{p_0} f)^p\, dx \leqslant C' C^p_{p/p_0} \int |f|^p\, dx.$$

Choosing now $L = 2C_{p/p_0}\|f\|_p$, we have obtained

$$\int |Tf|^p \leqslant C \|f\|_p^p$$

as we wanted.

Clearly Theorems 1 and 2 together imply that a sublinear operator of weak type (H^{p_1}, p_1) and (p_2, p_2), $p_1 < 1 < p_2$, must be bounded in L_p, $1 < p < p_2$.

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