STUDIA MATHEMATICA, T. LXXXII. (1985)

Strict dual of $C^b(X, E)$

by

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Abstract. A representation of the strict dual of the space $C^b(X, E)$ of all bounded continuous functions from a completely regular space X into a Hausdorff topological vector space E is obtained.

1. Introduction. The Riesz-type representations of functionals on spaces of vector-valued continuous functions have been studied by several authors including [3, 6, 8, 9, 11, 14, 16]. In most of these works, at some point, the density of the algebraic tensor product $C^b(X)\otimes E$ in the space $C^b(X, E)$ equipped with the strict topology β_0 was important and, moreover, this seemed essential for obtaining the corresponding results. However, although $C^b(X)\otimes E$ is always β_0-dense in $C^b(X, E)$ for a locally convex E, and also for *some* concrete classes of not necessarily locally convex spaces E, the general case remains open. This had been obstructing research (cf. [11, 14]) for quite a while until Kalton (cf. [7]) realized, using some idea of "submeasure convergence", that at least if X is compact (and then β_0 is the uniform topology), the "representation theory" can avoid the "density problem". A version of this works fine also if X is an arbitrary completely regular space and is used in this paper to prove that $C^b(X)\otimes E$ is always dense in $C^b(X, E)$ with its weak topology (i.e. the weakest one defined by the dual of $(C^b(X, E), \beta_0)$). This suffices to obtain the results given in the abstract.

2. Preliminaries. Throughout this paper X will denote a completely regular space, E a real Hausdorff topological vector space, $C^b(X, E)$ the space of all bounded continuous E-valued functions on X. We will denote by $C^b(X)$ the space $C^b(X, \mathbf{R})$ and by $I(X)$ the subset of $C^b(X)$ of all functions ψ satisfying $0 \leqslant \psi \leqslant 1$. The algebraic tensor product $C^b(X)\otimes E$ is the subspace of $C^b(X, E)$ spanned by the functions of the form $f\otimes e$, $f\otimes e(x) = f(x)e$, where $f\in C^b(X)$, $e\in E$. The *uniform topology* u on $C^b(X, E)$ is the vector topology which has as a base at zero the family of all sets of the form $\{f\in C^b(X, E): f(X)\subset W\}$, where W is a neighbourhood of zero in E. The *strict topology* β_0 is the linear topology which has as a base at zero all sets of the form $\{f\in C^b(X, E): gf(X)\subset W\}$, where W is a neighbourhood of zero in E and g is a bounded real function on X vanishing at infinity.

Let Y be a completely regular space. We will denote by $\mathcal{K}(Y)$, $\mathcal{Z}(Y)$, and $c\mathcal{Z}(Y)$ the families of all compact, zero, and cozero subsets of Y, respectively. By $\mathcal{B}(Y)$ we will denote the algebra of subsets of Y generated by $\mathcal{Z}(Y)$. We refer to [5] for general facts about $\mathcal{Z}(Y)$.

Let f be a function on X into E or $\boldsymbol{R}$ and $U \in c\mathcal{Z}(X)$. $f \prec U$ means that there exists a $Z \in \mathcal{Z}(X)$ such that $Z \subset U$ and $\operatorname{supp} f \subset Z$, where $\operatorname{supp} f = \{x\colon f(x) \neq 0\}$.

We recall that if $A \in \mathcal{K}(X)$ or $A \in \mathcal{Z}(X)$, and $B \in c\mathcal{Z}(X)$, $A \subset B$, then:

(a) there are sets $O \in \mathcal{Z}(X)$, $Z \in \mathcal{Z}(X)$ such that $A \subset O \subset Z \subset B$;

(b) there is a function $\psi \in I(X)$ such that $\psi \prec B$ and $\psi(A) = \{1\}$.

If E is a topological vector space, then its topology may be generated by some family of F-seminorms. This implies that E has a base at zero consisting of zero or cozero and balanced sets.

3. The topology γ_0 on $C^b(X, E)$. A *positive Baire measure m* on X is a finite, real-valued, nonnegative, finitely additive set function on $\mathcal{B}(X)$ such that if $B \in \mathcal{B}(X)$ then $m(B) = \sup\{m(Z)\colon Z \in B, Z \in \mathcal{Z}(X)\}$. The measure m is called *tight* if for every $\varepsilon > 0$ there exists a compact set K such that $m(B) \leqslant \varepsilon$ for any set $B \in \mathcal{B}(X)$ which is disjoint from K. The family of all positive, tight Baire measures on X will be denoted by $M_t^+(X)$.

By μ_t is denoted the vector topology on $C^b(X, E)$ which has as a base at zero the family of all sets of the form

$$(*) \qquad \{f \in C^b(X, E)\colon m(\{x\colon f(x) \notin W\}) \leqslant \varepsilon\},$$

where $m \in M_t^+(X)$, W is a neighbourhood of zero in E which belongs to $\mathcal{B}(E)$ and ε is a positive number.

Let $\mathcal{T}$ be the family of all linear topologies τ on $C^b(X, E)$ satisfying $\tau|_H \leqslant \mu_t|_H$ for any u-bounded subset H of $C^b(X, E)$. We define γ_0 as $\sup \mathcal{T}$.

LEMMA 1. *If T is a linear functional on $C^b(X, E)$, then the following statements are equivalent*:

(a) $T \in (C^b(X, E), \gamma_0)'$,

(b) $Tf_\alpha \to 0$ *for every net* $\{f_\alpha\} \subset C^b(X, E)$ *which is u-bounded and μ_t-convergent to zero.*

Proof. The implication (a)$\Rightarrow$(b) is obvious. (b) implies that the weak topology $\sigma(T)$ induced on $C^b(X, E)$ by T belongs to $\mathcal{T}$. Thus $\sigma(T) \leqslant \gamma_0$, so T is γ_0-continuous.

LEMMA 2. *The space $C^b(X) \otimes E$ is γ_0-dense in $C^b(X, E)$.*

Proof. Fix $f \in C^b(X, E)$. Let R be the family of all functions of the form $\sum_{i=1}^{n} \psi_i \otimes f(x_i)$, where $\psi_i \in I(X)$, $x_i \in X$, $\psi_i(x_i) = 1$ and $\operatorname{supp}\psi_i \cap \operatorname{supp}\psi_j = \emptyset$ if $i \neq j$, $i, j = 1, \ldots, n$, $n \in N$. It is easy to see that R is u-bounded. Let V be a μ_t-neighbourhood of zero of the form $(*)$. We can assume that W is balanced. Choose a balanced neighbourhood of zero W_1 in E such that $W_1 + W_1 \subset W$ and $W_1 \in c\mathcal{Z}(E)$. By the tightness of m we can find $K \in \mathcal{K}(X)$ such that $m(B) \leqslant \varepsilon/2$ for every $B \in \mathcal{B}(X)$, $B \cap K = \emptyset$. The set $f(K)$ is compact, so $f(K) \subset S + W_1$ for some finite subset S of E. It follows that there are sets $B_1, \ldots, B_k \in \mathcal{B}(X)$ such that $K \subset \bigcup_{i=1}^{k} B_i$, $B_i \cap B_j = \emptyset$ if $i \neq j$ and $f(x) - f(x') \in W$ for any $x, x' \in B_i$, $i = 1, \ldots, k$. Let $Z_i \in \mathcal{Z}(X)$, $U_i \in c\mathcal{Z}(X)$ be such that $Z_i \subset B_i \subset U_i$ and $m(U_i \setminus Z_i) \leqslant \varepsilon/(2k)$. We can find sets $O_i \in c\mathcal{Z}(X)$ such that $Z_i \subset O_i \subset U_i$ and $O_i \cap O_j = \emptyset$ if $i \neq j$, $i = 1, \ldots, k$. There exist functions $\psi_i \in I(X)$ such that $\psi_i(Z_i) = \{1\}$, $\operatorname{supp}\psi_i = O_i$, $i = 1, \ldots, k$. Choose $x_i \in Z_i$, $i = 1, \ldots, k$. Then the function $h = \sum_{i=1}^{k} \psi_i \otimes f(x_i)$ belongs to R and

$$m(\{x\colon (f-h)(x) \notin W\}) \leqslant \sum_{i=1}^{k} m(U_i \setminus Z_i) + m(X \setminus \bigcup_{i=1}^{k} U_i) \leqslant k\varepsilon/(2k) + \varepsilon/2 = \varepsilon.$$

Therefore $f - g \in V$, so $C^b(X) \otimes E$ is γ_0-dense in $C^b(X, E)$.

4. The dual of $(C^b(X, E), \beta_0)$.

THEOREM. *Every β_0-continuous linear functional on $C^b(X, E)$ is γ_0-continuous.*

Proof. Let $T \in (C^b(X, E), \beta_0)'$. There exists a β_0-neighbourhood of zero $V = \{f\colon gf(X) \subset W_1\}$ in $C^b(X,E)$ such that

$$(1) \qquad |Tf| \leqslant 1 \quad \text{for any } f \in V.$$

We may assume that W_1 is balanced and belongs to $c\mathcal{Z}(E)$. Since $\beta_0 \leqslant u$, we can find a u-neighbourhood of zero $G = \{f\colon f(X) \subset W_2\}$ in $C^b(X, E)$ such that W_2 is balanced and

$$(2) \qquad G \subset V, \quad W_2 \subset W_1, \quad W_2 \in \mathcal{Z}(E).$$

We first observe that

(3) for every $\varepsilon > 0$, there is a $K_\varepsilon \in \mathcal{K}(X)$ such that, if $f \in G$ and $f(K_\varepsilon) = \{0\}$, then $|Tf| \leqslant \varepsilon$.

Indeed, let K_ε be a compact subset of X such that $\{x\colon |g(x)| \geqslant \varepsilon\} \subset K_\varepsilon$. For any $f \in G$, $f(X) \subset W_1$ by (2). If, moreover, $f(K_\varepsilon) = \{0\}$, then $gf(X) \subset \varepsilon W_1$. Therefore by (1), $|Tf| \leqslant \varepsilon$.

We define $F(U) = \sup\{|Tf|\colon f \in G, f \prec U\}$ for any $U \in c\mathcal{Z}(X)$. Obviously F is positive, finite and if $U_1, U_2 \in c\mathcal{Z}(X)$, $U_1 \subset U_2$, then $F(U_1) \leqslant F(U_2)$. We will show that

$$(4) \qquad F(U_1 \cup U_2) \leqslant F(U_1) + F(U_2) \quad \text{for any } U_1, U_2 \in c\mathcal{Z}(X).$$

Let $U_1, U_2 \in c\mathscr{Z}(X)$, $f \in G$ and $f \prec U_1 \cup U_2$. There is a $Z \in \mathscr{Z}(X)$ such that $\operatorname{supp} f \subset Z \subset U_1 \cup U_2$. Fix $\varepsilon > 0$. Let K_ε be such as in (3). Put $K = Z \cap K_\varepsilon$, $H_1 = K \cap U_1$, $H_2 = K \setminus H_1$. For any $x \in H_i$ there are sets $U^x \in c\mathscr{Z}(X)$, $Z^x \in \mathscr{Z}(X)$ such that $x \in U^x \subset Z^x \subset U_i$, $i = 1, 2$. The family $\{U^x: x \in K\}$ is an open cover of K. By the compactness of K we can find a finite subset S of K such that $K \subset \bigcup \{U^x: x \in S\}$. Let $Z_i' = \bigcup \{Z^x: x \in S \cap H_i\}$, $i = 1, 2$. Then $Z_i' \in \mathscr{Z}(X)$, $Z_i' \subset U_i$, $i = 1, 2$. There are sets $Z_i \in \mathscr{Z}(X)$ and $O_i \in c\mathscr{Z}(X)$ such that $Z_i' \subset O_i \subset Z_i \subset U_i$, $i = 1, 2$. Let $\psi_0, \psi_1, \psi_2 \in I(X)$ be functions with supports $U_1 \cup U_2 \setminus (Z_1' \cup Z_2')$, O_1, O_2, respectively. We define functions $\varphi_i = \psi_i(\psi_0+\psi_1+\psi_2)^{-1}$, $i = 0, 1, 2$. Let $f_i = \varphi_i f$, $i = 0, 1, 2$. Then $f_1 \prec U_1$, $f_2 \prec U_2$ and $f_0(K_\varepsilon) = \{0\}$. Moreover, $f_i \in G$, $i = 1, 2$, and $f = f_0+f_1+f_2$, so that

$$|Tf| \leqslant |Tf_0|+|Tf_1|+|Tf_2| \leqslant \varepsilon+F(U_1)+F(U_2).$$

This implies $F(U_1 \cup U_2) \leqslant F(U_1)+F(U_2)$. Suppose additionally that $U_1 \cap U_2 = \emptyset$. Fix $\varepsilon > 0$. Let $f_1, f_2 \in G$ be such that $Tf_i \geqslant F(U_i)-\varepsilon/2$, $i = 1, 2$. Then $f = f_1+f_2 \prec U_1 \cup U_2$, $f \in G$ and $Tf \geqslant F(U_1)+F(U_2)-\varepsilon$. Thus

(5) $$F(U_1 \cup U_2) = F(U_1)+F(U_2) \quad \text{for any } U_1, U_2 \in c\mathscr{Z}(X),\ U_1 \cap U_2 = \emptyset.$$

From (3) it immediately follows that

(6) for every $\varepsilon > 0$, there exists a $K_\varepsilon \in \mathscr{K}(X)$ such that $F(U) \leqslant \varepsilon$ if $U \in c\mathscr{Z}(X)$ and $U \cap K_\varepsilon = \emptyset$.

Moreover,

(7) for every $\varepsilon > 0$, and $U \in c\mathscr{Z}(X)$ there is a $Z \in \mathscr{Z}(X)$, $Z \subset U$ such that $F(U \setminus Z) \leqslant \varepsilon$.

Indeed, if this statement fails to be true for some $\varepsilon > 0$, then by induction we can find a sequence $\{f_n\} \subset G$ such that $\operatorname{supp} f_i \cap \operatorname{supp} f_j = \emptyset$ for $i \neq j$ and $Tf_j > \varepsilon$, $i, j = 1, 2, \ldots$ But $f^n = f_1 + \ldots + f_n$ belongs to G for every $n \in N$ and $Tf^n > n\varepsilon$. This contradicts (1).

We define $m(B) = \inf\{F(U): U \in c\mathscr{Z}(X),\ U \supset B\}$ for $B \subset X$. It is easy to see that the family $\mathscr{B}$ of all subsets B of X such that for any given $\varepsilon > 0$ there are $Z \in \mathscr{Z}(X)$, $U \in c\mathscr{Z}(X)$, $Z \subset B \subset U$ satisfying $m(U \setminus Z) \leqslant \varepsilon$ is an algebra. By (7), $c\mathscr{Z}(X) \subset \mathscr{B}$, so $\mathscr{B}(X) \subset \mathscr{B}$. The function F restricted to $\mathscr{B}(X)$ is a positive Baire measure on X. From (6) it immediately follows that m is tight.

We will now show that T is γ_0-continuous. Let $\{f_\alpha\}_{\alpha \in A}$ be a u-bounded net in $C^b(X, E)$ which is μ_t-convergent to zero. There is a $\delta > 1$ such that $\{f_\alpha\} \subset \delta G$. Fix $\varepsilon > 0$. Let $Z_\alpha = \{x: f_\alpha(x) \notin \varepsilon W_1\}$, $U_\alpha = \{x: f_\alpha(x) \notin \varepsilon W_2\}$. Then $Z_\alpha \in \mathscr{Z}(X)$, $U_\alpha \in c\mathscr{Z}(X)$ and $Z_\alpha \subset U_\alpha$. We can find functions $\psi_\alpha \in I(X)$ such that $\psi_\alpha \prec U_\alpha$ and $\psi_\alpha(Z_\alpha) = \{1\}$, $\alpha \in A$. Let $h_\alpha = \psi_\alpha f_\alpha$ and $k_\alpha = (1-\psi_\alpha) f_\alpha$. Then $h_\alpha \prec U_\alpha$ and $\delta^{-1} h_\alpha \in G$, so that $|Th_\alpha| \leqslant \delta m(U_\alpha)$. Moreover, $k_\alpha(X) \subset \varepsilon W_1$, and so $k_\alpha \in s\varepsilon V$ where $s = \sup\{|g(x)|: x \in X\}$. Thus $|Tf_\alpha| \leqslant |Th_\alpha|+|Tk_\alpha| \leqslant \delta m(U_\alpha) + s\varepsilon$. This implies that $\lim_\alpha Tf_\alpha = 0$, and so, by Lemma 1, T is γ_0-continuous.

COROLLARY 1. *The space $C^b(X) \otimes E$ is $\sigma(Y, Y')$-dense in $C^b(X, E)$, where* $Y = (C^b(X, E), \beta_0)$.

Proof. By the Theorem, $\sigma(Y, Y') \leqslant \gamma_0$. Now, the statement immediately follows from Lemma 2.

COROLLARY 2. $(C^b(X, E), \beta_0)' = M_t(\mathrm{Bo}(X), E')$.

(For the definition of $M_t(\mathrm{Bo}(X), E')$ see [11].)

Proof. This corollary follows immediately from [11], Theorem 4.8 and Corollary 1.

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Received March 19, 1984 (1964)

STUDIA MATHEMATICA, T. LXXXII. (1985)

An atomic theory of ergodic H^p spaces

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Abstract. Let T be an invertible measure-preserving ergodic transformation on a probability space. We define elementary functions associated with T, called "atoms", and we use them to define ergodic Hardy spaces H^p for $p \leqslant 1$. From this atomic definition we obtain maximal function characterizations of H^p. We identify the duals of H^p and of H^1, and finally we obtain interpolation theorems between H^p and L_q, $p \leqslant 1 < q$.

Introduction. In this paper we study the Hardy spaces induced by an invertible, ergodic, measure-preserving transformation on a probability space X.

In [2], Coifman and Weiss studied the space $H^1(X)$, which they defined as the space of functions in $L_1(X)$ whose ergodic Hilbert transform is in $L_1(X)$. Their main results are that, as in the classical case, H^1 can be characterized in terms of maximal operators and that the dual of H^1 can be identified with the space of functions of bounded mean oscillation. (See [4] for the case $H^1(\boldsymbol{R}^n)$).

It was found later that $H^p(\boldsymbol{R}^n)$ can be defined in terms of elementary functions called "atoms" [1], this atomic characterization being very useful in studying interpolation, duality, etc.

Since the methods of [2] do not seem to work for $p < 1$, we use an "atomic" approach. We define $H^{p,q}(X)$ for $1/2 < p \leqslant 1$, $p < q$, as the spaces of functions that can be written in terms of (p, q) atoms. In the first section we show that $H^{p,q}$ can be characterized in terms of maximal operators as in the case $p = 1$. As a corollary we show that $H^{p,q}$ depends only on p, i.e. $H^{p,q} = H^{p,\infty}$, so that we may write simply H^p.

In the second section we use our atoms to study the dual of H^p. One easily sees then that the dual of H^1 is BMO, obtaining another proof of the result in [2]. For $p < 1$ the analogy with the case $H^p(\boldsymbol{R}^n)$ breaks down since the dual of $H^p(X)$ $(p < 1)$ is made only of multiples of the functional induced by the measure on X, while in the classical case H^{p*} is a space of Lipschitz functions. For ergodic H^p spaces, defined by an ergodic action of $\boldsymbol{R}$ in X, this result was obtained by Muhly in [6], but his methods are entirely different and do not seem to be applicable to the discrete case. Our "atomic" proof