Strict dual of $C^b(X, E)$

by

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Abstract. A representation of the strict dual of the space $C^b(X, E)$ of all bounded continuous functions from a completely regular space $X$ into a Hausdorff topological vector space $E$ is obtained.

1. Introduction. The Riesz-type representations of functionals on spaces of vector-valued continuous functions have been studied by several authors including [3, 6, 8, 9, 11, 14, 16]. In most of these works, at some point, the density of the algebraic tensor product $C^b(X) \otimes E$ in the space $C^b(X, E)$ equipped with the strict topology $\beta_0$ was important and, moreover, this seemed essential for obtaining the corresponding results. However, although $C^b(X) \otimes E$ is always $\beta_0$-dense in $C^b(X, E)$ for a locally convex $E$, and also for some concrete classes of not necessarily locally convex spaces $E$, the general case remains open. This had been obstructing research (cf. [11, 14]) for quite a while until Kalton (cf. [7]) realized, using some idea of "submeasure convergence", that at least if $X$ is compact (and then $\beta_0$ is the uniform topology), the "representation theory" can avoid the "density problem". A version of this works fine also if $X$ is an arbitrary completely regular space and is used in this paper to prove that $C^b(X) \otimes E$ is always dense in $C^b(X, E)$ with its weak topology (i.e. the weakest one defined by the dual of $(C^b(X, E), \beta_0)$). This suffices to obtain the results given in the abstract.

2. Preliminaries. Throughout this paper $X$ will denote a completely regular space, $E$ a real Hausdorff topological vector space, $C^b(X, E)$ the space of all bounded continuous $E$-valued functions on $X$. We will denote by $C^b(X)$ the space $C^b(X, \mathbb{R})$ and by $I(X)$ the subset of $C^b(X)$ of all functions $\psi$ satisfying $0 \leq \psi \leq 1$. The algebraic tensor product $C^b(X) \otimes E$ is the subspace of $C^b(X, E)$ spanned by the functions of the form $f \otimes e$, $f \otimes e(x) = f(x)e$, where $f \in C^b(X)$, $e \in E$. The uniform topology $\gamma$ on $C^b(X, E)$ is the vector topology which has as a base at zero the family of all sets of the form \{f \in C^b(X, E); f(x) \in W\}, where $W$ is a neighbourhood of zero in $E$. The strict topology $\beta_0$ is the linear topology which has as a base at zero all sets of the form \{f \in C^b(X, E); g(f)(x) \in W\}, where $W$ is a neighbourhood of zero in $E$ and $g$ is a bounded real function on $X$ vanishing at infinity.
Let $Y$ be a completely regular space. We will denote by $x(Y), \mathcal{Z}(Y)$, and $cZ(Y)$ the families of all compact, zero, and cozero subsets of $Y$, respectively. By $\mathcal{G}(Y)$ we will denote the algebra of subsets of $Y$ generated by $\mathcal{Z}(Y)$. We refer to [5] for general facts about $\mathcal{Z}(Y)$.

Let $f$ be a function on $X$ into $E$ or $R$ and $U \subseteq \mathcal{Z}(X), f \upharpoonright U$ means that there exists a $Z \subseteq \mathcal{Z}(X)$ such that $f \upharpoonright \mathcal{Z}(X) = U$ and $supp f \subseteq Z$, where $supp f = \{x; f(x) \neq 0\}$.

We recall that if $A \subseteq x(Y)$ or $A \subseteq \mathcal{Z}(X)$, and $B \subseteq \mathcal{Z}(X)$, then:

(a) there are sets $O \subseteq \mathcal{Z}(X), Z \subseteq \mathcal{Z}(X)$ such that $A = O \subseteq Z \subseteq B$;

(b) there is a function $\psi \in I(X)$ such that $\psi < B$ and $\psi(A) = \{1\}$.

If $E$ is a topological vector space, then its topology may be generated by some family of $F$-seminorms. This implies that $E$ has a base at zero consisting of zero or cozero and balanced sets.

3. The topology $\gamma_0$ on $C^0(X, E)$. A positive Baire measure $m$ on $X$ is a finite, real-valued, nonnegative, finite, finite additive set function on $\mathcal{G}(E)$ such that $m(B) = m(B \setminus \mathcal{Z}(X))$. The measure $m$ is called tight if for every $\epsilon > 0$ there exists a compact set $K$ such that $m(K) < \epsilon$ for any set $B \subseteq \mathcal{G}(E)$ which is disjoint from $K$. The family of all positive, tight Baire measures on $X$ will be denoted by $M^*_0(X)$.

By $\mu_\epsilon$ is denoted the vector topology on $C^0(X, E)$ which has a base at zero the family of all sets of the form

$$\{f \in C^0(X, E); m(\{x; f(x) \geq \epsilon \}) < \epsilon\},$$

where $m \in M^*_0(X), W$ is a neighborhood of zero in $E$ which belongs to $\mathcal{G}(E)$ and $\epsilon$ is a positive number.

Let $\mathcal{F}$ be the family of all linear topologies $\tau$ on $C^0(X, E)$ satisfying $\tau \subseteq \mathcal{G}(E)$. The measure $m$ is called a topological measure $\gamma_0$ on $X$.

**Definition.** Every positive $\beta_0$-continuous linear functional on the space $C^0(X, E)$ is $\gamma_0$-continuous.

4. The dual of $C^0(X, E), \beta_0$. We define $T \in (C^0(X, E), \beta_0)$. There exists a $\beta_0$-neighbourhood of zero $T \subseteq (C^0(X, E), \beta_0)$. We may assume that $W_1$ belongs to $\mathcal{G}(E)$ and $\gamma_0$-convergent to zero.

**Proof.** Let $T \subseteq (C^0(X, E), \beta_0)$. There exists a $\beta_0$-neighbourhood of zero $T \subseteq (C^0(X, E), \beta_0)$ such that

$$\|T \| \leq 1 \text{ for any } f \in \mathcal{F}.$$

We may assume that $W_1$ is balanced and belongs to $\mathcal{G}(E)$. Since $\beta_0 \subseteq \mathcal{G}(E)$, we can find a $\beta_0$-neighbourhood of zero $G = \{f; f \in W_1 \}$. Then $G \subseteq C^0(X, E)$ such that $W_2$ is balanced and

$$G \subseteq W_2 \subseteq W_2 \subseteq \mathcal{G}(E).$$

We first observe that

(3) for every $\epsilon > 0$, there is a $K_\epsilon \subseteq \mathcal{X}(X)$ such that, if $f \in \mathcal{F}$ and $f(K_\epsilon) = [0]$, then $\|T \| \leq \epsilon$.

Indeed, let $K_\epsilon$ be a compact subset of $X$ such that $\{x; |g(x)| \geq \epsilon \} \subseteq K_\epsilon$. For any $f \in G$, $f \in W_2$ by (2). If, moreover, $f(K) = [0], \|f \| \leq 1$. Therefore by (1), $\|T \| \leq \epsilon$.

We define $F(U) = \sup \|T \| f \in U \} f \in G, \mathcal{G}(X), \mathcal{G}(X), \mathcal{G}(X)$, Obviously $F$ is positive and finite and if $U_1, U_2 \subseteq \mathcal{G}(X), U_1 \subseteq U_2$, then $F(U_1) \leq F(U_2)$. We will show that

(4) $F(U_1 \cup U_2) \leq F(U_1) + F(U_2)$ for any $U_1, U_2 \subseteq \mathcal{G}(X)$.
Let \( U_1, U_2 \in c_0^X(X), f \in G \) and \( f \leq U_1 \cup U_2 \). There is a \( Z \in c_0^X(X) \) such that \( \operatorname{supp} f \subseteq Z \subseteq U_1 \cup U_2 \). Fix \( \varepsilon > 0 \). Let \( K_1 \) be as in (3). Put \( K = Z \cup K_{1}, H_1 = K \cup U_1, H_2 = K \setminus H_1 \). For any \( x \in H_1 \) there are sets \( U^* \in c_0^X(X), Z^* \in c_0^X(X) \) such that \( x \in U^* \subseteq Z^* \subseteq U_1 \), \( i = 1, 2 \). The family \( \{ U^* : x \in K \} \) is an open cover of \( K \). By the compactness of \( K \) we can find a finite subset \( S \) of \( K \) such that \( K \subseteq \bigcup \{ U^* : x \in S \} \). Let \( Z_i = \bigcup \{ Z^* : x \in S \cap H_i \}, i = 1, 2 \). Then \( Z_i \in c_0^X(X), Z_i \subseteq U_i \), \( i = 1, 2 \). There are sets \( Z_i \in c_0^X(X) \) and \( O_i \in c_0^X(X) \) such that \( Z_i \subseteq O_i \subseteq Z_i \subseteq U_i \), \( i = 1, 2 \). Let \( \psi_0, \psi_1, \psi_2 \in F(X) \) be functions with supports \( U_1 \cup U_2 \cup \{ Z_1 \cup Z_2 \}, O_1 \), \( O_2 \), respectively. We define functions \( f_i = \psi_0 \psi_1 + \psi_2 \), \( i = 0, 1, 2 \). Then \( f_1 < U_1, f_2 < U_2 \) and \( f = f_0 + f_1 + f_2 \), so that
\[
|T_f| \leq |T_{f_0}| + |T_{f_1}| + |T_{f_2}| \leq \varepsilon + F(U_1) + F(U_2).
\]
This implies \( F(U_1 \cup U_2) \leq F(U_1) + F(U_2) \). Suppose additionally that \( U_1 \cap U_2 = \varnothing \). Fix \( \varepsilon > 0 \). Let \( f_i, g \in G \) be such that \( |T_{f_i}| > F(U_i) - \varepsilon / 2, i = 1, 2 \). Then \( f = f_1 + f_2 < U_1 \cup U_2 \), \( f \in G \) and \( |T_f| > F(U_1) + F(U_2) - \varepsilon \). Thus (5)
\[
F(U_1 \cup U_2) = F(U_1) + F(U_2),
\]
for any \( U_1, U_2 \in c_0^X(X), U_1 \cap U_2 = \varnothing \).

From (3) it immediately follows that
\[
\text{for every } \varepsilon > 0, \text{ there exists a } K_\varepsilon \in c_0^X(X) \text{ such that } F(U) \leq \varepsilon \text{ if } U \in c_0^X(X) \text{ and } U \cap K_\varepsilon = \varnothing.
\]

Moreover,
\[
\text{for every } \varepsilon > 0, \text{ and } U \in c_0^X(X) \text{ there is a } Z \in c_0^X(X), Z \subseteq U \text{ such that } F(U) \leq \varepsilon \text{ and } U \cap Z = \varnothing.
\]

Indeed, if this statement fails to be true for some \( \varepsilon > 0 \), then by induction we can find a sequence \( \{ f_n \} \subset G \) such that \( \operatorname{supp} f_n \cap \operatorname{supp} f_j = \varnothing \) for \( i \neq j \) and \( |T_{f_n}| > \varepsilon / n, i = 1, 2, \ldots \) But \( f_n = f_1 + \ldots + f_n \) belongs to \( G \) for every \( n \in \mathbb{N} \) and \( |T_{f_n}| \rightarrow \infty \). This contradicts (1).

We define \( m(B) = \inf \left\{ |F(U) : U \in c_0^X(X), U \supseteq B \} \right\} \) for \( B \subset X \). It is easy to see that the family \( \mathcal{A} \) of all subsets \( B \) of \( X \) such that for any \( \varepsilon > 0 \) there are \( Z \in c_0^X(X), U \in c_0^X(X), Z \subseteq B \subset X \) satisfying \( m(U, Z) \leq \varepsilon \) is an algebra. By (7), \( c_0^X(X) \subset \mathcal{A} \), so \( \mathcal{A} \subseteq \mathcal{A} \). The function \( F \) restricted to \( \mathcal{A} \) is a positive Baire measure on \( X \). From (6) it immediately follows that \( m \) is tight.

We will now show that \( T \) is \( \gamma_0 \)-continuous. Let \( \{ f_n \}_{n \in \mathbb{N}} \) be an \( \varepsilon \)-bounded net in \( C^0(X, E) \) which is \( \mu \)-convergent to zero. There is a \( \delta > 1 \) such that \( |f_n| \leq \delta \varepsilon \). Fix \( \varepsilon > 0 \). Let \( Z_\varepsilon = \{ x : f_n(x) \notin W_1 \}, U_\varepsilon = \{ x : f_n(x) \notin W_2 \} \). Then \( Z_\varepsilon \in c_0^X(X), U_\varepsilon \in c_0^X(X) \) and \( Z_\varepsilon \subseteq U_\varepsilon \). We can find functions \( \psi_n \in F(X) \) such that \( \psi_n \leq U_\varepsilon \) and \( \psi_n(Z_\varepsilon) = \{ 0 \}, n \in \mathbb{N} \). Let \( h_n = \psi_n f_n \) and \( k_n = (1 - \psi_n) f_n \). Then \( h_n \leq U_\varepsilon \) and \( \delta^{-1} h_n \in G \), so that \( |T_{h_n}| \leq \delta m(U_\varepsilon) \). Moreover, \( k_n \in W_1 \), and so \( k_n \in W_\varepsilon \) where \( s = \sup \{|g(x)| : x \in X\} \). Thus \( |T_{k_n}| \leq |T_{h_n}| + |T_{k_n}| \leq \delta m(U_\varepsilon) + s \varepsilon \). This implies \( T_{k_n} \leq 0 \), and so, by Lemma 1, \( T \) is \( \gamma_0 \)-continuous.

**Corollary 1.** The space \( C^0(X) \otimes_\varepsilon E \) is \( \sigma(Y, \gamma)^\varepsilon \)-dense in \( C^0(X, E) \), where \( Y = (C^0(X, E), \beta_0) \).

**Proof.** By the Theorem, \( \sigma(Y, \gamma)^\varepsilon \leq \gamma_0 \). Now, the statement immediately follows from Lemma 2.

**Corollary 2.** \( (C^0(X, E), \beta_0) = M_f(\text{Bo}(X), E) \).

(For the definition of \( M_f(\text{Bo}(X), E) \) see [11].)

**Proof.** This corollary follows immediately from [11], Theorem 4.8 and Corollary 1.

**References**


An atomic theory of ergodic $H^p$ spaces

by

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Abstract. Let $T$ be an invertible measure-preserving ergodic transformation on a probability space. We define elementary functions associated with $T$, called "atoms", and we use them to define ergodic Hardy spaces $H^p$ for $p \leq 1$. From this atomic definition we obtain maximal function characterizations of $H^p$. We identify the duals of $H^p$ and of $H^q$, and finally we obtain interpolation theorems between $H^p$ and $L_p$, $p \leq q$.

Introduction. In this paper we study the Hardy spaces induced by an invertible, ergodic, measure-preserving transformation on a probability space $X$.

In [2], Coifman and Weiss studied the space $H^1(X)$, which they defined as the space of functions in $L_1(X)$ whose ergodic Hilbert transform is in $L_1(X)$. Their main results are that, as in the classical case, $H^1$ can be characterized in terms of maximal operators and that the dual of $H^1$ can be identified with the space of functions of bounded mean oscillation. (See [4] for the case $H^1(R^d)$.)

It was found later that $H^p(R^d)$ can be defined in terms of elementary functions called "atoms" [1], this atomic characterization being very useful in studying interpolation, duality, etc.

Since the methods of [2] do not seem to work for $p < 1$, we use an "atomic" approach. We define $H^{p,q}(X)$ for $1/2 < p \leq 1$, $p < q$, as the spaces of functions that can be written in terms of $(p, q)$ atoms. In the first section we show that $H^{p,q}$ can be characterized in terms of maximal operators as in the case $p = 1$. As a corollary we show that $H^{p,\infty}$ depends only on $p$, i.e. $H^{p,\infty} = H^p$, so that we may write simply $H^p$.

In the second section we use our atoms to study the dual of $H^p$. One easily sees then that the dual of $H^1$ is BMO, obtaining another proof of the result in [2]. For $p < 1$ the analogy with the case $H^1(R^d)$ breaks down since the dual of $H^p(X)$ ($p < 1$) is made only of multiples of the functional induced by the measure on $X$, while in the classical case $H^{p,\infty}$ is a space of Lipschitz functions. For ergodic $H^p$ spaces, defined by an ergodic action of $R$ in $X$, this result was obtained by Muhly in [6], but his methods are entirely different and do not seem to be applicable to the discrete case. Our "atomic" proof