On the a.e. divergence of the arithmetic means
of double orthogonal series

by

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Abstract. The classical coefficient test for the $(C, 1)$-summability of single orthogonal series
is due to Men'shov (1926) and Kaczmarz (1927). The first named author has extended the
Men'shov-Kaczmarz theorem for double orthogonal series in [5] giving sufficient conditions for
the $(C, 1, 1)$- and $(C, 1, 0)$-summability. The second named author has proved in [10] that the
condition ensuring the $(C, 1, 1)$-summability is necessary in the case of nonincreasing dyadic
blocks of the coefficients if all double ONS are taken into account. Now we prove that the
coefficient test for the $(C, 1, 0)$-summability is also necessary in the same sense. Besides, we
present counterexamples showing that the Kolmogorov type and Kaczmarz type results ob-
tained in [5] are the best possible ones.

1. Introduction. Let $(X, \mathcal{F}, \mu)$ be a positive measure space and
\[ \varphi = \{ \varphi_k(x); \quad i, k = 1, 2, \ldots \} \]
an orthonormal system (in abbreviation: ONS) on
\(X\). We will consider the double orthogonal series
\[ a = \{ a_{ik}; \quad i, k = 1, 2, \ldots \} \]
be a double sequence of real numbers
\( (\text{coefficients}) \) for which
\[ \sum_{i=1}^{m} \sum_{k=1}^{n} a_{ik}^2 < \infty. \]

By the Riesz-Fischer theorem, there exists a function
\[ f(a, \varphi; x) \in L^2(X, \mathcal{F}, \mu) \]
such that the rectangular partial sums
\[ s_{mn}(a, \varphi; x) = \sum_{i=1}^{m} \sum_{k=1}^{n} a_{ik} \varphi_k(x) \quad (m, n = 1, 2, \ldots) \]
are a.e. bounded and converges to \( f(a, \varphi; x) \) in \( L^2 \)-metric:
\[ \lim_{m,n \to \infty} \int_X \left| s_{mn}(a, \varphi; x) - f(a, \varphi; x) \right|^2 \, d\mu(x) = 0. \]

This research was completed while the first named author was a visiting professor at the
Indiana University, Bloomington.
We will study the a.e. convergence behavior of the following arithmetic means of the rectangular partial sums:

\[ \sigma^{10}_{mn}(a, \varphi; x) = \frac{1}{m} \sum_{i=1}^{m} \sum_{k=1}^{n} s_{k}(a, \varphi; x) = \frac{1}{m} \sum_{i=1}^{m} \left( 1 - \frac{i-1}{m} \right) a_{k} \varphi_{a}(x) \]

and

\[ \sigma^{11}_{mn}(a, \varphi; x) = \frac{1}{mn} \sum_{i=1}^{m} \sum_{k=1}^{n} s_{k}(a, \varphi; x) = \frac{1}{mn} \sum_{i=1}^{m} \sum_{k=1}^{n} \left( 1 - \frac{i-1}{m} \right) \left( 1 - \frac{k-1}{n} \right) a_{k} \varphi_{a}(x) \quad (m, n = 1, 2, \ldots). \]

The following two theorems were proved in [5].

**Theorem A.** If

\[ \sum_{i=1}^{\infty} \sum_{k=1}^{i} a_{k} \left( \log \log (i+3) \right)^{2} \left( \log (k+1) \right)^{2} < \infty, \]

then for every double ONS \( \varphi = \{ \varphi_{a}(x) \} \)

\[ \lim_{m,n \to \infty} \sigma^{10}_{mn}(a, \varphi; x) = f(a, \varphi; x) \quad a.e. \]

**Theorem B.** If

\[ \sum_{i=1}^{\infty} \sum_{k=1}^{i} a_{k} \left( \log \log (i+3) \right)^{2} \left( \log \log (k+3) \right)^{2} < \infty, \]

then for every double ONS \( \varphi = \{ \varphi_{a}(x) \} \)

\[ \lim_{m,n \to \infty} \sigma^{11}_{mn}(a, \varphi; x) = f(a, \varphi; x) \quad a.e. \]

In this paper the logarithms are to the base 2.

During the proofs of Theorems A and B, a Kolmogorov type and a Kaczmaz type result (cf. [2, pp. 118 and 119] concerning single orthogonal series, respectively) were obtained in [5].

**Theorem C.** If

\[ \sum_{i=1}^{\infty} \sum_{k=1}^{i} a_{k} \left( \log \log \{ i, k \} + 3 \right)^{2} < \infty, \]

then for every double ONS \( \varphi = \{ \varphi_{a}(x) \} \)

\[ \lim_{m,n \to \infty} \left[ \sigma^{11}_{mp, nq}(a, \varphi; x) - \sigma^{11}_{mp, nq}(a, \varphi; x) \right] = 0 \quad a.e. \]

**Theorem D.** If condition (1.5) is satisfied, then for every double ONS \( \varphi = \{ \varphi_{a}(x) \} \)

\[ \lim_{m,n \to \infty} \max_{2^{p} \leq m \leq 2^{p+1}} \max_{2^{q} \leq n \leq 2^{q+1}} \left| \sigma^{11}_{mp, nq}(a, \varphi; x) - \sigma^{11}_{mp, nq}(a, \varphi; x) \right| = 0 \quad a.e. \]

The main point is that the sufficient conditions (1.3) and (1.4) in certain cases are also necessary and condition (1.5) is the best possible if all double ONS \( \varphi = \{ \varphi_{a}(x) \} \) are taken into account on a particular measurable space.

To be more specific, from now on (except Lemma 1) let \((X, \mathcal{F}, \mu)\) be either the unit interval \( I = (0, 1) \) or the unit square \( S = (0, 1) \times (0, 1) \) (in the latter case we will write \((x_1, x_2)\) rather than \(x\)) with the \(\sigma\)-algebra of Borel subsets and Lebesgue measure denoted by \(|\cdot|\). It will be clear from the context whether \(|\cdot|\) means the measure on the real line or plane.

Let us set

\[ A_{p,q}^{+} = \left\{ \sum_{i=2^{p+1}}^{2^{p+1}} \sum_{k=2^{q+1}}^{2^{q+1}} a_{k}^{12} \right\}^{1/2} \quad (p, q = 0, 1, \ldots), \]

\[ A_{p,-1}^{-} = \left\{ \sum_{k=2^{p+1}}^{2^{p+1}} a_{k}^{12} \right\}^{1/2} \quad (q = 0, 1, \ldots), \]

\[ A_{p,-1}^{-} = \left\{ \sum_{i=2^{p+1}}^{2^{p+1}} a_{k}^{12} \right\}^{1/2} \quad (p = 0, 1, \ldots), \]

and

\[ A_{1,1}^{-} = |a_{11}|. \]

We agree that by \(2^{-1}\) we mean 0 in this paper. With this agreement, formula (1.6) for \(p = -1\) or \(q = -1\) comprises the subsequent formulas all.

The following two theorems where proved in [10].

**Theorem E.** If for \(p, q = -1, 0, 1, \ldots\)

\[ A_{p,q}^{+} \geq \max \{ A_{p+1,-1}^{-}, A_{p+1,1}^{-} \} \]

and condition (1.4) is not satisfied, then there exists a double ONS \( \Phi = \{ \Phi_{a}(x_1, x_2) \} \) on \( S \) such that

\[ \lim_{m,n \to \infty} \sup_{a_{mn}} \left| \sigma^{11}_{mp, nq}(a, \Phi; x_1, x_2) \right| = \infty \quad a.e. \]

**Theorem F.** Let \( \{ a_{i}; i = 1, 2, \ldots \} \) and \( \{ b_{k}; k = 1, 2, \ldots \} \) be non-decreasing sequences of positive numbers, and let

\[ \lim_{i \to \infty} \frac{a_{i}}{i \cdot \log \log (i+3)} = 0. \]
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(i) Then there exist a double ONS $\Phi = \{\Phi_n(x)\}$ on 1 and a double sequence $a = [a_{nk}]$ of real numbers such that

\[
\sum_{i=1}^{n_k} \sum_{k=1} a_i^2 x_1^i x_2^i x_3^i < \infty,
\]

\[
\lim_{n \to \infty} s_{2P, B}^2(a, \Phi; x) \text{ exists a.e. on } 1,
\]

and

\[
\limsup_{n \to \infty} |a_{n1}^{11} s_{2P, B}^2(a, \Phi; x)| = \infty \text{ a.e. on } (0, 1/2).
\]

(ii) Furthermore, there exist a double ONS $\Psi = \{\Psi_n(x)\}$ on 1 and a double sequence $b = [b_{nk}]$ of real numbers such that condition (1.8) is satisfied with $b_{nk}$ instead of $a_{nk},$

\[
\lim_{n \to \infty} a_{n1}^{11} s_{2P, B}^2(b, \Psi; x) \text{ exists a.e. on } 1,
\]

and

\[
\limsup_{n \to \infty} \left| s_{2P, B}^2(b, \Psi; x) \right| = \infty \text{ a.e. on } (0, 1/2).
\]

2. New results. We will prove that, in the case of nonincreasing dyadic blocks, condition (1.3) is not only sufficient but also necessary for the conclusion in Theorem A.

**Theorem 1.** If for $p = -1, 0, 1, \ldots; k = 1, 2, \ldots,$

\[
A_{nk} = \left\{ \sum_{i=1}^{n_k} a_i^{1/2} \right\} \geq \max \left\{ A_{n-1, k}, A_{n, k+1} \right\}
\]

and condition (1.3) is not satisfied, then there exists a double ONS $\Phi = \{\Phi_n(x_1, x_2)\}$ on S such that

\[
\limsup_{n \to \infty} \sup_{m \in \mathbb{N}} |s_{2P, B}^2(a, \Phi; x_1, x_2)| = \infty \text{ a.e.}
\]

In accordance with our agreement, for $p = -1$ in (2.1) we have $A_{1, k} = [a_{1k}]$ ($k = 1, 2, \ldots$).

The next theorem reveals that condition (1.5) in Theorem D is the best possible.

**Theorem 2.** If $\{a_i; i = 1, 2, \ldots\}$ and $\{b_i; k = 1, 2, \ldots\}$ are nondecreasing sequences of positive numbers and condition (1.7) is satisfied, then there exists a double ONS $\Phi = \{\Phi_n(x)\}$ on 1 and a double sequence $a = [a_{nk}]$ of real numbers such that condition (1.8) is satisfied and

\[
\limsup_{n \to \infty} \left[ \max_{2P, B} \left\{ \left| s_{2P, B}^2(a, \Phi; x) - a_{n1}^{11} s_{2P, B}^2(a, \Phi; x) \right| \right\} \right] = \infty \text{ a.e. on } (0, 1/2).
\]

3. Auxiliary results. We begin with a few definitions and notations. By an interval $(a, b)$ we mean either open interval $(a, b)$, or one of the half-closed intervals $[a, b)$ and $(a, b]$, or closed interval $[a, b]$. By a rectangle $R = (a_1, b_1) \times (a_2, b_2)$ we mean a rectangle with sides parallel to the coordinate axes.

A function $f$ defined on $I$ (or $S$) is said to be a step function if $I$ (or $S$) can be represented as the union of finitely many disjoint intervals (rectangles) such that $f$ is constant on each of these intervals (rectangles). A subset $H$ of $I$ (or $S$) is said to be simple if $H$ is the union of finitely many disjoint intervals (rectangles).

Given a function $f$ defined on $I$ and a subinterval $J = (a_b, b)$ of $I,$ we set

\[
f(J; x) = \begin{cases} f(x-a) & \text{for } x \in J, \\ 0 & \text{otherwise.} \end{cases}
\]

If $H$ is a subset of $I$, then by $H(J)$ we denote the set into which $H$ is carried over by the linear transformation $x = (a-b)x + a.$ Similarly, given a function $g$ defined on $S$ and a subrectangle $R = (a_1, b_1) \times (a_2, b_2)$ of $S,$ we set

\[
g(R; x_1, x_2) = \begin{cases} g \left( x_1 - a_1, x_2 - a_2 \right) & \text{for } (x_1, x_2) \in R, \\ 0 & \text{otherwise.} \end{cases}
\]

Finally, if $H$ is a subset of $S,$ then by $H(R)$ we denote the set into which $H$ is carried over by the linear transformation $x_i = (b_i-a_i)x_i + a_i$ and $x_2 = (b_2-a_2)x_2 + a_2.$

Now we present five lemmas. The extension of the Rademacher–Men’shov inequality is due to Agnew [1] (see also [4]).

**Lemma 1.** For every ONS $\Phi = [\Phi_n(x)],$ sequence $a = [a_{nk}]$ of real numbers and $M, N = 1, 2, \ldots,$

\[
\left\{ \max_{m = m_0 + 1}^{m_0 + M} \sup_{n = n_0 + 1}^{n_0 + N} \left| s_{2P, B}^2(a, \Phi; x_1, x_2) \right| \right\} \leq \sum_{i = i_0 + 1}^{i_0 + M} \sum_{j = j_0 + 1}^{j_0 + N} a_{ij}^2.
\]

The next lemma is due to Men’shov [3] (see also [9]). In the sequel, by $C_1, C_2, \ldots$ we denote positive absolute constants.

**Lemma 2.** For every $p = 1, 2, \ldots, 2p,$ there exist an ONS $\{f(p; x)\}$ and a simple set $E(p) \subseteq I$ such that

\[
a) \quad |E(p)| \geq C_1,
\]

\[
b) \quad |f(p; x)| \leq C_2 \quad \text{for } x \in I \text{ and } i = 1, 2, \ldots, 2p;
\]

\[
c) \quad f(x) \in E(p) \text{ there exists an integer } m = m(p; x) \leq m < 2p, \text{ such that } f_i(p; x) \geq 0 \text{ for each } i = 1, 2, \ldots, m \text{ and }\]

\[
\sum_{i=1}^{m} f_i(p; x) \geq \sum_{i=1}^{m} C_1 \sqrt{p} \log p.
\]
We note that this lemma was originally proved in the case where \( p = q^2 \), \( q \geq 2 \) is an integer. However, Lemma 2 obviously follows from this particular case.

Let \( \mathcal{R} = \{ r_i(x_i)r_j(x_j) : i, k = 1, 2, \ldots; x_i, x_j \in S \} \)
be the two-dimensional Rademacher system (concerning the one-dimensional Rademacher system we refer, e.g., to [11, p. 212]).

**Lemma 3.** If condition (1.2) is not satisfied, then
\[
\limsup_{n \to \infty} \|s_n^p(a, \mathcal{R}; x, x)\| = \infty \quad a.e.
\]

This lemma is an easy extension of a result of Zygmund [11, pp. 205 and 212] from the one-dimensional case to the two-dimensional case. We omit the proof.

In the remaining part of this section we will consider the single orthogonal series

\[
(3.1) \quad \sum_{i=1}^{m} a_i \varphi_i(x)
\]

where \( \varphi = \{ \varphi_i(x) : i = 1, 2, \ldots \} \) is an ONS on \( I \) and \( a = \{ a_i : i = 1, 2, \ldots \} \)

is an ordinary sequence of real numbers with

\[
(3.2) \quad \sum_{i=1}^{m} a_i^2 < \infty.
\]

We will use the following notations for the partial sums of (3.1) and the first arithmetic means of them:

\[
s_n(a, \varphi; x) = \sum_{i=1}^{m} a_i \varphi_i(x)
\]

and

\[
s_n(a, \varphi; x) = \frac{1}{m} \sum_{i=1}^{m} s_i(a, \varphi; x) = \sum_{i=1}^{m} \left( 1 - \frac{i-1}{m} \right) a_i \varphi_i(x) \quad (m = 1, 2, \ldots, \).
\]

The next lemma was proved in [6].

**Lemma 4.** If \( |a_i| \geq |a_{i+1}| \) for \( i = 1, 2, \ldots, \) and

\[
\sum_{i=1}^{m} a_i^2 [\log(i+1)]^2 = \infty,
\]

then there exists an ONS \( \varphi = \{ \varphi_i(x) \} \) of step functions on \( I \) such that

\[
\limsup_{n \to \infty} \|s_n(a, \varphi; x)\| = \infty \quad a.e.
\]

We note that actually only the a.e. divergence of the orthogonal series (3.1) was proved in [6] instead of unbounded divergence expressed by (3.3). But (3.3) follows from this divergence theorem in a routine way (cf. the proof of Lemma 5 below).

**Lemma 5.** If for \( p = 0, 1, \ldots, \)

\[
\sum_{i=1}^{2^{p+1}} a_i^2 \geq \sum_{i=1}^{2^{p+1}} a_i^2
\]

and

\[
\sum_{i=1}^{m} a_i^2 \left[ \log \log(i+3) \right]^2 = \infty,
\]

then there exists an ONS \( \varphi = \{ \varphi_i(x) \} \) of step functions on \( I \) such that

\[
\limsup_{p \to \infty} \|s_{p}(a, \varphi; x)\| = \infty \quad a.e.
\]

**Proof of Lemma 5.** Combining [7, Theorem 2] and [8, Theorem 7] yields the following weaker version: under the conditions of Lemma 5, there exists an ONS \( \varphi = \{ \varphi_i(x) \} \) of step functions on \( I \) such that

\[
\limsup_{p \to \infty} s_{p}(a, \varphi; x) \text{ fails to exist a.e.}
\]

To prove the stronger statement (3.4), we distinguish two cases.

*Case 1:* (3.2) is not satisfied. Then by Zygmund's theorem quoted above, the one-dimensional Rademacher system can be taken in the capacity of \( \varphi \).

*Case 2:* (3.2) is satisfied. Then first we construct a nonincreasing sequence \( \lambda = \{ \lambda_i : i = 1, 2, \ldots \} \) of positive numbers tending to 0 such that for \( p = 0, 1, \ldots, \)

\[
\sum_{i=1}^{2^{p+1}} \lambda_i^2 a_i^2 \geq \sum_{i=1}^{2^{p+1}} \lambda_i^2 a_i^2
\]

and

\[
\sum_{i=1}^{m} \left( \sum_{i=1}^{2^{p+1}} \lambda_i^2 a_i^2 \right) [\log(p+2)]^2 = \infty.
\]

Thus, by the above weaker version, there exists an ONS \( \varphi = \{ \varphi_i(x) \} \) of step functions on \( I \) such that for \( \lambda \lambda = \{ \lambda_i a_i \} \)

\[
\limsup_{p \to \infty} s_{p}(\lambda a, \varphi; x) \text{ fails to exist a.e.}
\]

An Abel transformation yields

\[
s_{p}(\lambda a, \varphi; x) = \sum_{i=1}^{2^{p-1}} (\lambda_i - \lambda_{i+1}) s_{i}(a, \varphi; x) + \lambda_{2p} s_{p}(a, \varphi; x).
\]
By (3.2)
\[
\sum_{i=1}^{m} (\lambda_i - \lambda_{i+1}) \int \phi_i(a, \varphi; x) dx \leq \sum_{i=1}^{m} (\lambda_i - \lambda_{i+1}) \left( \int \phi_i^2 \right)^{1/2} < \infty.
\]
Hence B. Levi's theorem implies that the series
\[
\sum_{i=1}^{m} (\lambda_i - \lambda_{i+1}) s_i(a, \varphi; x)
\]
converges a.e.

By (3.5) and (3.6),
\[
\lim_{p \to \infty} \lambda_{sp} s_{sp}(a, \varphi; x) \text{ fails to exist a.e.}
\]
which is equivalent to (3.4) to be proved.

4. Proof of Theorem 1. The methods applied during the proof are similar to those which were elaborated in [10].

First we make a reduction. Instead of Theorem 1, it is enough to prove the following.

**Theorem 1'.** Under the conditions of Theorem 1, there exist an ONS \( \Psi = \{\Psi_k(x_1, x_2)\} \) of step functions on \( S \) and a subset \( H \) of \( S \) with \( |H| > 0 \) such that

\[
\limsup_{n \to \infty} |\Psi_n(a) - \Psi_h(a, \varphi; x_1, x_2)| = \infty \quad \text{for } (x_1, x_2) \in H.
\]

First we show how Theorem 1' implies Theorem 1.

Let us assume that Theorem 1' has been proved. Then there exist an increasing sequence \( \{r_p: p = 1, 2, \ldots, r_1 = 0\} \) of integers and a sequence \( \{H_p: p = 1, 2, \ldots\} \) of simple subsets of \( S \) such that for \( p = 1, 2, \ldots \),

\[
|H_p| \geq C_p,
\]
and for \((x_1, x_2) \in H_p\)

\[
\max_{r_p \times s_p \times r_p+1} \left| \sum_{i=1}^{r_p} s_{ip}(a, \varphi; x_1, x_2) \left( 1 - \frac{i-1}{m} \right) \phi_i(x_1, x_2) \right| \geq \sum_{i=1}^{r_p} \sum_{k=1}^{r_p} |\phi_k| M_k
\]
where
\[
Q = \{|(i, k): i = 1, 2, \ldots, m; k = 1, 2, \ldots, n\} \quad (m, n = 1, 2, \ldots)
\]

and
\[
M_k = \max_{(x_1, x_2) \neq 0} |\Psi_k(x_1, x_2)| \quad (i, k = 1, 2, \ldots).
\]

Our goal is to construct an ONS \( \Phi = \{\Phi_k(x_1, x_2)\} \) of step functions on \( S \) and a sequence \( \{E_p: p = 1, 2, \ldots\} \) of simple subsets of \( S \) such that these sets are stochastically independent, for \( p = 1, 2, \ldots \)

\[
\max_{(x_1, x_2) \in E_p} |\Phi_k(x_1, x_2)| \geq C_p
\]
for \((x_1, x_2) \in E_p\)

\[
\max_{(x_1, x_2) \in E_p} |\Phi_k(x_1, x_2)| \leq M_k
\]
which is actually (4.3) with \( \Phi_k \) instead of \( \Psi_k \).

We will proceed by induction on \( p \). If \( p = 1 \), then let

\[
\Phi_k(x_1, x_2) = \Psi_k(x_1, x_2) \quad \text{for } i, k = 1, 2, \ldots, r_2 \text{ and } E_1 = H_1.
\]

Conditions (4.4)-(4.6) are obviously satisfied.

Next let \( r_0 \geq 2 \) be an integer and assume that the step functions \( \{\Phi_k(x_1, x_2): i, k = 1, 2, \ldots, r_{p-1}\} \) and the simple sets \( \{E_p: p = 1, 2, \ldots, r_{p-1}\} \) have been defined in such a way that these functions are orthonormal on \( S \), these sets are stochastically independent, and relations (4.4)-(4.6) are satisfied for \( p = 1, 2, \ldots, r_{p-1} \). We can divide \( S \) into a finite number of disjoint rectangles \( \{R_s: s = 1, 2, \ldots, \sigma\} \) such that the functions \( \{\Phi_k(x_1, x_2): i, k = 1, 2, \ldots, r_{p-1}\} \) are constant on each \( R_s \) and the sets \( \{E_p: p = 1, 2, \ldots, r_{p-1}\} \) are the unions of certain \( R_s \). Let \( R'_s \) and \( R''_s \) denote the two halves of \( R_s \), for example, if \( R_s = \langle a_1, b_1 \rangle \times \langle a_2, b_2 \rangle \), then let \( R'_s = \langle a_1, a_1 + b_1/2 \rangle \times \langle a_2, b_2 \rangle \) and \( R''_s = \langle a_1 + b_1/2, b_1 \rangle \times \langle a_2, b_2 \rangle \). We set for \( i, k = 1, 2, \ldots, r_{p_0+1} \) when \( \max(i, k) > r_{p_0} \),

\[
\Phi_k(x_1, x_2) = \sum_{s=1}^{\sigma} [\Psi_s(R'_s, x_1, x_2) - \Psi_s(R''_s, x_1, x_2)]
\]
and

\[
E_{p_0} = \bigcup_{s=1}^{\sigma} [E_{p_0}(R'_s) \cup E_{p_0}(R''_s)].
\]

It is easy to verify that the step functions \( \{\Phi_k(x_1, x_2): i, k = 1, 2, \ldots, r_{p_0+1}\} \) form an ONS on \( S \), the simple sets \( \{E_p: p = 1, 2, \ldots, r_{p_0+1}\} \) are stochastically independent, and conditions (4.4)-(4.6) are satisfied for \( p = p_0 \) due to (4.2) and (4.3).
The above induction scheme shows that the ONS \( \Phi = \{ \Phi_k(x_1, x_2) \} \) and the sequence \( \{ E_p \} \) of stochastically independent sets can be defined so that conditions (4.4)-(4.6) are satisfied for every \( p = 1, 2, \ldots \).

Putting (4.5) and (4.6) together, we can conclude for \((x_1, x_2) \in E_p\)

\[
\max_{r_1 \in S_1, r_2 \in S_2} |\sigma_{1,2}^0(a, \Phi; x_1, x_2)| \geq p \quad (p = 1, 2, \ldots).
\]

Setting

\[
E = \lim_{p \to \infty} \sup E_p
\]

(4.4) implies \(|E| = 1\) via the Borel–Cantelli lemma. If \((x_1, x_2) \in E\), then we have (4.7) for infinitely many \(p\). Consequently, (2.2) is satisfied which was to be proved.

Proof of Theorem 1. We may assume that condition (1.2) is satisfied. Otherwise, (4.1) immediately follows from Lemma 3 even with \(H = S\).

Our starting point is that if condition (2.1) is satisfied, while (1.3) is not, then

\[
\sum_{p=1}^{m} \sum_{k=1}^{n} A_{pk}^2 \left[ \log(p+3)^2 \log(k+1)^2 \right] = \infty.
\]

We will distinguish three cases:

(a) \(\sum_{p=1}^{m} A_{p}^2 \left[ \log(p+2)^2 \right] = \infty\),

(b) \(\sum_{k=1}^{n} A_{k}^2 \left[ \log(k+1)^2 \right] = \sum_{k=1}^{n} a_{k}^2 \left[ \log(k+1)^2 \right] = \infty\),

(c) for every \(r = 1, 2, \ldots\),

\[
\sum_{p=1}^{m} \sum_{k=1}^{n} A_{pk}^2 \left[ \log(p+2)^2 \log(k+1)^2 \right] = \infty.
\]

Case (a). By Lemma 5, there exists an ONS \( \phi = \{ \phi_i(x) \} \) of step functions on \(I\) such that

\[
\lim_{p \to \infty} \sup |\sigma_{1,2}(d^{(1)}, \phi; x)| = \infty \quad \text{a.e.}
\]

where \(d^{(1)} = \{ \delta_{i} : i = 1, 2, \ldots \} \).

On the other hand condition (1.2) clearly implies

\[
\sum_{i=1}^{\infty} a_{i}^2 < \infty,
\]

and thus Kolmogorov's theorem (see, e.g., [2, p. 118]) is applicable to obtain

\[
\lim_{p \to \infty} |\sigma_{1,2}(d^{(1)}, \phi; x) - \sigma_{1,2}(d^{(1)}, \phi; x)| = 0 \quad \text{a.e.}
\]

This and (4.9) give that

\[
\lim_{p \to \infty} |\sigma_{1,2}(d^{(1)}, \phi; x)| = \infty \quad \text{a.e.}
\]

We take an arbitrary sequence \( \{R_{ik} : i = 1, 2, \ldots ; k = 2, 3, \ldots \} \) of disjoint rectangles such that

\[
\bigcup_{i=1}^{\infty} \bigcup_{k=2}^{\infty} R_{ik} \subseteq S_2 \setminus S \quad \text{where} \quad S_2 = (0, 2) \times (0, 2).
\]

Set for \(i = 1, 2, \ldots ; k = 2, 3, \ldots \),

\[
\psi_{ik}(x_1, x_2) = \begin{cases} a_{i1} \delta_{i} & \text{for} \ (x_1, x_2) \in S, \\ 0 & \text{for} \ (x_1, x_2) \in S_2 \setminus S, \end{cases}
\]

while for \(i = 1, 2, \ldots ; k = 2, 3, \ldots \),

\[
\psi_{ik}(x_1, x_2) = \begin{cases} |R_{ik}|^{-1/2} & \text{for} \ (x_1, x_2) \in R_{ik}, \\ 0 & \text{for} \ (x_1, x_2) \in S_2 \setminus R_{ik}. \end{cases}
\]

It is not hard to check that \(\psi = \{ \psi_{ik}(x_1, x_2) : i = 1, 2, \ldots \} \) is an ONS of step functions on \(S_2\) and for almost every \((x_1, x_2) \in S\)

\[
\lim_{p \to \infty} |\sigma_{1,2}^0(a, \psi; x_1, x_2)| = \infty.
\]

Finally, the system \(\Psi = \{ \psi_{ik}(x_1, x_2) \} \) defined for \((x_1, x_2) \in S\) by

\[
\Phi_{ik}(x_1, x_2) = 2\psi_{ik}(2x_1, 2x_2) \quad (i, k = 1, 2, \ldots)
\]

and \(H = S_{1/2} = (0, \frac{1}{2}) \times (0, \frac{1}{2})\) meet all requirements stated in Theorem 1.

Case (b). By Lemma 4, there exists an ONS \(\phi = \{ \phi_i(x) \} \) of step functions on \(I\) such that

\[
\lim_{n \to \infty} \sup \left|s_n(a_{i1}, \phi; x)\right| = \infty \quad \text{a.e.}
\]

where \(a_{i1} = \{ a_{i1} : k = 1, 2, \ldots \} \).

Let \(\{R_{ik} : i = 2, 3, \ldots ; k = 1, 2, \ldots \} \) be again disjoint rectangles whose union is contained in \(S_2 \setminus S\) (cf. (4.10)). Set for \(k = 1, 2, \ldots \),

\[
\psi_{ik}(x_1, x_2) = \begin{cases} a_{i1} \delta_{i} & \text{for} \ (x_1, x_2) \in S, \\ 0 & \text{for} \ (x_1, x_2) \in S_2 \setminus S, \end{cases}
\]

while for \(i = 2, 3, \ldots ; k = 1, 2, \ldots \), set (4.11).

Clearly, \(\psi = \{ \psi_{ik}(x_1, x_2) : i = 1, 2, \ldots \} \) is an ONS of step functions on \(S_2\) and for almost every \((x_1, x_2) \in S\)

\[
\lim_{n \to \infty} |\sigma_{1,2}^0(a, \psi; x_1, x_2)| = \infty.
\]

Setting (4.12) and \(H = S_{1/2}\) completes the proof in this case.
Case (c). Without loss of generality, we may assume that the coefficients 
$a_k$ are rational numbers and different from zero. In fact, we can choose a sequence $\tilde{a} = \{a_\tilde{a_k} \neq 0; \tilde{k} = 1, 2, \ldots\}$ of rational numbers such that for 
p = -1, 0, 1, \ldots, k = 1, 2, \ldots,
$$A_{p_k} = \left\{ \sum_{i=1}^{2^p+1} a_i \right\}^{1/2} \geq \max \{A_{p+1,k}, A_{p+1+1}\}$$
(we recall that $A_{p+1,k} = \|a_d\|$ for $p = -1$),
$$\sum_{i=1}^{m} \sum_{k=1}^{n} a_k \log \log (i+3))^2 \log (k+1))^2 = \infty,$
and
$$\sum_{i=1}^{m} \sum_{k=1}^{n} |a_k - \tilde{a}_k| < \infty.$$  

Then for every ONS $\varphi = \{\varphi_k(x_1, x_2)\}$ on $S$,
$$\sum_{i=1}^{m} \sum_{k=1}^{n} |a_k - \tilde{a}_k| \int_0^1 \{|\varphi_k(x_1, x_2)| d\varphi_1 \leq \sum_{i=1}^{m} \sum_{k=1}^{n} |a_k - \tilde{a}_k| < \infty.$$  

Thus,
$$\lim_{m,n=\infty} \{s_{mn}(a, \varphi; x_1, x_2) - s_{mn}(\tilde{a}, \tilde{\varphi}; x_1, x_2)\} \text{ exists a.e.,}$$
a fortiori,
$$\lim_{m,n=\infty} \{s_{mn}^{(a)}(a, \varphi; x_1, x_2) - s_{mn}^{(a)}(\tilde{a}, \tilde{\varphi}; x_1, x_2)\} \text{ exists a.e.}$$

Consequently, if there exists an ONS $\Phi = \{\varphi_k(x_1, x_2)\}$ on $S$ and a subset $H$ of $S$ such that
$$\limsup_{m,n=\infty} |s_{mn}^{(a)}(a, \Phi; x_1, x_2)| = \infty \text{ for } (x_1, x_2) \in H,$$
then also
$$\limsup_{m,n=\infty} |s_{mn}^{(a)}(a, \Phi; x_1, x_2)| = \infty \text{ for almost every } (x_1, x_2) \text{ in } H.$$  

So, we may assume from now on that the coefficients $a_k$ are rational numbers different from zero. From (2.1) and (4.8) it follows that for every 
r = 1, 2, \ldots,
$$\sum_{r=1}^{m} \sum_{q=1}^{n} p^2 q^2 r^{2^m+1} A_{2^m+1, 2^m+1} = \infty.$$  

Thus, there exist an increasing sequence $\{r_j; j = 1, 2, \ldots; r_j \geq 2\}$ of integers}

and a nonincreasing sequence $\{s_j; j = 1, 2, \ldots\}$ of positive numbers tending
to zero such that for 
j = 1, 2, \ldots,
$$\sum_{j=1}^{\infty} s_j \sum_{p=q}^{r_{j+1}} \sum_{q=r_{j+1}}^{r_{j+1}} p^2 q^2 r^{2^m+1} A_{2^m+1, 2^m+1} = \infty$$
and
$$s_j \sum_{p=r_{j+1}}^{r_{j+1}} \sum_{q=r_{j+1}}^{r_{j+1}} p^2 q^2 r^{2^m+1} A_{2^m+1, 2^m+1} \leq 1.$$  

Let $j = 1, 2, \ldots$ be fixed and let $\{R_{r_{j+1}}^{(a)}; r_j \leq p, q < r_{j+1}\}$ be disjoint rectangles in $S$ with

$$|R_{r_{j+1}}^{(a)}| = s_j^2 p^2 q^2 r^{2^m+1} A_{2^m+1, 2^m+1}.$$  

By (4.14), this is possible. We apply Lemma 2 separately for $2^{m-1}$ and $2^m$, and set for $2^m \leq i < 2^{m+1}$, $2^m \leq k < 2^{m+1}$
$$f_{r_{j+1}}^{(a)}(x_1, x_2) = f_{r_{j+1}}(2^{m-1}; x_1) f_{r_{j+1}}(2^{m-1}; x_2)$$
(here $(x_1, x_2) \in S$) and for $r_j \leq p, q < r_{j+1}$
$$H_{r_{j+1}}^{(a)} = E(2^m) \times E(2^m).$$  

Finally, we set for $2^m \leq i < 2^{m+1}$, $2^m \leq k < 2^{m+1}$
$$f_{r_{j+1}}^{(a)}(x_1, x_2) = |R_{r_{j+1}}^{(a)}| \sum_{r_{j+1}}^{r_{j+1}} R_{r_{j+1}}^{(a)}(x_1, x_2)$$
and for $r_j \leq p, q < r_{j+1}$
$$H_{r_{j+1}}^{(a)} = H_{r_{j+1}}^{(a)}(R_{r_{j+1}}^{(a)}).$$

It is easy to see that for every $j = 1, 2, \ldots$ the step functions
$$f_{r_{j+1}}^{(a)}(x_1, x_2); 2^m \leq i < 2^{m+1}$$
form an ONS on $S$ and the simple sets $\{H_{r_{j+1}}^{(a)}; r_j \leq p, q < r_{j+1}\}$ are disjoint. In addition, (a)-(c) in Lemma 2 result in the following properties: for $r_j \leq p, q < r_{j+1}$
$$|H_{r_{j+1}}^{(a)}| \geq C_1 s_j^2 p^2 q^2 r^{2^m+1} A_{2^m+1, 2^m+1},$$
$$f_{r_{j+1}}^{(a)}(x_1, x_2) = 0 \text{ for } (x_1, x_2) \in H_{r_{j+1}}^{(a)},$$
$$\text{if } (i, k) \notin [2^m, 2^{m+1})^2 [2^m, 2^{m+1}).$$  

Furthermore, for $(x_1, x_2) \in H_{r_{j+1}}^{(a)}$ there exist integers $m=m(x_1, x_2)$ and $n=n(x_1, x_2)$,
$$2^m \leq m < 2^{m+1}, 2^m \leq n < 2^{m+1},$$
such that
$$f_{r_{j+1}}^{(a)}(x_1, x_2) \geq 0 \text{ for } 2^m \leq i < 2^{m+1}, 2^m \leq k < 2^{m+1},$$
and
$$\sum_{r_{j+1}}^{r_{j+1}} \sum_{k=2^m}^{2^{m+1}} A_{r_{j+1}} f_{r_{j+1}}^{(a)}(x_1, x_2) \geq C_3 s_j.$$
Now we take into account that the $a_k$ are rational numbers and different from zero. Therefore for $j = 1, 2, \ldots$ there exists a positive integer $Q_j$ such that for every $i, k$, and $s$ with $2^s < i < 2^{s+1}$, $2^s < s < 2^{s+1}$,

$$ \frac{a_k}{A_k} = \frac{P(i, k, s)}{Q_j} $$

where the $P(i, k, s)$ are also integers and $\varphi$.

We consider a decomposition of $S$ into disjoint rectangles $\{R_n; n = 1, 2, \ldots, Q_j\}$ of equal measure:

$$ |R_n| = \frac{1}{Q_j} \quad (n = 1, 2, \ldots, Q_j). $$

We set for $2^s < i < 2^{s+1}, 2^s < s < 2^{s+1}$,

$$ g^{\varphi}_{ijk}(x_1, x_2) = \sum_{a_k = 0}^{\infty} \sum_{k = -1}^{\infty} f_{ijk}(R_n; x_1, x_2) $$

where $v(2^s, k) = 0$ and $v(2^s, k) = \sum_{i = 2^s+1}^{2^{s+1}} P(i, s, k) (i > 2^s)$, and for $r_j \leq p,$

$$ F_{ijk} = \bigcup_{x_1} H^{\varphi}_{ijk}(R_n). $$

We agree to denote by $\varphi$ the set of ordered pairs of positive integers (the so-called lattice points), while for $j = 1, 2, \ldots$ we set

$$ N_j = \{(i, k) \in \varphi; \; a(r) < i \leq a(r+1), \; \beta(r) < k \leq \beta(r+1)\} $$

where

$$ a(m) = 2^m \quad \text{and} \quad \beta(m) = 2^m \quad (m = 0, 1, \ldots). $$

Now it is routine to check that for every $j = 1, 2, \ldots$ the step functions $\{g^{\varphi}_{ijk}(x_1, x_2); (i, k) \in N_j\}$ are orthonormal on $S$, and the sets $\{F_{ijk}; r_j \leq p, q < r_{j+1}\}$ are simple and disjoint. From (4.15) (4.18) it follows that for every pair $(p, q)$, $r_j \leq p, q < r_{j+1}$,

$$ F_{ijk} \supseteq C_i \mathbb{R} \times \mathbb{R} \times 2^{s+1} x 2^{s+1}, $$

$$ \sum_{i = 2^{s+1}}^{2^{s+2}} g^{\varphi}_{ijk}(x_1, x_2) = 0 \quad \text{for} \; (x_1, x_2) \in F_{ijk}, $$

if $(i, k) \notin (2^{2s}, 2^{2s+1}) \times (2^s, 2^{s+1})$,

and for $(x_1, x_2) \in F_{ijk}$ there exist integers $m = m(x_1, x_2), n = n(x_1, x_2),$ $2^s < m < 2^{s+1}, 2^s < n < 2^{s+1},$ such that

$$ a_k g^{\varphi}_{ijk}(x_1, x_2) \geq 0 \quad \text{for} \; 2^{2s} < i \leq 2^{2s}, 2^s < k \leq n, $$

and

$$ \left| \sum_{i = 2^{2s}+1}^{2^{2s+1}} \sum_{k = 2^s}^{2^{s+1}} a_k g^{\varphi}_{ijk}(x_1, x_2) \right| \geq $$

$$ \frac{C_i}{j}. $$

Next, we will define an ONS $\{\phi_{ijk}(x_1, x_2); (i, k) \in \varphi; j = 1, 2, \ldots, \}$ of step functions on $S$ and a sequence $\{G_j; j = 1, 2, \ldots, \}$ of stochastically independent simple subsets of $S$ such that for every $j = 1, 2, \ldots$,

$$ |G_j| = \sum_{p=q}^{r_j-1} \sum_{q=r_j}^{r_{j+1}-1} |F_{ijk}|, $$

for $(x_1, x_2) \in G_j$ there exist integers $p, q, m, n$ and $s$ such that $r_j \leq p, q < r_{j+1}.$

$$ \sum_{i = 2^{2m}}^{2^{2m+1}} \sum_{k = 2^m}^{2^{m+1}} a_k \phi_{ijk}(x_1, x_2) \geq $$

$$ \frac{C_i}{s_j}, $$

and

$$ \phi_{ijk}(x_1, x_2) = 0 \quad \text{for} \; (x_1, x_2) \in G_j, $$

if $(i, k) \notin N_j$ but $(i, k) \notin (2^{2m}, 2^{2m+1}) \times (2^m, 2^{m+1})$.

The construction will be done by induction on $j$. If $j = 1$, then we set

$$ G_1 = \bigcup_{p=q}^{r_1} \bigcup_{q=r_1}^{r_2-1} F_{ijk}, $$

and for $(i, k) \notin N_1$,

$$ \phi_{ijk}(x_1, x_2) = g^{\varphi}_{ijk}(x_1, x_2). $$

Then conditions (4.23) (4.26) practically coincide with (4.19) (4.22).

Now let $j_0 \geq 1$ be an integer and assume that the step functions $\{\phi_{ijk}(x_1, x_2); (i, k) \in \varphi; j = 1, 2, \ldots, j_0\}$ and the simple sets $\{G_j; j = 1, 2, \ldots, j_0\}$ have been defined in such a way that these functions are orthonormal on $S,$ these sets are stochastically independent, and relations (4.23) (4.26) are satisfied for $j = 1, 2, \ldots, j_0$.

We can split $S$ into a finite number of disjoint rectangles $\{R_s; s = 1, 2, \ldots, \} \supseteq \bigcup R_s$ such that the above step functions are constant on each $R_s$ and the above simple sets are the unions of certain $R_s$. Let $R_s'$ and $R_s''$ be disjoint subrectangles of $R_s$ with equal measure:

$$ |R_s'| = |R_s''| \quad (s = 1, 2, \ldots, \).$$
We define
\[ G_{j_0 + 1} = \bigcup_{n = 0}^{n_0 + 2 - 1} \bigcup_{v = n_0 + 1}^{3} \left[ F_P^{(i + 1)}(R_i^v) \cup F_P^{(j + 1)}(R_j^v) \right] \]
and for \((i, k) \in N_{j_0 + 1}\)
\[ \phi_{ak}(x_1, x_2) = \sum_{v = 1}^{n} [g_{ak}^{(i + 1)}(R_i^v; x_1, x_2) - g_{ak}^{(j + 1)}(R_j^v; x_1, x_2)]. \]

It is not hard to verify that the step functions \( \{ \phi_{ak}(x_1, x_2); (i, k) \in N; j = 1, 2, \ldots, j_0 + 1 \} \) are orthonormal on \( S \), the simple sets \( \{ G_j; j = 1, 2, \ldots, j_0 + 1 \} \) are stochastically independent, and (4.23)-(4.26) are satisfied also for \( j = j_0 + 1 \) due to (4.19)-(4.22).

This induction scheme shows that the ONS \( \{ \phi_{ak}(x_1, x_2); (i, k) \in \bigcup_{i = 1}^\infty N_i \} \) of step functions and the sequence \( \{ G_j; j = 1, 2, \ldots \} \) of stochastically independent simple sets can be defined so that conditions (4.23)-(4.26) are satisfied for every \( j = 1, 2, \ldots \)

By (4.13), either
\[
(4.27) \quad \sum_{j = 1}^{\infty} \sum_{r_2 = 1}^{r_2 + 1 - 1} \sum_{p \neq r_2}^{r_2 - 1} \sum_{q \neq r_2}^{r_2 - 1} \sum_{p \neq r_2}^{r_2 - 1} p^2 q^2 2^{m + 1} A_{j+1}^{2 p q + 2 p q + 1} = \infty.
\]
or
\[
(4.27) \quad \sum_{j = 1}^{\infty} \sum_{r_2 = 1}^{r_2 + 1 - 1} \sum_{p \neq r_2}^{r_2 - 1} \sum_{q \neq r_2}^{r_2 - 1} \sum_{p \neq r_2}^{r_2 - 1} p^2 q^2 2^{m + 1} A_{j+1}^{2 p q + 2 p q + 1} = \infty.
\]

For the sake of definiteness, assume that (4.27) is the case.

We define an ONS \( \{ \tilde{\phi}_{ak}(x_1, x_2); (i, k) \in \bigcup_{i = 1}^\infty \tilde{N}_i \} \) of step functions on \( S_2 \) as follows. Set for \((i, k) \in \bigcup_{i = 1}^\infty \tilde{N}_i \)
\[ \tilde{\phi}_{ak}(x_1, x_2) = \begin{cases} 
\phi_{ak}(x_1, x_2) & \text{for } (x_1, x_2) \in S, \\
0 & \text{for } (x_1, x_2) \in S_2 \setminus S,
\end{cases} \]
and for \((i, k) \in \tilde{N}_i \)
\[ \tilde{\phi}_{ak}(x_1, x_2) = \begin{cases} 
|R_{ak}^{i+1}|^{1/2} & \text{for } (x_1, x_2) \in R_{ak}, \\
0 & \text{for } (x_1, x_2) \in S_2 \setminus R_{ak},
\end{cases} \]
where \( \{ R_{ak}; (i, k) \in \tilde{N}_i \} \) are arbitrary disjoint rectangles in \( S_2 \setminus S \).

By (4.19), (4.23), and (4.27),
\[
\sum_{j = 1}^{\infty} |G_{j_0}| = \infty.
\]

Thus, by the Borel-Cantelli lemma, for \( G = \lim \sup G_j \) we have \( |G| = 1 \).

By (4.24)-(4.26) we can deduce that for every \( j = 1, 2, \ldots \) and \((x_1, x_2) \in G_2 \) there exist integers \( m_j = m_j(x_1, x_2) \) and \( n_j = n_j(x_1, x_2) \) such that \( z_j^2 < m_j, n_j < z_j^2 + 1 \) and
\[
\sum_{i \in \tilde{N}_i} \sum_{j = 1}^{n_j} \left( \frac{1 - (i - 1)/2^{m_j}}{2^{m_j}} \right) a_{ik} \tilde{\phi}_{ik}(x_1, x_2) \geq \frac{1}{2} \sum_{i \in \tilde{N}_i} \sum_{j = 1}^{n_j} a_{ik} \tilde{\phi}_{ik}(x_1, x_2) \geq C \frac{1}{2^{s_j}}
\]
since \( 1 - (i - 1)/2^{m_j} \geq \frac{1}{2} \) for \( i \leq 2^{m_j} - 1 \). If \((x_1, x_2) \in G \), then this estimate holds for infinitely many \( j \) and, consequently,
\[
(4.28) \quad \lim \sup_{j \to \infty} \max_{(x_1, x_2) \in G} \left| \sum_{i \in \tilde{N}_i} \sum_{j = 1}^{n_j} \left( \frac{1 - (i - 1)/2^{m_j}}{2^{m_j}} \right) a_{ik} \tilde{\phi}_{ik}(x_1, x_2) \right| = \infty.
\]

In the following, we slightly modify the definition of the functions \( \tilde{\phi}_{ak}(x_1, x_2) \). By (1.2),
\[
\sum_{j = 1}^{\infty} \sum_{i \in \tilde{N}_i} \left( \frac{1 - (i - 1)/2^{m_j}}{2^{m_j}} \right) a_{ik} \tilde{\phi}_{ik}(x_1, x_2) dx_1 dx_2 < \infty,
\]
whence
\[
\sum_{j = 1}^{\infty} \sum_{i \in \tilde{N}_i} \left( \frac{1 - (i - 1)/2^{m_j}}{2^{m_j}} \right) a_{ik} \tilde{\phi}_{ik}(x_1, x_2) dx_1 dx_2 < \infty,
\]

for almost every \((x_1, x_2) \in S_2 \). Thanks to this fact, the series
\[
(4.29) \quad \sum_{j = 1}^{\infty} \sum_{i \in \tilde{N}_i} \left( \frac{1 - (i - 1)/2^{m_j}}{2^{m_j}} \right) a_{ik} \tilde{\phi}_{ik}(x_1, x_2) dx_1 dx_2,
\]

involving the one-dimensional Rademacher functions \( r_j(t); j = 1, 2, \ldots \), converges for almost every \( t \) in \( I \) for almost every \( (x_1, x_2) \) in \( S_2 \) (see e.g. [11], p. 212). It is well-known that then series (4.29) also converges for almost every \((x_1, x_2) \) in \( S_2 \) for almost every \( t \) in \( I \). So, we can select a dyadically irrational number \( t_0 \) in \( I \) such that series (4.29) converges for almost every \((x_1, x_2) \) in \( S_2 \) for \( t = t_0 \).

We define a new system \( \psi = \{ \psi_{ak}(x_1, x_2); (i, k) \in \tilde{N}_2 \} \) on \( S_2 \) as follows:
\[
\psi_{ak}(x_1, x_2) = \begin{cases} 
r_{f(t_0)} a_{ik} \tilde{\phi}_{ik}(x_1, x_2) & \text{if } (i, k) \in \tilde{N}_2; j = 1, 2, \ldots; \\
\tilde{\phi}_{ak}(x_1, x_2) & \text{if } (i, k) \in \tilde{N}_i \setminus \tilde{N}_2;
\end{cases}
\]

and for \((i, k) \in \tilde{N}_i \setminus \tilde{N}_2 \)
\[ \psi_{ak}(x_1, x_2) = \begin{cases} 
r_{f(t_0)} a_{ik} \tilde{\phi}_{ik}(x_1, x_2) & \text{if } (i, k) \in \tilde{N}_2; j = 1, 2, \ldots; \\
\tilde{\phi}_{ak}(x_1, x_2) & \text{if } (i, k) \in \tilde{N}_i \setminus \tilde{N}_2;
\end{cases} \]
It is evident that \( \psi \) is an ONS of step functions on \( S_{2^*} \), by (4.28) for \((x_1, x_2) \in G \)

\[
\limsup_{j \to \infty} \max_{(m, n) \in \mathcal{J}^1} \left| \sum_{a \in \mathbb{N} \cap (2^j \mathbb{Z} + 1)} \sum_{a_2 \in \mathbb{N} \cap (2^j \mathbb{Z} + 1)} \left( 1 - \frac{1}{m} \right) a_{2^j} \psi_{2^j}(x_1, x_2) \right| = \infty
\]

and the series (cf. (4.29))

\[
\left| \sum_{j=1}^{m} \sum_{(a_1, \ldots, a_m) \in \mathcal{J}^1} a_{2^j} \psi_{2^j}(x_1, x_2) \right|\]

converges for almost every \((x_1, x_2) \in S_{2^*} \).

Given a pair \((m, n) \in \mathcal{A} \cap (2^j \mathbb{Z} + 1) \), let \(j(m, n)\) be that positive integer for which \((m, n) \in \mathcal{A}_{2^j} \). By definition, for \((x_1, x_2) \in S_{2^*} \)

\[
\sigma^{2^j}(a, \psi; x_1, x_2) = \sum_{a \in \mathbb{N} \cap (2^{j+1} \mathbb{Z} + 1)} \sum_{\mathcal{A}_{2^j}} \left( 1 - \frac{1}{m} \right) a_{2^j} \psi_{2^j}(x_1, x_2) + R(m, n; x_1, x_2)
\]

where

\[
R(m, n; x_1, x_2) = \sum_{j=1}^{m} \sum_{(a_1, \ldots, a_m) \in \mathcal{J}^1} \left( 1 - \frac{1}{m} \right) a_{2^j} \psi_{2^j}(x_1, x_2).
\]

Our next goal is to show that the limit

\[
\lim_{j \to \infty} R(m, n; x_1, x_2) = 0\]

exists for almost every \((x_1, x_2) \in S_{2^*} \). To this end, we introduce the notation

\[
s_{2^j}(u; x_1, x_2) = \sum_{a \in \mathbb{N} \cap (2^{j+1} \mathbb{Z} + 1)} \sum_{\mathcal{A}_{2^j}} a_{2^j} \psi_{2^j}(x_1, x_2),
\]

and apply an Abel transformation to obtain

\[
\sum_{a \in \mathbb{N} \cap (2^j \mathbb{Z} + 1)} \left( 1 - \frac{1}{m} \right) a_{2^j} \psi_{2^j}(x_1, x_2) = \sum_{a \in \mathbb{N} \cap (2^{j+1} \mathbb{Z} + 1)} s_{2^{j+1}}(u; x_1, x_2)
\]

\[
\sum_{a \in \mathbb{N} \cap (2^j \mathbb{Z} + 1)} \left( 1 - \frac{1}{m} \right) a_{2^j} \psi_{2^j}(x_1, x_2) = R^{(1)}(m, j; x_1, x_2) + R^{(2)}(m, j; x_1, x_2),
\]

This shows that

\[
R(m, n; x_1, x_2) = \sum_{j=1}^{m-n-1} R^{(1)}(m, j; x_1, x_2) + \sum_{j=1}^{m-n-1} R^{(2)}(m, j; x_1, x_2).
\]

First, we deal with \( R^{(1)}(m, j; x_1, x_2) \). Setting for \( j = 1, 2, \ldots \)

\[
\delta_j(x_1, x_2) = \max_{a \in \mathbb{N} \cap (2^{j+1} \mathbb{Z} + 1)} \left| s_{2^{j+1}}(u; x_1, x_2) \right|
\]

obviously

\[
\left| \sum_{j=1}^{m-n-1} R^{(1)}(m, j; x_1, x_2) \right| \leq \sum_{j=1}^{m-n-1} \sum_{a \in \mathbb{N} \cap (2^{j+1} \mathbb{Z} + 1)} \delta_j(x_1, x_2).
\]

Thus, for \( k = 2, 3, \ldots \)

\[
\delta^{(1)}(x_1, x_2) = \max_{a \in \mathbb{N} \cap (2^{j+1} \mathbb{Z} + 1)} \left| \sum_{j=1}^{m-n-1} R^{(1)}(m, j; x_1, x_2) \right|
\]

\[
\leq \sum_{j=1}^{m-n-1} \sum_{a \in \mathbb{N} \cap (2^{j+1} \mathbb{Z} + 1)} \delta_j(x_1, x_2) \leq \frac{1}{m} \sum_{j=1}^{m-n-1} \sum_{a \in \mathbb{N} \cap (2^{j+1} \mathbb{Z} + 1)} \delta_j(x_1, x_2)
\]

since \( r_{j+1} = r_{j+1} + 1 \). In virtue of Lemma 1, for \( j = 1, 2, \ldots \)

\[
\delta^{(2)}(x_1, x_2) = \sum_{a \in \mathbb{N} \cap (2^{j+1} \mathbb{Z} + 1)} \alpha_a \delta_a
\]

whence, on the basis of (1.2) and (4.35),

\[
\delta^{(2)}(x_1, x_2) \leq 4 \sum_{a \in \mathbb{N} \cap (2^{j+1} \mathbb{Z} + 1)} \alpha_a \delta_a
\]

By B. Levi's theorem, for almost every \((x_1, x_2) \in S_{2^*} \),

\[
\lim_{j \to \infty} \delta^{(3)}(x_1, x_2) = 0.
\]
Secondly, we treat $R^{(3)}(m; j; x_1, x_2)$. By definition, for $l = 2, 3, \ldots$,
\begin{equation}
\sum_{j=1}^{l-1} R^{(3)}(m; j; x_1, x_2) = \sum_{j=1}^{l-1} \sum_{x_1, x_2 \in S_2} \frac{1}{x_1, x_2 \in S_2} \left[ \varphi(x_1, x_2) \right]^{(j)} \left( \frac{1}{x_1, x_2} \right) \left( \frac{1}{x_1, x_2} \right) \left( \frac{1}{x_1, x_2} \right)
\end{equation}
\begin{equation}
= \frac{1}{x_1, x_2 \in S_2} \left[ \varphi(x_1, x_2) \right]^{(j)} \left( \frac{1}{x_1, x_2} \right) \left( \frac{1}{x_1, x_2} \right) \left( \frac{1}{x_1, x_2} \right)
\end{equation}
Since series (4.31) converges a.e. the limit
\begin{equation}
\lim_{l \to \infty} R^{(3)}(m; j; x_1, x_2)
\end{equation}
exists for almost every $(x_1, x_2) \in S_2$.

On the other hand, for $l = 2, 3, \ldots$,
\begin{equation}
\delta^{(k)}(x_1, x_2) = \max_{a(\phi) \subseteq \varphi(x_1, x_2)} |R^{(k)}(m; l; x_1, x_2)|
\end{equation}
\begin{equation}
\leq \sum_{j=1}^{l-1} \frac{1}{x_1, x_2 \in S_2} \delta_j(x_1, x_2) \leq \frac{1}{x_1, x_2 \in S_2} \sum_{j=1}^{l-1} \delta_j(x_1, x_2)
\end{equation}
(cf. (4.35)) This, (1.2), and (4.36) imply, in the same way as in the case of (4.37), that for almost every $(x_1, x_2) \in S_2$
\begin{equation}
\lim_{l \to \infty} \delta^{(k)}(x_1, x_2) = 0.
\end{equation}

Collecting (4.34), (4.35), (4.38), (4.40)–(4.42) together, we can establish the existence of the limit (4.33) for almost every $(x_1, x_2) \in S_2$. On account of (4.30) and (4.32), we can conclude
\begin{equation}
\lim_{n \to \infty} \sup_{(m, n) \in S} \varphi^{(k)}(m, n; x_1, x_2) = \infty
\end{equation}
for almost every $(x_1, x_2) \in S$. Since $G \subseteq S$ and $|G| = 1$, this relation holds for almost every $(x_1, x_2) \in S$.

Finally, setting for $i, k = 1, 2, \ldots$,
\begin{equation}
\Psi^{(k)}(x_1, x_2) = 2\varphi^{(k)}(x_1, x_2), \quad (x_1, x_2) \in S
\end{equation}
we get (4.1) with $H = S_{1/i}$ and this completes the proof of Theorem I'.

5. Proof of Theorem 2. We begin with the definition of an increasing sequence \{ $M_r$ : $r = 1, 2, \ldots$ \} such that for $r = 2, 3, \ldots$,
\begin{equation}
\frac{C_r \log M_r}{8r^2 + 1} \geq r + C + \frac{1}{x_1, x_2 \in S_2} 2M_r^2
\end{equation}
where
\begin{equation}
\gamma(r) = 2^{2m^2}, \quad \beta(r) = 2^r.
\end{equation}

Then we define the double sequence $a = \{ a_{ik} : i, k = 1, 2, \ldots \}$ of coefficients by
\begin{equation}
a_{ik} = \begin{cases}
\frac{1}{rM_r \lambda_{\gamma(r)+1} \gamma(r)+1} & \text{for } i = 2^r+1, 2^r+2, \ldots, \gamma(r)+1;
\end{cases}
\end{equation}
\begin{equation}
k = 2^r+1; \quad r = 1, 2, \ldots
\end{equation}
\begin{equation}
0 & \text{otherwise.}
\end{equation}

Clearly, condition (1.8) is satisfied:
\begin{equation}
\sum_{k=1}^{\infty} \sum_{r=1}^{M_r^2} a_{ik} \lambda_{\gamma(r)+1} \gamma(r)+1 \leq \sum_{k=1}^{\infty} \sum_{r=1}^{M_r^2} a_{ik} \lambda_{\gamma(r)+1} \gamma(r)+1
\end{equation}
\begin{equation}
= \sum_{k=1}^{\infty} \sum_{r=1}^{M_r^2} \lambda_{\gamma(r)+1} \gamma(r)+1 \leq \sum_{k=1}^{\infty} \frac{1}{x_1, x_2 \in S_2} 2M_r^2 \lambda_{\gamma(r)+1} \gamma(r)+1
\end{equation}
\begin{equation}
< \infty.
\end{equation}

The main part of the proof is the definition of the ONS $\{ \Psi^{(k)}(x_1, x_2) : i, k = 1, 2, \ldots \}$. To this end, we apply Lemma 2 with $p = M_r$ for every $r = 1, 2, \ldots$, denoting by $\{ f_i(r; x) : i = 1, 2, \ldots, 2M_r^2 \}$ and $E(r)$ the resulting ONS of step functions on $I$ and simple subset of $I$, respectively. By (a)–(c) in Lemma 2, we have for $r = 1, 2, \ldots$,
\begin{equation}
|E(r)| \geq C_1
\end{equation}
\begin{equation}
|f_i(r; x)| \leq C_2 \quad \text{for } x \in I \text{ and } i = 1, 2, \ldots, 2M_r^2
\end{equation}
\begin{equation}
\text{furthermore, for } x \in E(r) \text{ there exists an integer } m = m(r; x), M_r^2 \leq m < 2M_r^2,
\end{equation}
\begin{equation}
such that \quad f_i(r; x) = 0 \text{ for } i = 1, 2, \ldots, m \text{ and}
\end{equation}
\begin{equation}
\sum_{i=1}^{m} f_i(r; x) = C_3 M_r \log M_r.
\end{equation}

First, we define an ONS $\varphi = \{ \varphi_{ik}(x) : i, k = 1, 2, \ldots \}$ of step functions on $I_2 = (0, 2)$ and a sequence $\{ H_r : r = 1, 2, \ldots \}$ of stochastically independent simple subsets of $I$ in such a way that the following conditions are satisfied: for every $r = 1, 2, \ldots$,
\begin{equation}
|H_r| \geq C_1
\end{equation}
\begin{equation}
|\varphi_{ik}(x) \leq C_2 \text{ for } x \in I_2, i = 2^r+1, 2^r+2, \ldots, \gamma(r)+1; k = 2^r+1;
\end{equation}
\begin{equation}
\text{for } x \in H_r \text{ there exists an integer } m = m(r; x), M_r^2 \leq m < 2M_r^2,
\end{equation}
\begin{equation}
such that \quad \varphi_{ik}(x) = 0 \text{ for } s = 1, 2, \ldots, m \text{ and}
\end{equation}
\begin{equation}
\sum_{s=1}^{m} \varphi_{ik}(x) \geq C_3 M_r \log M_r.
\end{equation}
and
\[ \varphi_k(x) = 0 \quad \text{for } x \notin I \text{ if } (i, k) \neq (2^r + 1, 2^r + 1) \]
with \( s = 1, 2, \ldots, 2M_2^i; r = 1, 2, \ldots \)

In the special case \( r = 1 \), we set for \( s = 1, 2, \ldots, 2M_2^i \),
\[ \varphi_{2^r+1, s+1}^{2^r+1, s+1}(x) = f_i(1, x) \quad \text{and} \quad H_1 = E(1). \]

Now let \( r_0 \) be a positive integer and assume that the step functions \( \{\varphi_{2^r+1, s+1}^{2^r+1, s+1}(x) : s = 1, 2, \ldots, 2M_2^i; r = 1, 2, \ldots, r_0 \} \) and the simple subsets \( \{H_r: r = 1, 2, \ldots, r_0\} \) of \( I \) have been defined so that these functions form an ONS on \( I \), these sets are stochastically independent, and properties (5.5)–(5.8) are satisfied for \( r = 1, 2, \ldots, r_0 \).

We will define the step functions
\[ \{\varphi_{2^r+1, s+1}^{2^r+1, s+1}(x) : s = 1, 2, \ldots, 2M_2^{r_0+1}\} \]
of the \((r_0+1)\)st block and the simple set \( H_{r_0+1} \) in the following way. We divide \( I \) into a finite number of disjoint intervals \( \{J_p : p = 1, 2, \ldots, P\} \) such that the functions of the first \( r_0 \) blocks are constant on each \( J_p \) and each set \( \{H_r : r = 1, 2, \ldots, r_0\} \) is the union of certain \( J_p \). Denoting by \( J'_p \) and \( J''_p \) the two halves of the intervals \( J_p \), we set for \( s = 1, 2, \ldots, 2M_2^{r_0+1} \),
\[ \varphi_{2^r+1, s+1}^{2^r+1, s+1}(x) = \sum_{p=1}^{P} \left[ f_i(r_0+1; J'_p; x) - f_i(r_0+1; J''_p; x) \right] \]
and
\[ H_{r_0+1} = \bigcup_{p=1}^{P} \left[ E(r_0+1; J'_p) \cup E(r_0+1; J''_p) \right]. \]

It is easy to see that the step functions
\[ \{\varphi_{2^r+1, s+1}^{2^r+1, s+1}(x) : s = 1, 2, \ldots, 2M_2^r; r = 1, 2, \ldots, r_0+1\} \]
form an ONS on \( I \), the simple sets \( \{H_r : r = 1, 2, \ldots, r_0+1\} \) are stochastically independent, and properties (5.5)–(5.7) are satisfied for \( r = r_0+1 \), due to (5.2)–(5.4).

This induction scheme shows that the ONS
\[ \{\varphi_{2^r+1, s+1}^{2^r+1, s+1}(x) : s = 1, 2, \ldots, 2M_2^r; r = 1, 2, \ldots\} \]
and the sequence \( \{H_r : r = 1, 2, \ldots\} \) of stochastically independent sets can be defined so that conditions (5.5)–(5.7) hold true.

Then we consider an arbitrary decomposition \( \{J_k\} \) of the interval \((1, 2)\) into disjoint subintervals and set for
\[ (i, k) \neq (2^r + 1, 2^r + 1) \quad \text{with} \quad s = 1, 2, \ldots, 2M_2^i; r = 1, 2, \ldots, \]
\[ \varphi_k(x) = \begin{cases} |J_k|^{-1/2} & \text{for } x \in J_k, \\ 0 & \text{for } x \notin J_k \end{cases} \]
while we extend the functions \( \varphi_k(x) \) defined previously from \( I \) onto \( I_2 \) simply by setting \( \varphi_k(x) = 0 \) for \( x \in [1, 2) \). Thus, the entire ONS \( \{\varphi_k(x) : i, k = 1, 2, \ldots\} \) is defined on \( I_2 \) so that conditions (5.5)–(5.7) and (5.8) are satisfied.

If \( x \in H \), for some \( r = 2, 3, \ldots \), then from (5.1), (5.6)–(5.8) we can deduce that
\[ \sigma_{2^r+1}^{1} (a, \varphi; x) - \sigma_{2^r+1}^{1} (a, \varphi; x) \]
\[ \geq \sum_{i=1}^{2M_2^i} \left( 1 - \frac{2^r}{2^2 - 1} \right) \left( 1 - \frac{2^r}{2^2 - 1} \right) \varphi_{2^r+1, s+1}^{2^r+1, s+1}(x) - C_2 \sum_{i=1}^{2M_2^i} |J_k|^{-1/2} \]
\[ \geq \frac{1}{8rM_2^i} \sum_{i=1}^{2M_2^i} \varphi_{2^r+1, s+1}^{2^r+1, s+1}(x) - C_2 \sum_{i=1}^{2M_2^i} |J_k|^{-1/2} \]
\[ \geq C_3 \log M_2^i \sum_{i=1}^{2M_2^i} |J_k|^{-1/2} - C_2 \sum_{i=1}^{2M_2^i} |J_k|^{-1/2} \]
\[ \geq r. \]

Thanks to stochastic independence, for \( H = \lim sup H \), we have \( |H| = 1 \). If \( x \in H \), then (5.9) is satisfied for infinitely many \( r \), a tortiori,
\[ \lim_{r \to \infty} \max_{2^r < x < 2^{r+1}} \left( \sigma_{2^r+1}^{1} (a, \varphi, x) - \sigma_{2^r+1}^{1} (a, \varphi, x) \right) = \infty. \]

Since \( H \subset (0, 1) \), this relation is satisfied for almost every \( x \) in \( I \).

It remains only to contract the functions into \( I \):
\[ \varphi_k(x) = \sqrt{2} \varphi_k(2x) \quad \text{for } x \in I \text{ and } i, k = 1, 2, \ldots; \]
and this completes the proof of Theorem 2.

References

Remarks concerning the paper

"On a class of Hausdorff compacta and GSG Banach spaces"

by

WOLFGANG M. RUESS (Essen) and CHARLES P. STEGALL (Linz)

Abstract. The main result of the above paper by O. I. Reznov which appeared in Studia Math. 71 (1981) is incorrect.

We show, by counterexample, that the main result of the paper [7] is not correct. Moreover, it is clear that the mistake is not technical in nature, but fundamental: there is an improper use of interpolation techniques, not only those of [2] but of interpolation methods in general, e.g. [1]. We emphasize, however, that all the results of [7] that are true are easily obtained by applying the results and techniques of [2–6]; for example, Theorem 0.1 of [7] follows easily from [6].

Example. We consider the Banach spaces $l_1$ and $l_\infty$ in their natural duality.

Let $B = \{ x \in l_\infty \mid |x_i| \leq i^{-2} \}$. This is a norm compact convex circled subset of $l_\infty$.

Let $A = \bigcap_{i=1}^{\infty} (nB + n^{-1}D)$, where $D$ is the unit ball of $l_\infty$. Now, consider the sequence $a = (i^{-1})_i$. Since $1 \leq n/i + 1/n$ for all $i, n \in \mathbb{N}$, it follows that $1/i \leq n/i^2 + 1/n$ for all $i, n \in \mathbb{N}$, and we have $a \in A$. We shall compute the distance (in the norm generated by $A$) from the element $a$ to the linear span $\bigcup \{ \lambda B \mid \lambda \geq 0 \}$ of $B$ in $l_\infty$. Suppose $b = (b_i) \in B$, $\lambda \geq 0$, $s \geq 0$ are such that $a - \lambda b \in sA$. We must have $s > 0$, because $a \notin \text{span } B$. Then there exist $b_i = (b_{ni}) \in B$ and $x_n = (x_{ni}) \in D$ such that

$$
\frac{1}{i} - \lambda b_i = a \left( nb_{ni} + \frac{1}{n} x_{ni} \right)
$$

for all $i, n \in \mathbb{N}$.

Since, for $i \geq 2i$,

$$
\frac{1}{i} - \lambda b_i \geq \frac{1}{i} > 0,
$$

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