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On the a.e. divergence of the arithmetic means of double orthogonal series

by

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Abstract. The classical coefficient test for the $(C, 1)$-summability of single orthogonal series is due to Men'shov (1926) and Kaczmarz (1927). The first named author has extended the Men'shov–Kaczmarz theorem for double orthogonal series in [5] giving sufficient conditions for the $(C, 1, 1)$- and $(C, 1, 0)$-summability. The second named author has proved in [10] that the condition ensuring the $(C, 1, 1)$-summability is necessary in the case of nonincreasing dyadic blocks of the coefficients if all double ONS are taken into account. Now we prove that the coefficient test for the $(C, 1, 0)$-summability is also necessary in the same sense. Besides, we present counterexamples showing that the Kolmogorov type and Kaczmarz type results obtained in [5] are the best possible ones.

1. Introduction. Let $(X, \mathscr{F}, \mu)$ be a positive measure space and $\varphi = \{\varphi_{ik}(x)\colon i, k = 1, 2, \ldots\}$ an orthonormal system (in abbreviation: ONS) on X. We will consider the double orthogonal series

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik} \varphi_{ik}(x) \tag{1.1}$$

where $a = \{a_{ik}\colon i, k = 1, 2, \ldots\}$ is a double sequence of real numbers (coefficients) for which

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 < \infty. \tag{1.2}$$

By the Riesz–Fischer theorem, there exists a function $f(a, \varphi; x) \in L^2(X, \mathscr{F}, \mu)$ such that the rectangular partial sums

$$s_{mn}(a, \varphi; x) = \sum_{i=1}^{m} \sum_{k=1}^{n} a_{ik} \varphi_{ik}(x) \quad (m, n = 1, 2, \ldots)$$

of series (1.1) converge to $f(a, \varphi; x)$ in L^2-metric:

$$\lim_{m,n\to\infty} \int_X [s_{mn}(a, \varphi; x) - f(a, \varphi; x)]^2 \, d\mu(x) = 0.$$

This research was completed while the first named author was a visiting professor at the Indiana University, Bloomington.

We will study the a.e. convergence behavior of the following arithmetic means of the rectangular partial sums:

$$\sigma_{mn}^{10}(a, \varphi; x) = \frac{1}{m}\sum_{i=1}^{m} s_{in}(a, \varphi; x) = \sum_{i=1}^{m}\sum_{k=1}^{n}\left(1-\frac{i-1}{m}\right) a_{ik}\varphi_{ik}(x)$$

and

$$\sigma_{mn}^{11}(a, \varphi; x) = \frac{1}{mn}\sum_{i=1}^{m}\sum_{k=1}^{n} s_{ik}(a, \varphi; x)$$
$$= \sum_{i=1}^{m}\sum_{k=1}^{n}\left(1-\frac{i-1}{m}\right)\left(1-\frac{k-1}{n}\right) a_{ik}\varphi_{ik}(x) \qquad (m, n = 1, 2, \ldots).$$

The following two theorems were proved in [5].

THEOREM A. *If*

$$(1.3) \qquad \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}^2 [\log\log(i+3)]^2 [\log(k+1)]^2 < \infty,$$

then for every double ONS $\varphi = \{\varphi_{ik}(x)\}$

$$\lim_{m,n\to\infty} \sigma_{mn}^{10}(a, \varphi; x) = f(a, \varphi; x) \qquad a.e.$$

THEOREM B. *If*

$$(1.4) \qquad \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}^2 [\log\log(i+3)]^2 [\log\log(k+3)]^2 < \infty,$$

then for every double ONS $\varphi = \{\varphi_{ik}(x)\}$

$$\lim_{m,n\to\infty} \sigma_{mn}^{11}(a, \varphi; x) = f(a, \varphi; x) \qquad a.e.$$

In this paper the logarithms are to the base 2.

During the proofs of Theorems A and B, a Kolmogorov type and a Kaczmarz type result (cf. [2, pp. 118 and 119] concerning single orthogonal series, respectively) were obtained in [5].

THEOREM C. *If*

$$(1.5) \qquad \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}^2 [\log\log(\max\{i, k\}+3)]^2 < \infty,$$

then for every double ONS $\varphi = \{\varphi_{ik}(x)\}$

$$\lim_{p,q\to\infty} [s_{2^p,2^q}(a, \varphi; x) - \sigma_{2^p,2^q}^{11}(a, \varphi; x)] = 0 \qquad a.e.$$

THEOREM D. *If condition* (1.5) *is satisfied, then for every double ONS* $\varphi = \{\varphi_{ik}(x)\}$

$$\lim_{p,q\to\infty} \Big[\max_{2^p\le m\le 2^{p+1}}\ \max_{2^q\le n\le 2^{q+1}} |\sigma_{mn}^{11}(a, \varphi; x) - \sigma_{2^p,2^q}^{11}(a, \varphi; x)|\Big] = 0 \qquad a.e.$$

The main point is that the sufficient conditions (1.3) and (1.4) in certain cases are also necessary and condition (1.5) is the best possible if all double ONS $\varphi = \{\varphi_{ik}(x)\}$ are taken into account on a particular measure space.

To be more specific, from now on (except Lemma 1) let $(X, \mathscr{F}, \mu)$ be either the unit interval $I = (0, 1)$ or the unit square $S = (0, 1)\times(0, 1)$ (in the latter case we will write (x_1, x_2) rather than x) with the σ-algebra of Borel subsets and Lebesgue measure denoted by $|\cdot|$. It will be clear from the context whether $|\cdot|$ means the measure on the real line or plane.

Let us set

$$(1.6) \qquad A_{pq}^* = \Big\{\sum_{i=2^p+1}^{2^{p+1}}\ \sum_{k=2^q+1}^{2^{q+1}} a_{ik}^2\Big\}^{1/2} \qquad (p, q = 0, 1, \ldots),$$
$$A_{-1,q}^* = \Big\{\sum_{k=2^q+1}^{2^{q+1}} a_{1k}^2\Big\}^{1/2} \qquad (q = 0, 1, \ldots),$$
$$A_{p,-1}^* = \Big\{\sum_{i=2^p+1}^{2^{p+1}} a_{i1}^2\Big\}^{1/2} \qquad (p = 0, 1, \ldots),$$

and

$$A_{-1,-1}^* = |a_{11}|.$$

We agree that by 2^{-1} we mean 0 in this paper. With this agreement, formula (1.6) for $p = -1$ or/and $q = -1$ comprises the subsequent formulas all.

The following two theorems where proved in [10].

THEOREM E. *If for* $p, q = -1, 0, 1, \ldots,$

$$A_{pq}^* \geqslant \max\{A_{p+1,q}^*, A_{p,q+1}^*\}$$

and condition (1.4) *is not satisfied, then there exists a double ONS* $\Phi = \{\Phi_{ik}(x_1, x_2)\}$ *on* S *such that*

$$\limsup_{m,n\to\infty} |\sigma_{mn}^{11}(a, \Phi; x_1, x_2)| = \infty \qquad a.e.$$

THEOREM F. *Let* $\{\lambda_i\colon i = 1, 2, \ldots\}$ *and* $\{\nu_k\colon k = 1, 2, \ldots\}$ *be non-decreasing sequences of positive numbers, and let*

$$(1.7) \qquad \lim_{i\to\infty} \frac{\lambda_i}{\log\log(i+3)} = 0.$$

(i) *Then there exist a double ONS* $\Phi = \{\Phi_{ik}(x)\}$ *on* I *and a double sequence* $a = \{a_{ik}\}$ *of real numbers such that*

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}^2 \lambda_i^2 \nu_k^2 < \infty, \tag{1.8}$$

$$\lim_{p,q\to\infty} s_{2^p,2^q}(a, \Phi; x) \quad \text{exists a.e. on } I,$$

and

$$\limsup_{p,q\to\infty} |\sigma_{2^p,2^q}^{11}(a, \Phi; x)| = \infty \quad \text{a.e. on } (0, \tfrac{1}{2}).$$

(ii) *Furthermore, there exist a double ONS* $\Psi = \{\Psi_{ik}(x)\}$ *on* I *and a double sequence* $b = \{b_{ik}\}$ *of real numbers such that condition* (1.8) *is satisfied with* b_{ik} *instead of* a_{ik},

$$\lim_{p,q\to\infty} \sigma_{2^p,2^q}^{11}(b, \Psi; x) \quad \text{exists a.e. on } I,$$

and

$$\limsup_{p,q\to\infty} |s_{2^p,2^q}(b, \Psi; x)| = \infty \quad \text{a.e. on } (0, \tfrac{1}{2}).$$

2. New results. We will prove that, in the case of nonincreasing dyadic blocks, condition (1.3) is not only sufficient but also necessary for the conclusion in Theorem A.

THEOREM 1. *If for* $p = -1, 0, 1, \ldots; k = 1, 2, \ldots,$

$$A_{pk} = \Big\{\sum_{i=2^p+1}^{2^{p+1}} a_{ik}^2\Big\}^{1/2} \geqslant \max\{A_{p+1,k}, A_{p,k+1}\} \tag{2.1}$$

and condition (1.3) *is not satisfied, then there exists a double ONS* $\Phi = \{\Phi_{ik}(x_1, x_2)\}$ *on* S *such that*

$$\limsup_{m,n\to\infty} |\sigma_{mn}^{10}(a, \Phi; x_1, x_2)| = \infty \quad \text{a.e.} \tag{2.2}$$

In accordance with our agreement, for $p = -1$ in (2.1) we have $A_{-1,k} = |a_{1k}|$ $(k = 1, 2, \ldots)$.

The next theorem reveals that condition (1.5) in Theorem D is the best possible.

THEOREM 2. *If* $\{\lambda_i: i = 1, 2, \ldots\}$ *and* $\{\nu_k: k = 1, 2, \ldots\}$ *are nondecreasing sequences of positive numbers and condition* (1.7) *is satisfied, then there exist a double ONS* $\Phi = \{\Phi_{ik}(x)\}$ *on* I *and a double sequence* $a = \{a_{ik}\}$ *of real numbers such that condition* (1.8) *is satisfied and*

$$\limsup_{p,q\to\infty}\Big[\max_{2^p\leqslant m\leqslant 2^{p+1}}\ \max_{2^q\leqslant n\leqslant 2^{q+1}} |\sigma_{mn}^{11}(a, \Phi; x) - \sigma_{2^p,2^q}^{11}(a, \Phi; x)|\Big] \tag{2.3}$$
$$= \infty \quad \text{a.e. on } (0, \tfrac{1}{2}).$$

3. Auxiliary results. We begin with a few definitions and notations. By an interval $\langle a, b\rangle$ we mean either open interval (a, b), or one of the half-closed intervals $[a, b)$ and $(a, b]$, or closed interval $[a, b]$. By a rectangle $R = \langle a_1, b_1\rangle \times \langle a_2, b_2\rangle$ we mean a rectangle with sides parallel to the coordinate axes.

A function f defined on I (or S) is said to be a *step function* if I (or S) can be represented as the union of finitely many disjoint intervals (rectangles) such that f is constant on each of these intervals (rectangles). A subset H of I (or S) is said to be *simple* if H is the union of finitely many disjoint intervals (rectangles).

Given a function f defined on I and a subinterval $J = \langle a, b\rangle$ of I, we set

$$f(J; x) = \begin{cases} f\left(\dfrac{x-a}{b-a}\right) & \text{for } x \in J, \\ 0 & \text{otherwise.}\end{cases}$$

If H is a subset of I, then by $H(J)$ we denote the set into which H is carried over by the linear transformation $\bar{x} = (b-a)x + a$. Similarly, given a function g defined on S and a subrectangle $R = \langle a_1, b_1\rangle \times \langle a_2, b_2\rangle$ of S, we set

$$g(R; x_1, x_2) = \begin{cases} g\left(\dfrac{x_1-a_1}{b_1-a_1}, \dfrac{x_2-a_2}{b_2-a_2}\right) & \text{for } (x_1, x_2) \in R, \\ 0 & \text{otherwise.}\end{cases}$$

Finally, if H is a subset of S, then by $H(R)$ we denote the set into which H is carried over by the linear transformation $\bar{x}_1 = (b_1-a_1)x_1 + a_1$ and $\bar{x}_2 = (b_2-a_2)x_2 + a_2$.

Now we present five lemmas. The extension of the Rademacher–Men'shov inequality is due to Agnew [1] (see also [4]).

LEMMA 1. *For every ONS* $\varphi = \{\varphi_{ik}(x)\}$, *sequence* $a = \{a_{ik}\}$ *of real numbers and* $M, N = 1, 2, \ldots,$

$$\int_X \Big[\max_{1\leqslant m\leqslant M}\ \max_{1\leqslant n\leqslant N} |s_{mn}(a, \varphi; x)|\Big]^2 d\mu(x) \leqslant [\log 2M]^2 [\log 2N]^2 \sum_{i=1}^{M}\sum_{k=1}^{N} a_{ik}^2.$$

The next lemma is due to Men'shov [3] (see also [9]). In the sequel, by $C_1, C_2, \ldots$ we denote positive absolute constants.

LEMMA 2. *For every* $p = 2, 3, \ldots,$ *there exist an ONS* $\{f_i(p; x): i = 1, 2, \ldots, 2p\}$ *of step functions on* I *and a simple set* $E(p) \subseteq I$ *such that*

(a) $|E(p)| \geqslant C_1$,

(b) $|f_i(p; x)| \leqslant C_2$ *for* $x \in I$ *and* $i = 1, 2, \ldots, 2p$;

(c) *for* $x \in E(p)$ *there exists an integer* $m = m(p; x)$, $p \leqslant m < 2p$, *such that* $f_i(p; x) \geqslant 0$ *for each* $i = 1, 2, \ldots, m$ *and*

$$\sum_{i=1}^{m} f_i(p; x) \geqslant \sum_{i=1}^{m-1} f_i(p; x) \geqslant C_3\sqrt{p}\log p.$$

We note that this lemma was originally proved in the case where $p = q^2$, $q \geqslant 2$ is an integer. However, Lemma 2 obviously follows from this particular case.

Let

$$\mathcal{R} = \{r_i(x_1) r_k(x_2)\colon\ i, k = 1, 2, \ldots; (x_1, x_2) \in S\}$$

be the two-dimensional Rademacher system (concerning the one-dimensional Rademacher system we refer, e.g., to [11, p. 212]).

LEMMA 3. *If condition* (1.2) *is not satisfied, then*

$$\limsup_{m,n\to\infty} |\sigma_{mn}^{10}(a, \mathcal{R}; x_1, x_2)| = \infty \quad a.e.$$

This lemma is an easy extension of a result of Zygmund [11, pp. 205 and 212] from the one-dimensional case to the two-dimensional case. We omit the proof.

In the remaining part of this section we will consider the single orthogonal series

$$\sum_{i=1}^{\infty} a_i \varphi_i(x) \tag{3.1}$$

where $\varphi = \{\varphi_i(x)\colon\ i = 1, 2, \ldots\}$ is an ONS on I and $a = \{a_i\colon\ i = 1, 2, \ldots\}$ is an ordinary sequence of real numbers with

$$\sum_{i=1}^{\infty} a_i^2 < \infty. \tag{3.2}$$

We will use the following notations for the partial sums of (3.1) and the first arithmetic means of them:

$$s_m(a, \varphi; x) = \sum_{i=1}^{m} a_i \varphi_i(x)$$

and

$$\sigma_m(a, \varphi; x) = \frac{1}{m} \sum_{i=1}^{m} s_i(a, \varphi; x) = \sum_{i=1}^{m} \left(1 - \frac{i-1}{m}\right) a_i \varphi_i(x) \quad (m = 1, 2, \ldots).$$

The next lemma was proved in [6].

LEMMA 4. *If* $|a_i| \geqslant |a_{i+1}|$ *for* $i = 1, 2, \ldots$, *and*

$$\sum_{i=1}^{\infty} a_i^2 [\log(i+1)]^2 = \infty,$$

then there exists an ONS $\varphi = \{\varphi_i(x)\}$ *of step functions on* I *such that*

$$\limsup_{m\to\infty} |s_m(a, \varphi; x)| = \infty \quad a.e. \tag{3.3}$$

We note that actually only the a.e. divergence of the orthogonal series (3.1) was proved in [6] instead of unbounded divergence expressed by (3.3). But (3.3) follows from this divergence theorem in a routine way (cf. the proof of Lemma 5 below).

LEMMA 5. *If for* $p = 0, 1, \ldots,$

$$\sum_{i=2^p+1}^{2^{p+1}} a_i^2 \geqslant \sum_{i=2^{p+1}+1}^{2^{p+2}} a_i^2$$

and

$$\sum_{i=1}^{\infty} a_i^2 [\log\log(i+3)]^2 = \infty,$$

then there exists an ONS $\varphi = \{\varphi_i(x)\}$ *of step functions on* I *such that*

$$\limsup_{p\to\infty} |s_{2^p}(a, \varphi; x)| = \infty \quad a.e. \tag{3.4}$$

Proof of Lemma 5. Combining [7, Theorem 2] and [8, Theorem 7] yields the following weaker version: under the conditions of Lemma 5, there exists an ONS $\varphi = \{\varphi_i(x)\}$ of step functions on I such that

$$\lim_{p\to\infty} s_{2^p}(a, \varphi; x) \quad \text{fails to exist a.e.}$$

To prove the stronger statement (3.4), we distinguish two cases.

Case 1: (3.2) is not satisfied. Then by Zygmund's theorem quoted above, the one-dimensional Rademacher system can be taken in the capacity of φ.

Case 2: (3.2) is satisfied. Then first we construct a nonincreasing sequence $\lambda = \{\lambda_i\colon\ i = 1, 2, \ldots\}$ of positive numbers tending to 0 such that for $p = 0, 1, \ldots,$

$$\sum_{i=2^p+1}^{2^{p+1}} \lambda_i^2 a_i^2 \geqslant \sum_{i=2^{p+1}+1}^{2^{p+2}} \lambda_i^2 a_i^2$$

and

$$\sum_{p=0}^{\infty} \Big(\sum_{i=2^p+1}^{2^{p+1}} \lambda_i^2 a_i^2\Big) [\log(p+2)]^2 = \infty.$$

Thus, by the above weaker version, there exists an ONS $\varphi = \{\varphi_i(x)\}$ of step functions on I such that for $\lambda a = \{\lambda_i a_i\}$

$$\lim_{p\to\infty} s_{2^p}(\lambda a, \varphi; x) \quad \text{fails to exist a.e.} \tag{3.5}$$

An Abel transformation yields

$$s_{2^p}(\lambda a, \varphi; x) = \sum_{i=1}^{2^p-1} (\lambda_i - \lambda_{i+1}) s_i(a, \varphi; x) + \lambda_{2^p} s_{2^p}(a, \varphi; x). \tag{3.6}$$

By (3.2)

$$\sum_{i=1}^{\infty}(\lambda_i-\lambda_{i+1})\int_0^1 |s_i(a,\varphi;x)|\,dx \leqslant \sum_{i=1}^{\infty}(\lambda_i-\lambda_{i+1})\Big\{\sum_{j=1}^{i} a_j^2\Big\}^{1/2}$$
$$\leqslant \lambda_1\Big\{\sum_{j=1}^{\infty} a_j^2\Big\}^{1/2}<\infty.$$

Hence B. Levi's theorem implies that the series

$$\sum_{i=1}^{\infty}(\lambda_i-\lambda_{i+1})\,s_i(a,\varphi;x) \quad \text{converges a.e.}$$

By (3.5) and (3.6),

$$\lim_{p\to\infty}\lambda_{2^p}s_{2^p}(a,\varphi;x) \quad \text{fails to exist a.e.}$$

which is equivalent to (3.4) to be proved.

4. Proof of Theorem 1. The methods applied during the proof are similar to those which were elaborated in [10].

First we make a reduction. Instead of Theorem 1, it is enough to prove the following.

THEOREM 1′. *Under the conditions of Theorem 1, there exist an ONS $\Psi=\{\Psi_{ik}(x_1,x_2)\}$ of step functions on S and a subset H of S with $|H|>0$ such that*

$$\text{(4.1)}\qquad \limsup_{m,n\to\infty}|\sigma_{mn}^{10}(a,\Psi;x_1,x_2)|=\infty \quad \textit{for } (x_1,x_2)\in H.$$

First we show how Theorem 1′ implies Theorem 1.

Let us assume that Theorem 1′ has been proved. Then there exist an increasing sequence $\{r_p\colon p=1,2,\ldots,r_1=0\}$ of integers and a sequence $\{H_p\colon p=1,2,\ldots\}$ of simple subsets of S such that for $p=1,2,\ldots,$

$$\text{(4.2)}\qquad |H_p|\geqslant C_4,$$

and for $(x_1,x_2)\in H_p$

$$\text{(4.3)}\qquad \max_{r_p<m,n\leqslant r_{p+1}}\Big|\sum_{(i,k)\in Q_{mn}\setminus Q_{r_p,r_p}}\Big(1-\frac{i-1}{m}\Big)a_{ik}\Psi_{ik}(x_1,x_2)\Big|$$
$$\geqslant p+\sum_{i=1}^{r_p}\sum_{k=1}^{r_p}|a_{ik}|M_{ik}\qquad\Big(\sum_{i=1}^{0}\sum_{k=1}^{0}=0\Big),$$

where

$$Q_{mn}=\{(i,k)\colon i=1,2,\ldots,m;\ k=1,2,\ldots,n\}\qquad(m,n=1,2,\ldots)$$

and

$$M_{ik}=\max_{(x_1,x_2)\in S}|\Psi_{ik}(x_1,x_2)|\qquad(i,k=1,2,\ldots).$$

Our goal is to construct an ONS $\Phi=\{\Phi_{ik}(x_1,x_2)\}$ of step functions on S and a sequence $\{E_p\colon p=1,2,\ldots\}$ of simple subsets of S such that these sets are stochastically independent, for $p=1,2,\ldots$

$$\text{(4.4)}\qquad |E_p|\geqslant C_4,$$

for $(x_1,x_2)\in E_p$

$$\text{(4.5)}\qquad \max_{r_p<m,n\leqslant r_{p+1}}\Big|\sum_{(i,k)\in Q_{mn}\setminus Q_{r_p,r_p}}\Big(1-\frac{i-1}{m}\Big)a_{ik}\Phi_{ik}(x_1,x_2)\Big|\geqslant p+\sum_{i=1}^{r_p}\sum_{k=1}^{r_p}|a_{ik}|M_{ik}$$

(this is actually (4.3) with Φ_{ik} instead of Ψ_{ik}), and

$$\text{(4.6)}\qquad \max_{(x_1,x_2)\in S}|\Phi_{ik}(x_1,x_2)|\leqslant M_{ik}\qquad(i,k=1,2,\ldots).$$

We will proceed by induction on p. If $p=1$, then let

$$\Phi_{ik}(x_1,x_2)=\Psi_{ik}(x_1,x_2)\quad\text{for } i,k=1,2,\ldots,r_2\quad\text{and}\quad E_1=H_1.$$

Conditions (4.4)–(4.6) are obviously satisfied.

Now let $p_0\geqslant 2$ be an integer and assume that the step functions $\{\Phi_{ik}(x_1,x_2)\colon i,k=1,2,\ldots,r_{p_0}\}$ and the simple sets $\{E_p\colon p=1,2,\ldots,p_0-1\}$ have been defined in such a way that these functions are orthonormal on S, these sets are stochastically independent, and relations (4.4)–(4.6) are satisfied for $p=1,2,\ldots,p_0-1$. We can divide S into a finite number of disjoint rectangles $\{R_s\colon s=1,2,\ldots,\sigma\}$ such that the functions $\{\Phi_{ik}(x_1,x_2)\colon i,k=1,2,\ldots,r_{p_0}\}$ are constant on each R_s and the sets $\{E_p\colon p=1,2,\ldots,p_0-1\}$ are the unions of certain R_s. Let R_s' and R_s'' denote the two halves of R_s, for example, if $R_s=\langle a_1,b_1\rangle\times\langle a_2,b_2\rangle$, then let $R_s'=\langle a_1,(a_1+b_1)/2]\times\langle a_2,b_2\rangle$ and $R_s''=((a_1+b_1)/2,b_1\rangle\times\langle a_2,b_2\rangle$. We set for $i,k=1,2,\ldots,r_{p_0+1}$ when $\max\{i,k\}>r_{p_0}$,

$$\Phi_{ik}(x_1,x_2)=\sum_{s=1}^{\sigma}[\Psi_{ik}(R_s';x_1,x_2)-\Psi_{ik}(R_s'';x_1,x_2)]$$

and

$$E_{p_0}=\bigcup_{s=1}^{\sigma}[H_{p_0}(R_s')\cup H_{p_0}(R_s'')].$$

It is easy to verify that the step functions $\{\Phi_{ik}(x_1,x_2)\colon i,k=1,2,\ldots,r_{p_0+1}\}$ form an ONS on S, the simple sets $\{E_p\colon p=1,2,\ldots,p_0\}$ are stochastically independent, and conditions (4.4)–(4.6) are satisfied for $p=p_0$ due to (4.2) and (4.3).

The above induction scheme shows that the ONS $\Phi = \{\Phi_{ik}(x_1, x_2)\}$ and the sequence $\{E_p\}$ of stochastically independent sets can be defined so that conditions (4.4)–(4.6) are satisfied for every $p = 1, 2, \ldots$

Putting (4.5) and (4.6) together, we can conclude for $(x_1, x_2) \in E_p$

(4.7) $$\max_{r_p < m, n \leq r_{p+1}} |\sigma_{mn}^{10}(a, \Phi; x_1, x_2)| \geq p \quad (p = 1, 2, \ldots).$$

Setting
$$E = \limsup_{p \to \infty} E_p,$$
(4.4) implies $|E| = 1$ via the Borel–Cantelli lemma. If $(x_1, x_2) \in E$, then we have (4.7) for infinitely many p. Consequently, (2.2) is satisfied which was to be proved.

Proof of Theorem 1′. We may assume that condition (1.2) is satisfied. Otherwise, (4.1) immediately follows from Lemma 3 even with $H = S$.

Our starting point is that if condition (2.1) is satisfied, while (1.3) is not, then
$$\sum_{p=-1}^{\infty} \sum_{k=1}^{\infty} A_{pk}^2 [\log(p+3)]^2 [\log(k+1)]^2 = \infty.$$

We will distinguish three cases:

(a) $\sum_{p=0}^{\infty} A_{p1}^2 [\log(p+2)]^2 = \infty$,

(b) $\sum_{k=1}^{\infty} A_{-1,k}^2 [\log(k+1)]^2 = \sum_{k=1}^{\infty} a_{1k}^2 [\log(k+1)]^2 = \infty$,

(c) for every $r = 1, 2, \ldots$,

(4.8) $$\sum_{p=r}^{\infty} \sum_{k=r}^{\infty} A_{pk}^2 [\log(p+2)]^2 [\log(k+1)]^2 = \infty.$$

Case (a). By Lemma 5, there exists an ONS $\varphi = \{\varphi_i(x)\}$ of step functions on I such that

(4.9) $$\limsup_{p \to \infty} |s_{2^p}(a^{(1)}, \varphi; x)| = \infty \quad \text{a.e.}$$

where $a^{(1)} = \{a_{i1}: i = 1, 2, \ldots\}$.

On the other hand condition (1.2) clearly implies
$$\sum_{i=1}^{\infty} a_{i1}^2 < \infty,$$
and thus Kolmogorov's theorem (see, e.g. [2, p. 118]) is applicable to obtain
$$\lim_{p \to \infty} [s_{2^p}(a^{(1)}, \varphi; x) - \sigma_{2^p}(a^{(1)}, \varphi; x)] = 0 \quad \text{a.e.}$$

This and (4.9) give that
$$\limsup_{p \to \infty} |\sigma_{2^p}(a^{(1)}, \varphi; x)| = \infty \quad \text{a.e.}$$

We take an arbitrary sequence $\{R_{ik}: i = 1, 2, \ldots; k = 2, 3, \ldots\}$ of disjoint rectangles such that

(4.10) $$\bigcup_{i=1}^{\infty} \bigcup_{k=2}^{\infty} R_{ik} \subseteq S_2 \setminus S \quad \text{where} \quad S_2 = (0, 2) \times (0, 2).$$

Set for $i = 1, 2, \ldots$,
$$\psi_{i1}(x_1, x_2) = \begin{cases} \varphi_i(x_1) & \text{for } (x_1, x_2) \in S, \\ 0 & \text{for } (x_1, x_2) \in S_2 \setminus S; \end{cases}$$
while for $i = 1, 2, \ldots; k = 2, 3, \ldots$,

(4.11) $$\psi_{ik}(x_1, x_2) = \begin{cases} |R_{ik}|^{-1/2} & \text{for } (x_1, x_2) \in R_{ik}, \\ 0 & \text{for } (x_1, x_2) \in S_2 \setminus R_{ik}. \end{cases}$$

It is not hard to check that $\psi = \{\psi_{ik}(x_1, x_2): i, k = 1, 2, \ldots\}$ is an ONS of step functions on S_2 and for almost every (x_1, x_2) in S
$$\limsup_{p \to \infty} |\sigma_{2^p, 2^p}^{10}(a, \psi; x_1, x_2)| = \infty.$$

Finally, the system $\Psi = \{\Psi_{ik}(x_1, x_2)\}$ defined for $(x_1, x_2) \in S$ by

(4.12) $$\Psi_{ik}(x_1, x_2) = 2\psi_{ik}(2x_1, 2x_2) \quad (i, k = 1, 2, \ldots)$$

and $H = S_{1/2} = (0, \frac{1}{2}) \times (0, \frac{1}{2})$ meet all requirements stated in Theorem 1′.

Case (b). By Lemma 4, there exists an ONS $\varphi = \{\varphi_i(x)\}$ of step functions on I such that
$$\limsup_{n \to \infty} |s_n(a_{(1)}, \varphi; x)| = \infty \quad \text{a.e.}$$
where $a_{(1)} = \{a_{1k}: k = 1, 2, \ldots\}$.

Let $\{R_{ik}: i = 2, 3, \ldots; k = 1, 2, \ldots\}$ be again disjoint rectangles whose union is contained in $S_2 \setminus S$ (cf. (4.10)). Set for $k = 1, 2, \ldots$,
$$\psi_{1k}(x_1, x_2) = \begin{cases} \varphi_k(x_2) & \text{for } (x_1, x_2) \in S, \\ 0 & \text{for } (x_1, x_2) \in S_2 \setminus S; \end{cases}$$
while for $i = 2, 3, \ldots; k = 1, 2, \ldots$, set (4.11).

Clearly, $\psi = \{\psi_{ik}(x_1, x_2): i, k = 1, 2, \ldots\}$ is an ONS of step functions on S_2 and for almost every (x_1, x_2) on S
$$\limsup_{n \to \infty} |\sigma_{nm}^{10}(a, \psi; x_1, x_2)| = \infty.$$

Setting (4.12) and $H = S_{1/2}$ completes the proof in this case.

Case (c). Without loss of generality, we may assume that the coefficients a_{ik} are rational numbers and different from zero. In fact, we can choose a sequence $\tilde{a} = \{\tilde{a}_{ik} \neq 0\colon\ i, k = 1, 2, \ldots\}$ of rational numbers such that for $p = -1, 0, 1, \ldots;\ k = 1, 2, \ldots,$

$$\tilde{A}_{pk} = \Big\{\sum_{i=2^p+1}^{2^{p+1}} \tilde{a}_{ik}^2\Big\}^{1/2} \geqslant \max\{\tilde{A}_{p+1,k}, \tilde{A}_{p,k+1}\}$$

(we recall that $\tilde{A}_{-1,k} = |\tilde{a}_{1k}|$ for $p = -1$),

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} \tilde{a}_{ik}^2 [\log\log(i+3)]^2 [\log(k+1)]^2 = \infty,$$

and

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} |a_{ik} - \tilde{a}_{ik}| < \infty.$$

Then for every ONS $\varphi = \{\varphi_{ik}(x_1, x_2)\}$ on S,

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} |a_{ik} - \tilde{a}_{ik}| \int_0^1\int_0^1 |\varphi_{ik}(x_1, x_2)|\, dx_1\, dx_2 \leqslant \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} |a_{ik} - \tilde{a}_{ik}| < \infty.$$

Thus,

$$\lim_{m,n\to\infty} [s_{mn}(a, \varphi; x_1, x_2) - s_{mn}(\tilde{a}, \varphi; x_1, x_2)] \quad \text{exists a.e.,}$$

a fortiori,

$$\lim_{m,n\to\infty} [\sigma_{mn}^{10}(a, \varphi; x_1, x_2) - \sigma_{mn}^{10}(\tilde{a}, \varphi; x_1, x_2)] \quad \text{exists a.e.}$$

Consequently, if there exists an ONS $\Phi = \{\Phi_{ik}(x_1, x_2)\}$ on S and a subset H of S such that

$$\limsup_{m,n\to\infty} |\sigma_{mn}^{10}(\tilde{a}, \Phi; x_1, x_2)| = \infty \qquad \text{for } (x_1, x_2) \in H,$$

then also

$$\limsup_{m,n\to\infty} |\sigma_{mn}^{10}(a, \Phi; x_1, x_2)| = \infty \qquad \text{for almost every } (x_1, x_2) \text{ in } H.$$

So, we may assume from now on that the coefficients a_{ik} are rational numbers different from zero. From (2.1) and (4.8) it follows that for every $r = 1, 2, \ldots,$

$$\sum_{p=r}^{\infty}\sum_{q=r}^{\infty} p^2 q^2 2^{p+q} A_{2^{p+1}-1, 2^{q+1}-1}^2 = \infty.$$

Thus, there exist an increasing sequence $\{r_j\colon\ j = 1, 2, \ldots;\ r_1 \geqslant 2\}$ of integers

and a nonincreasing sequence $\{s_j\colon\ j = 1, 2, \ldots\}$ of positive numbers tending to 0 such that for $j = 1, 2, \ldots$

$$\sum_{j=1}^{\infty} s_j^2 \sum_{p=r_j}^{r_{j+1}-1}\sum_{q=r_j}^{r_{j+1}-1} p^2 q^2 2^{p+q} A_{2^{p+1}-1, 2^{q+1}-1}^2 = \infty \tag{4.13}$$

and

$$s_j^2 \sum_{p=r_j}^{r_{j+1}-1}\sum_{q=r_j}^{r_{j+1}-1} p^2 q^2 2^{p+q} A_{2^{p+1}-1, 2^{q+1}-1}^2 \leqslant 1. \tag{4.14}$$

Let $j = 1, 2, \ldots$ be fixed and let $\{R_{pq}^{(j)}\colon\ r_j \leqslant p, q < r_{j+1}\}$ be disjoint rectangles in S with

$$|R_{pq}^{(j)}| = s_j^2 p^2 q^2 2^{p+q} A_{2^{p+1}-1, 2^{q+1}-1}^2.$$

By (4.14), this is possible. We apply Lemma 2 separately for 2^{p-1} and 2^{q-1}, and set for $2^p \leqslant i < 2^{p+1}$, $2^q \leqslant k < 2^{q+1}$

$$\tilde{f}_{ik}^{(j)}(x_1, x_2) = f_{i-2^p+1}(2^{p-1}; x_1) f_{k-2^q+1}(2^{q-1}; x_2)$$

(here $(x_1, x_2) \in S$) and for $r_j \leqslant p, q < r_{j+1}$

$$\tilde{H}_{pq}^{(j)} = E(2^{p-1}) \times E(2^{q-1}).$$

Finally, we set for $2^p \leqslant i < 2^{p+1}$, $2^q \leqslant k < 2^{q+1}$

$$f_{ik}^{(j)}(x_1, x_2) = |R_{pq}^{(j)}|^{-1/2} \tilde{f}_{ik}^{(j)}(R_{pq}^{(j)}; x_1, x_2)$$

and for $r_j \leqslant p, q < r_{j+1}$

$$H_{pq}^{(j)} = \tilde{H}_{pq}^{(j)}(R_{pq}^{(j)}).$$

It is easy to see that for every $j = 1, 2, \ldots$ the step functions $\{f_{ik}^{(j)}(x_1, x_2)\colon\ 2^{r_j} \leqslant i, k < 2^{r_{j+1}}\}$ form an ONS on S and the simple sets $\{H_{pq}^{(j)}\colon\ r_j \leqslant p, q < r_{j+1}\}$ are disjoint. In addition, (a)–(c) in Lemma 2 result in the following properties: for $r_j \leqslant p, q < r_{j+1}$

$$|H_{pq}^{(j)}| \geqslant C_1 s_j^2 p^2 q^2 2^{p+q} A_{2^{p+1}-1, 2^{q+1}-1}^2, \tag{4.15}$$

$$f_{ik}^{(j)}(x_1, x_2) = 0 \quad \text{for } (x_1, x_2) \in H_{pq}^{(j)} \quad \text{if } (i, k) \notin [2^p, 2^{p+1}) \times [2^q, 2^{q+1}), \tag{4.16}$$

further, for $(x_1, x_2) \in H_{pq}^{(j)}$ there exist integers $m = m(x_1, x_2)$ and $n = n(x_1, x_2)$, $2^p < m < 2^{p+1}$, $2^q < n < 2^{q+1}$, such that

$$f_{ik}^{(j)}(x_1, x_2) \geqslant 0 \quad \text{for } 2^p \leqslant i \leqslant m,\ 2^q \leqslant k \leqslant n, \tag{4.17}$$

and

$$\sum_{i=2^p}^{m-1}\sum_{k=2^q}^{n} A_{ik} f_{ik}^{(j)}(x_1, x_2) \geqslant \frac{C_3}{s_j}. \tag{4.18}$$

Now we take into account that the a_{ik} are rational numbers and different from zero. Therefore, for $j = 1, 2, \ldots$ there exists a positive integer Q_j such that for every i, k, and s with $2^s < i \leqslant 2^{s+1}$, $2^{r_j} \leqslant s$, $k < 2^{r_{j+1}}$,

$$\frac{a_{ik}^2}{A_{sk}^2} = \frac{P(i, k, s)}{Q_j}$$

where the $P(i, k, s)$ are also integers and $\neq 0$.

We consider a decomposition of S into disjoint rectangles $\{R_n\colon n = 1, 2, \ldots, Q_j\}$ of equal measure:

$$|R_n| = 1/Q_j \quad (n = 1, 2, \ldots, Q_j).$$

We set for $2^s < i \leqslant 2^{s+1}$, $2^{r_j} \leqslant s$, $k < 2^{r_{j+1}}$,

$$g_{ik}^{(j)}(x_1, x_2) = \frac{A_{sk}}{a_{ik}} \sum_{n=\nu(i-1,k)+1}^{\nu(i,k)} f_{sk}(R_n; x_1, x_2)$$

where $\nu(2^s, k) = 0$ and $\nu(i, k) = \sum_{l=2^s+1}^{i} P(l, s, k)$ $(i > 2^s)$, and for $r_j \leqslant p$, $q < r_{j+1}$

$$F_{pq}^{(j)} = \bigcup_{n=1}^{Q_j} H_{pq}^{(j)}(R_n).$$

We agree to denote by $\mathcal{N}$ the set of ordered pairs of positive integers (the so-called lattice points), while for $j = 1, 2, \ldots$ we set

$$\mathcal{N}_j = \{i, k) \in \mathcal{N}\colon \alpha(r_j) < i \leqslant \alpha(r_{j+1}),\ \beta(r_j) < k \leqslant \beta(r_{j+1})\}$$

where

$$\alpha(m) = 2^{2^m} \quad \text{and} \quad \beta(m) = 2^m \quad (m = 0, 1, \ldots).$$

Now it is routine to check that for every $j = 1, 2, \ldots$ the step functions $\{g_{ik}^{(j)}(x_1, x_2)\colon (i, k) \in \mathcal{N}_j\}$ are orthonormal on S, and the sets $\{F_{pq}^{(j)}\colon r_j \leqslant p, q < r_{j+1}\}$ are simple and disjoint. From (4.15)–(4.18) it follows that for every pair (p, q), $r_j \leqslant p$, $q < r_{j+1}$,

$$|F_{pq}^{(j)}| \geqslant C_1 s_j^2 p^2 q^2 2^{p+q} A^2_{2^{p+1}-1, 2^{q+1}-1}, \tag{4.19}$$

$$g_{ik}^{(j)}(x_1, x_2) = 0 \quad \text{for } (x_1, x_2) \in F_{pq}^{(j)} \text{ if } (i, k) \notin (2^{2^p}, 2^{2^{p+1}}] \times (2^q, 2^{q+1}], \tag{4.20}$$

and for $(x_1, x_2) \in F_{pq}^{(j)}$ there exist integers $m = m(x_1, x_2)$, $n = n(x_1, x_2)$, $2^p < m < 2^{p+1}$, $2^q < n < 2^{q+1}$, such that

$$a_{ik} g_{ik}^{(j)}(x_1, x_2) \geqslant 0 \quad \text{for } 2^{2^p} < i \leqslant 2^m,\ 2^q < k \leqslant n, \tag{4.21}$$

and

$$\sum_{i=2^{2^p}+1}^{2^{m}-1} \sum_{k=2^q+1}^{n} a_{ik} g_{ik}^{(j)}(x_1, x_2) \geqslant \frac{C_3}{j}. \tag{4.22}$$

Next, we will define an ONS $\{\varphi_{ik}(x_1, x_2)\colon (i, k) \in \bigcup_{j=1}^{\infty} \mathcal{N}_j\}$ of step functions on S and a sequence $\{G_j\colon j = 1, 2, \ldots\}$ of stochastically independent simple subsets of S such that for every $j = 1, 2, \ldots$,

$$|G_j| = \sum_{p=r_j}^{r_{j+1}-1} \sum_{q=r_j}^{r_{j+1}-1} |F_{pq}^{(j)}|, \tag{4.23}$$

for $(x_1, x_2) \in G_j$ there exist integers p, q, m, and n such that $r_j \leqslant p$, $q < r_{j+1}$, $2^p < m < 2^{p+1}$, $2^q < n < 2^{q+1}$,

$$a_{ik}\varphi_{ik}(x_1, x_2) \text{ is of constant sign for every } 2^{2^p} < i \leqslant 2^m,\ 2^q < k \leqslant n, \tag{4.24}$$

$$\Big|\sum_{i=2^{2^p}+1}^{2^{m}-1} \sum_{k=2^q+1}^{n} a_{ik}\varphi_{ik}(x_1, x_2)\Big| \geqslant \frac{C_3}{s_j}, \tag{4.25}$$

and

$$\varphi_{ik}(x_1, x_2) = 0 \quad \text{for } (x_1, x_2) \in G_j \text{ if } (i, k) \in \mathcal{N}_j \text{ but } (i, k) \notin (2^{2^p}, 2^{2^{p+1}}] \times (2^q, 2^{q+1}]. \tag{4.26}$$

The construction will be done by induction on j. If $j = 1$, then we set

$$G_1 = \bigcup_{p=r_1}^{r_2-1} \bigcup_{q=r_1}^{r_2-1} F_{pq}^{(0)}$$

and for $(i, k) \in \mathcal{N}_1$

$$\varphi_{ik}(x_1, x_2) = g_{ik}^{(1)}(x_1, x_2).$$

Then conditions (4.23)–(4.26) practically coincide with (4.19)–(4.22).

Now let $j_0 \geqslant 1$ be an integer and assume that the step functions $\{\varphi_{ik}(x_1, x_2)\colon (i, k) \in \mathcal{N}_j; j = 1, 2, \ldots, j_0\}$ and the simple sets $\{G_j\colon j = 1, 2, \ldots, j_0\}$ have been defined in such a way that these functions are orthonormal on S, these sets are stochastically independent, and relations (4.23)–(4.26) are satisfied for $j = 1, 2, \ldots, j_0$.

We can split S into a finite number of disjoint rectangles $\{R_s\colon s = 1, 2, \ldots, \sigma\}$ such that the above step functions are constant on each R_s and the above simple sets are the unions of certain R_s. Let R_s' and R_s'' be disjoint subrectangles of R_s with equal measure:

$$|R_s'| = |R_s''| \quad (s = 1, 2, \ldots, \sigma).$$

We define

$$G_{j_0+1} = \bigcup_{s=1}^{\sigma} \bigcup_{p=r_{j_0+1}}^{r_{j_0+2}-1} \bigcup_{q=r_{j_0+1}}^{r_{j_0+2}-1} [F_{pq}^{(j_0+1)}(R'_s) \cup F_{pq}^{(j_0+1)}(R''_s)]$$

and for $(i, k) \in \mathcal{N}_{j_0+1}$

$$\varphi_{ik}(x_1, x_2) = \sum_{s=1}^{\sigma} [g_{ik}^{(j_0+1)}(R'_s;\, x_1, x_2) - g_{ik}^{(j_0+1)}(R''_s;\, x_1, x_2)].$$

It is not hard to verify that the step functions $\{\varphi_{ik}(x_1, x_2)\colon (i, k) \in \mathcal{N}_j, j = 1, 2, \ldots, j_0+1\}$ are orthonormal on S, the simple sets $\{G_j\colon j = 1, 2, \ldots, j_0+1\}$ are stochastically independent, and (4.23)–(4.26) are satisfied also for $j = j_0+1$ due to (4.19)–(4.22).

This induction scheme shows that the ONS $\{\varphi_{ik}(x_1, x_2)\colon (i, k) \in \bigcup_{j=1}^{\infty} \mathcal{N}_j\}$ of step functions and the sequence $\{G_j\colon j = 1, 2, \ldots\}$ of stochastically independent simple sets can be defined so that conditions (4.23)–(4.26) are satisfied for every $j = 1, 2, \ldots$

By (4.13), either

$$\sum_{j=1}^{\infty} s_{2j}^2 \sum_{p=r_{2j}}^{r_{2j+1}-1} \sum_{q=r_{2j}}^{r_{2j+1}-1} p^2 q^2 2^{p+q} A^2_{2^{p+1}-1, 2^{q+1}-1} = \infty \tag{4.27}$$

or

$$\sum_{j=1}^{\infty} s_{2j-1}^2 \sum_{p=r_{2j-1}}^{r_{2j}-1} \sum_{q=r_{2j-1}}^{r_{2j}-1} p^2 q^2 2^{p+q} A^2_{2^{p+1}-1, 2^{q+1}-1} = \infty.$$

For the sake of definiteness, assume that (4.27) is the case.

We define an ONS $\{\tilde{\varphi}_{ik}(x_1, x_2)\colon i, k = 1, 2, \ldots\}$ of step functions on S_2 as follows. Set for $(i, k) \in \bigcup_{j=1}^{\infty} \mathcal{N}_{2j}$

$$\tilde{\varphi}_{ik}(x_1, x_2) = \begin{cases} \varphi_{ik}(x_1, x_2) & \text{for } (x_1, x_2) \in S, \\ 0 & \text{for } (x_1, x_2) \in S_2 \setminus S; \end{cases}$$

and for $(i, k) \in \mathcal{N} \setminus \bigcup_{j=1}^{\infty} \mathcal{N}_{2j}$

$$\tilde{\varphi}_{ik}(x_1, x_2) = \begin{cases} |R_{ik}|^{-1/2} & \text{for } (x_1, x_2) \in R_{ik}, \\ 0 & \text{for } (x_1, x_2) \in S_2 \setminus R_{ik}, \end{cases}$$

where $\{R_{ik}\colon (i, k) \in \mathcal{N} \setminus \bigcup_{j=1}^{\infty} \mathcal{N}_{2j}\}$ are arbitrary disjoint rectangles in $S_2 \setminus S$.

By (4.19), (4.23), and (4.27),

$$\sum_{j=1}^{\infty} |G_{2j}| = \infty.$$

Thus, by the Borel–Cantelli lemma, for $G = \limsup_{j\to\infty} G_j$ we have $|G| = 1$.

By (4.24)–(4.26) we can deduce that for every $j = 1, 2, \ldots$ and $(x_1, x_2) \in G_{2j}$ there exist integers $m_j = m_j(x_1, x_2)$ and $n_j = n_j(x_1, x_2)$ such that $2^{r_{2j}} < m_j,\ n_j < 2^{r_{2j+1}}$ and

$$\left| \sum_{i=\alpha(r_{2j})+1}^{2^{m_j}} \sum_{k=\beta(r_{2j})+1}^{n_j} \left(1 - \frac{i-1}{2^{m_j}}\right) a_{ik} \tilde{\varphi}_{ik}(x_1, x_2) \right| \geqslant \frac{1}{2} \left| \sum_{i=\alpha(r_{2j})+1}^{2^{m_j-1}} \sum_{k=\beta(r_{2j})+1}^{n_j} a_{ik} \tilde{\varphi}_{ik}(x_1, x_2) \right| \geqslant \frac{C_3}{2 s_{2j}}$$

since $1 - (i-1)/2^{m_j} \geqslant \frac{1}{2}$ for $i \leqslant 2^{m_j-1}$. If $(x_1, x_2) \in G$, then this estimate holds for infinitely many j, and, consequently,

$$\limsup_{j\to\infty} \max_{(m,n)\in\mathcal{N}_{2j}} \left| \sum_{i=\alpha(r_{2j})+1}^{m} \sum_{k=\beta(r_{2j})+1}^{n} \left(1 - \frac{i-1}{m}\right) a_{ik} \tilde{\varphi}_{ik}(x_1, x_2) \right| = \infty. \tag{4.28}$$

In the following, we slightly modify the definition of the functions $\tilde{\varphi}_{ik}(x_1, x_2)$. By (1.2),

$$\sum_{j=1}^{\infty} \int_0^2 \int_0^2 \Big[\sum_{(i,k)\in\mathcal{N}_{2j}} a_{ik} \tilde{\varphi}_{ik}(x_1, x_2) \Big]^2 dx_1\, dx_2 < \infty,$$

whence

$$\sum_{j=1}^{\infty} \Big[\sum_{(i,k)\in\mathcal{N}_{2j}} a_{ik} \tilde{\varphi}_{ik}(x_1, x_2) \Big]^2 < \infty$$

for almost every (x_1, x_2) in S_2. Thanks to this fact, the series

$$\sum_{j=1}^{\infty} r_j(t) \sum_{(i,k)\in\mathcal{N}_{2j}} a_{ik} \tilde{\varphi}_{ik}(x_1, x_2), \tag{4.29}$$

involving the one-dimensional Rademacher functions $\{r_j(t)\colon j = 1, 2, \ldots\}$, converges for almost every t in I for almost every (x_1, x_2) in S_2 (see e.g. [11], p. 212]). It is well-known that then series (4.29) also converges for almost every (x_1, x_2) in S_2 for almost every t in I. So, we can select a dyadically irrational number t_0 in I such that series (4.29) converges for almost every (x_1, x_2) in S_2 for $t = t_0$.

We define a new system $\psi = \{\psi_{ik}(x_1, x_2)\colon i, k = 1, 2, \ldots\}$ on S_2 as follows:

$$\psi_{ik}(x_1, x_2) = \begin{cases} r_j(t_0)\, a_{ik} \tilde{\varphi}_{ik}(x_1, x_2) & \text{if } (i, k) \in \mathcal{N}_{2j};\ j = 1, 2, \ldots; \\ \tilde{\varphi}_{ik}(x_1, x_2) & \text{if } (i, k) \in \mathcal{N} \setminus \bigcup_{j=1}^{\infty} \mathcal{N}_{2j}. \end{cases}$$

It is evident that ψ is an ONS of step functions on S_2, by (4.28) for $(x_1, x_2)\in G$

(4.30) $$\limsup_{j\to\infty} \max_{(m,n)\in\mathcal{N}_{2j}} \left|\sum_{i=\alpha(r_{2j})+1}^{m} \sum_{k=\beta(r_{2j})+1}^{n} \left(1-\frac{i-1}{m}\right) a_{ik}\psi_{ik}(x_1, x_2)\right| = \infty$$

and the series (cf. (4.29))

(4.31) $$\sum_{j=1}^{\infty} \sum_{(i,k)\in\mathcal{N}_{2j}} a_{ik}\psi_{ik}(x_1, x_2)$$

converges for almost every (x_1, x_2) in S_2.

Given a pair $(m, n)\in\mathcal{N}\setminus\bigcup_{j=1}^{\infty}\mathcal{N}_{2j}$, let $j(m, n)$ be that positive integer for which $(m, n)\in\mathcal{N}_{2j(m,n)}$. By definition, for $(x_1, x_2)\in S$

(4.32) $$\sigma_{mn}^{10}(a, \psi; x_1, x_2) = \sum_{i=\alpha(r_{2j(m,n)})+1}^{m} \sum_{k=\beta(r_{2j(m,n)})+1}^{n} \left(1-\frac{i-1}{m}\right) a_{ik}\psi_{ik}(x_1, x_2)+R(m, n; x_1, x_2)$$

where

$$R(m, n; x_1, x_2) = \sum_{j=1}^{j(m,n)-1} \sum_{(i,k)\in\mathcal{N}_{2j}} \left(1-\frac{i-1}{m}\right) a_{ik}\psi_{ik}(x_1, x_2).$$

Our next goal is to show that the limit

(4.33) $$\lim R(m, n; x_1, x_2) \quad \text{as } m, n\to\infty,\ (m, n)\in\mathcal{N}\setminus\bigcup_{j=1}^{\infty}\mathcal{N}_{2j},$$

exists for almost every (x_1, x_2) in S_2. To this end, we introduce the notation

$$s_{\alpha\beta}(j; x_1, x_2) = \sum_{i=\alpha(r_{2j})+1}^{\alpha} \sum_{k=\beta(r_{2j})+1}^{\beta} a_{ik}\psi_{ik}(x_1, x_2),\quad (\alpha, \beta)\in\mathcal{N}_{2j},$$

and apply an Abel transformation to obtain

$$\sum_{(i,k)\in\mathcal{N}_{2j}} \left(1-\frac{\alpha-1}{m}\right) a_{ik}\psi_{ik}(x_1, x_2) = \frac{1}{m}\sum_{\alpha=\alpha(r_{2j})+1}^{\alpha(r_{2j+1})-1} s_{\alpha,\beta(r_{2j+1})}(j; x_1, x_2)$$
$$+\left(1-\frac{\alpha(r_{2j+1})-1}{m}\right) s_{\alpha(r_{2j+1}),\beta(r_{2j+1})}(j; x_1, x_2)$$
$$= R^{(1)}(m, j; x_1, x_2)+R^{(2)}(m, j; x_1, x_2),\quad \text{say.}$$

This shows that

(4.34) $$R(m, n; x_1, x_2) = \sum_{j=1}^{j(m,n)-1} R^{(1)}(m, j; x_1, x_2)+\sum_{j=1}^{j(m,n)-1} R^{(2)}(m, j; x_1, x_2).$$

First, we deal with $R^{(1)}(m, j; x_1, x_2)$. Setting for $j = 1, 2, \ldots,$

$$\delta_j(x_1, x_2) = \max_{\alpha(r_{2j})<\alpha\leqslant\alpha(r_{2j+1})} |s_{\alpha,\beta(r_{2j+1})}(j; x_1, x_2)|,$$

obviously

$$\left|\sum_{j=1}^{j(m,n)-1} R^{(1)}(m, j; x_1, x_2)\right| \leqslant \sum_{j=1}^{j(m,n)-1} \frac{\alpha(r_{2j+1})}{m}\delta_j(x_1, x_2).$$

Thus, for $l = 2, 3, \ldots,$

(4.35) $$\delta_l^{(1)}(x_1, x_2) = \max_{\alpha(r_{2l})<m\leqslant\alpha(r_{2l+1})} \left|\sum_{j=1}^{l-1} R^{(1)}(m, j; x_1, x_2)\right|$$
$$\leqslant \sum_{j=1}^{l-1} \frac{\alpha(r_{2j+1})}{\alpha(r_{2l})}\delta_j(x_1, x_2) \leqslant \frac{1}{\alpha(r_{2l-1})}\sum_{j=1}^{l-1}\delta_j(x_1, x_2)$$

since $r_{2l}\geqslant r_{2l-1}+1$. In virtue of Lemma 1, for $j = 1, 2, \ldots,$

(4.36) $$\int_0^2\int_0^2 \delta_j^2(x_1, x_2)\,dx_1\,dx_2 \leqslant 4\cdot 2^{2r_{2j+1}} r_{2j+1}^2 \sum_{(i,k)\in\mathcal{N}_{2j}} a_{ik}^2,$$

whence, on the basis of (1.2) and (4.35),

(4.37) $$\sum_{l=2}^{\infty}\int_0^2\int_0^2 [\delta_l^{(1)}(x_1, x_2)]^2\,dx_1\,dx_2$$
$$\leqslant 4\sum_{l=2}^{\infty}\frac{l}{\alpha(r_{2l-1})}\sum_{j=1}^{l-1} 2^{2r_{2j+1}} r_{2j+1}^2 \sum_{(i,k)\in\mathcal{N}_{2j}} a_{ik}^2$$
$$= 4\sum_{j=1}^{\infty}\sum_{(i,k)\in\mathcal{N}_{2j}} a_{ik}^2\, 2^{2r_{2j+1}} r_{2j+1}^2 \sum_{l=j+1}^{\infty}\frac{l}{\alpha(r_{2l-1})}$$
$$\leqslant C_5\sum_{j=1}^{\infty}\sum_{(i,k)\in\mathcal{N}_{2j}} a_{ik}^2\, 2^{2r_{2j+1}} r_{2j+1}^2 \frac{j+1}{\alpha(r_{2j+1})}$$
$$\leqslant C_6\sum_{j=1}^{\infty}\sum_{(i,k)\in\mathcal{N}_{2j}} a_{ik}^2 < \infty.$$

By B. Levi's theorem, for almost every (x_1, x_2) in S_2

(4.38) $$\lim_{l\to\infty}\delta_l^{(1)}(x_1, x_2) = 0.$$

Secondly, we treat $R^{(2)}(m, j; x_1, x_2)$. By definition, for $l = 2, 3, \ldots$,

$$\sum_{j=1}^{l-1} R^{(2)}(m, j; x_1, x_2) = \sum_{j=1}^{l-1} s_{\alpha(r_{2j+1}),\beta(r_{2j+1})}(j; x_1, x_2) - \frac{\alpha(r_{2j+1})-1}{m}\sum_{j=1}^{l-1} s_{\alpha(r_{2j+1}),\beta(r_{2j+1})}(j; x_1, x_2) = R^{(3)}(l; x_1, x_2) + R^{(4)}(m, l; x_1, x_2), \quad \text{say.} \tag{4.39}$$

Since series (4.31) converges a.e., the limit

$$\lim_{l\to\infty} R^{(3)}(l; x_1, x_2) \tag{4.40}$$

exists for almost every (x_1, x_2) in S_2.

On the other hand, for $l = 2, 3, \ldots$,

$$\delta_l^{(4)}(x_1, x_2) = \max_{\alpha(r_{2l}) < m \leqslant \alpha(r_{2l+1})} |R^{(4)}(m, l; x_1, x_2)| \leqslant \sum_{j=1}^{l-1} \frac{\alpha(r_{2j+1})}{\alpha(r_{2l})}\delta_j(x_1, x_2) \leqslant \frac{1}{\alpha(r_{2l-1})}\sum_{j=1}^{l-1}\delta_j(x_1, x_2) \tag{4.41}$$

(cf. (4.35)). This, (1.2), and (4.36) imply, in the same way as in the case of (4.37), that for almost every (x_1, x_2) in S_2

$$\lim_{l\to\infty} \delta_l^{(4)}(x_1, x_2) = 0. \tag{4.42}$$

Collecting (4.34), (4.35), (4.38), (4.40)–(4.42) together, we can establish the existence of the limit (4.33) for almost every (x_1, x_2) in S_2. On account of (4.30) and (4.32), we can conclude

$$\limsup_{m,n\to\infty} |\sigma_{mn}^{10}(a, \psi; x_1, x_2)| = \infty$$

for almost every (x_1, x_2) in G. Since $G \subset S$ and $|G| = 1$, this relation holds for almost every (x_1, x_2) in S.

Finally, setting for $i, k = 1, 2, \ldots$,

$$\Psi_{ik}(x_1, x_2) = 2\psi_{ik}(2x_1, 2x_2), \quad (x_1, x_2) \in S,$$

we get (4.1) with $H = S_{1/2}$ and this completes the proof of Theorem 1′.

5. Proof of Theorem 2. We begin with the definition of an increasing sequence $\{M_r\colon r = 1, 2, \ldots; M_1 = 2\}$ such that for $r = 2, 3, \ldots$,

$$\frac{C_2 \log M_r}{8r\lambda_{\gamma(r)+1}\nu_{\beta(r)+1}} \geqslant r + C_1 \sum_{\varrho=1}^{r-1} 2M_\varrho^2 \tag{5.1}$$

where

$$\gamma(r) = 2^{2M_r^2} \quad \text{and (as before)} \quad \beta(r) = 2^r.$$

Then we define the double sequence $a = \{a_{ik}\colon i, k = 1, 2, \ldots\}$ of coefficients by

$$a_{ik} = \begin{cases} \dfrac{1}{rM_r\lambda_{\gamma(r)+1}\nu_{\beta(r)+1}} & \text{for } i = 2^1+1, 2^2+1, \ldots, \gamma(r)+1; \\ & \qquad k = 2^r+1;\ r = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, condition (1.8) is satisfied:

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}^2\lambda_i^2\nu_k^2 = \sum_{r=1}^{\infty}\sum_{s=1}^{2M_r^2} a_{2^s+1,2^r+1}^2 \lambda_{2^s+1}^2 \nu_{2^r+1}^2 = \sum_{r=1}^{\infty}\sum_{s=1}^{2M_r^2} \frac{\lambda_{\beta(s)+1}^2}{r^2M_r^2\lambda_{\gamma(r)+1}^2} \leqslant 2\sum_{r=1}^{\infty}\frac{1}{r^2} < \infty.$$

The main part of the proof is the definition of the ONS $\{\Phi_{ik}(x_1, x_2)\colon i, k = 1, 2, \ldots\}$. To this end, we apply Lemma 2 with $p = M_r$ for every $r = 1, 2, \ldots$, denoting by $\{f_i(r; x)\colon i = 1, 2, \ldots, 2M_r^2\}$ and $E(r)$ the resulting ONS of step functions on I and simple subset of I, respectively. By (a)–(c) in Lemma 2, we have for $r = 1, 2, \ldots$,

$$|E(r)| \geqslant C_1, \tag{5.2}$$

$$|f_i(r; x)| \leqslant C_2 \quad \text{for } x \in I \text{ and } i = 1, 2, \ldots, 2M_r^2; \tag{5.3}$$

furthermore, for $x \in E(r)$ there exists an integer $m = m(r; x)$, $M_r^2 \leqslant m < 2M_r^2$, such that $f_i(r; x) \geqslant 0$ for $i = 1, 2, \ldots, m$ and

$$\sum_{i=1}^{m} f_i(r; x) \geqslant C_3 M_r \log M_r. \tag{5.4}$$

First, we define an ONS $\varphi = \{\varphi_{ik}(x)\colon i, k = 1, 2, \ldots\}$ of step functions on $I_2 = (0, 2)$ and a sequence $\{H_r\colon r = 1, 2, \ldots\}$ of stochastically independent simple subsets of I in such a way that the following conditions are satisfied: for every $r = 1, 2, \ldots$,

$$|H_r| \geqslant C_1, \tag{5.5}$$

$$|\varphi_{ik}(x_1, x_2)| \leqslant C_2 \quad \text{for } x \in I_2,\ i = 2^1+1, 2^2+1, \ldots, \gamma(r)+1;\ k = 2^r+1; \tag{5.6}$$

for $x \in H_r$ there exists an integer $m = m(r; x)$, $M_r^2 \leqslant m < 2M_r^2$, such that $\varphi_{2^s+1,2^r+1}(x) \geqslant 0$ for $s = 1, 2, \ldots, m$ and

$$\sum_{s=1}^{m} \varphi_{2^s+1,2^r+1}(x) \geqslant C_3 M_r \log M_r, \tag{5.7}$$

and

(5.8) $\varphi_{ik}(x)=0$ for $x\in I$ if $(i,k)\neq(2^s+1,2^r+1)$ with $s=1,2,\ldots,2M_r^2$; $r=1,2,\ldots$

In the special case $r=1$, we set for $s=1,2,\ldots,2M_1^2$,

$$\varphi_{2^s+1,3}(x)=f_s(1;x) \quad\text{and}\quad H_1=E(1).$$

Now let r_0 be a positive integer and assume that the step functions $\{\varphi_{2^s+1,2^r+1}(x)\colon s=1,2,\ldots,2M_r^2;\ r=1,2,\ldots,r_0\}$ and the simple subsets $\{H_r\colon r=1,2,\ldots,r_0\}$ of I have been defined so that these functions form an ONS on I, these sets are stochastically independent, and properties (5.5)–(5.8) are satisfied for $r=1,2,\ldots,r_0$.

We will define the step functions

$$\{\varphi_{2^s+1,2^{r_0+1}+1}(x)\colon s=1,2,\ldots,2M_{r_0+1}^2\}$$

of the (r_0+1)st block and the simple set H_{r_0+1} in the following way. We divide I into a finite number of disjoint intervals $\{J_p\colon p=1,2,\ldots,P\}$ such that the functions of the first r_0 blocks are constant on each J_p and each set $\{H_r\colon r=1,2,\ldots,r_0\}$ is the union of certain J_p. Denoting by J_p' and J_p'' the two halves of the intervals J_p, we set for $s=1,2,\ldots,2M_{r_0+1}^2$,

$$\varphi_{2^s+1,2^{r_0+1}+1}(x)=\sum_{p=1}^{P}[f_s(r_0+1;J_p';x)-f_s(r_0+1;J_p'';x)]$$

and

$$H_{r_0+1}=\bigcup_{p=1}^{P}[E(r_0+1;J_p')\cup E(r_0+1;J_p'')].$$

It is easy to see that the step functions

$$\{\varphi_{2^s+1,2^r+1}(x)\colon s=1,2,\ldots,2M_r^2;\ r=1,2,\ldots,r_0+1\}$$

form an ONS on I, the simple sets $\{H_r\colon r=1,2,\ldots,r_0+1\}$ are stochastically independent, and properties (5.5)–(5.7) are satisfied for $r=r_0+1$, due to (5.2)–(5.4).

This induction scheme shows that the ONS

$$\{\varphi_{2^s+1,2^r+1}(x)\colon s=1,2,\ldots,2M_r^2;\ r=1,2,\ldots\}$$

and the sequence $\{H_r\colon r=1,2,\ldots\}$ of stochastically independent sets can be defined so that conditions (5.5)–(5.7) hold true.

Then we consider an arbitrary decomposition $\{J_{ik}\}$ of the interval $(1,2)$ into disjoint subintervals and set for

$$(i,k)\neq(2^s+1,2^r+1)\quad\text{with}\quad s=1,2,\ldots,2M_r^2;\ r=1,2,\ldots,$$

$$\varphi_{ik}(x)=\begin{cases}|J_{ik}|^{-1/2} & \text{for } x\in J_{ik},\\ 0 & \text{for } x\in I_2\setminus J_{ik};\end{cases}$$

while we extend the functions $\varphi_{ik}(x)$ defined previously from I onto I_2 simply by setting $\varphi_{ik}(x)=0$ for $x\in(1,2)$. Thus, the entire ONS $\{\varphi_{ik}(x)\colon i,k=1,2,\ldots\}$ is defined on I_2 so that conditions (5.5)–(5.7) and (5.8) are satisfied.

If $x\in H_r$ for some $r=2,3,\ldots$, then from (5.1), (5.6)–(5.8) we can deduce that

$$\text{(5.9)}\quad |\sigma^{11}_{2\gamma(r)-1,2\beta(r)-1}(a,\varphi;x)-\sigma^{11}_{\gamma(r),\beta(r)}(a,\varphi;x)|$$
$$\geqslant\sum_{s=1}^{2M_r^2}\left(1-\frac{2^s}{2\gamma(r)-1}\right)\left(1-\frac{2^r}{2^{r+1}-1}\right)a_{2^s+1,2^r+1}\varphi_{2^s+1,2^r+1}(x)-C_2\sum_{\varrho=1}^{r-1}2M_\varrho^2$$
$$\geqslant\frac{1}{8rM_r\lambda_{\gamma(r)+1}\nu_{\beta(r)+1}}\sum_{s=1}^{2M_r^2}\varphi_{2^s+1,2^r+1}(x)-C_2\sum_{\varrho=1}^{r-1}2M_\varrho^2$$
$$\geqslant\frac{C_3\log M_r^2}{8r\lambda_{\gamma(r)+1}\nu_{\beta(r)+1}}-C_2\sum_{\varrho=1}^{r-1}2M_\varrho^2\geqslant r.$$

Thanks to stochastic independence, for $H=\limsup_{r\to\infty}H_r$ we have $|H|=1$. If $x\in H$, then (5.9) is satisfied for infinitely many r, a fortiori,

$$\lim_{p,q\to\infty}\Big[\max_{2^p\leqslant m<2^{p+1}}\ \max_{2^q\leqslant n<2^{q+1}}|\sigma^{11}_{mn}(a,\varphi;x)-\sigma^{11}_{2^p,2^q}(a,\varphi;x)|\Big]=\infty.$$

Since $H\subset(0,1)$, this relation is satisfied for almost every x in I.

It remains only to contract the functions into I:

$$\Phi_{ik}(x)=\sqrt{2}\,\varphi_{ik}(2x)\quad\text{for } x\in I \text{ and } i,k=1,2,\ldots;$$

and this completes the proof of Theorem 2.

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Remarks concerning the paper "On a class of Hausdorff compacts and GSG Banach spaces"

by

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Abstract. The main result of the above paper by O. I. Reynov which appeared in Studia Math. 71 (1981) is incorrect.

We show, by counterexample, that the main result of the paper [7] is not correct. Moreover, it is clear that the mistake is not technical in nature, but fundamental: there is an improper use of interpolation techniques, not only those of [2] but of interpolation methods in general, e.g. [1]. We emphasize, however, that all the results of [7] that are true are easily obtained by applying the results and techniques of [2–6]; for example, Theorem 0.1 of [7] follows easily from [6].

EXAMPLE. We consider the Banach spaces l_1 and l_∞ in their natural duality.

Let $B = \{x \in l_\infty \mid |x_i| \leqslant i^{-2}\}$. This is a norm compact convex circled subset of l_∞.

Let $A = \bigcap_{n=1}^{\infty} (nB + n^{-1} D)$, where D is the unit ball of l_∞. Now, consider the sequence $a = (i^{-1})_i$. Since $1 \leqslant n/i + i/n$ for all $i, n \in \mathbf{N}$, it follows that $1/i \leqslant n/i^2 + 1/n$ for all $i, n \in N$, and we have $a \in A$. We shall compute the distance (in the norm generated by A) from the element a to the linear span $\bigcup \{\lambda B \mid \lambda \geqslant 0\}$ of B in l_∞. Suppose $b = (b_i)_i \in B$, $\lambda \geqslant 0$, $\alpha \geqslant 0$ are such that $a - \lambda b \in \alpha A$. We must have $\alpha > 0$, because $a \notin \operatorname{span} B$. Then there exist $b_n = (b_{n,i})_i \in B$ and $x_n = (x_{n,i})_i \in D$ such that

$$\frac{1}{i} - \lambda b_i = \alpha\left(n b_{n,i} + \frac{1}{n} x_{n,i}\right) \quad \text{for all } i, n \in \mathbf{N}.$$

Since, for $i \geqslant 2\lambda$,

$$\frac{1}{i} - \lambda b_i \geqslant \frac{\lambda}{i^2} > 0,$$