STUDIA MATHEMATICA, T. LXXXII. (1985)

On weighted inequalities for ergodic operators

by

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Abstract. The purpose of this note is to show that the method given by A. P. Calderón in Proc. Nat. Acad. Sci. 59 (1968) can be applied to study ergodic problems involving weights, in particular the weighted L^p boundedness of the maximal ergodic operator.

§ 1. Introduction. It is known, after a paper of A. P. Calderón [4], that a certain class of results in ergodic theory follows from the validity of similar properties for translation-invariant operators. In this way, as is pointed out in that paper, the ergodic maximal theorem and Cotlar's results become immediate consequences of the Hardy–Littlewood maximal theorem and the usual results for singular integrals. In this note we show that the method given by A. P. Calderón can also be applied to problems involving weights. In a recent paper, E. Atencia and A. de la Torre [3] gave a characterization of weights W for which the discrete ergodic maximal operator is bounded on $L^p(W)$, following the lines of [5]. We consider a family of measure-preserving transformations with parameter on a locally compact abelian group G. The special case of the ergodic maximal operator defined by a Vitali family of neighborhoods of zero (see § 2) follows from the characterization of weights for which the Hardy–Littlewood maximal operator on spaces of homogeneous type is bounded in L^p (see [2] or [6]). When $G = \mathbf{Z}$ the weighted L^p boundedness of the ergodic maximal operator given in [3] follows in a straightforward manner from the Muckenhoupt A_p condition. Other interesting cases falling into the scope of this result are the ergodic Hilbert transform, its generalizations, and the maximal operators of some convolution approximate identities.

We introduce in § 2–§ 4 the basic structures and definitions; the main result is contained in § 5, and the applications in § 6.

§ 2. Group properties. Let $(G, +, 0)$ be a locally compact abelian group

1980 *Mathematics subject classification.* Primary 28D05, 28A25, Secondary 42B25.
Key words and phrases. Ergodic theory, weights, maximal functions.

and ν a σ-finite Haar measure on the Borel subsets of G. Let $\{U_\alpha\colon \alpha>0\}$ be a regular Vitali family in the sense of Rivière (see [7]), i.e.

(2.1) U_α is an open neighborhood of 0 for every $\alpha>0$,

(2.2) $\bar{U}_\alpha$ is a compact set and $\bigcap_{\alpha>0}\bar{U}_\alpha=\{0\}$,

(2.3) if $\alpha<\beta$ then $U_\alpha\subset U_\beta$,

(2.4) $\bigcup_{\alpha>0}U_\alpha=G$,

(2.5) there exists a continuous nondecreasing function Φ defined on $\boldsymbol{R}^+$ onto $\boldsymbol{R}^+$ such that $\Phi(\alpha)>\alpha$,

$$U_\alpha-U_\alpha\subset U_{\Phi(\alpha)}\quad\text{and}\quad \nu(U_{\Phi(\alpha)})\leqslant A\nu(U_\alpha)$$

for some nonnegative A.

We can assume without loss of generality that the sets U_α are symmetric and that $\nu(U_\alpha)$ is left-continuous as a fuction of α.

§ 3. **Measure properties.** Let $(X,\mathscr{F},\mu)$ be a σ-finite measure space. We assume that there is a function T from $G\times X$ into X such that, for each t and s in G,

(3.1) $T(0,\cdot)$ is the identity on X,

(3.2) $T(t+s,\cdot)=T(t,T(s,\cdot))$,

(3.3) $E\in\mathscr{F}$ if and only if $T(t,E)\in\mathscr{F}$, and $\mu(T(t,E))=\mu(E)$,

(3.4) if f is a μ-measurable function defined on X, then $F=f\circ T$ is $\nu\times\mu$-measurable.

§ 4. **The ergodic maximal operator.** As usual, $\mathscr{C}(G)$ denotes the space of real-valued continuous functions on G and $L^1_{\mathrm{loc}}(G)$ the space of locally integrable functions on G. Let S be a sublinear operator from $L^1_{\mathrm{loc}}(G)$ into $\mathscr{C}(G)$ such that

(4.1) S commutes with translations,

(4.2) S is semi-local, that is, there exists a (symmetric) neighborhood V of 0 such that $\bar{V}$ is compact and $\operatorname{supp} Sg\subset V+\operatorname{supp} g$ for all $g\in L^1_{\mathrm{loc}}(G)$.

We define the ergodic operator $S^\#$ associated with S in the following way: let $f_1\in L^1(X,\mu)$, $f_2\in L^\infty(X,\mu)$ and $f=f_1+f_2$. Then $F(\cdot,x)\in L^1_{\mathrm{loc}}(G)$ and $S(F(\cdot,x))\in\mathscr{C}(G)$ for almost every x. Hence

$$S^\# f(X)=S(F(\cdot,x))(0)$$

is well defined for almost every $x\in X$. Given a sequence $\{S_n\}$ of operators satisfying (4.1) and (4.2), define

$$S^*g(t)=\sup_n|S_ng(t)|,\qquad S_N^*g(t)=\sup_{n\leqslant N}|S_ng(t)|,$$

$$S^{\#*}f(x)=\sup_n|S_n^\# f(x)|,\qquad S_N^{\#*}f(x)=\sup_{n\leqslant N}|S_n^\# f(x)|.$$

§ 5. **The result.** If $1<p<\infty$ and $\lambda>0$, we shall say that a weight function w defined on G belongs to the class $A_p^\lambda(S^*)$ if and only if

$$\int_G[S^*g(t)]^p w(t)\,d\nu(t)\leqslant\lambda\int_G|g(t)|^p w(t)\,d\nu(t)\tag{5.1}$$

for every ν-measurable function g defined on G.

THEOREM. *If $1<p<\infty$, $\lambda>0$ and W is a weight on X such that $W(T(\cdot,x))\in A_p^\lambda(S^*)$ for almost every x, then*

$$\int_X[S^{\#*}f(x)]^p W(x)\,d\mu(x)\leqslant A\lambda\int_X|f(x)|^p W(x)\,d\mu(x)\tag{5.2}$$

holds for every $f\in(L^1+L^\infty)(X,\mu)$.

Proof. First observe that $S_N^{\#*}f(x)=S_N^*f(T(\cdot,x))(0)$; thus on account of (3.2) and (4.1) we have

$$S_N^{\#*}f(T(t,x))=S_N^*f(T(\cdot,x))(t).\tag{5.3}$$

If V_n is the neighborhood of 0 associated to S_n by (4.2), U_α a member of the regular Vitali family and $E(N,\alpha)=\bigcup_{n=1}^N V_n+U_\alpha$, then the inequality

$$S_N^*f(T(\cdot,x))(t)\leqslant S_N^*\big(\chi_{E(N,\alpha)}(\cdot)f(T(\cdot,x))\big)(t)+S_N^*\big(\chi_{E(N,\alpha)'}(\cdot)f(T(\cdot,x))\big)(t)$$

follows from the sublinearity of S_N^*; here $E(N,\alpha)'$ denotes the complementary set of $E(N,\alpha)$. Note that if $t\in U_\alpha$, the last term vanishes. Then using (5.3) and the fact that a function $h(x)$ and its ergodic translate $h(T(t,x))$ are equimeasurable as functions of x, we obtain

$$\begin{aligned}\int_X[S_N^{\#*}f(x)]^p W(x)\,d\mu(x)&=\frac{1}{\nu(U_\alpha)}\int_{U_\alpha}\Big\{\int_X[S_N^{\#*}f(T(t,x))]^p W(T(t,x))\,d\mu(x)\Big\}d\nu(t)\\&\leqslant\frac{1}{\nu(U_\alpha)}\int_{U_\alpha}\Big\{\int_X\big[S_N^*\big(\chi_{E(N,\alpha)}(\cdot)f(T(\cdot,x))\big)(t)\big]^p W(T(t,x))\,d\mu(x)\Big\}d\nu(t).\end{aligned}$$

By applying Tonelli's theorem and the hypothesis on W, the right-hand side is bounded by

$$\frac{\lambda}{\nu(U_\alpha)}\int_X\Big\{\int_{E(N,\alpha)}|f(T(t,x))|^p W(T(t,x))\,d\nu(t)\Big\}d\mu(x).$$

Changing again the order of integration and choosing α large enough that $\bigcup_{n=1}^{N} V_n \subset U_\alpha$, we obtain from (2.5) the inequality

$$\int_X [S_N^{\#*} f(x)]^p W(x)\, d\mu(x) \leqslant A\lambda \int_X |f(x)|^p W(x)\, d\mu(x),$$

uniformly in N. Therefore, by a passage to the limit (5.2) follows.

§ 6. Applications. If a sufficient condition in order that a weight w defined on G belong to $A_p^\lambda(S^*)$ is known, then the result just proved can be applied to obtain sufficient conditions in order that the operator $S^{\#*}$ be bounded on $L^p(X, W d\mu)$. This is the case for the Muckenhoupt A_p condition when S^* is the Hardy–Littlewood maximal function operator. Let

$$S^* g(t) = \sup_{\alpha > 0} \frac{1}{\nu(U_\alpha)} \int_{t+U_\alpha} |g(s)|\, d\nu(s)$$

be the Hardy–Littlewood maximal operator defined by the regular Vitali family $\{U_\alpha\}$. The function $d(t) = \inf\{\nu(U_\alpha)\colon t \in U_\alpha\}$ if $t \neq 0$, $d(0) = 0$ defines a space of homogeneous type on G; indeed, as is easy to check,

$$d(t+s) \leqslant A[d(t) + d(s)],$$

and if $B_d(0, r) = \{t\colon d(t) < r\}$ we have

$$\nu(B_d(0, 2r)) \leqslant 2\nu(B_d(0, r)).$$

Moreover, the Hardy–Littlewood maximal function in the space (G, d, ν), defined by

$$Mg(t) = \sup_{r>0} \frac{1}{\nu(B_d(t, r))} \int_{B_d(t,r)} |g(s)|\, d\nu(s),$$

is equivalent to S^*. Applying the Muckenhoupt characterization extended to the setting of spaces of homogeneous type in [2] and [6], we can obtain sufficient conditions in order that a weight $w \in A_p^\lambda(S^*)$. In fact, if

$$A_p\colon \quad \left(\frac{1}{\nu(U_\alpha)} \int_{t+U_\alpha} w(s)\, d\nu(s)\right) \left(\frac{1}{\nu(U_\alpha)} \int_{t+U_\alpha} w(s)^{1/(1-p)}\, d\nu(s)\right)^{p-1} \leqslant C$$

holds for all $\alpha > 0$ and $t \in G$, then there exists λ, depending only on p and C, such that $w \in A_p^\lambda(S^*)$. Therefore, if $S^{\#*} f$ is the ergodic maximal function

$$S^{\#*} f(x) = \sup_{\alpha > 0} \frac{1}{\nu(U_\alpha)} \int_{U_\alpha} |f(T(s, x))|\, d\nu(s),$$

a sufficient condition for (5.2) is the following: there exists C such that, for almost every $x \in X$,

$$(6.1)\quad \left(\frac{1}{\nu(U_\alpha)} \int_{t+U_\alpha} W(T(s, x))\, d\nu(s)\right) \left(\frac{1}{\nu(U_\alpha)} \int_{t+U_\alpha} W(T(s, x))^{1/(1-p)}\, d\nu(s)\right)^{p-1} \leqslant C$$

holds for all $t \in G$ and $\alpha > 0$. If, in addition, G is countable, (6.1) is equivalent to

$$\left(\frac{1}{\nu(U_\alpha)} \int_{U_\alpha} W(T(s, x))\, d\nu(s)\right) \left(\frac{1}{\nu(U_\alpha)} \int_{U_\alpha} W(T(s, x))^{1/(1-p)}\, d\nu(s)\right)^{p-1} \leqslant C$$

for almost all x and all $\alpha > 0$, which in the case $G = \boldsymbol{Z}$ is condition A'_p given in [3].

Condition A_p above, on a weight w on G, remains sufficient for the weighted L^p boundedness of a wider class of singular integral operators and approximate identities on the space (G, d, ν) (see [1]). Therefore by an application of the result proved here, (6.1) is a sufficient condition for the weighted L^p boundedness of the associated ergodic operators.

Observe, finally, that the method of A. P. Calderón can also be adapted to obtain the weighted estimate for the operator

$$M_W f(x) = \sup_{\alpha > 0} \left(\int_{U_\alpha} W(T(s, x))\, d\nu(s)\right)^{-1} \int_{U_\alpha} |f(T(s, x))|\, W(T(s, x))\, d\nu(s)$$

where W is a weight such that $W(T(\cdot, x))$ satisfies a uniform doubling condition.

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Received June 4, 1984 (1982)