STUDIA MATHEMATICA, T. LXXXII. (1985)

# On the local linear independence of translates of a box spline

by

WOLFGANG DAHMEN (Bielefeld) and CHARLES A. MICCHELLI (Yorktown Heights, N. Y.)

**Abstract.** In an earlier paper [4] we studied the linear independence of the translates of a box spline over $\mathbf{R}^s$. In this paper, we show that the (global) linear independence of the translates implies their independence over any bounded domain. We approach this question by studying the class of polynomials spanned by translates of a box spline. Specifically, we show that its dimension equals the number of translates whose support contains some generic point when all translates are independent. As an application of this result we construct linear spline projectors which have optimal approximation rates.

**1. Preliminaries: notation and problem formulation.** For any vectors $x^1, \ldots, x^n$, $x^i = (x_1^i, \ldots, x_s^i)$, not necessarily distinct, such that $X = \{x^1, \ldots, x^n\} \subset \mathbf{R}^s \setminus \{0\}$ and

$$\langle X \rangle = \operatorname{span}\{X\} = \mathbf{R}^s, \tag{1.1}$$

the box spline $B(x|X)$ is defined by requiring that

$$\int_{\mathbf{R}^s} f(x) B(x|X)\, dx = \int_{[0,1]^n} f(t_1 x^1 + \ldots + t_n x^n)\, dt_1 \ldots dt_n \tag{1.2}$$

holds for any continuous function $f$ on $\mathbf{R}^s$. The box spline was introduced in [1] while several of its basic properties were given in [2].

An attractive feature of the box spline is that it unifies many classical finite elements thereby giving a deeper understanding of spline spaces on regular grids [2, 3, 7, 8, 11] (see e.g. [6] for more details and further references). In this context, one is led to consider the space

$$\mathcal{S}(X) = \operatorname{span}\{B(\cdot - \alpha|X) \colon \alpha \in Z^s\}. \tag{1.3}$$

Next we will briefly review some basic properties of the box spline $B(\cdot|X)$ and the space $\mathcal{S}(X)$ which will be frequently used (cf. [2]).

Using standard notation we write for $\alpha, \beta \in Z^s$, $\alpha = (\alpha_1, \ldots, \alpha_s)$, $|\alpha|$

$=\alpha_1+\ldots+\alpha_s$, $x^\alpha = x_1^{\alpha_1}\ldots x_s^{\alpha_s}$, $\alpha! = \alpha_1!\ldots\alpha_s!$, $\binom{\alpha}{\beta} = \alpha!/(\beta!(\alpha-\beta)!)$ and let $\Pi_k$ be the space of all (real) polynomials of (total) degree $\leqslant k$,

$$\Pi_k = \Pi_k(\boldsymbol{R}^s) = \{\sum_{|\alpha|\leqslant k} c_\alpha x^\alpha \colon c_\alpha \in \boldsymbol{R}\}.$$

The box spline is known to be a piecewise polynomial of total degree $\leqslant n-s$, where $|X| = n$ is the cardinality of $X$. Furthermore, for every set $Y \subseteq X$ with $\dim\langle Y\rangle = |Y| = s-1$ and any vector of the form $x = \sum_{x^i \notin Y} c_i x^i$, $c_i \in \{0, 1\}$, the $(s-1)$-dimensional hyperplanes $x+\langle Y\rangle$ (*cut region*) bound the sets on which $B(x|X)$ is a polynomial. We let $c(X)$ be the union of cut regions for all the multi-integer translates of the box spline $B(\cdot|X)$.

Essential information about $B(x|X)$ and the space $\mathscr{S}(X)$ can be given in terms of the number

(1.4) $\quad d(X) = \max\{m\colon \text{for all } Y \subseteq X,\ |Y| = m \text{ implies } \langle X\setminus Y\rangle = \boldsymbol{R}^s\}.$

It is known that

(1.5) $$B(\cdot|X) \in C^{d(X)-1}(\boldsymbol{R}^s)\setminus C^{d(X)}(\boldsymbol{R}^s)$$

and when

(1.6) $$X \subset Z^s$$

we have

(1.7) $$\Pi_{d(X)} \subset \mathscr{S}(X), \qquad \Pi_{d(X)+1} \not\subset \mathscr{S}(X).$$

Clearly, in general

(1.8) $$d(X) \leqslant |X|-s,$$

where, however, the inequality is usually strict, so that (1.7) does not describe all polynomials being spanned by elements of $\mathscr{S}(X)$. In fact, to describe the local polynomial structure of $\mathscr{S}(X)$ we need the following notions [2] (see also [4, 6]). Define for any $l \leqslant s$

(1.9) $\quad \mathscr{Y}_l(X)$

$$= \{Y \subset X\colon \dim\langle X\setminus Y\rangle = l-1 \text{ while } \dim\langle X\setminus V\rangle \geqslant l \text{ for all } V \subsetneqq Y\}$$

and consider the subspace of distributions

(1.10) $$D_l(X) = \{T\colon T \in \mathscr{D}'(\boldsymbol{R}^s),\ D_Y T = 0,\ Y \in \mathscr{Y}_l(X)\}$$

where $D_Y = \prod_{y\in Y} D_y$ and $D_y$ is the directional derivative in the direction $y$.

In the univariate case, $s = 1$, $D_1(X)$ reduces to $\Pi_{|X|-1}(\boldsymbol{R})$. In general, for $s > 1$, it was shown in [4] that $D_s(X)$ is a finite-dimensional subspace of polynomials. Moreover, when $X \subset Z^s$ Poisson's summation formula shows that the mapping

(1.11) $$(Af)(x) = \sum_{\alpha\in Z^s} f(\alpha) B(x-\alpha|X)$$

is one-to one and onto $D_s(X)$ [4]. This was first proved in C. de Boor and K. Höllig, *B-splines from parallelepipeds*, MRC Report # 2320, University of Wisconsin, 1982, by different means for $D_s(X)$ intersected with all polynomials as it was not observed there that $D_s(X)$ only contains polynomials (see [2] for a proof of this latter fact when $X \subset Z^s$). In particular, (1.11) implies

(1.12) $$D_s(X) \subset \mathscr{S}(X).$$

Conversely, recalling from [2] that for any $V \subset X$

$$D_V B(\cdot|X) = \nabla_V B(\cdot|X\setminus V)$$

where $\nabla_y f(\cdot) = f(\cdot) - f(\cdot - y)$, $\nabla_V = \prod_{y\in V} \nabla_y$, we see from (1.2) that $D_V B(x|X)$ vanishes off the cut regions whenever $V \in \mathscr{Y}_s(X)$. Hence the polynomial pieces of any function in $\mathscr{S}(X)$ must belong to $D_s(X)$. Thus $D_s(X)$ represents exactly all polynomials which are locally in $\mathscr{S}(X)$.

Thus a central objective of this paper is to provide more information about the space $D_s(X)$ in order to understand the local structure of $\mathscr{S}(X)$. Specifically, we wish to analyze the local linear independence of box splines, i.e. we determine when for any bounded domain $\Omega \subset \boldsymbol{R}^s$

(1.13) $$\sum_{\alpha\in Z^s} a_\alpha B(x-\alpha|X) = 0, \quad x \in \Omega,$$

implies $a_\alpha = 0$ for all $\alpha$ such that $\Omega \cap \operatorname{supp} B(\cdot - \alpha|X) \neq \emptyset$. It will be shown here that local independence is equivalent to global independence which means that

(1.14) $$\sum_{\alpha\in Z^s} a_\alpha B(x-\alpha|X) = 0 \quad \text{for all } x \in \boldsymbol{R}^s$$

implies $a_\alpha = 0$, $\alpha \in Z^s$.

Only in the special case $s = 2$ for sets $X$ composed of repetitions of at most the four directions $(1, 0)$, $(0, 1)$, $(1, 1)$, $(-1, 1)$ has this problem been analyzed [3, 5]. In this case, it is easy to count the number of translates $B(x-\alpha|X)$ whose support intersects any fixed point not on a cut region [5]. Local independence follows from (1.12) by exhibiting as many linearly independent elements in $D_2(X)$ as translates which do not vanish at a generic point [3, 5].

In order to use this approach for arbitrary spatial dimension $s \geqslant 1$ and any $X \subset Z^s$ we introduce for any given bounded domain $\Omega \subset \boldsymbol{R}^s$

$$b(x|\Omega) = \{\alpha \in Z^s\colon x \in \Omega + \alpha\}.$$

When $\Omega = \operatorname{supp} B(\cdot|X)$ we denote this function by $b(x|X)$. Thus for any $x \notin c(X)$, the cut regions of the functions $B(\cdot-\alpha|X)$, $\alpha \in Z^s$, $|b(x|X)|$ is the number of translates which are positive at $x$. Our central objective is to establish a relation between $|b(x|X)|$ and $\dim D_s(X)$. Both numbers were heretofore undetermined. For this purpose, we define for any $X \subset \boldsymbol{R}^s$ and $l \leqslant s$,

(1.15) $$\mathscr{B}_l(x) = \{Y \subset X\colon\ |Y| = l,\ \dim\langle Y\rangle = l\}.$$

We will show that

$$\dim D_s(X) = |\mathscr{B}_s(X)|$$

while, when in addition $X \subset Z^s$ and $x \notin c(X)$, then

$$|b(x|X)| = \sum_{Y \in \mathscr{B}_s(X)} |\det Y|$$

where for notational convenience we have denoted the $s \times s$ matrix whose columns are the elements of $Y$ also by $Y$.

Hence, in view of (1.12), a necessary and sufficient condition for local linear independence of the translates $B(\cdot-\alpha|X)$ is that

(1.16) $$|\det Y| = 1 \quad \text{for all } Y \in \mathscr{B}_s(X).$$

This is the condition that was shown in [4] to be equivalent to global independence. Thus local linear independence holds iff global independence does.

Furthermore, when (1.16) holds we will show that one may uniquely interpolate any given data at the lattice points in $b(x|X)$, $x \notin c(X)$, by elements in $D_s(X)$. This enables us to use methods similar to those in [10] to construct *local linear projectors* (quasi-interpolants) from $L_p(\boldsymbol{R}^s)$ onto $\mathscr{S}(X)$.

**2. Construction of a basis of $D_s(X)$.** The main result of this section is

THEOREM 2.1. *Let $X$ be any finite set in $\boldsymbol{R}^s \setminus \{0\}$ such that $\langle X\rangle = \boldsymbol{R}^s$. Then*

(2.1) $$\dim D_s(X) = |\mathscr{B}_s(X)|.$$

When $s = 2$, $e^1 = (1, 0)$, $e^2 = (0, 1)$ and

$$X_M = \{\underbrace{e^1, \ldots, e^1}_{m_1}, \underbrace{e^2, \ldots, e^2}_{m_2}, \underbrace{e^1+e^2, \ldots, e^1+e^2}_{m_3}, \underbrace{e^2-e^1, \ldots, e^2-e^1}_{m_4}\}$$

it is easy to see that $|\mathscr{B}_2(X_M)| = \sum_{1 \leqslant i \leqslant j \leqslant 4} m_i m_j$. Then Theorem 2.1 gives

$$\dim D_2(X_M) = \sum_{1 \leqslant i \leqslant j \leqslant 4} m_i m_j.$$

This result was proved in [5] by other means.

Before we start proving the general result, let us briefly point out that its validity is easily checked for some special choices of $X$.



First suppose $X$ is comprised of repetitions of the coordinate vectors $(e^i)_j = \delta_{ij}$, $i, j = 1, \ldots, s$, that is,

$$X = \{\underbrace{e^1, \ldots, e^1}_{q_1+1}, \ldots, \underbrace{e^s, \ldots, e^s}_{q_s+1}\}.$$

In this case, $D_s(X)$ is easily seen to be the set of all functions which are polynomials of degree $\leqslant q_i$ in the $i$th variable, $i = 1, \ldots, s$. Therefore $\dim D_s(X) = \prod_{i=1}^{s} (q_i+1)$ while clearly $|\mathscr{B}_s(X)| = \prod_{i=1}^{s} (q_i+1)$, also.

When $X$ consists of vectors in general position, i.e. any $s$ of its vectors span $\boldsymbol{R}^s$, then $|\mathscr{B}_s(X)| = \binom{|X|}{s}$. Furthermore, since in this case $d(X) = |X|-s$, (cf. (1.4)), (1.12) gives $D_s(X) = \Pi_{|X|-s}(\boldsymbol{R}^s)$ and this space has dimension $\binom{|X|}{s}$.

Before proving Theorem 2.1 we give another proof of the latter case when $X$ is in general position which illuminates the proof of the general case.

This proof uses induction on $|X|$ so that we suppose $X = \{y\} \cup X'$. Then $\mathscr{B}_s(X)$ is the disjoint union of $\mathscr{B}_s(X')$ and the set

$$\{\{y\} \cup Y\colon\ Y \subset X',\ |Y| = s-1\}.$$

For any $Y = \{y^1, \ldots, y^{s-1}\} \subseteq X'$ we choose a dual basis $\{\lambda, \lambda^1, \ldots, \lambda^{s-1}\}$ for $\{y\} \cup Y$, i.e. $\lambda \cdot y = 1$, $\lambda \cdot y^i = 0$, $i = 1, \ldots, s-1$, and $\lambda^i \cdot y = 0$, $i = 1, \ldots, s-1$. Also, let

(2.2) $$Q_Y(x) = (\lambda \cdot x)^{n-s},$$

where $|X| = n$. When $X$ is in general position, $\mathscr{Y}_s(X) = \{V \subset X\colon\ |V| = n-s+1\}$ and so we have $D_V Q_Y = 0$ for all $V \in \mathscr{Y}_s(X)$, that is, $Q_Y \in D_s(X)$. On the other hand, for $Y \subset X'$ as above we have $X' \setminus Y \in \mathscr{Y}_s(X')$ while

(2.3) $$D_{X' \setminus Y} Q_Y = (n-s)! \prod_{x \in X' \setminus Y} (x \cdot \lambda) \neq 0$$

so that $Q_Y \notin D_s(X')$. However, for any $Y' \subset X$, $|Y'| = s-1$, $Y' \neq Y$ it follows that $(X' \setminus Y') \cap Y \neq \emptyset$. This yields

(2.4) $$D_{X' \setminus Y'} Q_Y = 0, \qquad Y' \neq Y.$$

Hence the set $\{Q_Y\colon\ |Y| = s-1,\ Y \subseteq X'\}$ spans all homogeneous polynomials of degree $n-s$. Therefore appending this set to any basis for $D_s(X')$ we obtain by the induction assumption

$$\binom{n-1}{s-1} + \binom{n-1}{s} = \binom{n}{s}$$

linearly independent elements of degree $\leqslant n-s$ in $D_s(X)$. This shows inductively that $D_s(X) = \Pi_{n-s}$.

Such a simple construction does not work in general. However, the above remarks provide a guide for the proof of Theorem 2.1 which follows.

To this end, we find it useful to record some elementary properties of the set $\mathcal{Y}_s(X)$.

LEMMA 2.1. *Let $X$ be a finite subset of $\boldsymbol{R}^s\setminus\{0\}$ such that $\langle X\rangle = \boldsymbol{R}^s$. Then*

(a) *$V\in\mathcal{Y}_s(X)$ if and only if $V = X\setminus\langle Y\rangle$ for some $Y\in\mathcal{B}_{s-1}(X)$.*

(b) *For any $Y, Y'\in\mathcal{B}_{s-1}(X)$ with $\langle Y\rangle\neq\langle Y'\rangle$ the set $(X\setminus\langle Y'\rangle)\cap\langle Y\rangle$ contains some element of $\mathcal{Y}_{s-1}(X\cap\langle Y\rangle)$.*

Proof. (a) quickly follows from the definitions (1.9) and (1.15). The second assertion follows from the fact that the set

$$(X\cap\langle Y\rangle)\setminus((X\setminus\langle Y'\rangle)\cap\langle Y\rangle) = X\cap\langle Y\rangle\cap\langle Y'\rangle$$

spans a set of dimension $< s-1$.

As a consequence we state

PROPOSITION 2.1. *For finite $X\subset\boldsymbol{R}^s\setminus\{0\}$ such that $\langle X\rangle = \boldsymbol{R}^s$ and for some $y\in\boldsymbol{R}^s\setminus\{0\}$ let $V\in\mathcal{Y}_s(\{y\}\cup X)$. If $y\in V$ then*

$$V = \{y\}\cup(X\setminus\langle Y\rangle) \tag{2.5}$$

*for some $Y\in\mathcal{B}_{s-1}(X)$ such that $\langle\{y\}\cup Y\rangle = \boldsymbol{R}^s$. However, if $y\notin V$ then*

$$\dim\langle(X\cap\langle Y\rangle)\setminus V\rangle < s-1 \tag{2.6}$$

*for all $Y\in\mathcal{B}_{s-1}(X)$ such that $\langle\{y\}\cup Y\rangle = \boldsymbol{R}^s$, i.e. $V$ contains an element of $\mathcal{Y}_{s-1}(X\cap\langle Y\rangle)$ for all $Y$ as above.*

Proof. By Lemma 2.1 (a) we have $V = (\{y\}\cup X)\setminus\langle Y\rangle$ for some $Y\in\mathcal{B}_{s-1}(\{y\}\cup X)$. Suppose first that $y\in V$. This implies that $y\notin\langle Y\rangle$ and so $\langle\{y\}\cup Y\rangle = \boldsymbol{R}^s$ which proves (2.5). Now suppose $y\notin V$. Then $y\in\langle Y\rangle$ which gives $\langle\{y\}\cup Y\rangle\neq\boldsymbol{R}^s$. Hence whenever $Y'\in\mathcal{B}_{s-1}(X)$ such that $\langle\{y\}\cup Y'\rangle = \boldsymbol{R}^s$ we have $\langle Y'\rangle\neq\langle Y\rangle$. We can now use Lemma 2.1 (b) to conclude that

$$V = (\{y\}\cup X)\setminus\langle Y\rangle = X\setminus\langle Y\rangle$$

contains some element of $\mathcal{Y}_{s-1}(X\cap\langle Y\rangle)$, which finishes the proof.

Our next comments concern some general properties of the classes $D_s(X)$. We will use them later in an inductive proof of Theorem 2.1.

LEMMA 2.2. *Let $W\subset\boldsymbol{R}^s\setminus\{0\}$, $y\in\boldsymbol{R}^s$, with $\dim\langle W\rangle = s-1$ and $\langle W\cup\{y\}\rangle = \boldsymbol{R}^s$. Then for any linear mapping $A$ from $\boldsymbol{R}^s$ onto $\boldsymbol{R}^{s-1}$ such that $Ay = 0$ there exist $m$ polynomials $P_1(x),\ldots,P_m(x)$, $x\in\boldsymbol{R}^s$, where $m = \dim D_{s-1}(AW)$, such that*

$$D_y P_i(x) = 0,$$

$$D_V P_i(x) = 0 \quad \text{for all } V\in\mathcal{Y}_{s-1}(W),\ i = 1,\ldots,m.$$

Proof. According to [4], $D_{s-1}(AW)$ is a finite-dimensional subspace of polynomials. Let $Q_1(u),\ldots,Q_m(u)$, $u\in\boldsymbol{R}^{s-1}$, be a polynomial basis for $D_{s-1}(AW)$. Define $P_i(x) = Q_i(Ax)$. Then $D_y P_i = 0$ because $Ay = 0$ and since $V\in\mathcal{Y}_{s-1}(W)$ if and only if $AV\in\mathcal{Y}_{s-1}(AW)$ we see that $P_i\in D_{s-1}(W)$.

We now turn to the proof of Theorem 2.1, which we do by induction on the spatial dimension $s$ and the cardinality $|X|$ of the set $X\subset\boldsymbol{R}^s\setminus\{0\}$, $\langle X\rangle = \boldsymbol{R}^s$. For $s = 1$, $X\subset\boldsymbol{R}^1\setminus\{0\}$, we have $D_1(X) = \Pi_{|X|-1}$, $|\mathcal{B}_1(X)| = |X|$ and so the assertion is valid. Furthermore, it is also true for any $s\geqslant 1$ when $|X| = n = s$ because $D_s(X) = \Pi_0(\boldsymbol{R}^s)$ and $|\mathcal{B}_s(X)| = 1$ in this case.

Now we assume that (2.1) holds for all finite sets $X\subset\boldsymbol{R}^l\setminus\{0\}$, $|X|\geqslant l$, where $1\leqslant l < s$, and for all $X\subset\boldsymbol{R}^s\setminus\{0\}$, $|X|\leqslant n$, where $n\geqslant s$. To advance the induction hypothesis, we consider some vector $y$ and fixed set $X$ such that

$$X\subset\boldsymbol{R}^s\setminus\{0\},\quad |X| = n,\quad \langle X\rangle = \boldsymbol{R}^s,\quad y\in\boldsymbol{R}^s\setminus\{0\}, \tag{2.7}$$

and proceed to prove the assertion of Theorem 2.1 for the set $\{y\}\cup X$. Since

$$D_s(X)\subset D_s(X\cup\{y\}) \tag{2.8}$$

our goal is to construct $|\mathcal{B}_s(X\cup\{y\})| - |\mathcal{B}_s(X)|$ linearly independent elements in $D_s(\{y\}\cup X)\setminus D_s(X)$ which when appended to any basis of $D_s(X)$ will span all of $D_s(X\cup\{y\})$.

The construction of such polynomials is the objective of

THEOREM 2.2. *Let $X$ and $y$ be given by (2.7). For each $Y\in\mathcal{B}_{s-1}(X)$ such that $\langle\{y\}\cup Y\rangle = \boldsymbol{R}^s$ there exist $m_Y$ polynomials $Q_{Y,j}$, $j = 1,\ldots,m_Y$, where $m_Y = |\mathcal{B}_{s-1}(X\cap\langle Y\rangle)|$, satisfying the following conditions.*

(i) *For*

$$P_{Y,j} = D_{X\setminus\langle Y\rangle}Q_{Y,j},\quad j = 1,\ldots,m_Y,$$

*the set $\{P_{Y,j}\colon j = 1,\ldots,m_Y\}$ is a linearly independent set of polynomials such that*

$$D_y P_{Y,j} = 0,\quad D_V P_{Y,j} = 0 \quad \text{for all } V\in\mathcal{Y}_{s-1}(X\cap\langle Y\rangle),\ j = 1,\ldots,m_Y. \tag{2.9}$$

*Moreover,*

(ii) *$D_V Q_{Y,j} = 0$ for all $V\in\mathcal{Y}_{s-1}(X\cap\langle Y\rangle)$, $j = 1,\ldots,m_Y$.*

(iii) *$Q_{Y,j}\in D_s(\{y\}\cup X)\setminus D_s(X)$, $j = 1,\ldots,m_Y$.*

(iv) *Let $\mathcal{R}$ be any subset of $\mathcal{B}_{s-1}(X)$ such that $Y\in\mathcal{R}$ implies $\langle\{y\}\cup Y\rangle = \boldsymbol{R}^s$ while for any two $Y, Y'\in\mathcal{R}$, $\langle Y\rangle\neq\langle Y'\rangle$. Then the collection of polynomials*

$$\{Q_{Y,j}\colon j = 1,\ldots,m_Y,\ Y\in\mathcal{R}\}$$

*is linearly independent.*

Remark. Conditions (i), (ii) generalize the properties (2.3) and (2.4) in the special case that $X\cup\{y\}$ is in general position. In this case the polynomials $Q_{Y,j}$ can be chosen to be the polynomials given by (2.2). In fact, then for all $Y \in \mathscr{B}_{s-1}(X)$ one has $Y = X \cap \langle Y \rangle$ so that $m_Y = |\mathscr{B}_{s-1}(Y)| = 1$. Therefore the $P_{Y,j}$ are constants.

Proof of Theorem 2.2. Let $A$ be any linear mapping from $\mathbf{R}^s$ onto $\mathbf{R}^{s-1}$ such that $Ay = 0$. By our induction assumption we have

$$\dim D_{s-1}\big(A(X \cap \langle Y \rangle)\big) = |\mathscr{B}_{s-1}(X \cap \langle Y \rangle)|.$$

Thus applying Lemma 2.2 to $W = X \cap \langle Y \rangle$ guarantees the existence of linearly independent polynomials $P_{Y,j}$, $j = 1, \ldots, m_Y$ satisfying (2.9).

The construction of the polynomials $Q_{Y,j}$ satisfying (i), (ii) with respect to the polynomials $P_{Y,j}$ is based on the following observation.

Lemma 2.3. *Let $\mathscr{V}$ be a collection of sets $V \subset \mathbf{R}^s$ such that $S = \langle \bigcup \{V\colon V \in \mathscr{V}\} \rangle$ has dimension $s-1$ and $u \notin S$. If $P$ is any polynomial satisfying*

$$D_V P = 0, \quad V \in \mathscr{V},$$

*then there is a polynomial $Q$ such that*

$$D_u Q = P, \quad D_V Q = 0, \quad V \in \mathscr{V}.$$

Proof. Let $A\colon \mathbf{R}^s \to \mathbf{R}^s$ be any linear mapping satisfying

$$AS = \mathbf{R}^{s-1}, \quad Au = e^s.$$

Then for $v = Ax$ we may write for some $q \in N$

$$P(x) = P(A^{-1}v) = R(v) = \sum_{l=0}^{q} v_s^l R_j(v_1, \ldots, v_{s-1})$$

where $R_j$'s are polynomials in $v_1, \ldots, v_{s-1}$. We define

$$G(v) = \sum_{l=0}^{q} \frac{1}{l+1} v_s^{l+1} R_j(v_1, \ldots, v_{s-1})$$

and let

$$Q(x) = G(Ax).$$

Then

$$D_u Q(x) = \frac{\partial}{\partial v_s} G(v) = R(v) = P(x)$$

while by the definition of $A$

$$D_V Q(x) = D_{AV} G(v) = \sum_{l=0}^{q} \frac{1}{l+1} v_s^{l+1} D_{AV} R_j(v_1, \ldots, v_{s-1}).$$



However,

$$0 = D_V P(x) = D_{AV} R(v) = \sum_{l=0}^{q} v_s^l D_{AV} R_j(v_1, \ldots, v_{s-1}), \quad V \in \mathscr{V},$$

whence we conclude

$$D_{AV} R_j(v_1, \ldots, v_{s-1}) = 0, \quad V \in \mathscr{V}.$$

This completes the proof of Lemma 2.3.

We are now in a position to construct the polynomials $Q_{Y,j}$ satisfying (i), (ii) by induction on $|X \setminus \langle Y \rangle|$.

When $X \setminus \langle Y \rangle = \{u\}$, Lemma 2.3 provides the desired polynomials by choosing $P = P_{Y,j}$ and $\mathscr{V} = \mathscr{Y}_{s-1}(X \cap \langle Y \rangle)$.

To advance the induction on $|X \setminus \langle Y \rangle|$ assume $|X \setminus \langle Y \rangle| > 1$, $u \in X \setminus \langle Y \rangle$ and let

$$X' = X \setminus \{u\}.$$

By our induction hypothesis there exist polynomials $Q'_{Y,j}$, $j = 1, \ldots m_Y$, such that

$$P_{Y,j} = D_{X' \setminus \langle Y \rangle} Q'_{Y,j} \tag{2.11}$$

and

$$D_V Q'_{Y,j} = 0, \quad V \in \mathscr{Y}_{s-1}(X' \cap \langle Y \rangle). \tag{2.12}$$

Since $X \setminus \langle Y \rangle = \{u\} \cup (X' \setminus \langle Y \rangle)$, (i), (ii) will be proved as soon as we find polynomials $Q_{Y,j}$ satisfying

$$\begin{aligned} &D_u Q_{Y,j} = Q'_{Y,j}, \\ &D_V Q_{Y,j} = 0, \quad V \in \mathscr{Y}_{s-1}(X \cap \langle Y \rangle). \end{aligned} \tag{2.13}$$

Observing that $X \cap \langle Y \rangle = X' \cap \langle Y \rangle$, so that $\mathscr{Y}_{s-1}(X \cap \langle Y \rangle) = \mathscr{Y}_{s-1}(X' \cap \langle Y \rangle)$, Lemma 2.3 provides a solution of (2.13) by choosing $P = Q'_{Y,j}$ and $\mathscr{V} = \mathscr{Y}_{s-1}(X \cap \langle Y \rangle)$.

In order to confirm that the polynomials $Q_{Y,j}$ constructed above belong to $D_s(X \cup \{y\})$ let $W \in \mathscr{Y}_s(X \cup \{y\})$. If $y \in W$ Proposition 2.1 says that

$$W = (\{y\} \cup X) \setminus \langle Y' \rangle$$

for some $Y' \in \mathscr{B}_{s-1}(X)$, $\langle \{y\} \cup Y' \rangle = \mathbf{R}^s$. If $\langle Y \rangle = \langle Y' \rangle$ we obtain by (i)

$$D_W Q_{Y,j} = D_y D_{X \setminus \langle Y \rangle} Q_{Y,j} = D_y P_{Y,j} = 0 \tag{2.14}$$

where we have used (2.9) in the last equation. If $\langle Y' \rangle \neq \langle Y \rangle$, Lemma 2.1 (b) says $W = V \cup V'$, $V \cap V' = \emptyset$, $V \in \mathscr{Y}_{s-1}(X \cap \langle Y \rangle)$, so that (ii) yields

$$D_W Q_{Y,j} = D_{V'}(D_V Q_{Y,j}) = 0.$$

If $y \notin W$ the same reasoning applies by the second part of Proposition 2.1 so that $D_W Q_{Y,j} = 0$ for all $W \in \mathscr{Y}_s(X \cup \{y\})$, i.e. $Q_{Y,j} \in D_s(X \cup \{y\})$, $j = 1, \ldots, m_Y$.

Thus in order to complete the proof of (iii) we have to make sure that

$$Q_{Y,j} \notin D_s(X). \tag{2.15}$$

But by Lemma 2.1 (a) $X \setminus \langle Y \rangle \in \mathscr{Y}_s(X)$ so that (2.15) is an immediate consequence of (i).

As for (iv) suppose

$$\sum_{Y' \in \mathscr{R}} \sum_{j=1}^{m_{Y'}} c_{Y',j} Q_{Y',j}(x) = 0, \qquad x \in \boldsymbol{R}^s,$$

holds for some constants $c_{Y',j}$. Combining Lemma 2.1 (b) and (ii) yields for any fixed $Y \in \mathscr{R}$

$$0 = D_{X \setminus \langle Y \rangle} \Big( \sum_{Y' \in \mathscr{R}} \sum_{j=1}^{m_{Y'}} c_{Y',j} Q_{Y',j}(x) \Big) = \sum_{j=1}^{m_Y} c_{Y,j} P_{Y,j}(x).$$

Since by (i) the $P_{Y,j}$, $j = 1, \ldots, m_Y$, are linearly independent we conclude $c_{Y,j} = 0$, $j = 1, \ldots, m_Y$, finishing the proof of Theorem 2.2.

We are now in a position to construct a basis for $D_s(\{y\} \cup X)$. To this end, note that the above results suggest partitioning the set

$$\mathscr{A}(X|y) = \{Y \in \mathscr{B}_{s-1}(X) \colon \langle \{y\} \cup Y \rangle = \boldsymbol{R}^s\}$$

into equivalence classes $\mathscr{E}_1, \ldots, \mathscr{E}_r$, $r = r(X)$, where $Y$, $Y'$ belong to the same class if and only if $\langle Y \rangle = \langle Y' \rangle$. Let us denote by $\mathscr{R}(X|y) = \{Y_i \colon i = 1, \ldots, r,\ Y_i \in \mathscr{E}_i\}$ any set of representatives of the classes $\mathscr{E}_i$.

THEOREM 2.3. *Let* $\mathscr{G}$ *be any basis of* $D_s(X)$. *Then*

$$\{Q_{Y,j} \colon Y \in \mathscr{R}(X|y),\ j = 1, \ldots, m_Y\} \cup \mathscr{G} \tag{2.16}$$

*is a basis for* $D_s(\{y\} \cup X)$.

Proof. From Theorem 2.2 (iii), (iv) it is clear that the elements in the set (2.16) are linearly independent and in $D_s(\{y\} \cup X)$. Thus it remains to show that they span all of $D_s(\{y\} \cup X)$. Let $Q \in D_s(\{y\} \cup X)$. Since for each $Y \in \mathscr{R}(X|y)$ $\{y\} \cup X \setminus \langle Y \rangle \in \mathscr{Y}_s(X \cup \{y\})$, by Lemma 2.1 (a) we have

$$D_y D_{X \setminus \langle Y \rangle} Q(x) = 0,$$

i.e.,

$$R_Y(x) = D_{X \setminus \langle Y \rangle} Q(x)$$

satisfies

$$D_y R_Y(x) \equiv 0. \tag{2.17}$$

Moreover, for each set $W \in \mathscr{Y}_{s-1}(X \cap \langle Y \rangle)$ we have $W \cap (X \setminus \langle Y \rangle) = \emptyset$ and

$$(X \setminus \langle Y \rangle) \cup W \in \mathscr{Y}_s(\{y\} \cup X). \tag{2.18}$$



In fact,

$$(X \setminus \langle Y \rangle) \cup W = X \setminus Z,$$

where $Z = (X \cap \langle Y \rangle) \setminus W$. Hence $\dim \langle Z \rangle = s-2$ and since $\langle \{y\} \cup Y \rangle = \boldsymbol{R}^s$ we have $\{y\} \cup Z \in \mathscr{B}_{s-1}(\{y\} \cup X)$. Thus $(X \setminus \langle Y \rangle) \cup W = (\{y\} \cup X) \setminus (\{y\} \cup Z)$ and (2.18) follows from Lemma 2.1 (a). Hence

$$0 = D_{X \setminus \langle Y \rangle} D_W Q = D_W R_Y$$

which means that for each $Y \in \mathscr{R}(X|y)$

$$R_Y \in D_{s-1}(X \cap \langle Y \rangle). \tag{2.19}$$

We next show that (2.17) and (2.19) imply that there exist coefficients $c_{Y,j}$, $j = 1, \ldots, m_Y = |\mathscr{B}_{s-1}(X \cap \langle Y \rangle)|$, such that

$$R_Y = \sum_{j=1}^{m_Y} c_{Y,j} P_{Y,j} \tag{2.20}$$

where $P_{Y,j}$ are the polynomials in Theorem 2.2 (i). To see this let again $A \colon \boldsymbol{R}^s \to \boldsymbol{R}^s$ be any linear mapping satisfying

$$A(\langle Y \rangle) = \boldsymbol{R}^{s-1}, \qquad Ay = e^s,$$

and define

$$P(u) = R_Y(x)$$

where $u = Ax$. Since $W \in \mathscr{Y}_{s-1}(X \cap \langle Y \rangle)$ iff $AW \in \mathscr{Y}_{s-1}(A(X \cap \langle Y \rangle))$ we have for such $W$ according to (2.19)

$$0 = D_W R_Y(x) = D_{AW} P(u)$$

so that as in [4] by the Hilbert Nullstellensatz $P$ is a polynomial in the first $s-1$ variables $u_1, \ldots, u_{s-1}$, i.e. for some $r$

$$P(u_1, \ldots, u_s) = \sum_{0 \le i_1, \ldots, i_{s-1} \le r} a_{i_1, \ldots, i_{s-1}}(u_s) u_1^{i_1} \ldots u_{s-1}^{i_{s-1}}.$$

By (2.17) we obtain by the definition of $A$

$$0 = D_y R_Y(x) = \frac{\partial}{\partial u_s} P(u)$$

and hence $\dfrac{\partial}{\partial u_s} a_{i_1, \ldots, i_{s-1}}(u_s) = 0$. Hence $R_Y(x)$ is a polynomial and (2.20) follows from Lemma 2.2.

Thus, by Theorem 2.2 (i), (ii) the linear combination

$$L(x) = \sum_{Y \in \mathscr{R}(X|y)} \sum_{j=1}^{m_Y} c_{Y,j} Q_{Y,j}$$

where the $c_{Y,j}$ are given by (2.20) belongs to $D_s(\{y\}\cup X)\setminus D_s(X)$ and satisfies for $Y\in\mathscr{R}(X|y)$

$$D_{X\setminus\langle Y\rangle}L(x)=\sum_{j=1}^{m_Y}c_{Y,j}P_{Y,j}(x)=R_Y(x).$$

In other words,

$$D_{X\setminus\langle Y\rangle}(L-Q)\equiv 0,\qquad Y\in\mathscr{R}(X|y). \tag{2.21}$$

Now let $V$ be any set in $\mathscr{Y}_s(X)$, then Lemma 2.1 (a) says that $V=X\setminus\langle Y\rangle$ for some $Y\in\mathscr{B}_{s-1}(X)$. If $y\notin\langle Y\rangle$ then (2.21) shows that $D_V(L-Q)=0$ while for $y\in\langle Y\rangle$ we have $V=(\{y\}\cup X)\setminus\langle Y\rangle\in\mathscr{Y}_s(X\cup\{y\})$ and so $D_V(L-Q)$ is still zero because $L, Q\in D_s(X\cup\{y\})$. Hence we conclude that

$$L-Q\in D_s(X)$$

proving $D_s(X\cup\{y\})$ is spanned by the set (2.16). This completes the proof of Theorem 2.3.

In order to finish the proof of Theorem 2.1 we merely have to count the elements in the basis (2.16). Again by our induction assumption we have

$$|\mathscr{G}|=|\mathscr{B}_s(X)|. \tag{2.22}$$

Furthermore, we clearly have

$$|\mathscr{A}(X|y)|=|\mathscr{B}_s(\{y\}\cup X)|-|\mathscr{B}_s(X)|. \tag{2.23}$$

But since by definition for each equivalence class $\mathscr{E}_i$ in $\mathscr{A}(X|y)$, $|\mathscr{E}_i|=|\mathscr{B}_{s-1}(X\cap\langle Y_i\rangle)|$, we get

$$|\mathscr{A}(X|y)|=\sum_{i=1}^{r}|\mathscr{E}_i|=\sum_{Y\in\mathscr{R}(X|y)}m_Y$$

which combined with (2.16), (2.22) and (2.23) completes the proof of Theorem 2.1.

**3. A formula for $|b(x|X)|$.** The objective of this section is to determine the number of translates $B(\cdot-\alpha|X)$ containing any given point $x\in\boldsymbol{R}^s$ in their support as a function of $x$ and $X$. For any bounded set $\Omega\subset\boldsymbol{R}^s$ we define

$$b(x|\Omega)=\{\alpha\in Z^s:\ x\in\Omega+\alpha\} \tag{3.1}$$

while for $\Omega=\operatorname{supp}B(\cdot|X)$ we use, as in Section 1, the notation

$$b(x|X)=\{\alpha\in Z^s:\ x\in(\operatorname{supp}B(\cdot|X))+\alpha\}.$$

Our main result is then

THEOREM 3.1. *Let $X$ be a finite subset of $Z^s\setminus\{0\}$. Then*

$$|b(x|X)|=\sum_{Y\in\mathscr{B}_s(X)}|\det Y|$$



*holds for all $x$ not in $c(X)$, the cut regions for all translates of $B(\cdot|X)$ over $Z^s$.*

As a consequence of Theorems 2.1 and 3.1 as well as of some results of [4] we obtain the equivalence of local and global independence of the translates $B(x-\alpha|X)$, $\alpha\in Z^s$.

THEOREM 3.2. *Let $X$ be a finite subset of $Z^s\setminus\{0\}$. The following statements are equivalent:*

(i) $|\det Y|=1$ *for all* $Y\in\mathscr{B}_s(X)$.

(ii) *The translates $B(\cdot-\alpha|X)$ are locally linearly independent, i.e. for any domain $\Omega\subset\boldsymbol{R}^s$*

$$\sum_{\alpha\in Z^s}a_\alpha B(x-\alpha|X)=0,\qquad x\in\Omega,$$

*implies $a_\alpha=0$ if $(\operatorname{supp}B(\cdot-\alpha|X))\cap\Omega\neq\emptyset$.*

(iii) *The translates $B(\cdot-\alpha|X)$ are globally linearly independent, i.e.*

$$\sum_{\alpha\in Z^s}a_\alpha B(x-\alpha|X)=0,\qquad x\in\boldsymbol{R}^s,$$

*implies $a_\alpha=0$, $\alpha\in Z^s$.*

We start the proof of these theorems with some general observations. Referring back to definition (3.1) we see that in any neighborhood $N$ of $\boldsymbol{R}^s$ not intersected by any of the translates $\partial\Omega+\alpha$, $\alpha\in Z^s$, $|b(x|\Omega)|$ is constant and $N$ is covered by exactly $|b(x|\Omega)|$ translates $\Omega+\alpha$, $\alpha\in Z^s$.

PROPOSITION 3.1. *Let $\Omega$ be a bounded measurable subset of $\boldsymbol{R}^s$ such that $\partial\Omega$ has measure zero.*

(i) *If $|b(x|\Omega)|$ is constant for all $x\notin\bigcup_{\alpha\in Z^s}(\partial\Omega+\alpha)$, then*

$$|b(x|\Omega)|=\operatorname{vol}_s(\Omega)\quad \textit{for all } x\notin\bigcup_{\alpha\in Z^s}(\partial\Omega+\alpha).$$

(ii) *Moreover, if $|b(x|\Omega)|$ is constant for all $x\in\boldsymbol{R}^s$, then*

$$\operatorname{vol}_s(\Omega)=|Z^s\cap\Omega|.$$

Proof. Since $\Omega$ is bounded and measurable, the function

$$f=\sum_{\alpha\in Z^s}\chi_{\Omega+\alpha}$$

is measurable. In fact, $f(x)=|b(x|\Omega)|=b$, $x\notin\bigcup_{\alpha\in Z^s}(\partial\Omega+\alpha)$. Since $\partial\Omega$ has measure zero we obtain

$$\int_{[0,N]^s}f(x)\,dx=bN^s. \tag{3.2}$$

On the other hand, since $\Omega$ is bounded we get $|\{\alpha\in Z^s:\ (\Omega+\alpha)\cap[0,N]^s\neq\emptyset\}|=N^s+O(N^{s-1})$ whence we conclude

$$\int_{[0,N]^s}f(x)\,dx=\operatorname{vol}_s(\Omega)N^s+O(N^{s-1}). \tag{3.3}$$

Comparing (3.2) and (3.3) and sending $N$ to infinity proves the first part of the assertion. As for rest of the assertion, note that there are exactly $|b(0|\Omega)| = b$ lattice points $\alpha^1, \ldots, \alpha^b$ such that

$$0 \in \Omega + \alpha^i, \quad i = 1, \ldots, b.$$

Hence

$$-\alpha^i \in \Omega, \quad i = 1, \ldots, |b(0|\Omega)| = |b(x|\Omega)|,$$

which completes the proof.

Corollary 3.1. *Let $\Omega \subset \mathbf{R}^s$ be some bounded measurable subset of $\mathbf{R}^s$ such that $\partial\Omega$ has measure zero. Suppose there exist subsets $Z_i \subset Z^s$, $i = 1, \ldots, l$, such that*

$$Z_i \cap Z_j = \emptyset, \quad i \neq j, \quad \bigcup_{i=1}^{l} Z_i = Z^s, \tag{3.4}$$

*and for each $i = 1, \ldots, l$, the collection*

$$\{\Omega + \alpha \colon \alpha \in Z_i\}$$

*forms a tessellation of $\mathbf{R}^s$, i.e.*

$$\begin{aligned} &(\Omega+\alpha) \cap (\Omega+\beta) = \emptyset, \quad \alpha \neq \beta, \quad \alpha, \beta \in Z_i, \\ &\bigcup_{\alpha \in Z_i} (\Omega+\alpha) = \mathbf{R}^s. \end{aligned} \tag{3.5}$$

*Then*

$$l = |b(x|\Omega)| = \operatorname{vol}_s(\Omega) = |Z^s \cap \Omega|.$$

Proof. Only the first relation needs comment. But since for each $i$ any $x \in \mathbf{R}^s$ belongs by (3.5) to exactly one translate $\Omega + \alpha$, $\alpha \in Z_i$, the claim is obvious.

As an application of this observation we obtain the following fact about parallelepipeds.

Corollary 3.2. *Let $X \subset Z^s$, $|X| = s$, $\langle X \rangle = \mathbf{R}^s$. Then for any set $P$ of the form*

$$P = \Big\{\sum_{i=1}^{s} t_i x^i \colon 0 \leqslant t_i \leqslant 1,\ t_i \neq \varepsilon_i,\ i = 1, \ldots, s\Big\}$$

*where $\varepsilon_i \in \{0, 1\}$, $i = 1, \ldots, s$, one has*

$$|b(x|P)| = |\det X| = |P \cap Z^s|.$$

Proof. Let $P \cap Z^s = \{\alpha^1, \ldots, \alpha^l\}$. Then the sets $Z_i = \{\alpha^i + X\beta \colon \beta \in Z^s\}$, where we have again identified $X$ with the matrix whose columns are the elements in $X$, satisfy (3.4). Since the sets $\{P + \alpha, \alpha \in Z_i\}$, $i = 1, \ldots, l$, form a tessellation of $\mathbf{R}^s$, the assertion follows from Corollary 3.1.



For later use we observe below that Corollary 3.2 extends to regions which can be partitioned by parallelepipeds. Let us use the abbreviated notation $\operatorname{supp} B(\cdot|X) = B(X)$ for any $X \subset \mathbf{R}^s \setminus \{0\}$.

Corollary 3.3. *Let $X \subset Z^s \setminus \{0\}$ and choose subsets $Y_i \subset X$, $Y_i \in \mathscr{B}_s(X)$, $i = 1, \ldots, l$, and $\beta^i \in Z^s$ such that*

$$\operatorname{vol}_s\big((B(Y_i)+\beta^i) \cap (B(Y_j)+\beta^j)\big) = 0, \quad i \neq j. \tag{3.6}$$

*Then for $\Omega = \bigcup_{j=1}^{l} (B(Y_j)+\beta^j)$ one has for $x \notin c(X)$*

$$|b(x|\Omega)| = \sum_{j=1}^{l} |\det Y_j|.$$

Proof. Note that by (3.6) for $x \notin c(X)$

$$|b(x|\Omega)| = \sum_{j=1}^{l} \big|b(x|B(Y_j))\big|$$

whence the assertion follows from Corollaries 3.1 and 3.2.

Remark. Corollary 3.3 allows us to point out that $|b(x|\Omega)| = \text{const}$ for all $x \in \mathbf{R}^s$ does not imply that the hypothesis of Corollary 3.1 is satisfied. For instance, for $e^1 = (1, 0)$, $e^2 = (0, 1)$ and $X = \{e^1, e^2, e^1+e^2, e^2-e^1\} \subset Z^2$, $B(X)$ is a closed convex polygon. We remove from $B(X)$ a part of its boundary so that the resulting figure $\Omega$ is a disjoint union of parallelepipeds of the type appearing in Corollary 3.2. Therefore $|b(x|\Omega)|$ is everywhere constant even though $\Omega$ does not satisfy the hypothesis of Corollary 3.1. This example indicates that the key to proving Theorem 3.1 is the decomposition of $B(X)$ into parallelepipeds.

Theorem 3.3. *Let $X$ be a finite subset of $\mathbf{R}^s \setminus \{0\}$, $\langle X \rangle = \mathbf{R}^s$. Then for each $Y \in \mathscr{B}_s(X)$ there exists $\beta_Y \in \mathbf{R}^s$ such that for $Y, Y' \in \mathscr{B}_s(X)$*

$$\operatorname{vol}_s\big((B(Y)+\beta_Y) \cap (B(Y')+\beta_{Y'})\big) = 0, \quad Y \neq Y', \tag{3.7}$$

*while*

$$\bigcup_{Y \in \mathscr{B}_s(X)} (B(Y)+\beta_Y) = B(X). \tag{3.8}$$

*Moreover, each $\beta_Y$ has the form*

$$\beta_Y = \sum_{z \in X \setminus Y} c_z z$$

*where $c_z \in \{0, 1\}$.*

Proof. We proceed by induction on $|X|$. When $|X| = s$ there is nothing to prove. Suppose the assertion holds for any $X$, $s \leqslant |X| \leqslant n$, with $\langle X \rangle = \mathbf{R}^s$.

We will now prove it holds for the set $X_y = X \cup \{y\}$. For this purpose, we define

$$\Gamma = \{x \in B(X)\colon\ x+ty \notin \operatorname{int} B(X) \quad \text{for all } t > 0\}.$$

Then $\Gamma$ is a closed subset of the boundary of $B(X)$. Furthermore, any closed line segment in $B(X)$ containing a point of $\Gamma$ in its (relative) interior lies in $\Gamma$. Therefore $\Gamma$ is partitioned by some collection of $(s-1)$-faces of $B(X)$. By our induction hypothesis there is a collection of parallelepipeds

$$\mathscr{P}(X) = \{B(Y)+\beta_Y\colon\ Y \in \mathscr{B}_s(X)\}$$

satisfying (3.7) and (3.8) with respect to $X$. In particular, the $(s-1)$-faces of some of the parallelepipeds in $\mathscr{P}(X)$ being $(s-1)$-parallelepipeds must induce a partition of $\Gamma$. Let $\mathscr{G}$ denote this partition. We now construct a partition of $B(X_y)$ by appending to $\mathscr{P}(X)$ the following set of parallelepipeds $\mathscr{A}$. Each element of $\mathscr{G}$ has the form

$$B(V)+\beta_V$$

where $\beta_V \in \Gamma$ and $V \in \mathscr{B}_{s-1}(X)$ satisfies

$$\langle \{y\} \cup V \rangle = \mathbf{R}^s. \tag{3.9}$$

In addition, from the decomposition (3.8) it follows that

$$\beta_V = \sum_{u \in X\setminus V} c_u u, \quad c_u \in \{0, 1\}. \tag{3.10}$$

The corresponding parallelepipeds in $\mathscr{A}$ are then obtained by forming the sets

$$B(V \cup \{y\})+\beta_V = \{x\colon\ x = u+ty,\ u \in B(V)+\beta_V,\ 0 \leqslant t \leqslant 1\}.$$

Clearly, the set $\Omega = \bigcup \{P\colon\ P \in \mathscr{A} \cup \mathscr{P}(X)\}$ is contained in $B(X_y)$. In order to show that $B(X_y) \subseteq \Omega$ it is, in view of our induction hypothesis, sufficient to show that any $x \in B(X_y)\setminus B(X)$ is also in $\Omega$. Such an $x$ has the form $x = u+ty$, $u \in B(X)$, $0 \leqslant t \leqslant 1$. Let $L$ denote the line segment connecting $x$ and $u$. $L$ must intersect the boundary of $B(X)$ since otherwise $x \in B(X)$. Let $\{v\} = L \cap \partial B(X)$. By definition of $\Gamma$ we must have $v \in \Gamma$ and $x = v+t_0 y$ where $0 < t_0 \leqslant 1$. By definition, $x$ lies therefore in some element of $\mathscr{A}$, showing that $x \in \bigcup \{P\colon\ P \in \mathscr{A}\}$. This proves

$$\Omega = B(X_y).$$

The assertion follows now as in Section 2 by also observing that

$$\mathscr{B}_s(X_y) = \mathscr{B}_s(X) \cup \{\{y\} \cup V\colon\ V \in \mathscr{B}_{s-1}(X),\ \langle \{y\} \cup V \rangle = \mathbf{R}^s\}.$$

As an immediate consequence we have

COROLLARY 3.4. *Let $X \subset \mathbf{R}^s\setminus\{0\}$ be finite such that $\langle X \rangle = \mathbf{R}^s$. Then*

$$\operatorname{vol}_s(B(X)) = \sum_{Y \in \mathscr{B}_s(X)} |\det Y|.$$



We are now ready to complete the proofs of Theorems 3.1 and 3.2. Theorem 3.1 readily follows from Theorem 3.3 and Corollary 3.3.

Proof of Theorem 3.2. The equivalence of (i) and (iii) is known [4, 9], and (iii) trivially follows from (ii). To prove that (i) implies (ii) we recall that $D_s(X) = \operatorname{span}\{B(x-\alpha|X)\colon\ x \in \Omega,\ \alpha \in Z^s\}$ for any domain $\Omega$ such that $\Omega \cap c(X) = \emptyset$. Since by Theorem 2.1, Theorem 3.1 and (i)

$$\dim D_s(X) = |b(x|X)|, \quad x \in \Omega,$$

the functions $B(\cdot-\alpha|X)$, $\alpha \in b(x|X)$, $x \in \Omega$, are linearly independent on $\Omega$.

**4. Interpolation from $D_s(X)$.** Let us point out next an interesting relationship between the local linear independence of translates of box splines and the following interpolation property of $D_s(X)$.

THEOREM 4.1. *Let $X \subset Z^s\setminus\{0\}$ be a finite set such that $\langle X \rangle = \mathbf{R}^s$. Then for any $x \in \mathbf{R}^s\setminus c(X)$ the set $b(x|X)$ is unisolvent for $D_s(X)$, i.e. given any data $\{f_\alpha\colon\ \alpha \in b(x|X)\}$, there exists a unique polynomial $P$ in $D_s(X)$ such that*

$$P(\alpha) = f_\alpha, \quad \alpha \in b(x|X), \tag{4.1}$$

*if and only if*

$$|\det Y| = 1 \quad \textit{for all } Y \in \mathscr{B}_s(X). \tag{4.2}$$

Proof. If (4.2) is not satisfied, Theorems 2.1, 3.1 imply

$$|b(x|X)| > \dim D_s(X)$$

proving the necessity of (4.2).

Conversely, if (4.2) holds Theorems 2.1, 3.1 yield $\dim D_s(X) = |b(x|X)|$ so that it remains to show that the only element in $D_s(X)$ interpolating the zero data must vanish identically. Let $\Omega$ be some neighborhood in $\mathbf{R}^s$ on which all translates of $B(\cdot|X)$ are polynomials, i.e. $\Omega \cap c(X) = \emptyset$. Thus $b(y|X) = \{\alpha^1, \ldots, \alpha^b\}$ for all $y \in \Omega$. Assume now that $P \in D_s(X)$ satisfies

$$P(\alpha^i) = 0, \quad i = 1, \ldots, b.$$

By (1.11) we observe that the function

$$Q(x) = (AP)(x) = \sum_{\alpha \in Z^s} P(\alpha) B(x-\alpha|X)$$

is in $D_s(X)$. However, for any $y \in \Omega$ we get

$$Q(y) = \sum_{\alpha \in Z^s} P(\alpha) B(y-\alpha|X)|_\Omega = \sum_{\alpha \in b(y|X)} P(\alpha) B(y-\alpha|X) = 0.$$

Hence $Q = 0$ and therefore (1.11) or Theorem 3.2 yield $P(\alpha) = 0$, *for all* $\alpha \in Z^s$. Thus $P = 0$ which completes the proof of Theorem 4.1.

**5. A linear projector onto $\mathscr{S}(X)$.** In this section we shall apply the previous results to construct linear projectors (quasi-interpolants) from $L_p(\mathbf{R}^s)$

onto $\mathscr{S}(X)$. Furthermore, combining this with Theorem 4.1 leads to the construction of linear optimal order approximation schemes which only require function values at the lattice points.

Choose any point $z \notin c(X)$ and set $\varphi(x) = B(x+z|X)$. Since $\hat{\varphi}(0) = 1$ the expansion

$$(\hat{\varphi}(x))^{-1} = \sum_{\alpha \in Z_+^s} a_\alpha x^\alpha$$

is defined in some neighborhood of zero. For sufficiently smooth functions $f$ we introduce

$$(Lf)(x) = \sum_{|\alpha| \leqslant n-s} a_\alpha (-i)^{|\alpha|} D^\alpha f(x) = \big(\tilde{P}(-iD) f\big)(x).$$

Since $D_s(X) \subset \Pi_{n-s}$ we know in particular that

$$q(D)\big(\tilde{P}(x)\hat{\varphi}(x)\big)|_{x=0} = 0, \quad q \in D_s(X) \setminus \Pi_0 \tag{5.1}$$

and

$$\tilde{P}(0)\hat{\varphi}(0) = 1. \tag{5.2}$$

Also we have shown in [4] that the mapping

$$(Pf)(x) = \sum_{\alpha \in Z^s} (Lf)(\alpha)\varphi(x-\alpha) \tag{5.3}$$

reproduces $D_s(X)$, i.e. $Pq = q$ for all $q \in D_s(X)$. Note that the function

$$F(x) = (L\varphi)(x) = L\big(B(\cdot|X)\big)(x+z) \tag{5.4}$$

is a piecewise polynomial defined off the cut regions of $B(\cdot+z|X)$. In particular, for our choice of $z$, $F$ is well defined on $Z^s$. Thus the mapping

$$(Qf)(x) = \sum_{\alpha \in Z^s} (Lf)(\alpha+z) B(x-\alpha|X) \tag{5.5}$$

is defined on $\mathscr{S}(X)$ and by the usual procedure of using the Hahn-Banach Theorem we can extend $Q$ to all of $L_p(\boldsymbol{R}^s)$ for any $p$, $1 \leqslant p \leqslant \infty$ (cf. [4]).

THEOREM 5.1. *Let $X$ satisfy* (4.2). *Then $Q$ is a linear projector from $L_p(\boldsymbol{R}^s)$ onto $\mathscr{S}(X)$.*

We use the following lemma for the proof of this result.

LEMMA 5.1. *The mapping*

$$(Gf)(x) = \sum_{\alpha \in Z^s} f(\alpha) F(x-\alpha)$$

*reproduces $D_s(X)$.*

Proof. Using the Poisson summation formula, we have for any polynomial $q \in D_s(X)$

$$(Gq)(x) = \sum_{\alpha \in Z^s} e^{2\pi i \alpha \cdot x} \big(q(-iD+x)\hat{F}\big)(2\pi\alpha). \tag{5.6}$$



To simplify this sum we recall that in [4] we showed

$$\big(q(D)\hat{B}(x|X)\big)(2\pi\alpha) = 0, \quad \alpha \in Z^s \setminus \{0\}, \tag{5.7}$$

for all $q \in D_s(X)$. Obviously, whenever $q \in D_s(X)$ the polynomial $r(x) = q(cx+y)$ is for any $c \in \boldsymbol{C}$, $y \in \boldsymbol{C}^s$ also in $D_s(X)$. Thus (5.7) implies that

$$q(D)\big(e^{x\cdot y}\hat{B}(x|X)\big)(2\pi\alpha) = 0$$

for all $\alpha \in Z^s \setminus \{0\}$, $q \in D_s(X)$ and $y \in \boldsymbol{R}^s$. Hence it follows that also

$$q(D)\big(v(x)e^{ix\cdot y}\hat{B}(x|X)\big)(2\pi\alpha) = 0, \quad \alpha \in Z^s \setminus \{0\},$$

for any polynomial $v$. Since

$$\hat{F}(x) = \tilde{P}(x)e^{ix\cdot z}\hat{B}(x|X) \tag{5.8}$$

we see that (5.6) simplifies to

$$(Gq)(x) = \big(q(-iD+x)\hat{F}\big)(0)$$

which in view of (5.1), (5.2) and (5.7) gives $Gq = q$.

Proof of Theorem 5.1. Let $\Omega$ be *any* region in supp $\varphi$ on which all the translates of $\varphi$ are polynomials. On $\Omega$, as pointed out before, $b(x|\text{supp } \varphi) = b(\Omega)$ is constant and clearly $0 \in b(\Omega)$. Furthermore, for $x \in \Omega$, $b(\Omega) = b(x+z|X)$, so that according to Theorem 4.1, there is a unique polynomial $p_\Omega \in D_s(X)$ such that

$$p_\Omega(\alpha) = \delta_{0\alpha}, \quad \alpha \in b(\Omega).$$

Hence, by Lemma 5.1, we have for $x \in \Omega$

$$p_\Omega(x) = (Gp_\Omega)(x) = \sum_{\alpha \in Z^s} p_\Omega(\alpha) F(x-\alpha) = \sum_{\alpha \in b(\Omega)} p_\Omega(\alpha) F(x-\alpha) = F(x),$$

and so $F(\alpha) = \delta_{0\alpha}$, $\alpha \in b(\Omega)$. Since $F(\alpha) = 0$ for $\alpha \notin \text{supp } \varphi$ we get $F(\alpha) = \delta_{0\alpha}$ for all $\alpha \in Z^s$. Therefore $Q$ is a projector on $\mathscr{S}(X)$ as asserted.

In order to pass from the projector $Q$ to an approximation scheme involving only function values instead of derivatives we will determine a corresponding alternative representation of $(Lf(\cdot))(z)$ as a functional on $D_s(X)$. To this end, suppose again that (4.2) holds and choose as above $z \notin c(X)$ in some neighborhood of zero. Again by Theorem 4.1 there are unique polynomials $p_\alpha \in D_s(X)$, $\alpha \in b(z|X)$ satisfying

$$p_\alpha(\beta) = \delta_{\alpha\beta}, \quad \alpha, \beta \in b(z|X),$$

so that $q(x) = \sum_{\beta \in b(z|X)} q(\beta) p_\beta(x)$ for $q \in D_s(X)$. Thus

$$\lambda f = \sum_{\beta \in b(z|X)} f(\beta)(Lp_\beta)(z)$$

satisfies

$$\lambda q = (Lq)(z), \qquad q \in D_s(X).$$

Therefore the operator

$$(\tilde{Q}f)(x) = \sum_{\alpha \in Z^s} \lambda f(\cdot + \alpha) B(x - \alpha | X)$$

still reproduces $D_s(X)$ although $\tilde{Q}$ will in general no longer be a projector on $\mathcal{S}(X)$.

It is now a matter of routine (cf. [4]) to show that the operators

$$Q_h f = \sum_{\alpha \in Z^s} (Lf)(h(\alpha + z)) B\left(\frac{\cdot}{n} - \alpha | X\right)$$

and

$$\tilde{Q}_h f = \sum_{\alpha \in Z^s} \lambda f(h(\cdot + \alpha)) B\left(\frac{\cdot}{h} - \alpha | X\right)$$

realize the optimal approximation rate $O(h^{d(X)+1})$ with respect to any $L_p$-norm.

**References**

[1] C. de Boor and R. DeVore, *Approximation by smooth multivariate splines*, Trans. Amer. Math. Soc. 276 (1983), 775–785.
[2] C. de Boor and K. Höllig, *B-splines from parallelepipeds*, J. Analyse Math. 42 (1982/83), 99–115.
[3] —, —, *Bivariate box splines and pp. functions on a three direction mesh*, J. Comput. Appl. Math. 9 (1983), 13–28.
[4] W. Dahmen and C. A. Micchelli, *Translates of multivariate splines*, Linear Algebra Appl. 52/53 (1983), 217–234.
[5] —, —, *On the optimal approximation rates for criss-cross finite element spaces*, IBM RC 10 168, 1983, J. Comput. Appl. Math. 10 (1984), 255–273.
[6] —, —, *Recent progress in multivariate splines*, in: Approximation Theory IV, ed. C. K. Chui, L. L. Schumaker and J. Ward, Academic Press, New York 1983, 27–121.
[7] G. J. Fix and G. Strang, *A Fourier analysis of the finite element variational method*, in: C. I. M. E., II Ciclo, Erice 1971, Constructive Aspects of Functional Analysis, ed. G. Geymonat, Roma 1973, 793–840.
[8] P. O. Frederickson, *Quasi-interpolation, extrapolation and approximation in the plane*, in: Proc. Manitoba Conference Numerical Math., Winnipeg 1971, 159–167.
[9] Rong-Qing Jia, *Linear independence of translates of a box spline*, J. Approx. Theory 40 (1984), 158–160.



[10] I. J. Schoenberg, *Contributions to the problem of approximation of equidistant data by analytic functions*, Quart. Appl. Math. 4 (1946), 45–99, 112–141.
[11] P. Zwart, *Multivariate splines with nondegenerate partitions*, SIAM J. Numer. Anal. 10 (1979), 665–675.

UNIVERSITÄT BIELEFELD
Universitätsstrasse
4800 Bielefeld 1, West Germany

and

IBM T.J. WATSON RESEARCH CENTER
Yorktown Heights, New York 10598, U.S.A.

*Received March 27, 1984* (1967)
*Revised version July 24, 1984*