STUDIA MATHEMATICA, T. LXXXII. (1985)

The sequence entropy for Morse shifts and some counterexamples

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Abstract. We develop methods of computing sequence entropies (topological and measure-theoretic) for Morse shifts. A number of counterexamples are given.

(1) The sequence entropy $h_A^{\text{top}}(T)$ need not be monotonical in A.

(2) The formulas $h_A(T^k) = kh_A(T)$, $h_A^{\text{top}}(T^k) = kh_A^{\text{top}}(T)$, $k \geq 2$, are false in general (a solution to Saleski's question [17]).

(3) The formula $h_A^{\text{top}}(T \times S) = h_A^{\text{top}}(T) + h_A^{\text{top}}(S)$ is not true in general (a solution to Goodman's question [3]).

(4) The formula $h_A(T \times S) = h_A(T) + h_A(S)$ need not hold (contrary to what is obtained in Kushnirenko [9]).

Topological sequence entropy can distinguish between two strictly ergodic and measure-theoretically isomorphic dynamical systems.

It is studied when sequence entropy is an invariant of spectral isomorphism in the class of Morse shifts.

Introduction. A new metric invariant called the sequence entropy (or A-entropy) was introduced in [9]. Kushnirenko proved that T has discrete spectrum if and only if $h_A(T) = 0$ for every sequence A. He used the sequence entropy to distinguish between two spectrally isomorphic and zero entropy transformations.

From [8] it follows that if $h(T) > 0$ then $h_A(T) = K(A)h(T)$, where $K(A)$ does not depend on T. That is why the sequence entropy is uninteresting as a new metric invariant in the class of automorphisms with positive entropy.

Dekking in [2] considered the $\{l^n\}_{n=0}^{\infty}$-entropy for continuous substitutions. It is easy to see that the continuous substitution $0 \mapsto b$ in normal form (i.e. when the block b starts with zero) is just the Morse sequence $x = b \times b \times \ldots$

Many authors ([2], [3], [9], [16]) were interested in the following problem. Which of the well-known properties of entropy are valid for sequence entropy?

In the present paper we give a list of properties of entropy that do not hold for sequence entropy. The required examples arise from the class of Morse shifts. The way of computing the sequence entropy for the sequence $\{n_t\}_{t=0}^{\infty}$ is an easy adaptation of Dekking's considerations in [2]. However, calculating the $\{n_t\}_{t=0}^{\infty}$-entropy for Morse shifts is important for two reasons. First, the class of Morse shifts is larger than the class of continuous substitutions, and we get automorphisms with $\{n_t\}_{t=0}^{\infty}$-entropy equal to zero. Secondly, we get a possibility of controlling the size of the $\{n_{t_k}\}_{k=0}^{\infty}$-entropy for any subsequence $\{n_{t_k}\}_{k=0}^{\infty}$ of $\{n_t\}_{t=0}^{\infty}$.

In the first section we recall some properties of Morse shifts needed in the sequel.

In the second section we provide formulas for the $\{n_t\}_{t=0}^{\infty}$-entropy and $\{n_t\}_{t=0}^{\infty}$-topological entropy.

In the third section we give a condition equivalent to the $\{n_t\}_{t=0}^{\infty}$-entropy being zero. For a given Morse sequence $x = b^0 \times b^1 \times \ldots$ we mean by $M(x)$ the class of all Morse sequences $y = \beta^0 \times \beta^1 \times \ldots$ with $|\beta^i| = |b^i|$, $i \geqslant 0$. The $\{n_t\}_{t=0}^{\infty}$-entropy gives a natural equivalence relation $\mathrm{SE}(x)$ on $M(x)$ such that $y \in \mathrm{SE}(x)$ provided $h_{\{n_{t_k}\}}(y) = h_{\{n_{t_k}\}}(x)$ for every subsequence $\{n_{t_k}\}_{k=0}^{\infty}$ of $\{n_t\}_{t=0}^{\infty}$. We compare this relation with the relation on $M(x)$ introduced in [10] and show that usually $\mathrm{SE}(x)$ consists of a continuum of spectrally nonisomorphic Morse sequences.

In the fourth section we present a list of examples.

We show that the formulas $h_A(T^k) = kh_A(T)$, $h_A^{\mathrm{top}}(T^k) = kh_A^{\mathrm{top}}(T)$, $k \geqslant 2$, need not hold. Moreover, the examples presented give a negative answer to Saleski's question in [17]. In [2] Dekking showed that the A-entropy does not depend monotonically on A, i.e. it is possible to have $h_A(T) < h_B(T)$ when A is a subsequence of B. We give an analogous example for topological entropy. Goodman in [3] showed that for every sequence A and for every homeomorphism τ of a Lebesgue space X (X has finite dimension)

$$h_A^{\mathrm{top}}(\tau) \geqslant \sup_{\mu \in M} h_A^{\mu}(\tau).$$

Here M denotes the collection of all τ-invariant Borel probability measures. Equality holds in the case $h^{\mathrm{top}}(\tau) > 0$. Goodman gave an example with $h_A^{\mathrm{top}}(\tau) = \log 2$ and $\sup_{\mu \in M} h_A^{\mu}(\tau) = 0$, where τ had discrete spectrum. We prove that this is possible even for τ having a continuous part in its spectrum. Since the variational principle need not be valid we ask the following question: Does the equality $h_A(T) = h_A(\tau)$ imply $h_A^{\mathrm{top}}(T) = h_A^{\mathrm{top}}(\tau)$ and vice versa? Here, by (X, T, μ), (Y, τ, ν) we mean strictly ergodic systems. It turns out that in both cases the answer is negative. It can even happen that topological sequence entropy can distinguish between two strictly ergodic and measure-theoretically isomorphic dynamical systems. For topological sequence entropy we have $h_A^{\mathrm{top}}(T \times \tau) \leqslant h_A^{\mathrm{top}}(T) + h_A^{\mathrm{top}}(\tau)$. Goodman in [3] showed $h_A^{\mathrm{top}}(T \times T) = 2h_A^{\mathrm{top}}(T)$ and he asked whether $h_A^{\mathrm{top}}(T \times \tau) = h_A^{\mathrm{top}}(T) + h_A^{\mathrm{top}}(\tau)$. We provide an example for which the above formula fails. In Example 6 we compute the $\{2^t\}_{t=0}^{\infty}$-entropy for Kakutani sequences (see [5]). We can reduce this problem to considering some continuous transformation on the unit interval and its time averages. In particular, for almost all Kakutani sequences this entropy is constant and the supremum of values of the $\{2^t\}_{t=0}^{\infty}$-entropy is obtained for the Morse sequence $x_0 = 01 \times 01 \times \ldots$ In the above class of sequences we find (Example 7) two automorphisms T and τ for which the formula $h_A(T \times \tau) = h_A(T) + h_A(\tau)$ is false. This shows that Lemma 4 in [9] is not valid. The counterexamples in Examples 5 and 7 work due to the simple fact that the sequence entropies calculated in them are upper limits, and not limits, of suitable sequences. We also consider Abramov's formula for the skew product ([1]) and show that a generalization to sequence entropy is impossible. Finally, we prove that no Morse automorphisms have weakly mixing factors.

1. Notations. Let (X, B, μ) be a compact metric space with a Borel measure μ and let $T\colon X \to X$ be a homeomorphism preserving μ.

Let $A = \{n_t\}_{t=0}^{\infty}$ be an infinite sequence of natural numbers.

Denote by $\mathscr{Z}$ the collection of measurable partitions of X with finite entropy and by $\mathscr{C}$ the collection of all open covers of X.

The *sequence entropy* (or *A-entropy*) of T with respect to A is defined by

$$h_A(T, \xi) = \limsup_{t \in N} \frac{1}{t} H(T^{-n_0}\xi \vee \ldots \vee T^{-n_{t-1}}\xi), \quad \xi \in \mathscr{Z},$$

$$h_A(T) = \sup_{\xi \in \mathscr{Z}} h_A(T, \xi).$$

Let $\alpha \in \mathscr{C}$ and denote by $N(\alpha)$ the minimal cardinality of any subcover of α. Then the *topological sequence entropy* of T with respect to A is given by

$$h_A^{\mathrm{top}}(T, \alpha) = \limsup_{t \in N} \frac{1}{t} \log N(T^{-n_0}\alpha \vee \ldots \vee T^{-n_{t-1}}\alpha), \quad \alpha \in \mathscr{C},$$

$$h_A^{\mathrm{top}}(T) = \sup_{\alpha \in \mathscr{C}} h_A^{\mathrm{top}}(T, \alpha).$$

It is well known that if $\{\eta_k\}_{k=0}^{\infty}$ is a sequence of open partitions of X and $\eta_k \nearrow \varepsilon$ then

$$(1) \qquad h_A^{\mathrm{top}}(T) = \lim_{k \to \infty} h_A^{\mathrm{top}}(T, \eta_k), \qquad h_A(T) = \lim_{k \to \infty} h_A(T, \eta_k).$$

Let ξ be a partition of X. Denote

$$\xi_k = \xi \vee T^{-1}\xi \vee \ldots \vee T^{-k+1}\xi$$

$$\xi^k = T^{-n_0}\xi \vee \ldots \vee T^{-n_{k-1}}\xi$$

$$\xi_k^m = (\xi_k)^m, \quad k, m \geqslant 1.$$

A sequence $B = (b_0, \ldots, b_{k-1})$ of zeros and unities is called a *block*. We put $|B| = k$ and call it the *length* of B. Denote $B[i, j] = (b_i, \ldots, b_j)$, $B[i, i] = B[i]$, $\tilde{B} = (\tilde{b}_0, \ldots, \tilde{b}_{k-1})$, where $\tilde{b}_i = 1 - b_i$, $i = 0, \ldots, k-1$. If $C = (c_0, \ldots, c_{m-1})$ is another block then we define

$$B \times C = B^{(c_0)} B^{(c_1)} \ldots B^{(c_{m-1})}$$

where $B^{(0)} = B$, $B^{(1)} = \tilde{B}$.

Assume $|B| \leqslant |C|$. Then $\mathrm{fr}(B, C)$ denotes the frequency of B in C, i.e.

$$\mathrm{fr}(B, C) = \mathrm{card}\{0 \leqslant j \leqslant |C| - |B| \colon C[j, j+|B|-1] = B\}.$$

Let $b^0, b^1, \ldots$ be finite blocks with length at least two starting with zero, and let

(2) $$x = b^0 \times b^1 \times \ldots$$

Next, let

$$r_i^* = \min\left\{\frac{1}{\lambda_i}\,\mathrm{fr}(0, b^i), \frac{1}{\lambda_i}\,\mathrm{fr}(1, b^i)\right\}, \quad i \geqslant 0,\ \lambda_i = |b^i|.$$

Definition 1 ([6]). The sequence x defined by (2) is called a *generalized Morse sequence* (or shortly a *Morse sequence*) if

(i) infinitely many of the b^i's are different from $0\ldots0$,

(ii) infinitely many of the b^i's are different from $01\ldots010$,

(iii) $\sum_{i=0}^{\infty} r_i^* = \infty$.

Obviously (i) follows from (iii).

If x is a Morse sequence then each block B has the average relative frequency $m_x(B) = \lim_{n\to\infty} n^{-1}\,\mathrm{fr}(B, x[0, n-1])$ in x and $m_x(B) = m_x(\tilde{B})$ ([6]).

Let $X = \prod_{-\infty}^{\infty} \{0, 1\}$ and let T be the shift transformation on X. We then obtain an ergodic dynamical system $\theta(x) = (X, T, m_x)$ called a *Morse shift*. In the sequel all properties of $\theta(x)$ will be called properties of x.

We denote $c_t = b^0 \times \ldots \times b^{t-1}$, $n_t = |c_t| = \lambda_0 \cdot \ldots \cdot \lambda_{t-1}$, $t \geqslant 1$, $n_0 = 1$, $x_t = b^t \times b^{t+1} \times \ldots$ We observe that x_t is also a Morse sequence and put $m_{x_t} = m_t$.

Let

(3) $$p_0^t = m_t(00) + m_t(11), \quad p_1^t = m_t(01) + m_t(10).$$

The block representation $\{b^t\}_{t=0}^{\infty}$ of a Morse sequence x is called *regular* provided there exists $\varrho > 0$ such that

(4) $$\varrho < p_0^t,\ p_1^t < 1 - \varrho, \quad t \geqslant 0$$

(see [11]). If no confusion arises we will say that $x = b^0 \times b^1 \times \ldots$ is a regular Morse sequence.

Let $X(x)$ denote the set described in [11, § 3] such that $(X(x), T, m_x)$ is strictly ergodic and isomorphic to $\theta(x)$, and let

$$\zeta^{(t)} = \{\zeta_{t,k}\}, \quad t \geqslant 1,\ 0 \leqslant k \leqslant n_t - 1,$$

$$\zeta_{t,k} = \{y \in X(x) \colon y[-k+sn_t, n_t-k-1+sn_t] = c_t \text{ or } \tilde{c}_t,\ s = 0, \pm 1, \ldots\}$$

be the partition described there.

Let $\xi = (\xi_0, \xi_1)$ be the zero time partition of $X(x)$. The partition ξ is a generator of $(X(x), T, m_x)$ and we call it the *natural generator* for x.

Remark 1. If $x = b^0 \times b^1 \times \ldots$ is a Morse sequence then for any $t \in N$ we put $c_m^t = b^t \times b^{t+1} \times \ldots \times b^{t+m-1}$, $n_0^t = 1$, $n_m^t = \lambda_t \cdot \ldots \cdot \lambda_{t+m-1}$, $m > 0$, and $A_t = \{n_m^t\}_{m=0}^{\infty}$. We denote by $\xi(t)$ the natural generator for x_t.

Any nonempty atom of ξ_k^m $(m, k \geqslant 1)$ is said to be a *general cylinder* (for short g.c.). Note that any atom of ξ_k^m has the form

$$\{y \in X(x) \colon y[n_j, n_j+k-1] = B^j, \quad j = 0, \ldots, m-1\}$$
$$= T^{-n_0}(B^0) \cap \ldots \cap T^{-n_{m-1}}(B^{m-1})$$

where B^j is a block with length k, and we denote such an atom by

$$[B^0 \colon \ldots \colon B^{m-1}].$$

It is easy to see that a ξ_k^m-block $[B^0 \colon \ldots \colon B^{m-1}]$ is a g.c. iff there exists a natural p such that

(5) $$x[p+n_j, p+n_j+k-1] = B^j, \quad j = 0, \ldots, m-1.$$

We also have

$$m_x([B^0 \colon \ldots \colon B^{m-1}]) = \lim_{t\to\infty} \frac{1}{n_t}\,\mathrm{fr}([B^0 \colon \ldots \colon B^{m-1}], c_t).$$

Moreover, if $[B^0 \colon \ldots \colon B^{m-1}]$ is a g.c. then $[\tilde{B}^0 \colon \ldots \tilde{B}^{m-1}]$ is a g.c. and they have the same measure m_x.

Remark 2. For $t \in N$ we denote by $[B^0 \colon \ldots \colon B^{m-1}]_t$ the atoms for x_t.

Now, we recall a result of Kwiatkowski's paper [10]. Let B, C be blocks with $|B| = |C| = n \geqslant 1$, and let

$$s(k, B) = \mathrm{card}\{i \colon 0 \leqslant i \leqslant n-k-1, B[i] \neq B[i+k]\}, \quad k = 1, \ldots, n-1.$$

We write $B \stackrel{s}{=} C$ iff $s(k, B) = s(k, C)$, $k = 1, \ldots, n-1$.

Theorem 1 ([10]). *Let $x = b^0 \times b^1 \times \ldots$ and $y = \beta^0 \times \beta^1 \times \ldots$ be regular Morse sequences such that $|b^i| = |\beta^i| = \lambda_i$ and $\lambda_i \leqslant r$, $i \geqslant 0$. Then $\theta(x)$ and $\theta(y)$ are spectrally isomorphic iff $b^j \stackrel{s}{=} \beta^j$ for j large enough.*

Note that the "if" part of Theorem 1 is valid without the assumption of the regularity of x and y.

2. $\{n_t\}$-entropy for Morse shifts. In this section we give formulas for the $\{n_t\}$-topological entropy and the $\{n_t\}$-entropy for a Morse sequence $x = b^0 \times b^1 \times \dots$

We start with formulating the following

THEOREM 2. $h_A^{\text{top}}(x) = h_A^{\text{top}}(x, \xi)$, $\quad h_A(x) = h_A(x, \xi)$.

The proof of the theorem can be obtained as in Dekking's paper [2]. We use the following properties:

(6) $\quad x[sn_1, sn_1+n_1-1] = c_1$ or $\tilde{c}_1$, $\quad x[sn_1] = x_1[s]$, $\quad s \geqslant 1$;

(7) $\quad$ if $[a_0:\dots:a_m]$ is a g.c. then $[a_1:\dots:a_m]_1$ is a g.c., $a_i = 0, 1$, $0 \leqslant i \leqslant m$;

(8) $\quad 2N(\xi^m(1)) \geqslant N(\xi^{m+1}) \geqslant N(\xi^m(1))$;

(9) $\quad 2^t N(\xi^{m-t}(t)) \geqslant N(\xi^m) \geqslant N(\xi^{m-t}(t))$, $\quad 0 \leqslant t \leqslant m-1$, $m \in \mathbf{N}$;

(10) $\quad N(\xi_2^{m+1}) \leqslant 4mN(\xi^{m+1})$, $\quad m \in \mathbf{N}$;

(11) $\quad N(\xi_k^m) \leqslant 4(2^k)^t (m-t+1) N(\xi^m)$, $\quad t \in \mathbf{N}$, $n_t > k$, $k, m \in \mathbf{N}$.

The properties (6)–(11) are obtained as in [2]. Instead of Lemma 3 in [2], needed to proving $h_A(x) = h_A(x, \xi)$, we use

Remark 3. Let $M \in \mathbf{N}$. If $y \in \zeta_{t,k}$, $t \geqslant 1$, $k = 0, 1, \dots, n_t - 1$, then there is a natural p such that $y[0, M] = x[p, p+M]$ and $p = k (\text{mod } n_t)$.

Indeed, if $y \in \zeta_{t,k}$ then $y[0, M] = (c_t \times B)[k, M-k]$ for some block B. The block $c_t \times B$ appears in x. Thus it is not difficult to verify that it appears in x at a place which is a multiple of n_t. Hence there is j, $j = sn_t$ for some natural s, such that $x[j, j+|B|\,n_t - 1] = c_t \times B$.

Now, we are in a position to give formulas for the $\{n_t\}$-topological entropy.

Take $i \in \mathbf{N}$. We define by induction a finite sequence of natural numbers $\{m^{(i)}\}_{m=0}^{i}$ as follows:

0° $m^{(i)} = m_0^{(i)} + m_1^{(i)}$;

1° $0_k^{(i)} = 1$, $\quad k = 0, 1$;

2° If $m < i$ then

(12)
$$(m+1)_0^{(i)} = \begin{cases} m^{(i)} & \text{if } b^{i-(m+1)} \neq 01\dots01,\ 01\dots010, \\ m_1^{(i)} & \text{if } b^{i-(m+1)} = 01\dots01, \\ m_0^{(i)} & \text{if } b^{i-(m+1)} = 01\dots010, \end{cases}$$
$$(m+1)_1^{(i)} = \begin{cases} m^{(i)} & \text{if } b^{i-(m+1)} \neq 0\dots0, \\ m_1^{(i)} & \text{if } b^{i-(m+1)} = 0\dots0, \end{cases}$$

and proceeding as Dekking in [2] we get

THEOREM 3. $h_A^{\text{top}}(x) = \limsup_{i \in \mathbf{N}} \dfrac{\log i^{(i)}}{i}$.

At present, we will deal with the measure-theoretic entropy. First we introduce some notations. For a given Morse sequence $x = b^0 \times b^1 \times \dots$ we denote

$$h_t(00, 11) = \text{fr}(00, b^t) + \text{fr}(11, b^t), \qquad h_t(01, 10) = \text{fr}(01, b^t) + \text{fr}(10, b^t),$$

(13)
$$\Pi_0^t = \frac{1}{\lambda_t} h_t(00, 11),$$
$$\Pi_1^t = \frac{1}{\lambda_t} h_t(01, 10),$$
$$\Pi_{00}^t = \frac{1}{\lambda_t}(\text{fr}(00, b^t 0) + \text{fr}(11, b^t 0)),$$
$$\Pi_{10}^t = \frac{1}{\lambda_t}(\text{fr}(00, b^t 1) + \text{fr}(11, b^t 1)),$$
$$\Pi_{k1}^t = 1 - \Pi_{k0}^t, \qquad k = 0, 1,\ t \geqslant 0.$$

In this way we have defined a sequence of matrices

$$\Pi^t = \begin{bmatrix} \Pi_{00}^t & \Pi_{01}^t \\ \Pi_{10}^t & \Pi_{11}^t \end{bmatrix}.$$

Let us observe that, for every $t \geqslant 0$,

(14)
$$p_0^t = \Pi_0^t + \begin{cases} \dfrac{1}{\lambda_t} p_0^{t+1} & \text{if } b^t[\lambda_t - 1] = 0, \\ \dfrac{1}{\lambda_t} p_1^{t+1} & \text{if } b^t[\lambda_t - 1] = 1. \end{cases}$$

This gives the following useful formula:

(15)
$$p_0^t = \Pi_{00}^t \pm \frac{1}{\lambda_t} p_1^{t+1}.$$

Going back to (14), let us put there the factor $1 = p_0^{t+1} + p_1^{t+1}$ before Π_0^t. We then obtain

$$p_0^t = \begin{cases} \left(\Pi_0^t + \dfrac{1}{\lambda_t}\right) p_0^{t+1} + \Pi_0^t p_1^{t+1} & \text{if } b^t[\lambda_t - 1] = 0, \\ \Pi_0^t p_0^{t+1} + \left(\Pi_0^t + \dfrac{1}{\lambda_t}\right) p_1^{t+1} & \text{if } b^t[\lambda_t - 1] = 1. \end{cases}$$

This, by the definition of Π_{ij}^t, proves the following

PROPOSITION 1. $\sum_{j=0}^{1} p_j^{t+1} \Pi_{ji}^t = p_i^t$ *for every* $t \in \mathbf{N}$ *and* $i = 0, 1$.

By consecutive substitutions of (14) into itself for $t = 0, 1, \ldots$, one can obtain the following formulas of Kwiatkowski [10]:

$$(16)\quad \begin{aligned} p_0 &= \frac{h_0(00,11)}{\lambda_0} + \sum_{\substack{i\geqslant 2\\ i-1\in I}} \frac{h_{i-1}(00,11)}{n_i} + \sum_{\substack{i\geqslant 2\\ i-1\in II}} \frac{h_{i-1}(01,10)}{n_i},\\ p_1 &= \frac{h_0(01,10)}{\lambda_0} + \sum_{\substack{i\geqslant 2\\ i-1\in I}} \frac{h_{i-1}(01,10)}{n_i} + \sum_{\substack{i\geqslant 2\\ i-1\in II}} \frac{h_{i-1}(00,11)}{n_i}, \end{aligned}$$

where

$$I = \{i \geqslant 1:\ c_i[n_i-1] = 0\}, \qquad II = \{i \geqslant 1:\ c_i[n_i-1] = 1\}.$$

These formulas will be used in our study of Kakutani sequences in Example 6.

Now, applying Proposition 1 and proceeding in the same way as Dekking in [2], we get

THEOREM 4. $h_A(x) = \limsup_{t\in N} \{-t^{-1} \sum_{r=0}^{t-1} (\sum_{0\leqslant i,j\leqslant 1} p_i^{r+1} \Pi_{ij}^r \log \Pi_{ij}^r)\}$.

3. Zero $\{n_t\}$-entropy and properties of $\mathrm{SE}(x)$. In this section we answer the question when $h_{\{n_t\}}(x) = 0$. Next, we introduce an equivalence relation on $M(x)$ (see the Introduction) and compare it with the relation introduced in [10].

Let $x = b^0 \times b^1 \times \ldots$ be a Morse sequence and put

$$(17)\qquad r_t = -\sum_{0\leqslant i,j\leqslant 1} p_i^{t+1} \Pi_{ij}^t \log \Pi_{ij}^t, \qquad t \geqslant 0.$$

For every x we will consider the following sequence of numbers:

$$w_t = \begin{cases} \min(p_0^t, p_1^t) & \text{if } b^t \neq 0\ldots0,\ 01\ldots01,\ 01\ldots010,\\ \min(p_1^{t+1}, 1/\lambda_t) & \text{if } b^t = 0\ldots0,\ 01\ldots01,\\ \min(p_0^{t+1}, 1/\lambda_t) & \text{if } b^t = 01\ldots010. \end{cases}$$

LEMMA 1. *For every $\delta > 0$ there exists an $\varepsilon > 0$ such that*

(*) *if $r_t < \varepsilon$ then $w_t < \delta$;*

(**) *if $w_t < \varepsilon$ then $r_t < \delta$ for every $t \in N$.*

Proof. Let $\delta > 0$ and $\varepsilon > 0$. We observe that if $b^t \neq 0\ldots0,\ 01\ldots01,\ 01\ldots010$ then

$$r_t \geqslant \min_{0\leqslant i,j\leqslant 1} (-\Pi_{ij}^t \log \Pi_{ij}^t) > 0.$$

Therefore $0 < -\Pi_{ij}^t \log \Pi_{ij}^t < \varepsilon$ for some $i, j = 0, 1$. This means that $0 < \Pi_{ij}^t < \delta/2$ or $1 > \Pi_{ij}^t > 1-\delta/2$ for suitable $\varepsilon > 0$. The last inequality implies $0 < \Pi_{i'j'}^t < \delta/2$ for some $i', j' = 0, 1$, so we may assume $0 < \Pi_{ij}^t < \delta/2$.

It is easy to see that this implies $p_0^t < \delta$ or $p_1^t < \delta$. Indeed, from (14) it follows that

$$p_0^t \leqslant \frac{2}{\lambda_t} h_t(00, 11), \qquad p_1^t \leqslant \frac{2}{\lambda_t} h_t(01, 10),$$

and

$$\Pi_{ij}^t \geqslant \min\left(\frac{1}{\lambda_t} h_t(00, 11), \frac{1}{\lambda_t} h_t(01, 10)\right).$$

Now we consider the case $b^t = 0\ldots0,\ 01\ldots01$. Then

$$(18)\qquad r_t = -p_1^{t+1}\left(\frac{1}{\lambda_t}\log\frac{1}{\lambda_t} + \frac{\lambda_t-1}{\lambda_t}\log\frac{\lambda_t-1}{\lambda_t}\right).$$

If $r_t < \varepsilon$ then $p_1^{t+1} < \sqrt{\varepsilon}$ or

$$-\left(\frac{1}{\lambda_t}\log\frac{1}{\lambda_t} + \frac{\lambda_t-1}{\lambda_t}\log\frac{\lambda_t-1}{\lambda_t}\right) < \sqrt{\varepsilon}.$$

But the last inequality implies $\lambda_t > 1/\delta$ for a suitable $\varepsilon > 0$. The same reasoning for $b^t = 01\ldots010$ makes the proof of (*) complete.

Now let $\delta > 0$ and $\varepsilon > 0$. If $b^t \neq 0\ldots0,\ 01\ldots01,\ 01\ldots010$ then $\Pi_{ji}^t < 2p_i^t$, $i, j = 0, 1$ so $r_t < \delta$ whenever $\min(p_0^t, p_1^t) < \varepsilon$ for a suitable $\varepsilon > 0$.

Let $b^t = 0\ldots0$. Then r_t is as in (18). If $\lambda_t > 1/\varepsilon$ then $r_t < \delta$, and if $p_1^{t+1} < \varepsilon$ then also $r_t < \delta$ for a suitable $\varepsilon > 0$.

Proceeding in the same way in the remaining cases, we obtain the required result.

COROLLARY 1. *Let $x = b^0 \times b^1 \times \ldots$ be a Morse sequence. Then $h_{\{n_t\}}(x) = 0$ iff $\lim_{t\in B} w_t = 0$ for some sequence $B \subseteq N$ of density* 1.

Proof. It is sufficient to see that $h_A(x) = 0$ iff $\lim_{t\to\infty} t^{-1}\sum_{i=0}^{t-1} r_i = 0$. It is known that the last equality is equivalent to $\lim_{t\in B} r_t = 0$ for some sequence $B \subseteq N$ of density 1. Now, we apply Lemma 1.

COROLLARY 2. *Let $x = b^0 \times b^1 \times \ldots$ be a Morse sequence. Then x is regular iff for every subsequence $\{n_{t_k}\}$, $h_{\{n_{t_k}\}}(x) > 0$.*

Proof. Let us observe thet the process of grouping the b^i's gives a new representation of x: $x = \beta^0 \times \beta^1 \times \ldots$ Moreover, the products of the lengths of successive β^i's form a subsequence $\{n_{t_k}\}$ of $\{n_t\}$. Observe that the corresponding sequence defined in (3) is a subsequence $(p_0^{t_k}, p_1^{t_k})$ of (p_0^t, p_1^t).

This means that the process of grouping the b^i's gives a regular representation of x whenever so is the representation $\{b^t\}$. Thus, by Corollary 1 and (4), $h_{\{n_{t_k}\}}(x) > 0$.

If x is not regular then there is a subsequence $\{t_k\}$ such that $\min\{p_0^{t_k}, p_1^{t_k}\}\underset{k}{\to} 0$. We group the b^t's into a new product $x = \beta^0\times\beta^1\times\ldots$ such that $p_i^{\prime k} = p_i^{t_k}$, $k\geqslant 0$, $i = 0, 1$ ($p_i^{\prime k}$ are the numbers defined in (3) for $\{\beta^t\}$).

Grouping again if necessary, we may assume $\beta^i\neq 0\ldots 0$, $01\ldots 01$, $01\ldots 010$. This forces $\{w_t\}$ to converge to zero on a set of density 1, and so the corresponding sequence entropy is equal to 0.

At present, we shall examine the relation $\mathrm{SE}(x)$ for a given $x = b^0\times b^1\times\ldots$

Proposition 2. *Let $x = b^0\times b^1\times\ldots$ be a regular Morse sequence and $y\in M(x)$, $y = \beta^0\times\beta^1\times\ldots$ be spectrally isomorphic to x. Then $y\in\mathrm{SE}(x)$.*

Proof. Let us use for x the symbols p_i^t, Π_{ij}^t, r_t and for y the symbols $p_i^{\prime t}$, $\Pi_{ij}^{\prime t}$, $r_t^\prime$ to denote the numbers defined in (3), (13), (17).

According to the proof of Theorem 2 in [10], if x and y are spectrally isomorphic then

(19) $$|p_i^t - p_i^{\prime t}| < \varepsilon_t,\quad t\geqslant 0,\ i = 0, 1\quad\text{and}\quad \lim_{t\to\infty}\varepsilon_t = 0,$$

and the following formula can be obtained:

(20) $$[s(q, b^t) - s(q, \beta^t)][1-(p_1^{t+1}-p_0^{t+1})^2] = \lambda_t\delta_t,\quad \delta_t\underset{t}{\to} 0,\quad 0\leqslant q\leqslant\lambda_t - 1.$$

We do not assume that $\{\lambda_i\}$ is bounded, so we cannot make use of Theorem 1.

To compare Π_{ij}^t with $\Pi_{ij}^{\prime t}$ recall ((15)) that

(21) $$\Pi_{00}^t = p_0^t \pm\frac{1}{\lambda_t}p_1^{t+1},\qquad \Pi_{00}^{\prime t} = p_0^{\prime t}\pm\frac{1}{\lambda_t}p_1^{\prime t+1}.$$

It is sufficient to show that

(22) $$\lim_{t\to\infty}|r_t - r_t^\prime| = 0.$$

Suppose that (22) fails. Hence there is $c > 0$ and a sequence $\{t_k\}$ such that $\lim_{k\to\infty} t_k = \infty$ and

(23) $$|r_{t_k} - r_{t_k}^\prime|\geqslant c,\qquad k = 0, 1, \ldots$$

By (19), (21) and (23), $\{\lambda_{t_k}\}$ is a bounded sequence. In view of (20) and by the regularity of x, $s(\lambda_{t_k}-1, b^t) = s(\lambda_{t_k}-1, \beta^t)$ for k large enough, so if

$$\Pi_{00}^{t_k} = p_0^{t_k} - \frac{1}{\lambda_{t_k}}p_1^{t_k+1}$$

then

$$\Pi_{00}^{\prime t_k} = p_0^{\prime t_k} - \frac{1}{\lambda_{t_k}}p_1^{\prime t_k+1}$$

and consequently $\lim_{k\to\infty}|r_{t_k} - r_{t_k}^\prime| = 0$: a contradiction to (23).

From Proposition 2 and Corollary 2 (or directly from (19)) follows

Corollary 3. *If $x = b^0\times b^1\times\ldots$ is a regular Morse sequence and $y\in M(x)$ is not regular then x and y are not spectrally isomorphic.*

Now, we give a class of examples of Morse sequences x for which $\mathrm{SE}(x)$ consists of a continuum of classes of spectral equivalence.

Let $k\geqslant 4$ and let B be a block with length k such that there is a C with $|B| = |C|$, $s(1, B) = s(1, C)$, $s(k-1, B) = s(k-1, C)$ and $s(r, B)\neq s(r, C)$ for some r, $2\leqslant r\leqslant k-1$. For instance, if $k = 4$ one can put $B = 0110$, $C = 0010$.

Let $x = b^0\times b^1\times\ldots$ be a Morse sequence such that the set

$$\mathscr{J} = \{i\in N\colon\ b^i = B\ \text{and}\ b^{i+1}\neq 0\ldots 0,\ 01\ldots 01,\ 01\ldots 010,\ \lambda_{i+1}\leqslant k\}$$

is infinite. Let $y = \beta^0\times\beta^1\times\ldots$ be a Morse sequence obtained from x by replacing infinitely many of the b^i's, $i\in\mathscr{J}$, by C. We observe that $y\in\mathrm{SE}(x)$ because the sequences $\{\Pi^t\}$ and $\{p_0^t, p_1^t\}$ are the same for x and y.

If $\{\lambda_t\}$ is bounded and x is regular we can use Theorem 1 to verify that x and y are not spectrally isomorphic.

In general, we note that if $t\in\mathscr{J}$; then $1-(p_1^{t+1}-p_0^{t+1})^2\geqslant 1-((k-2)/k)^2$.

Assume that x and y are spectrally isomorphic. From (20) it then follows that $s(q, b^t) = s(q, \beta^t)$, $0\leqslant q\leqslant\lambda_{t-1}$, for $t\in\mathscr{J}$ large enough. This is impossible because $s(r, B)\neq s(r, C)$: a contradiction.

Finally, note that x could be regular or not, and in both cases $\mathrm{SE}(x)$ contains a continuum of spectrally nonisomorphic Morse sequences.

We do not know whether Proposition 2 is true without the assumption of the regularity of x. However, in general we prove that the Morse sequence x^{-1} (defined below) is always spectrally isomorphic to x and $x^{-1}\in\mathrm{SE}(x)$.

Let $x = b^0\times b^1\times\ldots$ be a Morse sequence. For every block $B = (b_0, \ldots b_{m-1})$ we define a block B^{-1} as follows:

$$B^{-1} = \begin{cases}(b_{m-1}, \ldots, b_0) & \text{if } b_{m-1} = 0,\\ (b_{m-1}, \ldots, b_0)^\sim & \text{if } b_{m-1} = 1.\end{cases}$$

Put

$$x^{-1} = (b^0)^{-1}\times(b^1)^{-1}\times\ldots$$

It is easy to see that x^{-1} is also a Morse sequence.

Now, we define a function $\varphi\colon X\to X$ by the formula $\varphi(z) = z^{-1}$ for every two-sided sequence $z\in X$, where $z^{-1}[i] = z[-i]$. It is clear that φ is continuous, invertible and $\varphi T^{-1} = T\varphi$. Moreover, $\varphi(X(x)) = X(x^{-1})$. To see this assume $z \in X(x)$ and $z[-j_t, k_t] = c_t$ or $\tilde{c}_t$ (for the definition of the numbers j_t, k_t see [10], [11]). Then $\varphi(z)[-k_t, j_t] = c_t^{-1}$ or $(c_t^{-1})^{\sim}$. Since $(B \times C)^{-1} = B^{-1} \times C^{-1}$ for all blocks B, C, our claim is established. Next, $(X(x), T^{-1}, m_x)$ and $(X(x^{-1}), T, m_{x^{-1}})$ are strictly ergodic, and so φ must be an isomorphism between them. But, for every block B, $B \overset{s}{=} B^{-1}$, and so from the remark after Theorem 1, (T, m_x), (T^{-1}, m_x) and $(T, m_{x^{-1}})$ are always spectrally isomorphic.

Put, in the proof of Proposition 2, $y = x^{-1}$. Since $s(\lambda_t - 1, b^t) = s(\lambda_t - 1, (b^t)^{-1})$ we need not use the regularity of x to have $y \in \mathrm{SE}(x)$. So we have proved $x^{-1} \in \mathrm{SE}(x)$ for every Morse sequence x.

Note, however, that Theorem 8 in [11] provides several examples showing that (T, m_x) is not metrically isomorphic to (T^{-1}, m_x).

Dekking in [2] gave an example for which $h_B(T) \neq h_B(T^{-1})$ for some sequence $B \subseteq N$ (obviously B is not of the form $\{n_t\}$).

4. Examples and remarks. In the remaining part of this paper we will use the results obtained to answer a few questions concerning sequence entropy.

EXAMPLE 1. *The formula* $h_A(T^k) = k h_A(T)$, $k \geq 0$, *need not hold.*

(a) $k = 2$.

Let $x = 01 \times 00 \times 01 \times 00 \times 00 \times 01 \times 00 \times 00 \times 00 \times 01 \times \ldots$

Next, we group x into new products as follows:

$$\text{I:} \quad x = 01 \times (00 \times 01 \times 00) \times (00 \times 01 \times 00 \times 00) \times \times (00 \times 01 \times 00 \times 00 \times 00) \times \ldots,$$

$$\text{II:} \quad x = (01 \times 00) \times (01 \times 00 \times 00) \times (01 \times 00 \times 00 \times 00) \times \times (01 \times 00 \times 00 \times 00 \times 00) \times \ldots$$

Assume that $\{n_t\}$, $\{m_t\}$ are the corresponding sequences of the products of lengths of the successive blocks in I and II respectively. Thus $m_t = 2n_t$, $t \geq 1$.

We see that the representation of x in I is regular, and so $h_{\{n_t\}}(x) > 0$. Moreover, it is clear from Corollary 1 that $h_{\{m_t\}}(x) = 0$. In addition we can assume $m_0 = 2$ (see Proposition 3 in [17]). Hence $h_{\{n_t\}}(T^2) \neq 2h_{\{n_t\}}(T)$.

Indeed, from the definition of sequence entropy it follows that $h_{\{kn_t\}}(T, Q) = h_{\{n_t\}}(T^k, Q)$ for every $Q \in \mathscr{Z}$, $k \geq 1$, so that

$$h_{\{kn_t\}}(T) = h_{\{n_t\}}(T^k), \quad k \geq 1. \tag{24}$$

Therefore, if $h_{\{n_t\}}(T^2) = 2h_{\{n_t\}}(T)$ then $0 = h_{\{m_t\}}(T) = h_{\{2n_t\}}(T) = h_{\{n_t\}}(T^2) = 2h_{\{n_t\}}(T) > 0$: a contradiction.

(b) $k \geq 3$.

Take $B \neq 0\ldots 0$, $01\ldots 01$, $01\ldots 010$, $|B| = k$ and put

$$x = B \times 00 \times B \times 00 \times 00 \times B \times 00 \times 00 \times 00 \times \ldots$$

Next, we take two representations of x:

$$\text{I:} \quad x = (B \times 00) \times (B \times 00 \times 00) \times (B \times 00 \times 00 \times 00) \times \ldots,$$

$$\text{II:} \quad x = (B \times 00 \times B) \times (00 \times 00 \times B) \times (00 \times 00 \times 00 \times B) \times \ldots$$

As in the preceding case, one can easily prove that $h_{\{n_t\}}(x) > 0$, $h_{\{m_t\}}(x) = 0$ and $m_t = kn_t$, $t \geq 1$. Furthermore, from (24), $0 = h_{\{n_t\}}(T^k) \neq kh_{\{n_t\}}(T) > 0$.

Remark 4. The above example gives a solution to the following question of Saleski:

Suppose $\{a(n)/b(n)\}_{n=1}^{\infty}$ is bounded away from 0 and ∞. Is it true that $h_A(T, \alpha) = 0$ iff $h_B(T, \alpha) = 0$?

It is not difficult to verify that the examples in Example 1 give a negative answer to this question because $h_A(T) = h_A(T, \xi)$ where ξ is the natural generator.

Note, however, that the question of Saleski remains open if we consider A increasing at most exponentially (i.e. $\limsup_{t \in N} (n_t)^{1/t} < \infty$). If, for instance, $A = \{k^t\}$, $k \geq 2$, and $B = \{k^{t+1}\}$, then the answer is positive and moreover $h_A(T^k) = h_A(T)$ for any automorphism T.

EXAMPLE 2. *The formula* $h_A^{\mathrm{top}}(T^k) = kh_A^{\mathrm{top}}(T)$, $k \geq 2$, *need not hold.*

We only consider the case $k = 2$. We have

$$h_{\{n_t'\}}^{\mathrm{top}}(T) = h_{\{n_t\}}^{\mathrm{top}}(T) \quad \text{where } n_t' = n_t \text{ for } t \geq 1. \tag{25}$$

Now, let $x = 01 \times 00 \times 01 \times 00 \times \ldots$ and we take two representations of x:

$$\text{I:} \quad x = (01 \times 00) \times (01 \times 00) \times \ldots,$$

$$\text{II:} \quad x = (01 \times 00 \times 01) \times (00 \times 01) \times (00 \times 01) \times \ldots$$

Theorem 3 and (12) imply

$$h_{\{n_t\}}^{\mathrm{top}}(x) = h_{\{2^t\}}^{\mathrm{top}}(01 \times 01 \times \ldots) = \log \tfrac{1}{2}(1 + \sqrt{5})$$

(see also [2]) and $h_{\{m_t\}}^{\mathrm{top}}(x) = \log 2$ because each block in this representation is different from $0\ldots 0$, $01\ldots 01$, $01\ldots 010$.

In addition, $m_t = 2n_t$, $t \geq 1$, and by (25) we can assume $m_0 = 2n_0$, so $h_{\{n_t\}}^{\mathrm{top}}(T^2) \neq 2h_{\{n_t\}}^{\mathrm{top}}(T)$.

EXAMPLE 3. *A-topological entropy does not depend monotonically on A.*

In [2] Dekking proved that it is possible to have $h_A(T) < h_{A'}(T)$ when

A is a subsequence of A'. We show that the same is true for topological sequence entropy.

Let $B \neq 0\dots 0,\ 01\dots 01,\ 01\dots 010$ and put

$$x = 00 \times B \times B \times 00 \times B \times B \times 00 \times \dots$$

Taking $i \in N$, we have three possibilities:

(a) $0^{(i)} = 2,$
$(3n+1)^{(i)} = 2(3n)^{(i)},$
$(3n+2)^{(i)} = 2(3n+1)^{(i)},$
$(3n+3)^{(i)} = (3n+2)^{(i)} + (3n+1)^{(i)};$

(b) $0^{(i)} = 2,$
$(3n+1)^{(i)} = 2(3n)^{(i)},$
$(3n+2)^{(i)} = (3n+1)^{(i)} + (3n)^{(i)},$
$(3n+3)^{(i)} = 2(3n+2)^{(i)};$

(c) $0^{(i)} = 2,$
$1^{(i)} = 3,$
$(3n+2)^{(i)} = 2(3n+1)^{(i)},$
$(3n+3)^{(i)} = 2(3n+2)^{(i)},$
$(3n+4)^{(i)} = (3n+3)^{(i)} + (3n+2)^{(i)}.$

In case (a) we have $(3n+r)^{(i)} = 2^{r+1}6^n$, $r = 0, 1, 2$. Taking into consideration (b) and (c), we get

$$h_A^{\text{top}}(x) = \tfrac{1}{3}\log 6.$$

Next, we group x into a new product

$$x = 00 \times (B \times B) \times 00 \times (B \times B) \times \dots$$

and we obtain

$$h_{A'}^{\text{top}}(x) = \tfrac{1}{2}\log 3$$

and consequently $h_A^{\text{top}}(x) < h_{A'}^{\text{top}}(x)$.

Remark 5. Let us observe that the x which we have just considered is a continuous substitution $0 \mapsto (00 \times B \times B)$. Such shifts were examined by Dekking in [2]. It is clear that the A and A' used here are different from the sequences in [2].

We are also able to strengthen Dekking's result showing $0 = h_A(T) < h_{A'}(T)$. Indeed, putting

$$x = 00 \times \underset{i_1}{\dots} \times 00 \times B \times \underset{i_2}{\dots} \times B \times 00 \times \underset{i_3}{\dots} \times 00 \times B \times \underset{i_4}{\dots} \times B \times \dots,$$

grouping

$$x = 00 \times \dots \times 00 \times (B \times \dots \times B) \times 00 \times \dots \times 00 \times (B \times \dots \times B) \times \dots$$

and assuming

$$\lim_{k\to\infty} \frac{i_k}{\sum_{j=1}^{k} i_j} = 1,$$

one can get $h_{A'}^{\text{top}}(x) > 0$ and $h_{A'}(x) > 0$. But from Theorem 3 we have $h_A^{\text{top}}(x) = 0$, and so all the more $h_A(x) = 0$. However, in our example A grows faster than exponentially, contrary to Dekking's example.

Example 4 (On the variational principle).

Goodman in [3] gave an example of an automorphism τ such that $h_A^{\text{top}}(\tau) = \log 2$ and $\sup_{\mu\in M} h_A^{\mu}(\tau) = 0$ for some sequence $A \subseteq N$. His τ had discrete spectrum.

We see that for every Morse sequence $x = b^0 \times b^1 \times \dots$ the set $X(x)$ is strictly ergodic, so that $\sup_{\mu\in M} h_{\{n_t\}}^{\mu}(T) = h_{\{n_t\}}(x)$. If x is not regular then we can assume that $b^i \neq 0\dots 0,\ 01\dots 01,\ 01\dots 010$ and $h_{\{n_t\}}(x) = 0$. But from Theorem 3 it follows that $h_{\{n_t\}}^{\text{top}}(x) = \log 2$. Therefore we obtain an automorphism x with Goodman's property although this automorphism has a partly continuous spectrum.

Suppose (X, T, μ), (Y, τ, ν) are strictly ergodic systems and A is a sequence in N. Does $h_A(T) = h_A(\tau)$ imply $h_A^{\text{top}}(T) = h_A^{\text{top}}(\tau)$ and vice versa?

Let $x = b^0 \times b^1 \times \dots$, $y = \beta^0 \times \beta^1 \times \dots$, $y \in M(x)$, be Morse sequences, b^i, $\beta^i \neq 0\dots 0,\ 01\dots 01,\ 01\dots 010$, x is not regular with $h_{\{n_t\}}(x) = 0$ and y is regular. Then $h_{\{n_t\}}(y) > 0$ and $h_{\{n_t\}}^{\text{top}}(x) = h_{\{n_t\}}^{\text{top}}(y) = \log 2$.

Now, let

$$x = \underset{i_0}{01\dots 01} \times \underset{i_1}{01\dots 01} \times \underset{i_2}{01\dots 01} \times \dots,$$

(the length of the first block is i_0 etc.),

$$y = \underset{i_0}{01\dots 0100} \times \underset{i_1}{01\dots 0100} \times \underset{i_2}{01\dots 0100} \times \dots,$$

and assume

$$\sum_{k=0}^{\infty} 1/i_k < \infty. \tag{26}$$

Writing $x = b^0 \times b^1 \times \dots$, $y = \beta^0 \times \beta^1 \times \dots$, we see that

(i) x and y are continuous (for the definition see [6]) because i_k, $k \geqslant 0$, are even;

(ii) $\sum_{k=0}^{\infty} d(b^k, \beta^k) = \sum_{k=0}^{\infty} \operatorname{card}\{i\colon b^k[i] \neq \beta^k[i]\}/\lambda_k = \sum_{k=0}^{\infty} 1/i_k < \infty.$

Thus from a recent result of Kwiatkowski ([12]) it follows that x and y

are metrically isomorphic. From (12) it follows that

$$h^{\mathrm{top}}_{\{n_t\}}(y) = \log 2, \qquad h^{\mathrm{top}}_{\{n_t\}}(x) = \log \tfrac{1}{2}(1+\sqrt{5}).$$

In this way we show that topological entropy can distinguish between two strictly ergodic and metrically isomorphic systems.

EXAMPLE 5 (a solution to Goodman's question).

In [3] Goodman showed that $h^{\mathrm{top}}_A(T\times T) = 2h^{\mathrm{top}}_A(T)$ and $h^{\mathrm{top}}_A(T\times T') \leqslant h^{\mathrm{top}}_A(T)+h^{\mathrm{top}}_A(T')$ and he asked whether $h^{\mathrm{top}}_A(T\times T') = h^{\mathrm{top}}_A(T)+h^{\mathrm{top}}_A(T')$.

Let $x = b^0\times b^1\times\ldots$, $y = \beta^0\times\beta^1\times\ldots$, $y\in M(x)$, be Morse sequences and denote by ξ, ξ' the natural generators for $(X(x), T, m_x)$ and $(X(y), T, m_y)$ respectively. Then $\xi\times\xi'$ is an open partition of $X(x)\times X(y)$. Moreover, $(\xi\times\xi')^m_k = \xi^m_k\times\xi'^m_k$ and

(27) $$N\big((\xi\times\xi')^m_k\big) = N(\xi^m_k)\,N(\xi'^m_k), \qquad m, k\geqslant 1.$$

From (11) it follows that

$$N\big((\xi\times\xi')^m_k\big) \leqslant [4(2^k)^t(m-t+1)]^2\,N(\xi^m)\,N(\xi'^m) = [4(2^k)^t(m-t+1)]^2\,N(\xi\times\xi')^m.$$

Hence

$$h^{\mathrm{top}}_{\{n_t\}}\big((T\times T), (\xi\times\xi')_k\big) = \limsup_{m\in N}\frac{1}{m}\log N\big((\xi\times\xi')^m_k\big)$$
$$\leqslant \limsup_{m\in N}\frac{1}{m}\log\{[4(2^k)^t(m-t+1)]^2\,N(\xi\times\xi')^m\}$$
$$= h^{\mathrm{top}}_{\{n_t\}}(T\times T, \xi\times\xi').$$

Since $(\xi\times\xi')_k$ refines $\xi\times\xi'$ and $(\xi\times\xi')_k\nearrow_k\varepsilon$, from (1) we get

$$h^{\mathrm{top}}_{\{n_t\}}(T\times T, \xi\times\xi') = h^{\mathrm{top}}_{\{n_t\}}(T\times T).$$

Put

$$x = \underbrace{000\times\ldots\times000}_{i_0}\times\underbrace{011\times\ldots\times\overset{\downarrow}{011}}_{i_1}\times\underbrace{000\times\ldots\times000}_{i_2}\times\underbrace{011\times\ldots\times\overset{\downarrow}{011}}_{i_3}\times\ldots$$

(the first block 000 is repeated i_0 times etc.),

$$y = \underbrace{011\times\ldots\times\overset{\downarrow}{011}}_{i_0}\times\underbrace{000\times\ldots\times000}_{i_1}\times\underbrace{011\times\ldots\times\overset{\downarrow}{011}}_{i_2}\times\underbrace{000\times\ldots\times000}_{i_3}\times\ldots$$

and assume

(28) $$\lim_{j\to\infty}\frac{i_j}{\sum_{s=0}^{j} i_s} = 1.$$

It is not difficult to verify that $h^{\mathrm{top}}_{\{3^t\}}(x) = h^{\mathrm{top}}_{\{3^t\}}(y) = \log 2$. Indeed, it is sufficient to compute $i^{(i)}$ for the subsequences denoted by the arrows and to apply (28).

We will prove

(29) $$h^{\mathrm{top}}_{\{3^t\}}(x) = h^{\mathrm{top}}_{\{3^t\}}(y) = h^{\mathrm{top}}_{\{3^t\}}(T\times T).$$

In order to prove (29) we shall estimate $i^{(i)}\cdot\bar{\imath}^{(\bar{\imath})}$ where $\bar{\imath}^{(\bar{\imath})}$ denotes the number defined in (12) for y.

Take $i\in N$. Then there are a $k\in N$ and an s, $0\leqslant s\leqslant i_{k+1}$, such that $i = s+i_k+\sum_{t=0}^{k-1} i_t$. Put $\sum_{t=0}^{k-1} i_t = u_{k-1}$. From (12) we get

$$i^{(i)} \leqslant 2^s\cdot i_k\cdot 2^{u_{k-1}}, \qquad \bar{\imath}^{(\bar{\imath})} \leqslant s\cdot 2^{i_k}\cdot 2^{u_{k-1}}$$

or one has to replace $i^{(i)}$ by $\bar{\imath}^{(\bar{\imath})}$ and vice versa. We have

$$\log\frac{i^{(i)}\cdot\bar{\imath}^{(\bar{\imath})}}{i} \leqslant \frac{2u_{k-1}}{i}\log 2+\frac{\log s}{i}+\frac{\log i_k}{i}+\frac{s+i_k}{i}\log 2.$$

If i tends to infinity then $\log(i^{(i)}\cdot\bar{\imath}^{(\bar{\imath})})/i$ tends to $\log 2$. Therefore (29) is valid.

EXAMPLE 6 ($\{2^t\}$-entropy of Kakutani sequences).

Putting $b^t = 00$ or 01, $t\geqslant 0$, we obtain the class of Morse sequences considered by Kakutani [5]. We have

(30) $$p_1^{t+1} = \begin{cases}\lambda_t(p_1^t-\Pi^t_{01}) & \text{if } b^t[\lambda_t-1] = 0,\\ \lambda_t(\Pi^t_{01}-p_1^t) & \text{if } b^t[\lambda_t-1] = 1,\end{cases}$$

where

$$\Pi^t = \begin{bmatrix}1 & 0\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix} \quad\text{or}\quad \Pi^t = \begin{bmatrix}0 & 1\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}.$$

Due to this, for every Kakutani sequence and for every $t\geqslant 0$,

(31) $$p_1^{t+1} = 2\min(p_1^t, 1-p_1^t).$$

From Theorem 4 it follows that

(32) $$h_{\{2^t\}}(x) = \log 2\,\limsup_{t\in N} t^{-1}\sum_{i=0}^{t-1} p_1^i.$$

In connection with (31) consider the transformation of the unit interval into itself, $R\colon [0,1]\to[0,1]$, $R(d) = 2\min(d, 1-d)$. Consider also the transformation P mapping every Kakutani sequence x to its p_1.

The formulas (16) imply that P is one-to-one. Only dyadic rational numbers are not in the image (they would correspond to sequences with $b^t = 00$ for all t sufficiently large, but such sequences have been excluded). The correspondence P conjugates the left shift S on the space of

Kakutani sequences with R. In the measure-theoretic sense P conjugates the one-side Bernoulli shift S with R on $[0, 1]$ considered with the Lebesgue measure (which is clearly R-invariant and ergodic).

By (31) for almost every x

$$h_{\{2^t\}}(x) = \tfrac{1}{2}\log 2.$$

Let us observe that $h_{\{2^t\}}$ takes its maximal value $\frac{2}{3}\log 2$ at $x_0 = 01\times\times 01\times\ldots$. Indeed, we see that x_0 is a fixed point under S. Hence $P(x_0)$ must be fixed under R, so $P(x_0) = \frac{2}{3}$, and next we analyse time averages under R, $t^{-1}\sum_{i=0}^{t-1} R^i(d)$.

Moreover, we also have

$$h^{\text{top}}_{\{2^t\}}(y) \leqslant h^{\text{top}}_{\{2^t\}}(01\times 01\times\ldots) = \log \tfrac{1}{2}(1+\sqrt{5})$$

for every Kakutani sequence y. Indeed, this is a simple consequence of the equality

$$(m+1)^{(i)} = m^{(i)} + \max(m_0^{(i)}, m_1^{(i)})$$

for the sequence $\{i^{(i)}\}$ defined in (12) for x_0.

Example 7. *The formula* $h_A(T\times T') = h_A(T)+h_A(T')$ *fails.*

Let

$$x = 01\times\ldots\times\underset{i_0}{\overset{j_0\ \downarrow\ k_0}{01}}\times\ldots\times 01\times 00\times\ldots\times\underset{i_1}{\overset{j_1\quad k_1}{00}}\times\ldots\times 00\times 01\times\ldots\times\underset{i_2}{\overset{j_2\ \downarrow\ k_2}{01}}\times\ldots\times 01\times\ldots,$$

$$y = 00\times\ldots\times\underset{i_0}{\overset{j_0\quad k_0}{00}}\times\ldots\times 00\times 01\times\ldots\times\underset{i_1}{\overset{j_1\ \downarrow\ k_1}{01}}\times\ldots\times 01\times 00\times\ldots\times\underset{i_2}{\overset{j_2\quad k_2}{00}}\times\ldots\times 01\times\ldots,$$

and let $\{\varepsilon_s\}_{s\geqslant 0}$ be a sequence of positive numbers such that

$$\lim_{s\to\infty}\varepsilon_s = 0. \tag{33}$$

Put

$$l_s = \sum_{r=0}^{s-1} i_r + j_s$$

and assume

(i) $j_s/l_s \underset{s}{\to} 1$,

(ii) $\sum_{i=0}^{k_s} 1/2^{2i+1} > \frac{2}{3}-\varepsilon_s$.

Observe that (ii) can be obtained from the simple fact $\sum_{i=0}^{\infty} 1/2^{2i+1} = \frac{2}{3}$.

First, we calculate $h_{\{2^t\}}(x)$.

From (ii), applying (16), we have, for every i with $l_s - j_s \leqslant i \leqslant l_s$ and s even, $p_1^i \geqslant \frac{2}{3}-\varepsilon_s$. So

$$l_s^{-1}\sum_{i=0}^{l_s} p_1^i \geqslant l_s^{-1}\sum_{i=l_s-j_s}^{l_s} p_1^i \geqslant (j/l_s)(\tfrac{2}{3}-\varepsilon_s).$$

By (32), (33) and (i),

$$h_{\{2^t\}}(x) = \tfrac{2}{3}\log 2.$$

By replacing in the above considerations x by y and taking s odd we also get $h_{\{2^t\}}(y) = \frac{2}{3}\log 2$.

Applying the reasoning from Example 5 we see that

$$h^{\text{top}}_{\{2^t\}}(T\times T) \leqslant \log 2.$$

Therefore from the variational principle

$$h_{\{2^t\}}(T\times T) \leqslant \log 2 < h_{\{2^t\}}(x) + h_{\{2^t\}}(y).$$

Remark 6. Example 7 shows that Lemma 4 in [9] is false. We have only $h_A(\tau\times\tau) = 2h_A(\tau)$ for every sequence A and automorphism τ.

Let us observe that here and in Example 5 we make use of the fact that the limits

$$\lim_{t\to\infty}\frac{1}{t}\log N(T^{-n_0}\xi\vee\ldots\vee T^{-n_{t-1}}\xi),\qquad \lim_{t\to\infty}\frac{1}{t}H(T^{-n_0}\xi\vee\ldots\vee T^{-n_{t-1}}\xi)$$

do not exist.

Example 8 (on Abramov's formula).

As a simple consequence of Abramov's formula ([1]) we get

Proposition 3. *Let T: $X\to X$ be an ergodic automorphism of a Lebesgue space (X, μ) and let τ: $Y\to Y$ be its factor automorphism via φ: $X\to Y$. If φ is of finite order, i.e.* card $\{\varphi^{-1}(y)\}$ *is finite and constant for a.e.* $y\in X$, *then* $h(T) = h(\tau)$.

For sequence entropy this proposition is false. Indeed, take $X = X(x)$ where $x = b^0\times b^1\times\ldots$ is a Morse sequence, and $Y = X(x)/\eta$ where η is a partition of $X(x)$ into the sets of the form $\{z, \tilde{z}\}_{z\in X(x)}$.

Then $(Y, T/\eta, m_x/\eta)$ is an ergodic system with discrete spectrum ([11]), so that all sequence entropies of T/η are equal to 0 ([9]).

Let us note that the well-known formula for the induced automorphism: $h(T_C) = h(T)/\mu(C)$ cannot hold for sequence entropy for ergodic automorphisms with discrete spectrum because the family of all induced automorphisms gives (up to isomorphism) all loosely Bernoulli automorphisms with zero entropy (see [15]).

Remark 8. *No Morse sequences have weakly mixing factors.*

Let T be an automorphism of a Lebesgue space. We say that T is *bounded* if there is a $c > 0$ such that $h_A(T) < c$ for every sequence A.

From [17] it easily follows that if T is bounded then T has no weakly mixing factors.

Now, let T be an automorphism and τ its bounded factor automorphism via φ. Then, if φ is of finite order, say n, then T has no weakly mixing factors. Indeed, it is sufficient to show that T is also bounded. To prove this we use the formula (see [17])

$$h_A(T, \sigma) \leqslant h_A(T, \sigma') + H(\sigma|\sigma'). \tag{34}$$

Let $\sigma = (\sigma_1, \ldots, \sigma_u)$ be any partition of X. We can construct a partition $\eta = \{\eta_{i_1 \ldots i_n}: i_j \neq i_k \text{ for } j \neq k,\ j, k = 1, \ldots, u\}$ where $\eta_{i_1 \ldots i_n} = \varphi\sigma_{i_1} \cap \ldots \cap \varphi\sigma_{i_n}$. Put $\sigma' = \varphi^{-1}\eta$. Then from (34)

$$h_A(T, \sigma) \leqslant h_A(\tau, \eta) + H(\sigma|\sigma').$$

But each atom of σ' intersects at most n atoms of σ, and so we have $h_A(T) \leqslant h_A(\tau) + \log n$.

Now, let $x = b^0 \times b^1 \times \ldots$ be a Morse sequence. Since the algebra of mirror-invariant sets gives a factor with discrete spectrum via a map of order two (see [11]), we have proved the result stated above.

Acknowledgements. I wish to acknowledge my indebtedness to Jan Kwiatkowski and Feliks Przytycki for raising some problems which led to the final form of this paper as well as for many helpful conversations.

References

[1] L. M. Abramov and V. A. Rokhlin, *The entropy of a skew product of measure preserving transformations*, Vestnik Leningrad. Univ. 1962, no. 7, 5–13 (in Russian).

[2] F. M. Dekking, *Some examples of sequence entropy as an isomorphism invariant*, Trans. Amer. Math. Soc. 259 (1980), 167–183.

[3] T. N. T. Goodman, *Topological sequence entropy*, Proc. London Math. Soc. (3) 29 (1974), 331–350.

[4] P. Hulse, *On the sequence entropy of transformations with quasi-discrete spectrum*, J. London Math. Soc. (2) 20 (1979), 128–136.

[5] S. Kakutani, *Ergodic theory of shift transformation*, in: Proc. Fifth Berkeley Symp. Math. Stat. Prob. II, 1967, 128–136.

[6] M. Keane, *Generalized Morse sequences*, Z. Wahrsch. Verw. Gebiete 10 (1968), 335–353.

[7] —, *Strong mixing g-measures*, Invent. Math. 16 (1972), 309–324.

[8] E. Krug and D. Newton, *On sequence entropy of automorphisms of a Lebesgue space*, Z. Wahrsch. Verw. Gebiete 24 (1972), 211–214.

[9] A. G. Kushnirenko, *On metric invariants of entropy type*, Uspekhi Mat. Nauk 22 (5) (1967), 57–61 (in Russian).

[10] J. Kwiatkowski, *Spectral isomorphism of Morse dynamical systems*, Bull. Acad. Polon. Sci. 29 (3-4) (1981), 105–114.

[11] —, *Isomorphism of regular Morse dynamical systems*, Studia Math. 72 (1982), 59–89.

[12] —, *Isomorphism of regular Morse dynamical systems induced by arbitrary blocks*, to appear.

[13] M. Lemańczyk, *The rank of regular Morse dynamical systems*, to appear.

[14] R. Nürnberg, *All generalized Morse sequences are loosely Bernoulli*, Math. Z. 182 (1983), 403–407.

[15] D. Ornstein, D. Rudolph and B. Weiss, *Equivalence of measure preserving transformations*, Mem. Amer. Math. Soc. 37 (262) (1982).

[16] B. S. Pitskel', *Some properties of A-entropy*, Mat. Zametki 5 (1969), 327–334 (in Russian).

[17] A. Saleski, *Sequence entropy and mixing*, J. Math. Anal. Appl. 60 (1977), 58–66.

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Received December 1, 1983
Revised version October 10, 1984 (1938)