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Some incorrigible functions

by

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Abstract. We say that a function f on the unit circle $\boldsymbol{T}$ is *p-incorrigible* if whenever g is a function whose Fourier coefficients satisfy $\sum_n |\hat{g}_n|^p < \infty$ then $\mu\{t\colon f(t) = g(t)\} = 0$. We consider random Fourier series of the form $X = \sum_n c_n D_n e^{2\pi i n t}$, where D_n are independent random variables, each uniformly distributed on the unit disc, and show that if $c_n = n^{-1/p} (1 \leqslant p < 2)$ then X is almost surely p-incorrigible while if $c_n = n^{-1/2}(\log n)^{-\gamma}$, with $\gamma > 2$, then X is almost surely p-incorrigible for all $1 \leqslant p < 2$.

1. Introduction. The celebrated theorem of Men'shov states that if f is a measurable function on the unit circle then f can be changed on a set of arbitrarily small measure so that the "corrected" function has a uniformly convergent Fourier series (see [1]). On the other hand, Katznelson [5] showed that there exists a continuous function on the unit circle which is "incorrigible" — that is, it cannot be represented by an absolutely convergent trigonometric series on any set of positive measure. This result was extended by Olevskii [6], who showed that there is a continuous function f on the unit circle $\boldsymbol{T}$ which is "p-incorrigible" for all $p < 2$ — that is, if g is an integrable function on the unit circle which agrees with f on a set of positive measure then the Fourier coefficients $\hat{g}_n$ satisfy $\sum |\hat{g}_n|^p = \infty$ for all $p < 2$. He also showed that for each $1 \leqslant p < 2$ there exists a function f in $\mathrm{Lip}(1/p - 1/2)$ which is "p-incorrigible".

More recently, Hrushchev, Kahane and Katznelson [3] have shown that almost every Brownian path on the unit circle is incorrigible. This gives slightly weaker information about the Lipschitz condition than the result of Olevskii, but on the other hand it shows that there are incorrigible functions f whose Fourier coefficients satisfy $\hat{f}_n = O\left((\log |n|)^{1/2}/|n|\right)$. This raises the question: are there incorrigible functions f whose Fourier coefficients satisfy $\hat{f}_n = O(1/|n|)$?

We shall show, using an argument similar to that of Hrushchev, Kahane and Katznelson, thet in a natural sense almost every function f whose Fourier coefficients satisfy $\hat{f}_n = O(1/|n|)$ is incorrigible. More generally, we shall show that if $1 \leqslant p < 2$ almost every function f whose Fourier coefficients satisfy $\hat{f}_n = O(1/|n|^p)$ is p-incorrigible, and that almost every function f whose Fourier coefficients satisfy $\hat{f}_n = O(1/|n|^{1/2}(\log|n|^{\gamma}))$ (where $\gamma > 2$) is p-incorrigible for $1 \leqslant p < 2$.

I do not know if there is an incorrigible continuous function f whose Fourier coefficients satisfy $\hat{f}_n = o(1/|n|)$ nor if for $1 < p < 2$ there is a p-incorrigible continuous function f whose Fourier coefficients satisfy $\hat{f}_n = o(1/|n|^p)$.

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2. Statement of the results. Let $(D_k(\omega))$ be a sequence of independent random variables, each uniformly distributed on the unit disc $\{z: |z| \leqslant 1\}$.

We set

$$X_p(\omega)(t) = \sum_k{}' \frac{D_k(\omega)e^{2\pi ikt}}{|k|^{1/p}} \quad \text{for } 1 \leqslant p < 2$$

(where $\sum_k{}'$ is the sum over all nonzero k), and we set

$$X_{2,\gamma}(\omega)(t) = \sum_k{}'' \frac{D_k(\omega)e^{2\pi ikt}}{|k|^{1/2}\log^{\gamma}|k|} \quad \text{for } \gamma > 1,$$

(where $\sum_k{}''$ is the sum over all k with $|k| > 1$).

Recall that for almost ω the series for X_p and $X_{2,\gamma}$ converge uniformly (cf. [4], Chapter VII, Theorem 1) and that the moduli of continuity satisfy

$$\omega_{X_p}(h) = O(h^{1/p}\log^{1/2}(1/h)) \quad \text{for } 1 \leqslant p < 2,$$

$$\omega_{X_{2,\gamma}}(h) = O(\log^{1-\gamma}(1/h)) \quad \text{for } \gamma > 1$$

([4], Chapter VII, Theorem 2).

We shall prove the following theorems:

THEOREM 1. *Suppose that $1 \leqslant p < 2$. Then for almost all ω, $X_p(\omega)$ is p-incorrigible.*

THEOREM 2. *Suppose that $\gamma > 2$ and that $1 \leqslant p < 2$. Then for almost all ω, $X_{2,\gamma}(\omega)$ is p-incorrigible.*

3. Some linear operators. Before giving the proofs of Theorems 1 and 2, let us introduce some linear operators which we shall need. Suppose that n is a positive integer. Let F_n denote the subgroup of $\boldsymbol{T}$ of order n:

$$F_n = \{e^{2\pi ik/n}: 1 \leqslant k \leqslant n\}.$$

If f is a function on $\boldsymbol{T}$ and $t \in \boldsymbol{T}$ we set

$$(Q_t(f))_j = n^{-1}\sum_{k=1}^{n} f(t+k/n)e^{-2\pi ijk/n} \quad \text{for } 1 \leqslant j \leqslant n,$$

so that $Q_t(f)$ is an element of $\boldsymbol{C}^n$.

Suppose now that f is integrable and that $(\hat{f}_l) \in l_p$, where $1 \leqslant p \leqslant 2$. Then

$$f(t) = \sum_l \hat{f}_l e^{2\pi ilt}$$

for almost all t. Thus for almost all t

$$(Q_t(f))_j = \sum_l \hat{f}_l e^{2\pi ilt}\Big(n^{-1}\sum_{k=1}^{n} e^{2\pi ikl/n}e^{-2\pi ijk/n}\Big) = \sum_{\nu} \hat{f}_{j+\nu n}e^{2\pi i(j+\nu n)t}.$$

In the case where $p = 1$, this implies that

$$\|Q_t(f)\|_{l_1} \leqslant \sum_l |\hat{f}_l| \quad \text{for almost all } t$$

so that

$$\int_{\boldsymbol{T}} \|Q_t(f)\|_{l_1}\,dt \leqslant \sum_l |\hat{f}_l|;$$

while in the case where $p = 2$

$$\int_{\boldsymbol{T}} \|Q_t(f)\|_{l_2}^2\,dt = \sum_l |\hat{f}_l|^2$$

so that, by interpolation,

$$\int_{\boldsymbol{T}} \|Q_t(f)\|_{l_p}^p\,dt \leqslant \sum_l |\hat{f}_l|^p$$

for $1 \leqslant p \leqslant 2$.

Note also that if $\omega_f(h) \leqslant Ch^{1/p}\log^{1/2}(1/h)$ and $|t-s| < h$ then

$$|(Q_t(f))_j - (Q_s(f))_j| \leqslant Ch^{1/p}\log^{1/2}(1/h)$$

so that

$$\|Q_t(f) - Q_s(f)\|_{l_p} \leqslant C(nh)^{1/p}\log^{1/2}(1/h).$$

Thus

$$\|Q_t(f) - Q_s(f)\|_{l_p} \leqslant \varepsilon \quad \text{if } |t-s| < 1/n^2,$$

for sufficiently large n.

Similarly if $\omega_f(h) \leqslant C \log^{1-\gamma}(1/h)$ (where $\gamma > 2$), $1 \leqslant p < 2$ and $|t-s| < h$ then

$$\|Q_t(f) - Q_s(f)\|_{l_p} \leqslant Cn^{1/p} \log^{1-\gamma}(1/h)$$

In particular,

$$\|Q_t(f) - Q_s(f)\|_{l_p} \leqslant \varepsilon \quad \text{if } |t-s| \leqslant 2^{-n}$$

for sufficiently large n.

4. Proof of Theorem 1. The proof is rather similar to those of Olevskii and of Hrushchev, Kahane and Katznelson. If $1 \leqslant p < 2$, let

$$A_p = A_p(\boldsymbol{T}) = \{f \in \mathscr{L}^1(\boldsymbol{T}): \|f\|_{A_p} = (\sum_n |\hat{f}_n|^p)^{1/p} < \infty\}.$$

If E is a measurable subset of $\boldsymbol{T}$ of positive measure, let $A_p(E)$ denote the restriction of A_p to E; we give $A_p(E)$ the quotient norm $\|\ \|_{A_p(E)}$.

Suppose that the result is false. Let μ denote the normalized Haar measure on $\boldsymbol{T}$. Then there exist $\eta > 0$ and $\lambda > 0$ such that

$$P^*\{\omega: \text{there exists a measurable } E \text{ with } \mu(E) > \lambda \text{ such that } \|X_p(\omega)\|_{A_p(E)} < \infty\} > \eta.$$

In the proofs that follow, K_j (where j is an integer) will denote a positive constant that depends only on λ and p (and in Theorem 2 on γ), and L_j (where j is an integer) will denote a positive absolute constant.

Let $\varepsilon > 0$; we shall choose ε later on. Let $\mathscr{T}_m$ denote the space of trigonometric polynomials of degree at most m, and let Ω_C denote those functions f for which $\omega_f(h) \leqslant Ch^{1/p} \log^{1/2}(1/h)$. *Let* $U_{m,C}$ denote the set of ω for which

(i) $\sum_k (X_p(\omega))^\wedge_k e^{2\pi ikt}$ converges uniformly to $X_p(\omega)$,

(ii) $X_p(\omega) \in \Omega_C$,

(iii) there exists a measurable E with $\mu(E) > \lambda$ and T in $\mathscr{T}_m$ such that $\|X_p(\omega) - T\|_{A_p(E)} < \varepsilon$.

Then there exist m and C such that

$$P^*(U_{m,C}) > \eta.$$

Suppose that $\omega \in U_{m,C}$. Let E be a measurable subset of $\boldsymbol{T}$ with $\mu(E) > \lambda$ and T an element of $\mathscr{T}_m$ such that

$$\|X_p(\omega) - T\|_{A_p(E)} < 2\varepsilon/\lambda.$$

Let f be a function such that $f|_E = 0$ and

$$\|X_p(\omega) - T - f\|_{A_p} < 2\varepsilon/\lambda.$$

We consider translates of the group F_n. If $t \in \boldsymbol{T}$, let

$$n(t) = |\{s: s \in F_n, s+t \in E\}|.$$

Then

$$\int_{\boldsymbol{T}} n(t)\, d\mu(t) = \sum_{s \in F_n} \int_{\boldsymbol{T}} \chi_E(s+t)\, d\mu(t) = n\mu(E) > n\lambda,$$

Let $G_n = \{t \in \boldsymbol{T}: n(t) \geqslant n\lambda/2\}$. Then

$$n\mu(G_n) + n\lambda/2 \geqslant \int_{\boldsymbol{T}} n(t)\, d\mu(t) > n\lambda,$$

so that $\mu(G_n) > \lambda/2$.

Since

$$\Big(\int_{\boldsymbol{T}} \|Q_t(X_p(\omega) - T - f)\|_{l_p}^p\, dt\Big)^{1/p} \leqslant \|X_p(\omega) - T - f\|_{A_p} < 2\varepsilon/\lambda,$$

it follows that there exists t in G_n such that

$$\|Q_t(X_p(\omega) - T - f)\|_{l_p} < \varepsilon.$$

There exists u in F_{n^2} such that $|t-u| < 1/n^2$. Thus if n is sufficiently large,

$$\|Q_u(X_p(\omega)) - Q_t(T) - Q_t(f)\|_{l_p} < 2\varepsilon.$$

Now $Q_t(T) \in U_m = \operatorname{span}\{e_1, \ldots, e_m, e_{n-m}, \ldots, e_n\}$, while

$$Q_t(f) \in V_J,$$

where $J = \{s \in F_n: s+t \in E\}$ and

$$V_J = \operatorname{span}\{\sum_{j=1}^{n} e^{-2\pi ijk/n} e_j: k \notin J\}.$$

Consequently

$$U_{m,C} \subseteq \bigcup_{u \in F_{n^2}} \bigcup_{\substack{J \subseteq \{1,\ldots,n\} \\ |J| \geqslant n\lambda}} A_{u,J},$$

where $A_{u,J} = \{\omega: Q_u(X_p(\omega)) \in U_m + V_J + 2\varepsilon B_p\}$ (and where B_p is the unit ball in $(\boldsymbol{C}^n, \|\ \|_{l_p})$). Note that there are not more than $n^2 2^n$ events $A_{u,J}$.

We now estimate the probability of $A_{u,J}$. We do this by combining density estimates and volume estimates.

Before doing this, we discard some of the coordinates. Provided that n is large enough, we can choose an integer s so that $s \geqslant n\lambda/6$ and $t = n-2s \geqslant n - n\lambda/3$. Let Π be the orthogonal projection of $\boldsymbol{C}^n$ onto $\operatorname{span}(e_{s+1}, \ldots, e_{n-s})$. Then

$$P(A_{u,J}) \leqslant P\{\omega: \Pi Q_u(X_p(\omega)) \in W + 2\varepsilon\Pi(B_p)\},$$

where $W=\Pi(U_m+V_j)$. Let $\dim W=r$; note that for large enough n, $r\leqslant n-n\lambda/2\leqslant t-n\lambda/6$.

We begin with the density estimates. Let $S_j(\omega)=\big(Q_u(X_p(\omega))\big)_j$. For almost all ω,

$$S_j(\omega)=\sum_v{}' \frac{D_{j+vn}}{(j+vn)^{1/p}}e^{2\pi i(j+vn)t}.$$

The random variables S_j are independent, and

$$\left(\frac{1}{K_1 n^{1/p}}\right)^2\leqslant E(|S_j|^2)\ (=v_j^2\ \text{say})\ \leqslant\left(\frac{K_1}{n^{1/p}}\right)^2$$

for $s+1\leqslant j\leqslant n-s$. Let us set $T_j=S_j/v_j$, so that $E(|T_j|^2)=1$.

The density of S_j is clearly less than or equal to $n^{2/p}$, and so the density of T_j is less than or equal to K_1^2, for $s+1\leqslant j\leqslant n-s$. Further, an easy calculation shows that each of the random variables $\mathrm{Re}(D_k)$ is subnormal, and so each random variable $\mathrm{Re}(T_j)$ is subnormal; since the distribution of T_j is rotationally invariant, this means that

$$P(|T_j|>\alpha)\leqslant 4e^{-\alpha^2/4}$$

(cf. [4], pp. 54–57). We can write the density f_j of T_j as

$$f_j(x,y)=g_j\big((x^2+y^2)^{1/2}\big).$$

It is easy to see that g_j is a decreasing function; this means that

$$g_j(\alpha)\leqslant 4\alpha^{-2}e^{-\alpha^2/16},$$

and so

$$g_j(\alpha)\leqslant K_2 e^{-\alpha^2/16}.$$

Now suppose that $h\geqslant 1$ and that

$$|S_{s+1}|^p+\ldots+|S_{n-s}|^p\geqslant h^p K_1^p;$$

by Hölder's inequality,

$$|S_{s+1}|^2+\ldots+|S_{n-s}|^2\geqslant h^2K_1^2t^{1-2/p}$$

and so

$$|T_{s+1}|^2+\ldots+|T_{n-s}|^2\geqslant h^2 t.$$

Thus the joint density of $(T_{s+1},\ldots,T_{n-s})$ is less than or equal to $K_2e^{-h^2t/16}$ and the joint density of $(S_{s+1},\ldots,S_{n-s})$ is less than or equal to

$$(K_3n^{1/p})^{2t}e^{-h^2t/16}.$$

Note that the joint density of $(S_{s+1},\ldots,S_{n-s})$ is always less than or equal to $(K_1n^{1/p})^{2t}$.

We now turn to the volume estimates. Let C_p denote the unit ball in W. If h is a positive integer, then a standard argument [2] shows that $(hK_1+2\varepsilon)C_p$ can be covered by not more than $(7hK_1/\varepsilon)^{2r}$ balls of radius ε, and so $(hK_1+2\varepsilon)C_p+2\varepsilon\Pi(B_p)$ can be covered by not more than $(7hK_1/\varepsilon)^{2r}$ balls of radius 3ε; the volume of each such ball is not greater than $(L_1\varepsilon/t^{1/p})^{2t}$.

Let us set

$$E_h=\{\omega\colon \Pi Q_t(X_p(\omega))\in(W+2\varepsilon\Pi(B_p))\cap hK_1\Pi(B_p)\}.$$

As

$$(W+2\varepsilon\Pi(B_p))\cap hK_1\Pi(B_p)\subseteq(hK_1+2\varepsilon)C_p+2\varepsilon\Pi(B_p),$$

it follows by combining the volume and density estimates that

$$P(E_1)\leqslant\left(\frac{7K_1}{\varepsilon}\right)^{2r}\left(\frac{L_1\varepsilon}{t^{1/p}}\right)^{2t}(K_1n^{1/p})^{2t}\leqslant K_3^t\varepsilon^{2t-2r},$$

and

$$P(E_{h+1}\setminus E_h)\leqslant\left(\frac{7(h+1)K_1}{\varepsilon}\right)^{2r}\left(\frac{L_1\varepsilon}{t^{1/p}}\right)^{2t}e^{-h^2t/16}\leqslant K_4^t(h+1)^{2t}e^{-h^2t/16}\varepsilon^{2t-2r}.$$

Now it is easily verified that

$$\sum_{h=1}^{\infty}(h+1)^{2u}e^{-h^2u/16}\leqslant L_2^u\qquad\text{for } u\geqslant 1$$

and so

$$P(A_{u,J})\leqslant K_5^t\varepsilon^{2t-2r}\leqslant(K_6\varepsilon^{\lambda/3})^n.$$

This means that $\eta\leqslant n^2(2K_6\varepsilon^{\lambda/3})^n$; but we can choose ε small enough and n large enough so that this is not so.

5. Proof of Theorem 2. Theorem 2 is proved by making fairly-straightforward modifications to the proof of Theorem 1, which we now describe. Let us set $\varrho=1/p-1/2$.

We let Ω_C denote those functions f for which $\omega_f(h)\leqslant\log^{1-\gamma}(1/h)$. Then

$$U_{m,C}\subseteq\bigcup_{u\in E_{2n}}\ \bigcup_{\substack{J\subseteq\{1,\ldots,n\}\\|J|\geqslant n\lambda}}A_{u,J}.$$

In this case there are not more than 4^n events $A_{u,J}$.

Easy calculations show that if $s+1\leqslant j\leqslant n-s$ then

$$\frac{1}{K_7^2n(\log n)^{2\gamma-1}}\leqslant v_j^2\leqslant\frac{K_7^2}{n(\log n)^{2\gamma-1}}.$$

For such j, the density of S_j is less than or equal to $n(\log n)^{2\gamma}$ and the density of T_j is less than or equal to $K_7 \log n$. Arguing as in Theorem 1,

$$g_j(\alpha) \leqslant K_8 (\log n) e^{-\alpha^2/16}.$$

If $|S_{s+1}|^p + \ldots + |S_{n-s}|^p \geqslant h^p K_7^\varrho$ then

$$|S_{s+1}|^2 + \ldots + |S_{n-s}|^2 \geqslant h^2 K_7^2 t^{-2\varrho}$$

and so

$$|T_{s+1}|^2 + \ldots + |T_{n-s}|^2 \geqslant h^2 n (\log n)^{2\gamma-1} t^{-2\varrho}.$$

Thus the joint density of $(T_{s+1}, \ldots, T_{n-s})$ is less than or equal to

$$(K_9 \log n)^t \exp(-h^2 n (\log n)^{2\gamma-1} t^{-2\varrho}/16)$$

and the joint density of $(S_{s+1}, \ldots, S_{n-s})$ is less than or equal to

$$(K_9 n (\log n)^{2\gamma})^t \exp(-h^2 n (\log n)^{2\gamma-1} t^{-2\varrho}/16).$$

Consequently

$$P(E_{n\varrho}) \leqslant \left(\frac{7K_1 n}{\varepsilon}\right)^{2r} \left(\frac{L_1 \varepsilon}{t^{1/p}}\right)^{2t} (n(\log n)^{2\gamma})^t$$

$$\leqslant K_{10}^t \left(\frac{\varepsilon}{n^\varrho}\right)^{2t-2r} (\log n)^{2\gamma t} \leqslant K_{10}^t \varepsilon^{2t-2r}$$

for large enough n, and

$$P(E_{(h+1)n\varrho} \backslash E_{hn\varrho}) \leqslant \left(\frac{7K_1 (h+1) n}{\varepsilon}\right)^{2r} \left(\frac{L_1 \varepsilon}{t^{1/p}}\right)^{2t} (K_9 n (\log n)^{2\gamma}) \times$$

$$\times \exp(-h^2 n^{2\varrho} n (\log n)^{2\gamma-1} t^{-2\varrho}/16)$$

$$\leqslant K_{11} (h+1)^{2r} \left(\frac{\varepsilon}{n^\varrho}\right)^{2t-2r} (\log n)^{2\gamma t} e^{-h^2 t/16}$$

$$\leqslant K_{11} (h+1)^{2r} \varepsilon^{2t-2r} e^{-h^2 t/16}$$

for sufficiently large n (independent of h). Thus

$$P(A_{u,J}) \leqslant K_{12}^t \varepsilon^{2t-2r} \leqslant (K_{13} \varepsilon^{\lambda/3})^n,$$

so that $\eta \leqslant (4K_{13} \varepsilon^{\lambda/3})^n$. Once again, a suitable choice of ε and n shows that this is not so.

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